Ky Fan's Section Theorem and its Applications in Topological Ordered Spaces

Q. Luo
Department of Mathematics, Zhaoqing University
Zhaoqing, Guangdong, 526061, P.R. China

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Abstract—In this paper, we obtain Ky Fan's section theorem, Ky Fan's lemma and discuss its applications in topological ordered spaces. © 2004 Elsevier Ltd. All rights reserved.

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1. INTRODUCTION AND PRELIMINARIES

Ky Fan's section theorem has numerous applications in various fields of pure and applied mathematics. In this paper, by Theorem 2 of [1], we obtain Ky Fan's section theorem, Ky Fan's lemma, saddle-point theorem, generalized Fan-Browder fixed-point theorem, Ky Fan's inequality, etc., in topological ordered spaces.

A semilattice is a partially ordered set $X$, with the partial ordering denoted by $\leq$, for which any pair $(x, x')$ of elements has a least upper bound, denoted by $x \lor x'$. It is easy to see that any nonempty finite subset $A$ of $X$ has a least upper bound, denoted by $\sup A$. In a partially ordered set $(X, \leq)$, two arbitrary elements $x$ and $x'$ do not have to be comparable. In the case $x \leq x'$, the set $[x, x'] = \{ y \in X : x \leq y \leq x' \}$ is called an order interval. Now assume that $(X, \leq)$ is a semilattice and $A \subseteq X$ is a nonempty finite subset, then the set $\Delta(A) = \bigcup_{a \in A} [a, \sup A]$ is well defined and it has the following properties.

(a) $A \subseteq \Delta(A)$.
(b) If $A \subseteq A'$, then $\Delta(A) \subseteq \Delta(A')$.

We shall say that a subset $E \subseteq X$ is $\Delta$-convex if, for any nonempty finite subset $A \subseteq E$, we have $\Delta(A) \subseteq E$.

For any $D \subset X$, $\mathcal{F}(D)$ denotes the family of all finite subsets of $D$, $\Delta(D) = \bigcup_{A \in \mathcal{F}(D)} \Delta(A)$. The following theorem is due to [1].

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THEOREM 1.1. Let $X$ be a topological semilattice with path-connected intervals, $X_0 \subseteq X$ a nonempty subset of $X$, and $R \subseteq X_0 \times X$ a binary relation satisfy the following.

(i) For each $x \in X_0$, the set $R(x) = \{y \in X : (x, y) \in R\}$ is not empty and closed in $R(X_0) = \bigcup_{x \in X_0} R(x)$.

(ii) There exists $x_0 \in X_0$ such that the set $R(x_0)$ is compact.

(iii) For any nonempty finite subset $A \subseteq X_0$:

$$\bigcup_{x \in A} [x, \sup A] \subseteq \bigcup_{x \in A} R(x).$$

Then, the set $\bigcap_{x \in X_0} R(x)$ is not empty.

Let $X$ be a nonempty set and $Y$ a topological space, $2^Y$ denotes the family of all subsets of $Y$. A mapping $G : X \rightarrow 2^Y$ is said to be transfer closed valued (e.g., see [2]) if, for each $x \in X$ and $y \notin G(x)$, there exist $x' \in X$ and an open neighborhood $N(y)$ of $y$ in $Y$ such that $y' \notin N(y)$. It is obvious that if a mapping $G : X \rightarrow 2^Y$ is transfer closed valued, then for each $x \in X$ and $y \notin G(x)$, there exists some $x' \in X$ such that $y \notin \text{cl} G(x')$, where $\text{cl} G(x)$ is the closure of $G(x)$. Then, $G : X \rightarrow 2^Y$ is transfer closed if and only if $\bigcap_{x \in X} G(x) = \bigcap_{x \in X} \text{cl} G(x)$.

Let $X, Y$ be two topological spaces, $T : X \rightarrow 2^Y$ is said to have the local intersection property (see [3]) if for each $x \in X$ with $T(x) \neq \emptyset$, there exists an open neighborhood $N(x)$ of $x$ such that $\bigcap_{x \in N(x)} T(x) \neq \emptyset$. It is not hard to derive that $T : X \rightarrow 2^Y$ has the local intersection property if and only if $X \cap T^{-1} = X \cap \text{cl} T^{-1}$ is transfer closed valued, i.e., $\bigcup_{y \in Y} T^{-1}y = \bigcup_{y \in Y} \text{int}(T^{-1}y)$.

