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Hypercontractive Semigroups and Two Dimensional Self-Coupled Bose Fields

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We present an abstract perturbation theory for operators of the form $H_0 + V$ obeying four properties: (1) H_0 is a positive self-adjoint operator on $L^2(M, \mu)$ with μ a probability measure so that e^{-tH_0} is a contraction on L^1 for each $t > 0$; (2) e^{-TH_0} is a bounded map of L^2 to L^4 for some T; (3) $V \in L^p(M, \mu)$ for some $p > 2$; (4) $e^{-tV} \in L^1$ for all $t > 0$. We then show that spatially cutoff Bose fields in two-dimensional space–time fit into this framework. Finally, we discuss some details of two-dimensional Bose fields in the abstract including coupling constant analyticity in the spatially cutoff case.

Contents

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1. Introduction

It is our purpose here to give a comprehensive treatment of the mathematical properties of spatially cutoff self-coupled Bose field Hamiltonians in two-dimensional space–time; every attempt has been made to keep this paper comprehensible to the mathematically sophisticated reader who has only a minimal background in quantum field theory.

We view the work we describe here as a synthesis of two lines of development both of which have their roots in the pioneering study of Nelson [45]. On the one hand, Glimm–Jaffe [19–22] and Rosen [47, 48] using Nelson's estimates and techniques suggested by Jaffe's study of the strongly cutoff (finite number of degrees of freedom) theory [33] have developed an impressive amount of information for $(\varphi^4)_2$ and $(\varphi^{2n})_2$ theories. On the other hand, Segal [52, 53] using essentially an abstraction of the properties Nelson isolated has provided a streamlined proof of semiboundedness, suggested a way of defining the Hamiltonian as a formal sum, and made a series of elegant comments. In Section 2, we will continue Segal's abstract study and obtain a series of stronger results, the most important of which is the essential self-adjointness of the operator sum $H_0 + V$. This provides a proof of essential self-adjointness for the $V = \int g(x) : P(\varphi(x)) : dx$ case considerably simpler than the existing proofs of Glimm–Jaffe and Rosen [19, 47].

Since we are able to develop the results in Section 2 without ever introducing an ultraviolet cutoff, we are motivated to attempt to prove that two-dimensional Boson self-coupled field theories have all the properties needed to apply the methods of Section 2 by an approach which uses a minimal number of cutoffs. We do this in Section 3 by borrowing a technique from quantum optics: The formalism of coherent vectors [38]. The results of Section 3 have on the whole, already been proven (explicitly or implicitly) by Nelson [45], Glimm [18], Jaffe–Glimm [19], and Segal [53]. Their proofs always involve putting in approximations and the use of Fatou's lemma. We have attempted, on the other hand, to deal as directly as possible with the objects of interest themselves. It is our feeling that any trend towards fewer cutoffs in the proofs must be regarded as a healthy trend.

In Section 4, we will consider special aspects of the theory of self-coupled Boson fields in two dimensional space–time. Some of these results have already been obtained by Glimm and Jaffe [20–22] in the φ^4 case, by Rosen [47, 48] in the φ^{2n} case and by Segal [53] in the abstract setting. In our study of the vacuum vector, we will present some new results which are natural in the L^p setting, and we will present a streamlined proof of uniqueness of the vacuum vector for the spatially cutoff theory [21]. We also present a discussion of analyticity in the coupling constant and of asymptotic perturbation series for this spatially cutoff theory.

We should like to conclude this introduction by attempting a comparison of our techniques (which originate with Segal) and those of Glimm–Jaffe.[1] First, we should remark that we only deal with one part of the results of Glimm–Jaffe. Their work can be divided into two parts; those parts which require "higher order estimates,"[2] i.e., inequalities of the form $A^n \leqslant B^m$ (n, m not both 1) and those which rely on first order estimates only. This division is not always well-defined; for example, the Glimm–Jaffe proof of essential self-adjointness [19, 20] relies on higher order estimates, while Rosen's proof [47] needs no higher order estimates. *Results that depend critically on higher order estimates cannot be obtained by the methods of this paper*, but we remark that a great deal of the theory only needs first order estimates. In any event, the next step in an abstract development along the lines laid down by Segal and extended by us in this paper should be to attempt to translate the higher order estimates into the abstract framework.

We have been unable to prove the essential self-adjointness of the Cannon–Jaffe locally correct Lorentz generator [8], although we feel the modified theory of Section 2E (Theorem 2.26) might be applicable. In any event, once one knows $H_0 + H_I$ is essentially self-adjoint (which we can prove), the Cannon–Jaffe proof of essential self-adjointness only depends on one elementary quadratic estimate and on the Kato–Rellich theory of regular perturbations.

What then are the main technical simplifications we have accomplished? First, by using fewer cutoffs, our proofs in Section 3 are simpler looking in many instances although these proofs are essentially the same as the highly cutoff proofs. Secondly, we have found an elementary Fock space derivation of Mehler's formula [42] avoiding special formulas for Hermite functions [27] or the theory of Brownian motion [13]. Most crucially, we have eliminated all need of path

[1] The remainder of this introduction is not needed for the bulk of this paper and is intended primarily for readers familiar with the details of the work of Glimm and Jaffe.

[2] The most famous such higher order estimate is the quadratic estimate

$$H_0^2 + H_I^2 \leqslant a[H(g) + b]^2$$

for φ^4 field theories [20].

integration and the Feynman–Kac formula. Once one understands our proofs, one can see that they are in some simple sense the skeletons of the Glimm–Jaffe proofs. We have replaced the need for Feynman–Kac with the Trotter formula and have replaced the positivity of the path space measure with the fact that e^{-tH_0} is a contraction on the L^p spaces. It is well known [44] that once one has established the countable additivity of the path space measure, the Feynman–Kac formula is nothing but the Trotter product formula. The contractivity of e^{-tH_0} follows from the positivity of the kernel of e^{-tH_0},[3] which is essentially all that is needed for the construction of a cylinder set measure on the paths. The place where path integration techniques are stronger than the Trotter product formula plus contractivity is where the full measure is needed rather than the measure on just the cylinder sets; establishing the countable additivity of the full measure is the deep part of the theory of path space integration and this fact is apparently not needed in the applications made *thus far* in field theory.

Finally, by developing the theory in an abstract setting, we gain all the advantages of abstraction: Our proofs have been pared down to the essentials; the abstract setting suggests certain questions which aren't obvious in the concrete situation but whose answer would be illuminating; and it is possible to extend the theory to other cases of interest. We will, in fact, raise certain questions about the cutoff vacuum which are natural in this setting—questions for which a positive answer would have important consequences. And it is clear that our methods will handle some infinite Wick series $V = \int g(x) \sum_{n=0}^{\infty} a_n{:}\,\varphi^n(x){:}\, dx$. We have not attempted to isolate precise conditions on the a_n which will allow us to carry the theory through; we remark that severe fall offof a_n as $n \to \infty$ is needed; for example, the a_n's given by $\cos x = \sum_{n=0}^{\infty} a_n x^n$ don't fall off rapidly enough to get $e^{-Vt} \in L^1$. We also note that our techniques can be modified to handle interactions of the form $\int_{|x|<r} g(x) \rightarrow\exp \alpha\varphi(x)\leftarrow dx$ for α sufficiently small; this theory will be the subject of a separate note by one of us [28].

In summary then, we have isolated the crucial mathematical ideas behind a segment of the important work of Glimm and Jaffe by elaborating on the techniques suggested by Segal. The result seems to us to be a theory not only of great physical interest, but also of considerable mathematical beauty.

[3] And it is equivalent once one knows $H_0 1 = 1$.

