On the Meromorphic Solutions of Some Algebraic Differential Equations

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1. INTRODUCTION

Consider a differential equation of the form

\[ u' = \frac{\sum_{k=0}^{n} A_k(z) u^k}{\sum_{l=0}^{m} B_l(z) u^l} \]  

(1)

where each \( A_k \) and \( B_k \) are meromorphic functions. In this paper the term "meromorphic" will always mean meromorphic in the whole complex plane.

Malmquist [15] long ago proved that if a differential equation of the form (1) with rational coefficients \( A_k, B_k \) admits a transcendental meromorphic solution, then the differential equation is actually a Riccati. Many authors later generalized this classical result, including Yosida [22], Laine [13, 14], Hille [12], Strelitz [18,19], Gackstatter and Laine [5], and Steinmetz [17]. These studies contain many results with the common theme: If a differential equation of the form (1) (or if a differential equation is in a more general class of differential equations) admits a meromorphic solution which, in some sense, is of larger growth than the coefficients of the differential equation, then the differential equation must actually be a Riccati (or the differential equation must belong to a certain smaller subclass of the original class).

Despite these studies, many questions about algebraic differential equations (defined here to be equations of the form \( Q(z, f, f', ..., f^{(n)}) = 0 \)
where $n$ is a positive integer and $Q$ is a polynomial in $f, f', ..., f^{(n)}$ with meromorphic coefficients) are open. For instance, there does not seem to be an example in the literature of a non-Riccati differential equation of the form (1) that admits a one-parameter family of meromorphic solutions. In this paper we will show that certain equations of the form

$$u' = \sum_{k=0}^{n} A_k(z) u^k$$

where each $A_k$ is meromorphic can possess at most a finite number of distinct meromorphic (entire) solutions. The arguments used in the above-mentioned studies do not work with this kind of question because a meromorphic solution of a non-Riccati differential equation of the form (1) cannot be of larger growth than all of the coefficients. An application of one of our results will be a theorem that is analogous to the classical Malmquist theorem. Our Theorems 1, 2, 3, and 4 and Corollary 1 will show that the Riccati differential equation occupies a special position in certain subclasses of Eqs. (2) in different ways than the already well-known way from the previous studies.

We will also prove some results that concern the growth of meromorphic solutions of Riccati equations. The Riccati equation has a long history; see, e.g., [2, 20] and the references contained in these works.

In this paper we will give several examples of equations of the form (2) that possess meromorphic solutions.

2. Statement of the Main Results

First we shall prove

**Theorem 1.** For a differential equation of the form

$$u' = \sum_{k=0}^{n} E_k(z) u^k$$

where each $E_k$ is entire ($E_n \neq 0$) we have the following conclusions:

(a) For $n \geq 0$, if $u_1, u_2, ..., u_n$ are distinct meromorphic solutions of Eq. (3) such that each $u_i$ has only finitely many poles, then there exists a single entire function $h$ such that for $i \neq j$,

$$u_i - u_j = R_{ij} e^h$$

where $R_{ij}$ is a rational function. If $u_i$ and $u_j$ are entire, then $R_{ij}$ is a nonzero constant $c_{ij}$. 
(b) Let \( n \geq 2 \). Then

(i) \( \text{Eq. (3)} \) can admit at most \( n \) distinct entire solutions, and

(ii) if \( u_1, u_2, \ldots, u_{n+1} \) are distinct meromorphic solutions of \( \text{Eq. (3)} \) such that each \( u_i \) has only finitely many poles, then \( E_n \) has only finitely many zeros.

(c) For \( n \geq 3 \), if \( \text{Eq. (3)} \) actually admits three distinct entire solutions then any meromorphic solution of \( \text{(3)} \) is necessarily entire (and hence \( \text{(3)} \) can admit at most \( n \) distinct meromorphic solutions from (b)(i)).

Immediately following from Theorem 1(b)(i) will be

**Corollary 1.** There exist at most \( n \) distinct meromorphic solutions of Eq. (3) when \( n \geq 3 \) and \( E_n = e^g \) where \( g \) is entire.

In Section 5 we will give several examples of equations of the form (2), most of which are to illustrate the various ways in which Theorem 1 and Corollary 1 are best possible. For instance, an equation of the form (3) when \( n = 2 \) and \( E_2 = 1 \) can possess two distinct entire solutions and a one-parameter family of meromorphic solutions. For \( n \geq 3 \), if Eq. (3) admits \( n \) distinct entire solutions then (3) cannot possess any other meromorphic solutions from Theorem 1. Thus Theorem 1 and Corollary 1 illustrate new ways in which the Riccati is a special differential equation in the class of Eqs. (3).

The late Einar Hille posed the following question to the first author: Is a meromorphic solution of a non-Riccati differential equation of the form (1) necessarily a rational function of the coefficients \( A_k, B_k \)? Two examples in Section 7 will show that the answer to this question, in general, is no. However, by applying Corollary 1 we will prove the following result which answers "almost yes" to Hille's question in a special case.

**Theorem 2.** Consider a differential equation of the form

\[
u' = \sum_{k=0}^{n} \exp(-q_k z) P_k(e^z) u^k
\]

where each \( q_k \) is a nonnegative integer, each \( P_k(z) \) is a polynomial in \( z \), \( P_n(z) = z^\alpha \), and \( n \geq 3 \). Then there exists a fixed positive integer \( m \) so that if \( u = h(z) \) is any meromorphic solution of (5) then \( h \) must be of the form

\[
h(z) = \exp\left(-\frac{\alpha}{m} \right) Q \left( \exp\left( \frac{z}{m} \right) \right)
\]

where \( \alpha \) is a nonnegative integer and \( Q \) is a polynomial.
Theorem 2 is analogous to the classical Malmquist theorem in that the possible meromorphic solutions of the differential equation (5) belong to the same class of functions as the coefficients of the differential equation.