By Theorem 1.1 and the definition of transfer closed, it is easy to obtain the following generalized KKM theorem.

THEOREM 1.2. Let $X$ be a topological semilattice with path-connected intervals, $X_0 \subseteq X$ a nonempty subset of $X$, and $R \subseteq X_0 \times X$ a binary relation satisfy the following.

(1) $G : X_0 \rightarrow 2^X$ is transfer closed valued, where $G(x) = \{y \in X : (x, y) \in R\}$ for each $x \in X_0$.

(2) There exists $x_0 \in X_0$ such that the set $\text{cl} G(x_0)$ is compact.

(3) For any nonempty finite subset $A \subseteq X_0$, $\bigcup_{x \in A} [x, \sup A] \subseteq \bigcup_{x \in A} G(x)$.

Then the set $\bigcap_{x \in X_0} G(x)$ is nonempty.

PROOF. Since $\text{cl} G(x)$ satisfies all the conditions of Theorem 1.1, then $\bigcap_{x \in X_0} \text{cl} G(x)$ is nonempty. Since $\bigcap_{x \in X_0} G(x) = \bigcap_{x \in X_0} \text{cl} G(x)$, hence, $\bigcap_{x \in X_0} G(x)$ is not empty. The proof is complete.

Let $X$ be a topological semilattice or a $\Delta$-convex subset of a topological semilattice, $f : X \rightarrow (-\infty, +\infty)$ is $\Delta$-quasi-concave if, for any nonempty finite subset $A = \{x_1, x_2, \ldots, x_n\} \subseteq X$, for any $y \in \Delta(A)$, $f(y) \geq \min\{f(x_1), f(x_2), \ldots, f(x_n)\}$. It is easy to see that if $f : X \rightarrow (-\infty, +\infty)$ is $\Delta$-quasi-concave if and only if the set $\{y \in X : f(y) = \lambda\}$ or the set $\{y \in X : f(y) > \lambda\}$ is $\Delta$-convex for any $\lambda \in (-\infty, +\infty)$.

DEFINITION 1.1. Let $X, Y$ be two topological spaces, $\varphi(x, y) : X \times Y \rightarrow (-\infty, +\infty)$ is said to be strongly path transfer lower semicontinuous relative to $x$ (in short, SPT l.s.c.) if, for each $(x, y) \in X \times Y$ and for all $\varepsilon > 0$, there exists an open neighborhood $N(x)$ of $x$ in $X$ and there exists $y^0 \in Y$ such that for any $x' \in N(x)$,

$$\varphi(x, y) < \varphi(x', y^0) + \varepsilon.$$

If $-\varphi(x, y)$ is SPT l.s.c. relative to $x$, then $\varphi(x, y)$ is said to be strongly path transfer upper semicontinuous relative to $x$ (in short, SPT u.s.c.).

It is easy to see that if for any $y \in Y$, $\varphi(\cdot, y)$ is l.s.c., then $\varphi(x, y)$ is SPT l.s.c. relative to $x$. The converse is not true.
EXAMPLE 1.1. Let $X = [0, 1]$, $Y = [0, 1]$, a function $\varphi(x, y)$ defined on $X \times Y$ by

$$
\varphi(x, y) = \begin{cases} 
1, & \text{if } x = y, \\
2, & \text{if } y = 0, x \neq 0, \\
0, & \text{otherwise}.
\end{cases}
$$

It is easy to see that $\varphi(x, y)$ is not l.s.c. on $X$ for each $y \in Y$, $\varphi(x, y)$ is SPT l.s.c. relative to $x$.

DEFINITION 1.2. Let $X$ be a nonempty $\Delta$-convex subset of a topological semilattice with path-connected intervals $M$. A set-valued mapping $G : X \to 2^M$ is said to be an order KKM mapping (OKKM) if, for each nonempty finite subset $\{x_1, x_2, \ldots, x_n\} \subset X$, we have

$$
\Delta(\{x_1, x_2, \ldots, x_n\}) \subset \bigcup_{i=1}^{n} G(x_i).
$$

2. KY FAN’S SECTION THEOREM AND ITS APPLICATIONS

THEOREM 2.1. KY FAN’S SECTION THEOREM. Let $X$ be a nonempty $\Delta$-convex subset of a topological semilattice with path-connected intervals $M$ and $C \subset X \times X$.