2. A Perturbation Theory for Hypercontractive Semigroups

In this section, we consider the following abstract setting: We have an operator H_0 on $L^2(M, \mu)$ with μ a probability measure so that e^{-tH_0} $(t > 0)$ is a contraction on all the L^p spaces and e^{-TH_0} is a bounded map from L^2 to L^4 for some $T > 0$. We are interested in defining $H_0 + V$ as a nice operators on L^2 for a set of perturbations V which are unbounded multiplication operators.

It was Nelson [45] who first emphasized the crucial nature of the contractive property of e^{-tH_0} in field theory models. Recently, Segal [52, 53] considered the precise problem we consider in this section. The reader should view Sections 2B and the first part of 2C as a systematization of Segal's work. We go on to extend Segal's results in two ways: (a) We prove resolvant convergence of $H_0 + V_n$ to $H_0 + V$ when V_n converges to V in a certain sense; (b) we prove $H_0 + V(z)$ is an analytic family in the sense of Kato [35], where $z \to V(z)$ is analytic in a certain sense we make precise in Section 2D.

A. *Hypercontractive Semigroups*

Let M be a fixed measure space with μ probability measure (positive measure of total mass 1) on M.

Definition. Let $H_0 \geqslant 0$ be a self-adjoint operator on $L^2(M, \mu)$. We say e^{-tH_0} $(t > 0)$ is a *hypercontractive semigroup* if and only if:

(a) e^{-tH_0} is a contraction on $L^1(M, \mu)$ for all $t > 0$.

(b) For some T (which we denote $T(H_0)$), e^{-TH_0} is a bounded map from $L^2(M, \mu)$ to $L^4(M, \mu)$.

When (a) but not necessarily (b) holds, we say e^{-tH_0} is contractive.

For the reader's convience, we first recall that condition (a) is implied if e^{-tH_0} has a "probability kernel"; that is:

Proposition 2.1. *Let e^{-tH_0} be positivity preserving (i.e., $f \geqslant 0$ pointwise implies $e^{-tH_0}f \geqslant 0$ pointwise) and suppose $e^{-tH_0}\Omega_0 = \Omega_0$ where Ω_0 is the function $\Omega_0(x) = 1$ for all x. Then e^{-tH_0} is a contraction on $L^1(M, \mu)$.*

Proof. Since e^{-tH_0} is positivity preserving, we need only prove $\| e^{-tH_0}\psi \|_1 \leqslant \| \psi \|_1$ for $\psi \geqslant 0$. But since $\psi \geqslant 0$ and $e^{-tH_0}\psi \geqslant 0$:

$$\begin{aligned} \| e^{-tH_0}\psi \|_1 &= \langle \Omega_0 , e^{-tH_0}\psi \rangle = \langle e^{-tH_0}\Omega_0 , \psi \rangle \\ &= \langle \Omega_0 , \psi \rangle = \| \psi \|_1 . \end{aligned}$$

Q.E.D.

Various interpolation theorems imply that hypercontractive semigroups have additional contractive properties. The Reisz–Thorin theorem [14] coupled with duality (e^{-tH_0} is its own dual) implies:

PROPOSITION 2.2. *Let e^{-tH_0} be a hypercontractive semigroup. Then*

(a) *e^{-tH_0} is a contraction on $L^p(M, \mu)$ for all $1 \leqslant p \leqslant \infty$ and all $t > 0$.*

(b) *Let $1 < a < b < \infty$ be given. Then there is a $T_{a,b}$ and a C so that*

$$\| e^{-tH_0}\psi \|_p \leqslant C \| \psi \|_q$$

for all $t > T_{a,b}$ and all $a < p, q < b$.

Proof. (a) By duality, e^{-tH_0} is a contraction on L^∞ and by Reisz–Thorin on all L^p.

(b) Since e^{-tH_0} is a bounded map from L^2 to L^4 and from L^∞ to L^∞, it follows from Reisz–Thorin that it is a bounded map from L^p to L^{2p} ($p \geqslant 2$). Thus, e^{-nTH_0} is a bounded map from L^2 to L^{2n}. Find n so that $a_0 = 2^n/2^n - 1 \leqslant a < b \leqslant 2^n = b_0$. Then e^{-2ntH_0} is a bounded map from L^{a_0} to L^{b_0} (using the above and its dual). Since $\| \psi \|_p \leqslant \| \psi \|_q$ if $p \leqslant q$ and e^{-tH_0} is a contraction on each L^p, (b) follows. Q.E.D.

In what follows, we will need control over e^{-tH_0} for t small. This we can obtain through the use of the Stein interpolation theorem[4] which we recall says [56]:

PROPOSITION 2.3 (Stein interpolation theorem). *Let $H(z)$ be a family of maps from finite linear combinations of characteristic functions of sets of finite measure on some measure space to its dual. Suppose*

(a) *$H(z)$ is analytic in the strip $0 < \operatorname{Re} z < T$, continuous on the boundaries.*

(b) *If $\operatorname{Re} z = 0$, $\| H(z)\psi \|_{p_0} \leqslant C_0 \| \psi \|_{q_0}$ and if $\operatorname{Re} z = T$, $\| H(z)\psi \|_{p_1} \leqslant C_1 \| \psi \|_{q_1}$.[5] Then for t real, $0 < t < T$ ($s = t/T$),*

$$\| H(t)\, \psi \|_{p_t} \leqslant C_t \| \psi \|_{q_t}$$

[4] It is a pleasure to thank Dr. C. Fefferman for valuable discussions about this theorem.

[5] The original theorem allows growth of $C_{0,1}$ in the $\operatorname{Im} z$ direction.

with $p_t^{-1} = sp_1^{-1} + (1-s)\,p_0^{-1}$; $q_t^{-1} = sq_1^{-1} + (1-s)\,q_0^{-1}$,

$$\log C_t = s \log C_1 + (1-s) \log C_0$$

Proof. See Ref. [56].

PROPOSITION 2.4 (Segal [53]). *Let* e^{-tH_0} *be a hypercontractive semigroup and let* $p(t, \delta)$ *be defined by*

$$p^{-1} = p(t, \delta)^{-1} + t\delta.$$

Then given $1 < a \leqslant b < \infty$, *there is a* δ *and* C (*both depending on* a *and* b) *so that*:

$$\| e^{-tH_0} \|_{p(t,\delta)} \leqslant C^t \| \psi \|_p$$

for all $a \leqslant p \leqslant b$ *and all* $t < T(H_0)$.

Proof. Consider first the case $a = b = 2$. Let $f(z) = e^{-zH_0}$ be defined for $0 \leqslant \operatorname{Re} z \leqslant T$. $f(z)$ is analytic in the strip, continuous on the boundary; $\|f(z)\psi\|_2 \leqslant \|\psi\|_2$ for $\operatorname{Re} z = 0$ and $\|f(z)\psi\|_4 \leqslant C\|\psi\|_2$ for $\operatorname{Re} z = T$ (where C is the norm of e^{-tH_0} as a map of L^2 to L^4).

Thus, by the Stein interpolation theorem, the result follows with $\delta = 1/4T$. The general a, b case follows from the $a = b = 2$ case, the contractive nature of e^{-tH_0} on L^∞ and the Reisz-Thorin theorem. Q.E.D.

Finally, by a remark of Stein, we have:

PROPOSITION 2.5. *Let* e^{-tH_0} *be a contractive semigroup. If*

$$|\arg z| \leqslant \frac{\pi}{2}\left(1 - \left|\frac{2}{p} - 1\right|\right),$$

then e^{-zH_0} *is a contraction of* L^p *to* L^p.

Proof. See Stein [57] for this result which follows from applying the Stein interpolation theorem to $\exp H_0(\eta e^{i\theta t}) = f(t)$. Q.E.D.