In Section 7 we show that (6) does not, in general, hold for meromorphic solutions of Eq. (5) when \( n = 2 \). Thus Theorem 2 exhibits another new way in which the Riccati equation occupies a special position in a subclass of Eqs. (3). Theorem 2 raises the open question: If \( u = \phi(z) \) is a meromorphic solution of a non-Riccati differential equation of the form (1) where each \( A_k \) and \( B_k \) are rational functions of \( e^z \), then is it necessarily true that \( \phi(z) = R(e^{\beta z}) \) for some rational function \( R \) and some rational number \( \beta \)? If we replace \( e^z \) by \( \exp(z^3) \) in this question then the answer is no by Example 7.1 in Section 7.

We next consider Eq. (2) with polynomial coefficients. It is well known [10, Satz 4.5] that all the solutions of a Riccati differential equation with polynomial coefficients are meromorphic. Thus the following result again places the Riccati differential equation in a special position in a subclass of (2).

**Theorem 3.** For \( n \geq 3 \), the differential equation

\[
    u' = \sum_{k=0}^{n} P_k(z) u^k
\]

where each \( P_k \) is a polynomial \((P_n \neq 0)\) can possess at most a finite number of distinct meromorphic solutions.

In Section 9 we give examples of equations of the form (7) that possess \( n \) distinct meromorphic solutions. It is natural to ask what is the maximum number of distinct meromorphic solutions that a differential equation of the form (7) can possess?

The next result addresses some special cases of (2).

**Theorem 4.** Consider a Bernoulli differential equation of the form

\[
    u' = A_1(z) u + A_n(z) u^n
\]

where \( A_1 \) is entire, \( A_n \neq 0 \) is meromorphic, and \( n \geq 3 \).

(a) If \( n \geq 5 \) then Eq. (8) can admit at most \( n \) distinct meromorphic solutions.

(b) If \( n = 4 \) and \( A_4 \) is entire (meromorphic) then Eq. (8) can admit at most four (seven) distinct meromorphic solutions.

(c) If \( n = 3 \) and \( A_3 \) is entire (meromorphic) then Eq. (8) can admit at most five (seven) distinct meromorphic solutions.
In this paper we will assume that the reader is familiar with R. Nevanlinna’s theory of meromorphic functions \([9, 16]\). Our proof of Theorem 4 will depend on Nevanlinna’s theory of ramified values \([16]\).

In Section 11 we will show that all the numbers (of distinct meromorphic solutions) in the conclusions of Theorem 4 are the best possible. In fact, in Section 11 we will find quite specific information about the possible equations of the form (8) and their respective meromorphic solutions when these maximal numbers are achieved.

Last, we turn our attention to the Riccati equation

\[ u' = A(z) + u^2 \quad (9) \]

where \(A(z)\) is meromorphic. We will use the abbreviation n.e. (nearly everywhere) to mean everywhere in \((0, +\infty)\) except perhaps for an exceptional set of finite linear measure. We will prove the following three results.

**Theorem 5.** Let \(\phi(r)\) be any positive function on \((0, +\infty)\) satisfying the condition

\[ \limsup_{r \to \infty} \frac{\log \log \phi(r)}{\log r} < 1. \quad (10) \]

If \(A(z)\) is a transcendental meromorphic function such that \(\delta(A, \infty) > 0\), then Eq. (9) can admit at most two distinct meromorphic solutions \(u_1, u_2\) that satisfy the condition

\[ T(r, u_i) = o(\phi(T(r, A))) \quad \text{n.e. as } r \to \infty \quad (11) \]

for \(i = 1, 2\).

**Corollary 2.** If \(A(z)\) is a transcendental meromorphic function such that \(\delta(A, \infty) > 0\), then Eq. (9) can admit at most two distinct meromorphic solutions of finite order.

**Theorem 6.** Let \(A(z) \neq 0\) be rational and suppose that Eq. (9) admits at least three distinct rational solutions. If \(u\) is any meromorphic solution of (9) then

\[ \frac{u(z)}{A(z)} \to \infty \quad \text{as } z \to \infty. \quad (12) \]

Theorem 5 and Corollary 2 are improvements of Theorem 5.1 and Corollary 5.2 of [2], respectively. If we delete “\(\delta(A, \infty) > 0\)” from the hypothesis of either Theorem 5 or Corollary 2 then the respective conclusion will no longer hold by Example 13.1 in Section 13.
Theorem 6 is an analogue to Theorem 5 and Corollary 2 for rational functions. Example 13.2 in Section 13 shows that the number “three” in Theorem 6 cannot be reduced. Also in Section 13 we give other examples that concern meromorphic solutions of Eq. (9).

This paper is organized as follows. In Sections 3, 4, 6, 8, 10, and 12 we prove Theorems 1–6 and Corollaries 1 and 2, while in Sections 5, 7, 9, 11, and 13 we mostly discuss and give examples concerning these results.

3. PROOF OF THEOREM 1

Assume first the hypothesis in (a). For \( i \neq j \), if \( z_0 \) is a zero of \( u_i - u_j \), then from the uniqueness of solutions of Eq. (3), see [11, p. 451], \( z_0 \) must be a pole of both \( u_i \) and \( u_j \). Hence \( u_i - u_j \) has only finitely many zeros. It follows that

\[
u_i - u_j = Q_{ij} \exp(h_{ij})
\]

where \( Q_{ij} \neq 0 \) is a rational function and \( h_{ij}(z) \) is an entire function. Then for any three distinct indices \( i, j, k \) we obtain from (13) that

\[
Q_{ij} \exp(h_{ij}) = Q_{ik} \exp(h_{ik}) + Q_{kj} \exp(h_{kj}).
\]

Since the function \( \exp(h_{ik} - h_{kj}) + Q_{kj}(Q_{ik})^{-1} \) has only finitely many zeros, we can conclude from Nevanlinna’s three-functions theorem [9, p. 47] and (14) that

\[
\exp(h_{ij}) = B_1 \exp(h_{ik}) = B_2 \exp(h_{kj}).
\]

where \( B_1, B_2 \) are nonzero constants. By using all the meromorphic solutions \( u_1, u_2, \ldots \), (13), (14), and (15), it follows from this argument that there exists a single entire function \( h \) such that (4) holds. In the particular case when \( u_i \) and \( u_j \) are both entire then it is easy to see that \( Q_{ij} \) in (13) will be a constant, and thus \( R_{ij} \) in (4) will be a constant. This proves (a).