(1) For any $y \in X$, the set $\{x \in X : (x, y) \notin C\}$ is either $\Delta$-convex or empty.

(2) The mapping $x \to \{y \in X : (x, y) \in C\}$ is transfer closed valued.

(3) For any $x \in X$, $(x, x) \notin C$.

(4) There exists $x_0 \in X$ such that the set $\text{cl} \{y \in X : (x_0, y) \in C\}$ is compact.

Then there exists $x^* \in X$ such that $x \times \{x^*\} \subset C$.

PROOF. Define $G : X \to 2^X$ by

$$
G(x) = \{y \in X : (x, y) \in C\}, \quad \text{for each } x \in X.
$$

Then, $\text{cl} G(x_0)$ is compact.

Suppose that there exists a finite subset $A_0 = \{x_1, x_2, \ldots, x_n\} \subset X$ such that

$$
\Delta(A_0) \notin \bigcup_{i=1}^{n} G(x_i),
$$

then there exists $z \in \Delta(A_0)$ and $z \notin \bigcup_{i=1}^{n} G(x_i)$. Hence, for each $i = 1, 2, \ldots, n$, $z \notin G(x_i)$, $(x_i, z) \notin C$, and hence, $A_0 \subset \{x \in X : (x, z) \notin C\}$, by (1), $\Delta(A_0) \subset \{x \in X : (x, z) \notin C\}$, then $z \in \Delta(A_0) \subset \{x \in X : (x, z) \notin C\}$, $(z, z) \notin C$, which contradicts (3). Therefore, for any nonempty finite subset $A \subset X$,

$$
\Delta(A) \subseteq \bigcup_{x \in A} G(x).
$$

By Theorem 1.2, $\bigcap_{x \in X} G(x) \neq \emptyset$. Take $y^* \in \bigcap_{x \in X} G(x) \subset X$, then $X \times \{y^*\} \subset C$.

REMARK 2.1. Obviously, if $X$ is compact, then (4) holds.

THEOREM 2.2. Let $X$ be a nonempty compact $\Delta$-convex subset of a topological semilattice with path-connected intervals $M$, $f : X \times X \to (-\infty, +\infty)$, $T : X \to 2^X$ nonempty compact valued with $x \in T(x)$ for any $x \in X$. And

(i) for each $y \in Y$, $f(\cdot, y) : X \to (-\infty, +\infty)$ is $\Delta$-quasi-convex;

(ii) $f(x, y)$ SPT u.s.c. relative to $y$. 
Then there exists \( y^* \in X \) such that

\[
\inf_{x \in X, y \in T(x)} f(x, y) \leq \inf_{x \in X} f(x, y^*) \leq \sup_{y \in F} \inf_{x \in X} f(x, y).
\]

**Proof.** Let \( r = \inf_{x \in X, y \in T(x)} f(x, y) \), \( A = \{(x, y) \in X \times X : f(x, y) \geq r\} \).

Since \( x \in T(x) \) for any \( x \in X \), we have \( f(x, x) \geq r \), \( (x, x) \in A \).

For any \( x \in X \), let \( G(x) = \{y \in X : (x, y) \in A\} = \{y \in X : f(x, y) \geq r\} \). For \( y \notin G(x) \), \( f(x, y) < r \), by (ii), take \( \varepsilon = r - f(x, y) \), there exist an open neighborhood \( N(y) \) of \( y \), and \( x_0 \in X \) such that

\[
f(x, y) < -f(x_0, y') + \varepsilon = -f(x_0, y') + r - f(x, y)
\]

for any \( y' \in N(y) \), hence, \( f(x_0, y') < r \), \( y' \notin G(x_0) \), \( G \) is transfer closed valued.

For any \( y \in Y \), the set \( \{x \in X : (x, y) \notin A\} = \{x \in X : f(x, y) < r\} \) is \( \Delta \)-convex. By Theorem 2.1, there exists \( y^* \in X \) such that \( X \times \{y^*\} \subset A \), i.e., \( f(x, y^*) \geq r = \inf_{x \in X, y \in T(x)} f(x, y) \) for any \( x \in X \), hence,

\[
\inf_{x \in X, y \in T(x)} f(x, y) \leq \inf_{x \in X} f(x, y^*) \leq \sup_{y \in F} \inf_{x \in X} f(x, y).
\]

**Theorem 2.3.** KY Fan’s Lemma. Let \( X \) be a nonempty compact \( \Delta \)-convex subset of a topological semilattice with path-connected intervals \( M \), and \( B \subset X \times X \).