B. L^∞ *Perturbations*

Following Segal [53], we first consider the case of $V \in L^\infty$ (V is not necessarily real), and derive estimates that only depend on

$$\| e^{-tV} \|_1 = \int e^{-t\operatorname{Re} V}\, d\mu = (\| e^{-tm^{-1}V} \|_m)^m.$$

The simplest estimate is the following:

PROPOSITION 2.6 (Segal [53]). *Let* $V \in L^\infty$ *and let* e^{-tH_0} *be a contractive semigroup. Let* $\infty \geqslant p > q \geqslant 1$ *with* $q^{-1} = p^{-1} + m^{-1}$. *Then*:

$$\| e^{-t(H_0+V)}\psi \|_q \leqslant \| e^{-tV} \|_m \| \psi \|_p .$$

Proof. Consider $A_n = (e^{-tH_0/n}e^{-tV/n})^n$. By Hölder's inequality $\| e^{-tV/n}\psi \|_{(p^{-1}+n^{-1}m^{-1})^{-1}} \leqslant \| e^{-tV/n} \|_{nm} \| \psi \|_p$. By repeated use of this sort of inequality and the contractive nature of $e^{-tH_0/n}$ on all L^r, we see that $\| A_n\psi \|_q \leqslant \| e^{-tV/n} \|_{nm}^n \| \psi \|_p = \| e^{-tV} \|_m \| \psi \|_p$. By the Trotter product formula [59, 10], $A_n\psi \xrightarrow{L^2} e^{-t(H_0+V)}\psi$ for say $\psi \in L^\infty$. Since the $A_n\psi$ are in a fixed L^q sphere, they have a weak L^q limit point which must be the unique L^2 limit point. Thus $\| e^{-t(H_0+V)}\psi \|_q \leqslant \| e^{-tV} \|_m \| \psi \|_p$.
Q.E.D.

Proposition 2.6 is very weak in that even the identity is a contraction from L^p to L^q if $p > q$. But using the strong inequalities for hypercontractive e^{-tH_0} contained in Proposition 2.3, we can conclude strong inequalities for $e^{-t(H_0+V)}$. First:

PROPOSITION 2.7 (Segal [53]). *Let* $V \in L^\infty$ *and let* e^{-tH_0} *be a hypercontractive semigroup. Let* $1 < p < \infty$. *Then there are a* C *and an* m (*dependent on* p *and* H_0 *but independent of* V *and* t) *so that*

$$\| e^{-t(H_0+V)}\psi \|_p \leqslant C^t(\| e^{-V} \|_m)^t \| \psi \|_p .$$

Proof. As in the proof of Proposition 2.6, we need only prove for t fixed and all n sufficiently large that we have

$$\|(e^{-tV/n}e^{-tH_0/n})^n \psi \|_p \leqslant C^t \| e^{-V} \|_m)^t \| \psi \|_p .$$

This follows if we can prove

$$\|(e^{-tV/n}e^{-tH_0/n}) \psi \|_p \leqslant C^{t/n}(\| e^{-V} \|_m)^{t/n} \| \psi \|_p$$

for n sufficiently large (t fixed). By Proposition 2.4, we can find δ (dependent on p) so that

$$\| e^{-tH_0/n}\psi \|_{(p^{-1}-t\delta/n)^{-1}} \leqslant C^{t/n} \| \psi \|_p .$$

By Hölder's inequality,

$$\| e^{-tV/n}\varphi \|_p \leqslant \| e^{-tV/n} \|_{n/t\delta} \| \varphi \|_{(p^{-1}-t\delta/n)^{-1}} .$$

Since $\| e^{-tV/n} \|_{n/t\delta} = (\| e^{-V} \|_{1/\delta})^{t/n}$ the proof is completed. Q.E.D.

In the case $p = 2$, the above result can be restated in the form:

COROLLARY 2.8 (Segal [53]). *Let H_0 be a hypercontractive semigroup and let $E(V) = \inf \operatorname{spec}(H_0 + V)$ for $V \in L^\infty$, real. Then, there is a D and an m (independent of V) so that*

$$E(V) \geqslant -D - \log \| e^{-V} \|_m .$$

Such lower boundedness theorems in the field theory context originated in the work of Nelson [45] and were refined by Glimm [18]; see also Federbush [15]. The exact form of Corollary 2.6 is stated in Segal's note [52] and a very elegant (alternate) proof is sketched based on the following lemma:

LEMMA 2.9 (Segal). *If $\| e^{-A/2}e^{-B}e^{-A/2} \| \leqslant C$ for A and B semi-bounded operators on a Hilbert space and if $A + B$ is essentially self-adjoint on $D(A) \cap D(B)$, then $\| e^{-(A+B)} \| \leqslant C$.*

Proof. Segal's proof is based on monotonicity of the square-root on positive operators and the Trotter formula. Alternately, we can prove this by noting that Löwner's theorem [41] implies $\log X \leqslant \log Y$ if $0 < X \leqslant Y$. Thus, $\| e^{-A/2}e^{-B}e^{-A/2} \| \leqslant C$ which implies $e^{-B} \leqslant Ce^{+A}$ also implies $-B \leqslant \log C + A$ or $-(A + B) \leqslant \log C$. Q.E.D.

Notes. (1) Our proof using the monotonicity of log, proves $A + B \geqslant -\log C$ as a statement of quadratic forms even if $A + B$ is not self-adjoint. (2) For the reader's convenience we point out that the monotonicity of logarithm follows directly from the Herglotz representation formula for the logarithm:

$$\log x = \int_{-\infty}^{0} \frac{(1 + yx)}{(1 + y^2)(y - x)}\, dy.$$

One can strengthen Proposition 2.7 to the form:

PROPOSITION 2.10. *Let e^{-tH_0} be a hypercontractive semigroup. Let $1 < p, q < \infty$. Then there is an S, an E and an m so that for all $V \in L^\infty$:*

$$\| e^{-(H_0+V)S}\psi \|_p \leqslant E \| e^{-SV} \|_m \| \psi \|_q .$$

Proof. Suppose $2q > p > q$ first. As in Proposition 2.4, pick C, δ so that for all r, $q \leqslant r \leqslant (2p^{-1} - q^{-1})^{-1}$,

$$\| e^{-tH_0}\psi \|_{(r^{-1}-t\delta)^{-1}} \leqslant C^t \| \psi \|_r ,$$

for all $t < T$. Let $m = (2S\delta)^{-1}$, where S is given by $q^{-1} = p^{-1} + S\delta/2$. As usual, we need only prove

$$\|(e^{-VS/n}e^{-H_0S/n})^n \psi \|_p \leqslant C \| e^{-SV} \|_m \| \psi \|_q$$

for all n sufficiently large. Pick n so $S/n < T$. Then

$$\| e^{-H_0S/n}\psi \|_{(q-S\delta/n)^{-1}} \leqslant C^{s/n} \| \psi \|_q .$$

By Holder's inequality

$$\| e^{-VS/n}e^{-H_0S/n}\psi \|_{(q^{-1}-S\delta/2n)^{-1}} \leqslant C^{S/n} \| e^{-VS/n}\|_{2n/S\delta} \| \psi \|_q .$$

Iterating this argument, we see

$$\|(e^{-VS/n}e^{-H_0S/n})^n \psi \|_{(q^{-1}-S\delta/2} \leqslant C^S \| e^{-VS/n}\|^n_{2n/S\delta} \| \psi \|_q$$

so that

$$\| e^{-(H_0+V)S}\psi \|_p \leqslant C^S \| e^{-VS} \|_m \| \psi \|_q .$$

For arbitrary $p > q$, we iterate the above argument. Q.E.D.

In summary then, we have seen L^∞ perturbations yield semigroups with properties only slightly weaker than hypercontractive semigroups: Instead of being contractions from L^p to L^p all $1 \leqslant p \leqslant \infty$, the perturbed semigroups are *bounded* from L^p to L^p for $1 < p < \infty$ (for $V \in L^\infty$, they are bounded also from L^∞ to L^∞ but the bounds depend on $\| V \|_\infty$) and $e^{-T(H_0+V)}$ is still a bounded map for L^2 to L^4 for some T. More crucially, the bounds only depend on powers of the quantities $\int e^{-Vs}\, d\mu$. This suggests the results of this section should extend to a wider class of perturbations.