To finish the proof of Theorem 1 we will use the following result which is easy to prove.

**LEMMA 1.** Let \( u_1 \) be a fixed meromorphic solution of Eq. (3) for \( n \geq 2 \). If \( u \) is any solution of (3), then \( w = u - u_1 \) is a solution of the differential equation

\[
w' = \sum_{k=1}^{n} F_k(z) w^k
\]

where \( F_n = E_n \) and each \( F_k \) is a polynomial in \( E_0, E_1, \ldots, E_n, u_1 \).
Now we suppose that \( n \geq 2 \) and that \( u_1, \ldots, u_{n+1} \) are exactly \( n + 1 \) distinct meromorphic solutions of Eq. (3) such that each \( u_i \) has only finitely many poles. Then \( w_i = u_i - u_1 \) \((i = 1, \ldots, n+1)\) are \( n+1 \) distinct meromorphic solutions of (16). From (4),

\[
w_1 \equiv 0, \quad w_i \equiv R_i e^h \quad \text{for } i = 2, \ldots, n+1
\]  

(17)

where \( h \) is an entire function and \( R_2, \ldots, R_{n+1} \) are distinct rational functions, each \( R_i \neq 0 \). By substituting \( w_i \) (for \( i = 2, \ldots, n+1 \)) into (16) we obtain

\[
\frac{R'_i}{R_i} = F_1 - h' + \sum_{k=2}^{n} F_k R_i^{k-1} e^{(k-1)h}. 
\]

(18)

Consider (18) as \( n \) equations in the \( n \) functions \( F_1 - h', F_2 e^h, F_3 e^{2h}, \ldots, F_n e^{(n-1)h} \). Since the determinant of this system is a Vandermonde determinant \([4, p. 301]\) it follows that \( E_n \exp((n-1)h) \) is a rational function, hence a polynomial. This proves (b)(ii). If all of the functions \( u_1, \ldots, u_{n+1} \) were entire, then from (4), each \( R_i \) in (17) would be a nonzero constant \( c_i \), and we would obtain from (18) that \( E_n \exp((n-1)h) \equiv 0 \) which is a contradiction. This proves (b)(i).

Now let \( n \geq 3 \), and let us suppose that \( u_1, u_2, u_3 \) are three distinct entire solutions of Eq. (3), and that \( u_4 \) is a meromorphic solution of (3) that is distinct from \( u_1, u_2, u_3 \). Set \( w_i = u_i - u_1 \) for \( i = 1, 2, 3, 4 \). Then from (4), there is an entire function \( h \) and distinct constants \( C_1 = 0, C_2, C_3 \) such that \( w_i = C_i e^h \) for \( i = 1, 2, 3 \). If \( z_0 \) is a zero of \( w_4 - w_i \) for some \( i = 1, 2, 3 \), then \( z_0 \) must be a regular point of both \( w_4 \) and \( w_i \) because \( w_i \) is entire. This is impossible by the uniqueness of solutions of (16). Hence \( w_4 - w_i = w_4 - C_i e^h \) has no zeros for each \( i = 1, 2, 3 \). Therefore, the function \( w_4 e^{-h} \) has the three Picard values \( C_1, C_2, C_3 \). By Picard’s theorem, \( w_4 e^{-h} \) must be a constant. But this implies that \( w_4 \) is entire, which means that \( u_4 \) is entire. This proves (c) and the proof of Theorem 1 is complete.

**Remark.** By using the proof of Theorem 1(a) we can actually prove the following more general statement:

For \( n \geq 0 \), if \( u_1, u_2, \ldots \) are distinct meromorphic solutions of Eq. (3) such that the poles of each \( u_i \) have exponent of convergence less than one, then there exists a single entire function \( h \) such that for \( i \neq j \),

\[
u_i - u_j = f_{ij} e^h
\]

where \( f_{ij} \) is a meromorphic function of order less than one.
Suppose that \( u \) is a meromorphic solution of Eq. (3) in this particular case. If \( z_0 \) were a pole of \( u \) then by inspection of (3) we would have \( E_n(z_0) = \exp(g(z_0)) = 0 \) which is impossible. Hence \( u \) is entire, and Corollary 1 now follows from Theorem 1(b)(i).

5. Discussion of Theorem 1 and Corollary 1

Theorem 1(b)(i) is an optimal result. For all \( n \geq 2 \) it is possible for an equation of the form (3) to possess \( n \) distinct entire solutions by the following example.

Example 5.1. If \( u_1, \ldots, u_n \) are distinct meromorphic functions such that \( u_i - u_j \) is a constant for all \( 1 \leq i, j \leq n \), then \( u_1, \ldots, u_n \) all are solutions of the differential equation

\[
u' = u' + (u - u_1) \cdots (u - u_n).
\]

Theorem 1(b)(i) is not true for \( n = 1 \) by the differential equation \( u' = u \). Theorem 1(b)(i) is an improvement and generalization of Theorem 5.5 of [2].

Theorem 1(c) is also an optimal result. For \( n \geq 3 \), if Eq. (3) admits \( n \) distinct entire solutions then there can be no other meromorphic solutions of (3) from Theorem 1. This property does not hold for \( n = 2 \) by the next example (and does not hold for \( n = 1 \) by the differential equation \( u' = u \)).