(i) For any \( y \in X \), the set \( \{x \in X : (x, y) \in B\} \) is nonempty \( \Delta \)-convex.

(ii) The mapping \( y \mapsto \{x \in X : (x, y) \in B\} \) has the local intersection property. Then there exists \( x^* \in X \) such that \( (x^*, x^*) \in B \).

**Proof.** Let \( C = X \times X \setminus B \), \( F(x) = \{y \in X : (x, y) \in B\} \), for each \( x \in X \), then

\[
\{y \in X : (x, y) \in C\} = \{y \in X : (x, y) \notin B\} = X \setminus \{y \in X : (x, y) \in B\} = X \setminus F(x).
\]

By (ii), \( F^{-1}(y) = \{x \in X : (x, y) \in B\} \) has the local intersection property, then \( X \setminus F(x) \) is transfer closed valued.

For each \( y \in X \), \( \{x \in X : (x, y) \notin C\} = \{x \in X : (x, y) \in B\} \) is nonempty \( \Delta \)-convex.

Suppose \( (x, x) \notin B \) for each \( x \in X \). Then \( (x, x) \in C \), by Theorem 2.1, there exists \( y^* \in X \) such that \( X \times \{y^*\} \subset C \), i.e., for each \( x \in X \), \( (x, y^*) \in C \), \( (x, y^*) \notin B \), the set \( \{x \in X : (x, y^*) \in B\} \) is empty, which contradicts (i). Therefore, there exists \( x^* \in X \) such that \( (x^*, x^*) \in B \). The proof is complete.

**Theorem 2.4.** Generalized Fan-Browder Fixed-Point Theorem. Let \( X \) be a nonempty compact \( \Delta \)-convex subset of a topological semilattice with path-connected intervals \( M \), \( F : X \to 2^X \) has the local intersection property with nonempty \( \Delta \)-convex valued. Then, \( F \) has a fixed point, i.e., there exists \( x^* \in X \) such that \( x^* \in F(x^*) \).

**Proof.** Let \( B = \{(x, y) \in X \times X : x \in F(y)\} \). For each \( y \in X \),

\[
\{x \in X : (x, y) \in B\} = \{x \in X : x \in F(y)\} = F(y)
\]

nonempty \( \Delta \)-convex. The mapping \( y \mapsto \{x \in X : (x, y) \in B\} = F(y) \) has the local intersection property. By Theorem 2.3, there exists \( x^* \in X \) such that \( (x^*, x^*) \in B \), i.e., \( x^* \in F(x^*) \).

**Corollary 2.1.** Let \( X \) be a nonempty compact \( \Delta \)-convex subset of a topological semilattice with path-connected intervals \( M \), \( F : X \to 2^X \) with nonempty \( \Delta \)-convex valued, and for each \( y \in X \), \( F^{-1}(y) \) is an open set. Then \( F \) has a fixed point.

**Proof.** We only need to prove that \( F \) has the local intersection property. For each \( x \in X \) with \( F(x) \neq \emptyset \), take \( y \in F(x) \), then \( x \in F^{-1}(y) \). Since \( F^{-1}(y) \) is open, there exists an open neighborhood \( N(x) \) of \( x \) in \( X \) such that \( N(x) \subset F^{-1}(y) \). Then, for any \( z \in N(x) \), \( z \in F^{-1}(y) \), \( y \in F(z) \), hence, \( y \in \bigcap_{x \in N(z)} F(z) \), and hence, \( \bigcap_{x \in N(z)} F(z) \neq \emptyset \), \( F \) has the local intersection property. By Theorem 2.4, \( F \) has a fixed point.
3. SADDLE-POINT THEOREM

THEOREM 3.1. SADDLE-POINT Theorem. Let $X$ and $Y$ be two nonempty compact $\Delta$-convex subsets of two topological semilattice with path-connected intervals $M$ and $E$, $f : X \times Y \to (-\infty, +\infty)$.

(i) For each $x \in X$, $f(x, \cdot) : Y \to (-\infty, +\infty)$ is $\Delta$-quasi-convex.
(ii) For each $y \in Y$, $f(\cdot, y) : X \to (-\infty, +\infty)$ is $\Delta$-quasi-concave.
(iii) $f(x, y)$ SPT l.s.c. relative to $y$; $f(x, y)$ SPT u.s.c. relative to $x$.