C. *Almost Semibounded L^p Perturbations* ($p > 2$)

Again following Segal, we extend some of our results from Section 2B to a class of non-L^∞ potentials. Not surprisingly, we will deal with potentials V so that $\int e^{-tV}\, d\mu < \infty$. Thus while V is not necessarily bounded below, it can only get large and negative on very small sets ($\mu(\{x \mid V(x) \leqslant -C\}) < D_t e^{-Ct}$ for all t), i.e., it is "almost bounded below." We will see that the condition $V \in L^p$ for some $p > 2$ enters naturally.

The idea of Segal is to use DuHamel's formula to prove $e^{-t(H_0+V_n)}$ converges to a semigroup whose generator he *defines* to be the (formal) sum $H_0 + V$ (the V_n are L^p approximations to V which lie in L^∞). We follow Segal's argument [53] for defining $H_0 + V$, but our proof that $H + V$ so defined is actually essentially self-adjoint on $D(H_0) \cap D(V)$ differs slightly from Segal's proof. If one bears in mind that two-dimensional field theories fit into the framework of almost semibounded L^p perturbations and also recalls that spatially cutoff field theoretic Hamiltonians have been proven essentially self-adjoint as operator sums by Glimm–Jaffe [20] and Rosen [47], then the general essential self-adjointness result is not really very surprising.

We first recall DuHamel's formula:

LEMMA 2.11. *Let A be positive operator on a Hilbertspace H. Let B, C be bounded self-adjoint operators. Then*

$$e^{-t(A+B)} = e^{-t(A+C)} + \int_0^t e^{-(t-u)(A+B)}(C - B)\, e^{-u(A+C)}\, du.$$

Proof. Let $\psi \in D(A)$. Then $f(t) = e^{-t(A+B)}\psi$ is the unique solution of $f(0) = \psi$; $f'(t) = -(A + B)f(t)$. Let

$$g(t) = e^{-t(A+C)}\psi + \int_0^t e^{-(t-u)(A+B)}(C - B)\, e^{-u(A+C)}\psi\, du.$$

An elementary computation shows $g'(t) = -(A + B)\, g(t)$ so $g(t) = f(t)$. Q.E.D.

To emphasize where hypercontractivity as opposed to contractivity enters our arguments, let us first state a result that only require e^{-tH_0} to be contractive.

PROPOSITION 2.12 (Segal). *Suppose $V \in L^p$ (some $p > 2$) and $e^{-tV} \in L^1$ all $t > 0$. Let*

$$\begin{aligned} V_n(x) &= V(x), && \text{if } |V(x)| \leqslant n, \\ &= n, && \text{if } |V(x)| > n. \end{aligned}$$

Let e^{-tH_0} be a contractive semigroup and let $H_n = H_0 + V_n$. Then for *any* $\infty \geqslant r > s \geqslant 1$, *with* $s^{-1} > r^{-1} + p^{-1}$ *and any fixed* $t > 0$ e^{-tH_n} *converges as maps of* $L^r \to L^s$ *(in the norm topology) uniformly on compact subsets of* $[0, \infty)$. *The operators* e^{-tH} *so defined are bounded maps of* L^r *to* L^s *for all* $r > s$ *and obey*

$$\| e^{-tH}\psi \|_s \leqslant \| e^{-tV} \|_m \| \psi \|_r$$

where $s^{-1} = r^{-1} + m^{-1}$. *For any* $r > s > 1$, *and any* $\psi \in L^r$, $e^{-tH_n}\psi \to e^{-tH}\psi$ *in the* weak L^s *topology.*

Proof (Segal). Pick real numbers j, k so $r > j > k > s$ and $k^{-1} = j^{-1} + p^{-1}$. By Proposition 2.4, $\| e^{-u(H_0+V_n)}\psi \|_j \leqslant C_t \| \psi \|_r$ for all n and all $0 \leqslant u \leqslant t$ for some constant C_t independent of n. Since $V_n \to V$ in L^p, $\|(V_n - V_m)\, e^{-u(H_0+V_n)}\psi \|_k \to 0$ uniformly for ψ in the unit ball of L^r (by Hölder's inequality). Finally, the $e^{-(t-u)(H_0+V_m)}$ are uniformly bounded as maps of L^k to L^s so the first statement is proven by using DuHamel's formula.

Now let r and s ($r > s > 1$) be arbitrary. We have $\| e^{-tH_n}\psi \|_s \leqslant \| e^{-tV} \|_m \| \psi \|_r$ so by the weak compactness of balls in L^s, $e^{-tH_n}\psi$ has a weak L^s-limit point. As in the proof of Proposition 2.6, this limit point is $e^{-tH}\psi$ which lies in a suitable L^s ball for the $\| e^{-tV} \|_m$ inequality to hold. Q.E.D.

Until we obtain e^{-tH} from some L^p to *itself*, we cannot hope to define H as an infinitesmal generator of e^{-tH}. If V is not in L^∞, the only way of obtaining a result of the form $\| e^{-tH}\psi \|_p \leqslant C \| \psi \|_p$ is to use the full hypercontractivity. Put differently, the property that allows us to obtain H as an operator on L^2 is the boundedness below estimates on the H_n's. We thus see why the field theoretic estimates of Nelson, Glimm and Federbush enter crucially in the self-adjointness proof.

PROPOSITION 2.13 (Segal). *Let* e^{-tH_0} *be a hypercontractive semigroup. Let* $V \in L^p$ *for some* $p > 2$; $e^{-Vt} \in L^1$ *for all* $t > 0$. *Let* H_n, V_n *be defined as in Proposition* 2.12. *Then for any* $1 > q > (1 - p^{-1})^{-1}$, e^{-tH_n} *converges* strongly *to* e^{-tH} *as maps from* $L^q \to L^q$. *The operators* e^{-tH} *are a strongly continuous semigroup on these* L^q *and obey*:

$$\| e^{-tH}\psi \|_q \leqslant C^t(\| e^{-V} \|_m)^t \| \psi \|_q ,$$

where C *and* m *are dependent only on* H_0 *and* q. *(Note that we will eventually improve this result.)*

Proof. For $\psi \in L^\infty$, $e^{-tH_n}\psi \to e^{-tH}\psi$ in norm, in any L^q, $q < (1 - p^{-1})^{-1}$. Since the e^{-tH_n} are uniformly bounded and L^∞ is dense in any L^q, the strong convergence follows. The strong continuity of the e^{-tH} follows from the uniformity of the L^2 convergence of $e^{-tH_n}\psi$ (for $\psi \in L^\infty$) on compacts, the strong continuity of the e^{-tH_n}, the uniform boundedness of the e^{-tH} and the density of L^∞ in L^2. The bounds follow from the bounds on the e^{-tH_n}. Q.E.D.

We are thus able to state the following:

Definition (Segal). Let $e^{-H_0 t}$ be a hypercontractive semigroup. Let $V \in L^p$ for some $p > 2$, $e^{-tV} \in L^1$ for all $t > 0$. We define the formal sum $H_0 + V$ as the unbounded operator on L^2 which is the infinitesimal generator of the strongly continuous exponentially bounded semigroup e^{-tH} defined above.