Example 5.2 [2, p. 379]. The differential equation

\[
u' = -\frac{1}{4} - e^{2z}/4 + u^2
\]

has the two entire solutions

\[
u_1(z) = \frac{1}{2}(1 + e^z), \quad \nu_2(z) = \frac{1}{2}(1 - e^z),
\]

and the general solution

\[
u(z) = \nu_1(z) - \frac{e^z}{1 + C \exp(-e^z)}, \quad C \in \mathbb{C}.
\]

Examples 5.1 and 5.2 plus the differential equation \( u' = u \) illustrate ways in which Corollary 1 is a best possible result. Some extra condition on \( E_n \) is necessary in Corollary 1. In fact, if we delete the words "and \( E_n = e^k \) where
g is entire” from Corollary 1 then the statement no longer holds by the following example.

**Example 5.3.** The differential equation $-2u' = \sin (2z) u^3$ has the five solutions $u_1(z) = 0$, $u_{2,3}(z) = \pm \csc z$, $u_{4,5}(z) = \pm i \sec z$.

The next example shows that Theorem 1(b)(i) is not true if the functions $E_k$ in (3) are allowed to be meromorphic nonentire.

**Example 5.4.** If $b \neq 0$ is a constant then the differential equation

$$u' = e^z + \frac{2(u-e^z)(u-e^z+b)(u-e^z-b)}{e^{4z} - b^2}$$

admits the five entire solutions $u_1(z) = e^z$, $u_{2,3}(z) = e^z \pm b$, $u_{4,5}(z) = e^z \pm e^{2z}$.

Last, to illustrate possibilities that can occur, we give two more examples of Eq. (3).

**Example 5.5.** Let $a \neq 0$, $b, c \neq 0$ be complex numbers such that $b^2 \neq 4ac$. If $\alpha_1, \alpha_2, \alpha_3$ are the three distinct roots of $ax^3 + bx^2 + cx - 0$ then the differential equation

$$u' = a + (b + 3ae^z) u + (c + (1 + 2b) e^z + 3ae^{2z}) u^2$$
$$+ (ce^z + be^{2z} + ae^{3z}) u^3$$

admits the three meromorphic solutions

$$u_i(z) = (\alpha_i - e^z)^{-1}, \quad i = 1, 2, 3.$$

**Example 5.6.** If $h$ is any nonconstant entire function and $g$ is any entire function, then the differential equation

$$u' = \left( g' - (e^h + 1)(h' - g') \right) e^{-g} u^2 + (h' - g')(e^h + 1) e^{-2g} u^3$$

admits the three meromorphic solutions $u_1 = 0$, $u_2 = e^g$, $u_3 = e^g(e^h + 1)^{-1}$.

6. **Proof of Theorem 2**

Suppose that $u = h(z)$ is a meromorphic solution of Eq. (5). By inspection of (5) we see that for any fixed integer $\ell$, $h(z + 2\pi i \ell')$ will also be a meromorphic solution of (5). Since by Corollary 1 there can only be a finite number of distinct meromorphic solutions of (5), it follows that $h(z + 2\pi im) \equiv h(z)$ for some positive integer $m$. Thus $h$ is periodic with
period $2\pi i m$. Of course $h$ is entire from inspection of (5). Therefore, there exists a function $F(\zeta)$ that is analytic in the punctured plane $\zeta \neq 0$ such that

$$h(z) = F(\exp(z/m)).$$ (19)

By using (19) and the change of variable $\zeta = \exp(z/m)$ in the differential equation (5) we will obtain

$$F'(\zeta) = \frac{m}{\zeta} \sum_{k=0}^{n} \zeta^{-m+1}P_k(\zeta^n)(F(\zeta))^k$$ (20)

for $\zeta \neq 0$. By applying Malmquist's theorem to (20) (e.g., see the form in [3]) we can conclude that $F(\zeta)$ does not have an essential singularity at $\zeta = \infty$. Similarly, by using the change of variable $\zeta = 1/\zeta$ in (20), we can conclude that $F(\zeta)$ does not have an essential singularity at $\zeta = 0$. Therefore, $F(\zeta)$ is a rational function. Since $h$ is entire, it easily follows from (19) that $h$ has the form (6).

We would like to acknowledge an idea from Steven B. Bank that was used in the proof of Theorem 2.

7. DISCUSSION OF THEOREM 2

Theorem 2 is not true for $n = 2$ ($n = 1$) by Example 5.2 (by the differential equation $u' = e^u$).

The following two examples show that the answer to Hille's question mentioned in Section 2 is no.

**Example 7.1.** The meromorphic function $u(z) = z^{-1}\exp(z^3)$ satisfies the differential equation

$$u' = \frac{3\exp(2z^3) - \exp(-z^3)u^3}{u}.$$

**Example 7.2.** The meromorphic function $u(z) = \exp(-z/2)$ satisfies the differential equation $u' = -\frac{1}{2}ezu^3$.

The next two examples are for illustrative purposes with respect to the open question posed after the statement of Theorem 2 in Section 2.

**Example 7.3.** The meromorphic function $u(z) = z^{-1}e^{z}(e^z - 1)^{-1}$ satisfies the Ricatti differential equation

$$u' = (1 - e^z)^{-1}u + e^{-z}(1 - e^z)u^2.$$
EXAMPLE 7.4. The meromorphic function \( u(z) = (e^z - 1)/(e^z - 2) \) satisfies the differential equation

\[
u' = \frac{-e^z + e^z u}{e^z + (4 - 2e^z) u},
\]

8. PROOF OF THEOREM 3

It is well known [15] that any meromorphic solution of Eq. (7) must be rational. Also, if \( z_0 \) is a pole of multiplicity \( k \) of a solution \( u \) of (7), then by inspection of (7) we see that \( z_0 \) must be a zero of \( P_n \) of multiplicity \( \geq k \). Hence \( uP_n \) will be a polynomial.