Then,
$$\inf_{y \in Y} \sup_{x \in X} f(x, y) = \sup_{x \in X} \inf_{y \in Y} f(x, y).$$

PROOF. It is easy to see
$$\sup_{x \in X} \inf_{y \in Y} f(x, y) \leq \inf_{y \in Y} \sup_{x \in X} f(x, y).$$

Suppose that
$$\sup_{x \in X} \inf_{y \in Y} f(x, y) < \inf_{y \in Y} \sup_{x \in X} f(x, y).$$

Then there exists $r \in (-\infty, +\infty)$ such that
$$\sup_{x \in X} \inf_{y \in Y} f(x, y) < r < \inf_{y \in Y} \sup_{x \in X} f(x, y).$$

And by (i),(ii), for any $x \in X$, $G(x) = \{y \in Y : f(x, y) < r\}$ nonempty $\Delta$-convex; for any $y \in Y$, $K(y) = \{x \in X : f(x, y) > r\} = \{x \in X : -f(x, y) < -r\}$ nonempty $\Delta$-convex.

For $x \in X$ with $G(x) \neq \emptyset$, take $y_0 \in G(x)$, i.e., $f(x, y_0) < r$, by (iii), for $\varepsilon = r - f(x, y_0) > 0$, there exist an open neighborhood $N(x)$ of $x$, and $y_1 \in Y$ such that
$$-f(x, y_0) < -f(x', y_1) + \varepsilon = -f(x', y_1) - f(x, y_0) + r, \quad \text{for any } x' \in N(x),$$

i.e., $f(x', y_1) < r$, hence, $y_1 \in G(x')$, for any $x' \in N(x)$, and hence, $\bigcap_{x' \in N(x)} G(x') \neq \emptyset$, $G$ has the local intersection property. Similarly, one can prove that $K$ has the local intersection property.

Let $C = X \times Y$, $F : C \to 2^C$, for $u = (x, y) \in C$, $F(u) = K(y) \times G(x)$, By Theorem 2.4, $F$ has a fixed point. i.e., there exists $u^* = (x^*, y^*) \in C$ such that
$$u^* = (x^*, y^*) \in K(y^*) \times G(x^*),$$

then $f(x^*, y^*) > r$ and $f(x^*, y^*) < r$, which is a contradiction. Hence,
$$\inf_{y \in Y} \sup_{x \in X} f(x, y) = \sup_{x \in X} \inf_{y \in Y} f(x, y).$$

COROLLARY 3.1. Let $X$ and $Y$ be two nonempty compact $\Delta$-convex subsets of two topological semilattice with path-connected intervals $M$ and $E$, $f : X \times Y \to (-\infty, +\infty)$.

(i) For each $x \in X$, $f(x, \cdot) : Y \to (-\infty, +\infty)$ is $\Delta$-quasi-convex and l.s.c.
(ii) For each $y \in Y$, $f(\cdot, y) : X \to (-\infty, +\infty)$ is $\Delta$-quasi-concave and u.s.c.

Then there exists $(x^*, y^*) \in X \times Y$ such that
$$f(x, y^*) \leq f(x^*, y^*) \leq f(x^*, y), \quad \text{for any } (x, y) \in X \times Y.$$
4. GENERALIZED KY FAN INEQUALITY

Let \( X \) be a \( \Delta \)-convex subset of a topological semilattice \( M \) and let \( \gamma \in (-\infty, +\infty) \). Suppose \( f(x, y) : X \times X \rightarrow (-\infty, +\infty) \), then \( f(x, y) \) is said to be ordered \( \gamma \)-diagonal quasi-convex (respectively, concave) in \( y \) if, for any nonempty finite subset \( A = \{y_1, y_2, \ldots, y_n\} \subset X \) and \( x_0 \in \Delta(A) \), we have that \( \gamma \leq \max_{y_i \in A} f(x_0, y_i) \) (respectively, \( \gamma \geq \min_{y_i \in A} f(x_0, y_i) \)).