Corollary 2.8 immediately extends:

Corollary 2.14. *Under the conditions of Proposition* 2.13, $-H + E(V)$ *is a dissipative operator for some* $E(V)$ *with* $E(V) \leqslant -D - \log \| e^{-V} \|_m$ *for some* m *and* D *independent of* V. *If* V *is real valued,* H *is self-adjoint and semibounded;* $E(V)$ *is a lower bound on the spectrum.*

What we have discussed so far is contained implicitly or explicitly in Segal's work [52, 53]. Segal's final and most crucial result is essential self-adjointness of the operator sum $H_0 + V$ in case V is real. We give a modified proof in the spirit of our development so far. We first recall three elementary Hilbert space facts:

Lemma 2.15. (a) *Let* D *be dense in Hilbert space* $\mathscr{H}$. *Let* A *be self-adjoint and semibounded on* $\mathscr{H}$. *Then for any* $t > 0$, $e^{-tA}D$ *is a core for* A.

(b) *Let* A_n, A *be self-adjoint semibounded operators on a Hilbert space* $\mathscr{H}$ *which are uniformly bounded below. If* $e^{-A_n t}$ *converges strongly to* e^{-At} *for all* $t > 0$, *then* $A_n e^{-A_n t}$ *converges strongly to* Ae^{-At} *for any* $t > 0$.

(c) *Let* A *be a self-adjoint operator and suppose* $\phi_n \in D(A)$, $\phi_n \to \phi$ *in norm,* $A\phi_n \to \psi$ *weakly.*[6] *Then* $\phi \in D(A)$ *and* $\psi = A\phi$.

Proof. (a) There is a straightforward proof using the spectral theorem but consider the following slick argument: if $\psi \in [e^{-tA}D]^\perp$, then $0 = \langle \psi, e^{-tA}\phi \rangle$ all $\phi \in D$ so $e^{-tA}\psi = 0$ (since D is dense), so

[6] That $A\phi_n$ is a weakly convergent *sequence* rather than a general net is crucial.

$\psi = 0$. Thus $e^{-tA}D$ is dense. Since it is a family of analytic vectors for A, it is a core by Nelson's theorem [43].

(b) Since the A_n are uniformly semibounded, the $\{e^{-tA_n}\phi\}$ are a family of analytic functions on $\{t \mid \operatorname{Re} t > 0\}$ which are uniformly bounded on compacts. By assumption, they converge pointwise on the real axis. Thus, by the Vitali convergence theorem, they converge uniformly on compacts so the convergence of the derivatives follows.

(c) Pass to a spectral representation for A. By passing to a subsequence, we can suppose $\phi_n(x) \to \phi(x)$ pointwise a.e. Since $A\phi_n$ is weakly convergent, there is a C with $\| A\phi_n \|^2 = \int x^2 \mid \phi_n(x)|^2 \, d\mu < C$. By Fatou's lemma, $\liminf x^2 \mid \phi_n(x)|^2 = x^2 \mid \phi(x)|^2$ is in L^1, i.e., $\phi \in D(A)$. Let η be a function of compact support in the spectral representation space. Then $\overline{x\eta(x)}\phi_n(x) \to \overline{x\eta(x)}\phi(x)$ in L^1, so $\langle \eta, A\phi_n \rangle \to \langle \eta, A\phi \rangle$ for a dense set of η's. Since the $\| A\phi_n \|$ are bounded, $A\phi_n \to A\phi$ weakly so $\psi = A\phi$. Q.E.D.

Our first main result is:

THEOREM 2.16 (Segal). *Let e^{-tH_0} be a hypercontractive semigroup. Suppose V is real, $V \in L^p$ for some $p > 2$ and $e^{-tV} \in L^1$ for all $t > 0$. Then the formal sum $H = H_0 + V$ is essentially self-adjoint on a subset $\mathscr{D}$ of $D(H_0) \cap D(V)$ and H* is just the closure of the usual operator sum $H_0 + V$.

Proof. Let $\mathscr{D} = e^{-tH}L^\infty$. By Lemma 2.15(a), $\mathscr{D}$ is a core for H. By Proposition 2.12, $\mathscr{D} \subset L^s$ for all $s < \infty$ and $e^{-tH_n}\psi \to e^{-tH}\psi$ weakly in L^q where $q^{-1} + p^{-1} = 1/2$ (for any $\psi \in L^\infty$). In particular, $\mathscr{D} \subset D(V)$ and $V_n e^{-tH_n}\psi \to Ve^{-tH}\psi$ weakly in L^2. By Lemma 2.15(b) and Proposition 2.13, $H_n e^{-tH_n}\psi \to He^{-tH}\psi$ in norm. Thus $H_0 e^{-tH_n}\psi \to (H - V)\, e^{-tH}\psi$ weakly and $e^{-tH_n}\psi \to e^{-tH}\psi$ in norm. By Lemma 2.15(c), $e^{-tH}\psi \in D(H_0)$ and $H_0(e^{-tH}\psi) = (H - V)(e^{-tH}\psi)$. Q.E.D.

Remark. By using the resolvent in place of the semigroups and results we prove below one can prove the formal sum $H_0 + V$ is always the closure of the operator sum, even if V isn't real. Also, one can prove the L^q generator of e^{-tH} is the closure of the ordinary L^q sum if $q^{-1} + p^{-1} > \frac{1}{2}$. Since the real V case on L^2 is what arises in application, we do not discuss the details.

The next question which naturally arises is whether $H_0 + V$ depends continuously on V in some sense. To prove a suitable result, we turn to proving norm convergence of semigroups (not uniformly in [0, 1] however!) via DuHamel's formula. The critical inequalities needed are that e^{-tH} takes L^r to L^s with $s \geqslant r$ if t is big enough; that is the analogs of Propositions 2.7 and 2.10.

PROPOSITION 2.17. *Let e^{-tH_0} be a hypercontractive semigroup. Let $1 < r, q < \infty$. Then there is an S, an E and an m so that for all V in some L^p ($p > 2$) with $e^{-tV} \in L^1$ all $t > 0$:*

$$\| e^{-(H_0+V)S}\psi \|_r \leqslant E \| e^{-SV} \|_m \| \psi \|_q .$$

For any r, there is an m and a C (independent of V) so that

$$\| e^{-t(H_0+V)}\psi \|_r \leqslant C^t(\| e^{-V} \|_m)^t \psi]_r .$$

Proof. Use the standard weak limit argument and Propositions 2.7 and 2.10. Q.E.D.

Proposition 2.15 also carries through:

PROPOSITION 2.18. *Under the hypotheses of Proposition 2.17, $e^{-z(H_0+V)}$ is a bounded map of $L^p \to L^p$ if $| \arg z | < (\pi/2)(1 - | 2/p - 1 |)$ with bounds of exponential growth in $| z |$ (for p, $\arg z$ fixed) at worst.*

LEMMA 2.19. *Let $V, W \in L^p$ ($p > 2$); e^{-tV}, $e^{-tW} \in L^1$, all $t > 0$. Let $q > p$. Then for any $\psi \in L^\infty$,*

$$e^{-t(H_0+V)}\psi = e^{-t(H_0+W)}\psi + \int_0^t e^{-(t-u)(H_0+V)}(W - V)\, e^{-u(H_0+W)}\psi \, du,$$

where the integral is a weak L^q integral.

Proof. Take derivatives as in Proposition 2.9. Q.E.D.

PROPOSITION 2.20. *Let $V_n \to V$ in L^p some $p > 2$. Suppose, for each fixed t, $\| e^{-tV_n} \|_1$ is uniformly bounded in n. Let $(1 - p^{-1})^{-1} < q < p$. Then for some S (depending only on H_0 and q) $e^{-S(H_0+V_n)} \to e^{-S(H_0+V)}$ in norm as operators from L^q to L^q.*

Proof. Pick T so that $e^{-T(H_0+V)}$ is a bounded map from L^q to L^r ($r^{-1} + p^{-1} = q^{-1}$) and from L^s to L^q($q^{-1} + p^{-1} = s^{-1}$), with bounds only depending on $\| e^{-SV} \|_m$. Then, for $T \leqslant t \leqslant 2T$, $e^{-t(H_0+V_n)}$ are bounded maps of L^q to L^r, L^s to L^q with bounds uniform in t and n. Let $S = 2T$ and let $\psi \in L^\infty$.