Now let \( u_1 \) be a fixed rational solution of (7) and let \( u \neq u_1 \) be any rational solution of (7). Then it can be found that \( w = P_n(u - u_1) \) will be a polynomial solution of a differential equation of the form

\[
(P_n(z))^{n-2}w' = \sum_{k=1}^{n-1} q_k(z) w^k + w^n \tag{21}
\]

where each \( q_k \) is a polynomial that is a function of \( u_1 \) and the polynomials \( P_j \). It follows from (21) that there is an integer \( m \) such that

\[
\deg(w) \leq m \tag{22}
\]

holds for all polynomial solutions \( w \) of (21).

Now suppose that \( u - u_1 \) (for \( u \neq u_1 \)) has a zero at \( z_0 \). From the uniqueness of solutions of (7) it follows that \( z_0 \) must be a pole of both \( u \) and \( u_1 \). Hence \( P_n(z_0) = 0 \). Therefore, when \( w \neq 0 \) we have the condition:

\[
\text{If } w(z_0) = 0 \quad \text{then } P_n(z_0) = 0. \tag{23}
\]

Now we make the assumption that there exists an infinite number of distinct rational solutions of (7). Since \( P_n \) has only a finite number of zeros, it follows from the two conditions (22) and (23) that there must exist an infinite number of distinct nonzero polynomial solutions \( w_0, w_1, \ldots \) of (21) which all have the same zeros counting multiplicities. Hence for each \( i, w_i \equiv C_i w_0 \) where \( C_i \) is a nonzero constant. Substitution of \( w_i \equiv C_i w_0 \) into Eq. (21) yields

\[
0 = q_i(z) w_0 - (P_n(z))^{n-2}w_0' + \sum_{k=2}^{n-1} q_k(z) C_i^{k-1} w_0^k + C_i^{n-1} w_0^n \tag{24}
\]

for each \( i \). For \( i = 1, \ldots, n \), (24) is a homogeneous system of \( n \) equations in the \( n \) functions \( q_1 w_0 - (P_n)^{n-2}w_0', q_2 w_0^2, \ldots, w_0^n \). Since the constants \( C_1, \ldots, C_n \) are all distinct, the determinant of this system is a nonvanishing Vander-
monde determinant. But this implies that \( w_0'' \equiv 0 \) which is a contradiction. Hence our assumption that (7) has an infinite number of distinct rational solutions is false. This proves Theorem 3.

9. DISCUSSION OF THEOREM 3

There are equations of the form (7) that admit \( n \) distinct rational solutions. Example 5.1 with \( u_1, \ldots, u_n \) all polynomials gives equations of the form (7) which possess \( n \) distinct polynomial solutions. Such equations cannot have any other meromorphic solutions from Corollary 1. Another example is given by

**Example 9.1.** The differential equation

\[ u' = 1 - 3zu + (3z^2 - 2)u^2 + (z - z^3)u^3 \]

is satisfied by the three functions \( u_1(z) = z^{-1}, u_2(z) = (z - 1)^{-1}, u_3(z) = (z + 1)^{-1}. \)

Thus the question arises whether an equation of the form (7) can admit more than \( n \) distinct rational solutions.

10. PROOF OF THEOREM 4

If \( g \) is a primitive of \( A_1 \) then the substitution \( u = ye^g \) transforms Eq. (8) into the differential equation

\[ y' = A_n(z) \exp((n - 1) g(z)) y^n. \tag{25} \]

If \( y_0 \neq 0 \) is a meromorphic solution of (25) then from integration of (25) we obtain

\[ y_0^{1-n} = (h - c_0)(1-n) \tag{26} \]

where \( h \) is a primitive of \( A_n \exp((n - 1) g) \) and \( c_0 \) is a finite constant. Thus \( h \) is nonconstant. If \( h \) has a \( c_0 \)-point (pole) at \( z_1 \) then from (26) we see that \( z_1 \) will have multiplicity at least \( n - 1 \). Hence by using Nevanlinna’s first fundamental theorem, \( h \) is a meromorphic function that satisfies the following two conditions as \( r \to \infty \) [9, p. 453]:

\[ \bar{N}(r, h) \leq \frac{1}{n-1} T(r, h) + O(1); \tag{27} \]

\[ \bar{N}(r, h, c_0) \leq \frac{1}{n-1} T(r, h) + O(1). \tag{28} \]
Suppose that \( n \geq 5 \). Then it follows from the Nevanlinna defect relation, (27), and (28) that there could be only one possible value of \( c_0 \) that satisfies (28). Thus there can be at most \( n-1 \) distinct functions \( y_0 \) that satisfy (26). Since \( u = y_0 e^x \) and \( u \equiv 0 \) satisfy (8), it follows that Eq. (8) can admit at most \( n \) distinct meromorphic solutions. This proves (a).

Now suppose \( n = 4 \). If \( A_n \) is meromorphic then \( h \) is meromorphic and we see from the Nevanlinna defect relation, (27), and (28) that there could be at most two values of \( c_0 \) so that (28) holds. By the previous reasoning we then obtain that Eq. (8) can admit at most seven distinct meromorphic solutions. If \( A_n \) is entire then \( h \) is entire and of course \( \overline{N}(r, h) = 0 \). In this case (28) could only hold for at most one value of \( c_0 \). Then (8) could admit at most four distinct meromorphic solutions. This proves (b).

The proof of (c) is completely analogous to the proof of (b).

### 11. Discussion of Theorem 4

In this section we will show that all of the different statements in Theorem 4 are sharp. In fact we will prove more. In Examples 11.1, 11.2, 11.3, and 11.4 below we will find quite specific information about the possible equations of the form (8) and their respective meromorphic solutions when the maximum number (according to Theorem 4) of distinct meromorphic solutions is achieved.

The first observation to make is the following

**Lemma 2.** If \( u \not\equiv 0 \) is a meromorphic solution of Eq. (8), then (8) will possess the \( n - 1 \) distinct meromorphic solutions \( c_1 u, c_2 u, \ldots, c_{n-1} u \) where \( c_1, \ldots, c_{n-1} \) are the \( (n - 1) \)-roots of unity.