**Lemma 4.1.** Let \( X \) be a \( \Delta \)-convex subset of a topological semilattice \( M \) and let \( \gamma \in (-\infty, +\infty) \). Suppose \( f(x, y) : X \times X \rightarrow (-\infty, +\infty) \), then \( f(x, y) \) is ordered \( \gamma \)-diagonal quasi-concave in \( y \) if and only if the mapping \( G(y) = \{x \in X : f(x, y) \leq \gamma\} \) is OKKM.

**Proof.** If \( f(x, y) \) is ordered \( \gamma \)-diagonal quasi-concave in \( y \), for any \( A = \{x_1, x_2, \ldots, x_n\} \subset X \) and \( x_0 \in \Delta(A) \), we have

\[
\min_{x_i \in A} f(x_0, x_i) \geq \gamma,
\]

then there exists \( j \) such that

\[
f(x_0, x_j) = \min_{x_i \in A} f(x_0, x_i) \leq \gamma,
\]

i.e., \( x_0 \in G(x_j) \), hence, \( \Delta(A) \subset \bigcup_{i=1}^{n} G(x_i) \), \( G \) is OKKM.

Conversely, if \( G \) is OKKM, for any \( A = \{x_1, x_2, \ldots, x_n\} \subset X \) and \( x_0 \in \Delta(A) \), then for any \( x_0 \in \Delta(A) \), \( x_0 \in \bigcup_{i=1}^{n} G(x_i) \), then there exists \( j \) such that \( x_0 \in G(x_j) \), hence,

\[
\min_{x_i \in A} f(x_0, x_i) \leq f(x_0, x_j) \leq \gamma,
\]

i.e., \( f(x, y) \) is ordered \( \gamma \)-diagonal quasi-concave in \( y \).

**Theorem 4.1.** Generalized Ky Fan Inequality. Let \( X \) be a nonempty compact \( \Delta \)-convex subset of a topological semilattice \( M \), \( \gamma \in (-\infty, +\infty) \). Suppose \( f(x, y) : X \times X \rightarrow (-\infty, +\infty) \).

(i) \( f(x, y) \) is ordered \( \gamma \)-diagonal quasi-concave in \( y \).

(ii) \( f(x, y) \) SPT l.s.c. relative to \( x \).

Then there exists \( x^* \in X \) such that \( f(x^*, y) \leq \gamma \) for any \( y \in X \).

**Proof.** For any \( y \in X \), let \( G(y) = \{x \in X : f(x, y) \leq \gamma\} \), by Lemma 4.1, \( G \) is OKKM. By (ii), it is easy to prove that \( G \) is transfer closed valued. hence, \( \bigcap_{x \in X} G(x) \neq \emptyset \), take \( x^* \in \bigcap_{y \in X} G(y) \), then \( f(x^*, y) \leq \gamma \), for any \( y \in X \).

**Corollary 4.1.** Let \( X \) be a nonempty compact \( \Delta \)-convex subset of a topological semilattice with path-connected intervals \( M \). \( \gamma \in (-\infty, +\infty) \). \( f : X \times X \rightarrow (-\infty, +\infty) \) satisfies

1. for any \( x \in X \), \( f(x, x) \leq \gamma \);
2. for each fixed \( x \in X \), \( y \rightarrow f(x, y) \) is \( \Delta \)-quasi-concave;
3. \( f(x, y) \) is SPT l.s.c. relative to \( x \).

Then there exists \( x_0 \in X \) such that \( \sup_{y \in X} f(x_0, y) \leq \gamma \).

**Proof.** We only need to prove that the mapping \( W : X \rightarrow 2^X \) is an OKKM mapping, where

\[
W(y) := \{x \in X : f(x, y) \leq \gamma\}, \quad \text{for each } y \in X.
\]

Suppose that \( W \) is not an OKKM mapping, then there exists \( A = \{y_1, y_2, \ldots, y_n\} \subset X \) such that

\[
\Delta(A) \not\subset \bigcup_{i=1}^{n} W(y_i),
\]

i.e., there exists \( x_0 \in \Delta(A) \), \( x_0 \not\in \bigcup_{i=1}^{n} W(y_i) \), hence, \( x_0 \not\in W(y_i) \), for each \( i = 1, 2, \ldots, n \), and hence, \( f(x_0, y_i) > \gamma \), for each \( i = 1, 2, \ldots, n \). By (2), we have

\[
f(x_0, x_0) \geq \min_{y_i \in A} f(x_0, y_i) > \gamma,
\]

which contradicts (1), so \( W \) is OKKM.
REFERENCES