Then

$$\|(e^{-2T(H_0+V)} - e^{-2T(H_0+V_n)})\,\psi\,\|_q$$
$$\leqslant \left\| \int_0^T e^{-(2T-u)(H_0+V)}(V_n - V)\, e^{-u(H_0+V_n)}\psi\, du \right\|_q$$
$$+ \left\| \int_T^{2T} e^{-(2T-u)(H_0+V)}(V_n - V)\, e^{-u(H_0+V_n)}\psi\, du \right\|_q$$
$$\leqslant T[(\sup_{T\leqslant t\leqslant 2T} \| e^{-t(H_0+tV)} \|_{s,q}) \| V_n - V \|_p (\sup_{0\leqslant u\leqslant T} \| e^{-u(H_0+V)} \|_{q,q})$$
$$+ (\sup_{0\leqslant t\leqslant 2T} \| e^{-t(H_0+V)} \|_{q,q}) \| V_n - V \|_p (\sup_{T\leqslant u\leqslant 2T} \| e^{-u(H_0+V_n)} \|_{q,r})] \| \psi \|_q ,$$

where $\| A \|_{\alpha,\beta}$ is the norm of A as a map from L^α to L^β. This proves the proposition since L^∞ is dense in L^q.

We can now state our second main result:

THEOREM 2.21. *Let e^{-tH_0} be a hypercontractive semigroup. Let $V_n \to V$ in L^p for some $p > 2$, with V_n real. Suppose $\| e^{-tV_n} \|_1$ is uniformly bounded in n for each t fixed. Then, for any t*

$$e^{-t(H_0+V_n)} \to e^{-t(H_0+V)}$$

in norm as maps of L^2 to L^2. The convergence is uniform on compact subsets of $(0, \infty)$. *In particular, for any $z \notin$* $\operatorname{spec}(H_0 + V)$,

$$(H_0 + V_n - z)^{-1} \to (H_0 + V - z)^{-1}$$

in $\| \quad \|_{2,2}$.

Proof. By Proposition 2.20, $e^{-S(H_0+V_n)} \to e^{-S(H_0+V)}$ as maps of L^2 to L^2 for some S. By the continuity of the functional calculus, $e^{-\alpha S(H_0+V_n)} \to e^{-\alpha S(H_0+V)}$ for any $\alpha \geqslant 0$ and the convergence is uniform on compacts of $(0, \infty)$. This proves the statement on convergence of the semigroups.

To prove convergence of the resolvents, we use the formula, $(H - z)^{-1} = \int_0^\infty e^{zt} e^{-Ht}$ for any z with

$$\| e^{-Ht} \|_{2,2} \leqslant C \exp[(-\operatorname{Re} z - \epsilon)\, t]$$

and the uniform boundedness below of the H_n's. Q.E.D.

Remarks. (1) We view this as an abstract form of Rosen's result on the resolvent convergence of cutoff $(\phi^{2n})_2$ Hamiltonians [47].

(2) This result holds in a more general context: First V real is not needed and the results hold as maps of L^q to L^q if $(1 - p^{-1})^{-1} < q < p$.

As a final result, we translate the second half of Proposition 2.17 into a statement about resolvents and derive a weak continuity requirement sufficient for use in Section 2D.

PROPOSITION 2.22. *Let e^{-tH_0} be a hypercontractive semigroup. Let $V \in L^p$ ($p > 2$) with $e^{-tV} \in L^1$ (all $t > 0$). Let C and m be as in Proposition* 2.17 *(dependent on r). If* $\operatorname{Re} z < -\log(C \| e^{-V} \|_m)$, *then*

$$(H_0 + V - z)^{-1} : L^r \to L^r$$

is a bounded map with bound only dependent on $\| e^{-V} \|_m$.

Proof. Proposition 2.17 and the formula $(H - z)^{-1} = \int_0^\infty z^{zt} e^{-Ht}\, dt$.
Q.E.D.

PROPOSITION 2.23. *Let e^{-tH_0} be a hypercontractive semigroup. Let $V_n \to V$ in L^p ($p > 2$) with $\| e^{-tV_n} \|_1$ uniformly bounded for each fixed $t > 0$. Then, for z with sufficiently negative real part, $(H_0 + V_n - z)^{-1} \to (H_0 + V - z)^{-1}$ strongly in L^q for fixed q with $q < p$.*

Proof. Let $\psi \in L^r$ with $r^{-1} + p^{-1} = q^{-1}$. Then for z very negative,

$$\begin{aligned}&(H_0 + V_n - z)^{-1} \psi - (H_0 + V - z)^{-1} \psi \\ &\qquad = (H_0 + V_n - z)^{-1}(V - V_n)(H_0 + V - z)^{-1} \psi.\end{aligned}$$

But $(H_0 + V - z)^{-1} \psi \in L^r$, $\| V - V_n \|_{q,r} \to 0$ and $(H_0 + V_n - z)^{-1}$ are uniformly bounded as maps of L^q to L^q. Since L^r is dense in L^q and the $\|(H_0 + V_n - z)^{-1} \|_{q,q}$ are bounded uniformly in n, strong convergence follows. Q.E.D.

D. *Analytic Perturbation Theory*

Our last main result in this abstract setting concerns analyticity of $H_0 + V_\lambda$ as an analytic function of λ in some suitable sense. We recall [35], a family $T(\lambda)$ of closed operators on a Banach space X with $z \in \rho(T(0))$ is called holomorphic at $\lambda = 0$ if and only if $z \in \rho(T(\lambda))$ if $|\lambda|$ is sufficiently small and $(T(\lambda) - z)^{-1}$ is a bounded holomorphic function. And we recall that for bounded operator-valued functions, norm holomorphy is equivalent to weak holomorphy [35].

THEOREM 2.24. *Let D be a domain in the complex plane. Let e^{-tH_0} be a hypercontractive semigroup and let $\lambda \to V(\lambda)$ be a map from D to L^p*

($p > 2$) which is analytic on D. Suppose $\| e^{-tV(\lambda)} \|_1$ is uniformly bounded on compacts in D. Then $H_0 + V(\lambda)$ (the closure of the operator sum) is a holomorphic family of operators on L^2.

Proof. Let $\phi, \psi \in L^\infty$ and consider for z very negative,

$$(\lambda - \lambda_0)^{-1}\langle\phi, [(H_0 + V_\lambda - z)^{-1} - (H_0 + V_{\lambda_0} - z)^{-1}]\psi\rangle$$
$$= \langle H_0 + \bar{V}_\lambda - z)^{-1}\phi, \left(\frac{V(\lambda_0) - V(\lambda)}{\lambda - \lambda_0}\right)(H_0 + V_{\lambda_0} - z)^{-1}\psi\rangle.$$

Since $(H_0 + \bar{V}_\lambda - z)^{-1}\phi$ converges in L^2 to $(H_0 + V_\lambda - z)^{-1}\phi$ (by Proposition 2.23), $(V(\lambda_0) - V(\lambda))/(\lambda - \lambda_0)$ converges in L^p to $-V'(\lambda_0)$ and since $(H_0 + V_{\lambda_0} - z)^{-1}\psi \in L^r$, $(r^{-1} + p^{-1} = \frac{1}{2})$, we see that $\langle\phi, (H_0 + V_\lambda - z)^{-1}\psi\rangle$ is analytic at $\lambda = \lambda_0$. Since the $\phi, \psi \in L^\infty$ are dense in L^2 and the $(H_0 + V_\lambda - z)^{-1}$ are uniformly bounded for λ near λ_0, the resolvent is weakly holomorphic and thus holomorphic. Q.E.D.