Obviously, \( u \equiv 0 \) is always a solution of Eq. (8).

**Example 11.1.** Suppose in Eq. (8) that either \( n \geq 5 \), or \( n = 4 \) and \( A_4 \) is entire. If \( u_1 \not\equiv 0 \) is any meromorphic function and \( A_n = u'_1(u_1)^{-n} \), then \( u = 0, c_1 u_1, \ldots, c_{n-1} u_1 \) (where \( c_1, \ldots, c_{n-1} \) are the \( (n - 1) \)-roots of unity) are \( n \) distinct meromorphic solutions of the differential equation \( u' = A_n u^n \). Because of the transformation (25) we can achieve \( n \) distinct meromorphic solutions of Eq. (8) for any choice of the entire function \( A_1 \). Thus in these two cases of Eq. (8) the maximal number of distinct meromorphic solutions in Theorem 4 is sharp. Furthermore, in view of Lemma 2, either Eq. (8) has only one meromorphic solution (namely, the trivial solution) or it has \( n \) distinct meromorphic solutions.

**Example 11.2.** Suppose in Eq. (8) that \( n = 4 \) and \( A_4 \) is meromorphic. From Theorem 4(b) and Lemma 2, there can only exist either one, four, or
seven distinct meromorphic solutions of Eq. (8). In view of Lemma 2, Eq. (8) will possess seven distinct meromorphic solutions if and only if (8) possesses two linearly independent meromorphic solutions $u_1$ and $u_2$.

Assume that $u_1$ and $u_2$ are linearly independent meromorphic solutions of (8). Then the equation $f' = -3A_1f - 3A_4$ is satisfied by $f_j = (u_j)^{-3}$ for $j = 1, 2$. Hence $f_1 - f_2 = \exp(-3g)$ where $g$ is a suitable primitive of $A_1$. Thus

$$\left( \frac{e^g}{u_1} \right)^3 + \left( -\frac{e^g}{u_2} \right)^3 = 1. \quad (29)$$

Baker [1] and Gross [7] found independently that if $F$ and $G$ are non-constant meromorphic functions that satisfy the identity $F^3 + G^3 \equiv 1$, then

$$F(z) = \frac{\sqrt[3]{3} + \wp(h(z))}{2\sqrt[3]{3} \wp(h(z))}, \quad G(z) = c \frac{\sqrt[3]{3} - \wp'(h(z))}{2\sqrt[3]{3} \wp(h(z))}, \quad (30)$$

where $h$ is nonconstant entire, $c$ is a cubic root of unity, and $\wp$ is the Weierstrass $\wp$-function with periods $\omega_1$, $\omega_2$ that are chosen so that $(\wp')^2 = 4\wp^3 - 1$. Hence from (29),

$$u_1 = F^{-1}e^g \quad \text{and} \quad u_2 = -G^{-1}e^g. \quad (31)$$

It can be verified that if $A_1$ is any entire function, $g$ is a primitive of $A_1$, and $A_4 = -F^2F^2 \exp(-3g) = G'G^2 \exp(-3g)$ where $F$ and $G$ are as in (30), then $u_1$ and $u_2$ in (31) are linearly independent meromorphic solutions of the equation $u' = A_1u + A_4u^2$.

Thus in view of Lemma 2, the number “seven” in Theorem 4(b) is sharp, and furthermore, we have found a form for all the equations and solutions where “seven” actually occurs.

**EXAMPLE 11.3.** Suppose in Eq. (8) that $n = 3$ and $A_3$ is entire. Our discussion of this case will be similar to Example 11.2.

There can only exist either one, three, or five distinct meromorphic solutions of Eq. (8), and (8) will possess five distinct meromorphic solutions if and only if (8) possesses two linearly independent meromorphic solutions. If $u_1$ and $u_2$ are linearly independent meromorphic solutions of Eq. (8), then we find that

$$\left( \frac{e^g}{u_1} \right)^2 + \left( i \frac{e^g}{u_2} \right)^2 = 1. \quad (32)$$

where $g$ is a suitable primitive of $A_1$. Gross [6] found that if $F$ and $G$ are
nonconstant meromorphic functions that satisfy the identity $F^2 + G^2 \equiv 1$, then

$$F = 2A(1 + A^2)^{-1}, \quad G = (1 - A^2)(1 + A^2)^{-1}, \quad (33)$$

where $A$ is a nonconstant meromorphic function. Hence from (32),

$$u_1 = F^{-1}e^g \quad \text{and} \quad u_2 = iG^{-1}e^g. \quad (34)$$

It can be verified that if $A_1$ is any entire function, $g$ is a primitive of $A_1$, and $A_3 = -F'F \exp(-2g) = G'G \exp(-2g)$ where $F$ and $G$ are as in (33), then $u_1$ and $u_2$ in (34) are linearly independent meromorphic solutions of the equation $u' = A_1 u + A_3 u^3$.

Thus the number "five" in Theorem 4(c) is sharp, and furthermore, we have found a form for all the equations and solutions where "five" actually occurs.

Remark. Example 5.3 is a special case of Example 11.3.

Example 11.4. Suppose in Eq. (8) that $n = 3$ and $A_3$ is meromorphic. From Theorem 4(c) and Lemma 2, there can only exist either one, three, five, or seven distinct meromorphic solutions. In view of Lemma 2, Eq. (8) will possess seven distinct meromorphic solutions if and only if (8) possesses three meromorphic solutions $u_1 \neq 0$, $u_2 \neq 0$, $u_3 \neq 0$ such that $u_i/u_j$ is nonconstant for $i \neq j$. From (26) and (25) we see that this will be the case exactly when there exists a nonconstant meromorphic function $h$ and three distinct finite constants $a_1, a_2, a_3$ such that $h - a_j$ is the square of a meromorphic function for $j = 1, 2, 3$. Then the differential equation $u' = h' u^3$ will be satisfied by the seven distinct meromorphic functions $u_0 \equiv 0$, $u_j = \pm (2a_j - 2h)^{-1/2}$, $j = 1, 2, 3$.