E. *Extensions*

Our goal in this addendum is to point out that some of our main results hold under modified conditions. In the first place, when V is bounded below (so $e^{-Vt} \in L^1$ is trivial), we only need $V \in L^2$:

THEOREM 2.25. *Let e^{-tH_0} be a hypercontractive semigroup. Let $V \geqslant 0$, $V \in L^2$. Then $H_0 + V$ is essentially self-adjoint on $D(H_0) \cap D(V)$. If $V_n \to V$ in L^2 and $V_n \geqslant 0$ all n, then $(H_0 + V_n + E)^{-1} \to (H_0 + V + E)^{-1}$ strongly if* $\operatorname{Re} E > 0$. *If $\lambda \to V(\lambda)$ is an analytic L^2-valued function on a domain D so that* $\inf_x (\operatorname{Re} V(\lambda)(x))$ *is uniformly bounded below on compacts, then $\lambda \to H_0 + V(\lambda)$ (closure of the operator sum) is an analytic family on D.*

Remarks. (1) One of us (R.H.K.) has used this theorem as a starting point for an analysis of two-dimensional field theories with an $:\exp \alpha\phi(x):$ $|\alpha|^2 < 2\pi$ interaction density [28].

(2) The basic idea is to establish that $e^{-t(H_0+V)}$ is a contraction on L^∞ by using the Trotter product formula. See Ref. [28] for details.

In most field theoretic applications we have $V \in L^p$ for all p (see Section 3C). It thus seems of interest to state the minimal conditions needed on H_0 for $H_0 + V$ to be essentially self-adjoint.

THEOREM 2.26. *Suppose $V \in L^p$ all $p < \infty$, V is real and $e^{-tV} \in L^1$ for all $t > 0$. Let M_0 be a semibounded self-adjoint operator on L^2 so that:*

(a) *For some $s > 2$ and some C*

$$\| e^{-tM_0} \|_s \leqslant e^{Ct} \| \psi \|_s ;$$

(b) *For some* $r > 2$ *and some* $T > 0$, e^{-TM_0} *is a bounded map of* L^2 *to* L^r.

Then $M_0 + V$ *is essentially self-adjoint on* $D(M_0) \cap D(V)$.

Proof. Let us only sketch the details. By using (a), the Reisz–Thorin theorem and the argument of Proposition 2.6, we establish that the $e^{-t(M_0+V_n)}$ are uniformly bounded as maps of L^s to L^k for $s > k \geqslant 2$. By a DuHamel argument, the semigroups converge in norm as maps of L^p to L^q for any $s > p > q > 2$. By (b) and Segal's lower boundedness argument (Lemma 2.9) the $e^{-(M_0+V)}$ are uniformly bounded as maps of L^2 to L^2. Our essential self-adjointness proof (Theorem 2.16) now carries through. Q.E.D.

Segal has remarked [53] that this results hold if $e^{-tV} \in L^1$ for some fixed $t = t_0$ (and thereby for all $0 \leqslant t \leqslant t_0$). The essential self-adjointness result also carries over without any real change in such a case

PROPOSITION 2.27. *Theorem* 2.16 *holds if the condition* $e^{-tV} \in L^1$ *for all* $t > 0$ *is replaced with* $e^{-tV} \in L^1$ *for some* $t > 0$.

Finally, we remark that we will prove the following abstract result in Section 4C on the relations between the L^q bounds of $e^{-t(H_0+V)}$.

PROPOSITION 2.28. *Let* e^{-tH_0} *be a hypercontractive semigroup. Let* $V \in L^p$, *real, for some* $p > 2$; *let* $e^{-tV} \in L^1$ *for all* $t > 0$. *Suppose* $E = \inf \sigma_{L^2}(H_0 + V)$ *is the bottom of the* L^2*-spectrum of* $H = H_0 + V$. *Then, for any* q, *there is a* C *so that*:

$$\| e^{-t(H_0+V)} \|_{q,q} \leqslant Ce^{-Et}$$

for all t *large. In particular, if* $\operatorname{Re} z < E$, *then* z *is in the* L^q*-resolvant set for* $H_0 + V$ *for any* $q < \infty$. *If* $(H_0 + V)\Omega = E\Omega$ *for some* $\Omega \in L^2$, *then* $\Omega \in L^p$ *for all* $q < \infty$, *so that* $\inf \sigma_{L^q}(H_0 + V) \cap \mathbf{R} = E$ *also.*

3. HYPERCONTRACTIVE SEMIGROUPS IN FOCK SPACE

To apply the theory of Section 2 to a formal sum $H_0 + V$ arising in field theory, we need to find a representation of Fock space as $L^2(M, \mu)$ in such a way that V is an unbounded multiplication operator and then we must prove four things: (1) $\| e^{-tH_0}\psi \|_1 \leqslant \| \psi \|_1$ for all $t > 0$, (2) $\| e^{-TH_0}\psi \|_4 \leqslant C \| \psi \|_2$ for some T and C; (3) $V \in L^p$ for some $p > 2$;

(4) $e^{-tV} \in L^1$ for all $t > 0$. Since the typical field theory interactions are heuristically unbounded operators affiliated with the abelian Von Neumann algebra of the time zero smeared fields, it is natural to obtain $L^2(M, \mu)$ by taking M as the spectrum of this algebra.

In Section 3A, we present a formalism for dealing with $L^p(M, \mu)$ directly; this formalism will ignore the underlying points of M, which as the spectrum of a Von Neuman algebra is quite complicated. In Section 3B, we will show the quantization of any one particle operator $\omega \geqslant mI$ (for some $m > 0$) is the generator of a hypercontractive semigroup. This result is not new; it is explicitly proven by Segal [43] and also follows from earlier results of Glimm [18] and a limiting argument. We provide this new proof because we feel it is more elegant to never put in cutoffs or make approximations with pure point spectra; to put it more strongly, we feel it is essential to begin to deal as directly as possible with $L^p(M, \mu)$. In Section 3C, we will turn to the Fock space associated with a massive Bose field in two-dimensional space–time and will present the continuous form of Nelson's argument that interactions $V = \int g(x) :P(\phi(x)): dx$ obey (4). We will also see that various self-adjointness statements for V are easy to prove in this formalism.

A. *Coherent Vectors*

To keep this paper as self-contained as possible and thereby accessible to as large and audience as possible, let us first review "abstract" Fock space in the formalism of Cook [11] and Segal [49]. We will restrict ourselves to the Bose case. Let $\mathscr{H}$ be a real Hilbert space and let $\mathscr{F}_1$ be its complexification. Let $\mathscr{F}_0 =$ (and $\mathscr{F}_n = (\otimes_n \mathscr{F}_1)_s$, the symmetrited n-fold tensor product of $\mathscr{F}_1$. (It is the symmetrization that makes this Bose–Einstein.) Fock space is $\mathscr{F} = \bigoplus_{n=0}^{\infty} \mathscr{F}_n$; thus $\mathscr{F}$ is just the symmetric tensor algebra over $\mathscr{F}_1$.