An example of such an $h$ is the Weierstrass $\wp$-function. Specifically, if we are given distinct finite constants $a_1, a_2, a_3$ such that $a_1 + a_2 + a_3 = 0$ then there exists a Weierstrass $\wp$-function $\wp(z)$ that satisfies the differential equation $(\wp')^2 = 4(\wp - a_1)(\wp - a_2)(\wp - a_3)$. Then the differential equation $u' = \wp' u^3$ has the seven distinct meromorphic solutions $u_0 \equiv 0$, $u_j = \pm (2a_j - 2\wp)^{-1/2}$, $j = 1, 2, 3$. It follows from (25) that if $A_1$ is any entire function, then there exists seven distinct meromorphic solutions of the equation $u' = A_1 u + \wp' \exp(-2g) u^3$ where $g$ is any primitive of $A_1$.

Thus the number "seven" in Theorem 4(c) is sharp.

Remark. Examples 11.1, 11.2, 11.3, and 11.4 show that all the numbers in the conclusions of Theorem 4 are the best possible.
12. Riccati Proofs

Proof of Theorem 5. Suppose that \( u_1, u_2, u_3 \) are three distinct meromorphic solutions of Eq. (9) that each satisfy the condition (11). Then the residue of \( 2u_1 \) at any pole of \( u_1 \) must be an integer [2, Proposition 2.4]. Hence there exists a meromorphic function \( y \neq 0 \) such that \( 2u_1 = y'y/y \). Therefore

\[
A = u_1' - u_1^2 = \frac{1}{2} \left( \frac{y''}{y} \right)' - \frac{1}{4} \left( \frac{y''}{y} \right)^2
\]

and we obtain

\[
m(r, A) = O(\log r + \log T(r, y)) \quad \text{n.e. as } r \to \infty
\]  

(35)

from Nevanlinna's fundamental estimate of the logarithmic derivative. Since \( \delta(A, \infty) > 0 \), we can say that \( T(r, A) \leq Cm(r, A) \) for some constant \( C > 0 \). Combining this with (35) gives

\[
T(r, A) = O(\log r + \log T(r, y)) \quad \text{n.e. as } r \to \infty.
\]  

(36)

If we set \( v = (u_1 - u_2)^{1} - (u_1 - u_3)^{1} \) then it is elementary to deduce that [2, pp. 371-372]

\[
v' + (y'/y)v = 0.
\]

Therefore

\[
v = (u_1 - u_2)^{1} - (u_1 - u_3)^{1} = D/y
\]  

(37)

for some constant \( D \neq 0 \). From (10), there exists a constant \( \lambda, 0 < \lambda < 1 \), such that \( \phi(r) \leq \exp(r^2) \) for all sufficiently large values of \( r \). By applying (37), (36), and (11) we will obtain n.e. as \( r \to \infty \) the following inequalities (for some real number \( M > 0 \)):

\[
T(r, y) = o(\phi(T(r, A))) + O(1) \leq o(\exp(T(r, y))^2)
\]

\[
\leq o(\exp((M \log T(r, y) + M \log r)^2))
\]

\[
\leq o(\exp(M^2(\log T(r, y))^2 + M \log r)) \leq o(r^M(T(r, y))^2).
\]

This yields

\[
T(r, y) = o(r^{M \left( 1 - \frac{1}{2} \right)}) \quad \text{n.e. as } r \to \infty.
\]

But then from (36) we obtain \( T(r, A) = O(\log r) \) n.e. as \( r \to \infty \). This is impossible because \( A(z) \) is transcendental. This proves Theorem 5.
Proof of Corollary 2. Suppose that Eq. (9) admits three distinct meromorphic solutions \( u_1, u_2, u_3 \), all of finite order. We can follow the proof of Theorem 5 up to (37). By (37), \( y \) has finite order. But then by (36), \( T(r, A) = O(\log r) \) n.e. as \( r \to \infty \), which again contradicts that \( A(z) \) is transcendental.

Proof of Theorem 6. It is well known [21, p. 284] that if an equation of the form (9) admits three distinct rational solutions, then all solutions of (9) will be rational. Suppose that \( u_1, u_2, u_3 \) are any three distinct rational solutions of (9). Then any pole of \( 2u_i, i = 1, 2, 3 \), is simple and has a residue that is an integer [2, Theorem 2.5]. Hence

\[
2u_1(z) = \sum_{k=1}^{m} \frac{\beta_k}{z - z_k} + p(z) \tag{38}
\]

where each \( \beta_k \) is an integer and \( p \) is a polynomial. Now set

\[
R(z) = (z - z_1)^{\beta_1}(z - z_2)^{\beta_2} \cdots (z - z_m)^{\beta_m},
\]

let \( q \) be a primitive of \( p \), and let \( h \) be a primitive of \( \text{Re}^q \). By using \( v = (u - u_1)^{-1}, u \neq u_1, \) we can easily deduce from (9) and (38) that

\[
u_i - u_1 = -\frac{\text{Re}^q}{h + c_i} \text{ for } i = 2, 3, \tag{39}\]

where \( c_2 \) and \( c_3 \) are distinct constants. By eliminating \( \text{Re}^q \) from the two equations in (39) we find that \( h \) is rational. Hence \( q' = p = 0 \). Then from (38), \( u_1(z) \to 0 \) as \( z \to \infty \). If \( u_1 \equiv 0 \) then \( A(z) \equiv 0 \) which we are assuming is not the case. Hence \( u_1 \neq 0 \), and as \( z \to \infty \),

\[
\frac{u_1(z)}{A(z)} = \left(\frac{u_1'(z)}{u_1(z)} - u_1(z)\right)^{-1} \to \infty,
\]

which is condition (12). This clearly proves Theorem 6.

13. Riccati Examples

If we delete "\( \delta(A, \infty) > 0 \)" from the hypothesis of either Theorem 5 or Corollary 2, then the respective conclusion will not hold by the following example.

Example 13.1 [2, p. 380]. The differential equation

\[
u' = A(z) + u^2 \text{ with } A(z) = -(e^z + 1)(e^z - 1)^{-2} \tag{40}\]
has exactly the meromorphic solutions \( u_1(z) = (e^z - 1)^{-1} \) and
\[
u(z) = u_1(z) + \frac{2(1 - e^{-z})^2}{C - 2z - 4e^{-z} + e^{-2z}}, \quad C \in \mathbb{C}.
\]

Choosing \( \phi(r) = \exp(r^\lambda) \) with \( 0 < \lambda < 1 \), we then have
\[
\lim_{r \to \infty} \frac{\log \log \phi(r)}{\log r} = \lambda < 1,
\]
\[
\phi(T(r, A)) = \exp \left( \left( \frac{2r}{\pi} \right)^\lambda (1 + o(1)) \right) \quad \text{as } r \to \infty,
\]
and for any meromorphic solution \( u \) of (40),
\[
T(r, u) = o(\phi(T(r, A))) \quad \text{as } r \to \infty.
\]

The number "three" in Theorem 6 is optimal because it is possible for two distinct rational solutions \( u_1, u_2 \) of Eq. (9) with \( A(z) \neq 0 \) to satisfy
\[
u_i(z) = O(A(z)) \quad \text{as } z \to \infty, \quad i = 1, 2,
\]
by the following example which comes from Proposition 6.10 of [2].

**Example 13.2.** Let \( \alpha \neq 0 \) be a constant, \( P \neq 0 \) be a polynomial, and \( S \) be the unique polynomial that satisfies \( S' + \alpha S = P \). It can be verified that
\[
u_1 = \frac{1}{2} \left( \frac{P'}{P} + \alpha \right) \quad \text{and} \quad \nu_2 = \frac{1}{2} \left( \frac{P'}{P} - \alpha \right) - \frac{S'}{S}
\]
both satisfy \( \nu' = A(z) + \nu^2 \) where \( A(z) = \nu_i(z) - (\nu_i(z))^2 \neq 0 \). As \( z \to \infty \) we have
\[
\frac{u_1(z)}{A(z)} \to \frac{2}{\alpha} \quad \text{and} \quad \frac{u_2(z)}{A(z)} \to \frac{2}{\alpha}.
\]

We make two more remarks concerning Theorem 6:

(i) We can have a one-parameter family \( \{ u \} \) of rational solutions of a differential equation (9) with \( A(z) \neq 0 \) such that each \( u \) satisfies \( u(z) = O(zA(z)) \) as \( z \to \infty \) [2, Example 6.7(e)].

(ii) If \( A(z) \) is a nonconstant polynomial then (9) can admit at most one rational solution \( u_1 \), and in this case, \( u_1(z) = o(A(z)) \) as \( z \to \infty \) is always true (see Section 4 in [2]). If \( A(z) \equiv C \neq 0 \) then (9) has two constant solutions and the rest of the solutions are transcendental meromorphic.
There are Riccati equations that do not possess any meromorphic solutions. For instance, by using the same argument [2, pp. 385–386] as that used to prove [2, Example 6.6(b)] we can prove the following more general result.

**Example 13.3.** Let $4\beta_i = 1 - n_i^2$ where $n_i$ is an integer $\geq 2$, $i = 1, 2, \ldots, m$, and $z_1, \ldots, z_m$ be distinct complex numbers. If $1 - m + \sum_{i=1}^{m} n_i^2$ is not the square of an integer then the Riccati differential equation

$$u' = \sum_{i=1}^{m} \frac{\beta_i}{(z - z_i)^2} + u^2$$

does not possess a meromorphic solution.

Last, suppose that $A(z)$ is transcendental meromorphic with finite order $\lambda$. The examples in the literature of meromorphic solutions $u$ of Eq. (9) all seem to satisfy either $\rho(u) = \lambda$ or $\rho(u) = \infty$. Here $\rho(f)$ denotes the order of a meromorphic function $f$. The next example shows that it is possible to have $\lambda < \rho(u) < \infty$.

**Example 13.4.** Let $m$ be a positive integer and $p$ be a nonconstant polynomial of degree $q$. Then the function

$$h(z) = \exp \left( (2\pi i)^{-2m} \int \frac{p'(z)(p(z))^{2m}}{e^{p(z)} - 1} \, dz \right)$$

is entire and has order $q(2m + 1)$ [8, pp. 293–294]. Let $f(z) = (\exp(p(z)) - 1)^{-1}$. Then $y = f(1 + \alpha h)^{-1}$ satisfies the differential equation

$$y' = \left( f' \frac{h'}{f} \right) y + h' \frac{y^2}{h}$$

for all $\alpha \in \mathbb{C}$. Now using the transformation [20, p. 77]

$$y = (c(z))^{-1}u - (2c(z))^{-1}b(z) - \frac{i}{2}(c(z))^{-2}c'(z)$$

where $b = f'/f - h'/h$ and $c = h'/fh$, we obtain Eq. (9) where

$$A = \frac{b^2}{4} + \frac{b'}{2} - \frac{3}{4} \left( \frac{c'}{c} \right)^2 - \frac{b}{2} \frac{c'}{c} + \frac{1}{2} \frac{c''}{c}.$$

It follows that $\rho(A) = q$ and Eq. (9) has a one-parameter family of meromorphic solutions $\{u_\alpha \}_{\alpha \in \mathbb{C}}$ where $\rho(u_\alpha) = q(2m + 1)$ if $\alpha \neq 0$. 
REFERENCES