Given $h \in \mathscr{H}$, define the creation operator $a^*(h)$ on $\mathscr{F}$ with $a^*(h) : \mathscr{F}_n \rightarrow \mathscr{F}_{n+1}$ by:

$$a^*(h)\, \psi_n = \sqrt{n+1}\, S_{n+1}(h \otimes \psi_n). \tag{3.1}$$

Here $\psi_n \in \mathscr{F}_n$ and S_m is the symmetrizer from $\otimes_m \mathscr{F}_1$ to $\mathscr{F}_m$. It is not hard to show that any $\psi_n \in \mathscr{F}_n$ is in $D([a^*(h)]^*)$ and that the adjoint of $a^*(h)$ which is denoted by $a(h)$ (the annihilation operator) acts on ψ_n by:

$$a(h)\, \psi_n = (\sqrt{n})^{-1} \langle h, \psi_n \rangle \tag{3.2}$$

where $\langle h, \ \rangle$ is defined on $\otimes_n \mathscr{F}_1$ by

$$\langle h, \phi_1 \otimes \cdots \otimes \phi_n \rangle = \sum_{i=1}^{n} \langle h, \phi_i \rangle \phi_1 \otimes \cdots \otimes \hat{\phi}_i \otimes \cdots \otimes \phi_n .$$

The vectors in any $\mathscr{F}_n$ are analytic vectors for $a(h)$ and $a^*(h)$ and on the incompleted sum, F, of the $\mathscr{F}_n$ (i.e., finite sums of vectors in some $\mathscr{F}_m$), the a's obey the canonical commutation relations:

$$[a(h), a^*(g)] = \langle h, g \rangle 1. \tag{3.3}$$

It is our desire to have 1 on the right side of (3.3) that led to the unantural looking $\sqrt{n+1}$ in (3.2). If one defines[7]:

$$\Phi(h) = a^*(h) + a(h), \tag{3.4a}$$

$$\pi(h) = (2i)^{-1}[a^*(h) - a(h)], \tag{3.4b}$$

then $\Phi(h)$ and $\pi(h)$ have all of F as analytic vectors, are symmetric and obey the Heisenberg relations:

$$[\Phi(h), \Phi(g)] = 0 = [\pi(h), \pi(g)], \tag{3.5a}$$

$$[\pi(h), \Phi(g)] = -i\langle h, g \rangle \tag{3.5b}$$

on F. As a result, Nelson's theory [43] tells us that $\pi(h)$ and $\Phi(g)$ are essentially self-adjoint on F and obey the Weyl relations:

$$e^{i\pi(h)} e^{i\Phi(g)} e^{-i\pi(h)} e^{-i\Phi(g)} = e^{i\langle h, g \rangle}. \tag{3.5c}$$

If one realizes $\mathscr{H}$ as $L^2(R, d\mu)$ so that $\mathscr{F}_n$ is the totally symmetric complex-valued functions on $\mathbf{R}^n$, then (3.1) and (3.2) take the explicit form:

$$(a^*(h)\,\psi_n)(x_1, \dots, x_{n+1}) = \frac{\sqrt{n+1}}{(n+1)!} \sum_{\pi \in \Sigma_{n+1}} h(x_{\pi(1)})\,\psi_n(x_{\pi(2)}, \dots, x_{\pi(n+1)})$$

$$= \frac{1}{\sqrt{n \times 1}} \sum_{i=1}^{n+1} h(x_i)\,\psi_n(x_1, \dots, \hat{x}_i, \dots, x_{n+1})$$

and

$$(a(h)\,\psi_n)(x_1, \dots, x_{n-1}) = \sqrt{n} \int h(x)\,\psi_n(x, x_1, \dots, x_{n-1})\, d\mu(x).$$

[7] It is more usual to put $1/\sqrt{2}$ in both Φ and π and we will use that conversion when we get down to concrete Fock space in Section 3c. To keep our formulas simple looking, we use (3.4) in Sections 3A and 3B.

Finally, one often writes Ω_0 for the element $1 \in \mathscr{F}_0$ and calls it the Fock vacuum. This completes our review of the Cook formalism.

Following Jaffe and Glimm [20], we represent $\mathscr{F}$ as $L^2(Q, d\mu)$ as follows: The operators $\{\Phi(g) \mid g \in \mathscr{H}\}$ generate a Von Neumann algebra, $\mathscr{M}$, which is a maximal Abelian Von Neumann algebra with Ω_0 as a cyclic vector. If we let Q be the spectrum of $\mathscr{M}$, then $\mathscr{M}$ is algebraically isomorphic to $C(Q) = L^\infty(Q)$ under the Gelfand map, and $\mathscr{F}$ can be represented uniquely as $L^2(Q, d\mu)$ if we require Ω_0 to go into the function 1 and that the isomorphism of $\mathscr{M}$ and $C(Q)$ be spatial.[8] Since Q is a totally disconnected space, it might be useful (and enlightening!) to find a Hilbert space dense in Q and replace μ with a (Gaussian) Hilbert space measure in one of the standard theories of Hilbert space integration [16, 24, 25]. We will avoid doing this or considering points of Q directly by employing the coherent vector formalism explained below. Since $\mu(Q) = \int 1 \, d\mu = \langle \Omega_0, \Omega_0 \rangle = 1$, we see that μ is a probability measure.

Given a finite dimensional subspace, V, of $\mathscr{H}$ and a C^∞ function of fast decrease, F, on that subspace, we define

$$\psi(F) = \left[\int dh\, F(h) \exp(i\,\Phi(h))\right] \Omega_0, \tag{3.6b}$$

where dh is Lebesgue measure on V which we leave implicit in the symbol $\psi(F)$. In analogy with quantum optics [36–38], we call such vectors (in L^2) coherent vectors. In a Hilbert space integration formalism for $L^2(Q)$, they are the cylinder functions. If one picks an orthonormal basis $h_1, \ldots, h_n$ in V and writes $F(t_1, \ldots, t_n) = F(\Sigma\, t_i h_i)$, then $\psi(F)$ can also be written:

$$\psi(F) = \hat{F}(\Phi(h_1), \ldots, \Phi(h_n))\, \Omega_0, \tag{3.6b}$$

where we have normalized the Fourier transform by

$$\hat{F}(u) = \int dh\, e^{i\langle u, h\rangle} F(h) \tag{3.7a}$$

and so its inverse is

$$\check{F}(h) = (2\pi)^{-n} \int du\, e^{-i\langle u, h\rangle} F(u). \tag{3.7b}$$

On account of (3.6b), one can immediately conclude:

[8] Equivalently, the state $\omega_0(A) = \langle \Omega_0, A\Omega_0 \rangle$ defines a positive linear functional $C(Q)$ and thus a measure on Q.

THEOREM 3.1. *$\psi(F) \in L^\infty(Q)$ and thus in each $L^p(Q)$. If $\hat{F} \geqslant 0$ (equivalently, if F is of positive type), then $\psi(F)$ is a positive Q-space function. If $\psi(F)$ is a coherent vector, then $|\psi(F)|$ (absolute value in Q-space) is a coherent vector, explicitly*:

$$|\psi(F)| = \psi(\widecheck{|\hat{F}|}). \tag{3.8}$$

Proof. Let us only prove the last statement, which we remark implies $\psi(F)$ is positive if and only if F is of positive type. Write $F = F_+ - F_-$ where $\hat{F}_+$, $\hat{F}_- \geqslant 0$ and $\hat{F}_+\hat{F}_- = 0$. Then $\hat{F}_+(\Phi(h_1),..., \Phi(h_n))\ \hat{F}_-(\Phi(h),..., \Phi(h_n)) = 0$ by the functional calculus so that $\psi(F_+)\,\psi(F_-) = 0$. Thus $\psi(F) = \psi(F_+) - \psi(F_-)$ with $\psi(F_+)\,\psi(F_-) = 0$, $\psi(F_+), \psi(F_-) \geqslant 0$ implies

$$|\psi(F)| = \psi(F_+) + \psi(F_-) = \psi(\widecheck{|\hat{F}|}) \quad \text{Q.E.D.}$$

Now, let $\mathfrak{A}$ be the algebra consisting of all operators of the form $\int dh\, F(h) \exp(i\Phi(h))$, where we integrate over all finite-dimensional spaces and $F \in \mathscr{S}$. Since $\mathscr{M}$ is *not* the norm closure of $\mathfrak{A}$, $\{\mathfrak{A}\Omega\}$ (i.e., all coherent vectors) is not dense in L^∞ but $\mathscr{M}$ is the weak closure of $\mathfrak{A}$ so that:

THEOREM 3.2. *For any $1 \leqslant p < \infty$, the set of coherent vectors is dense in L^p. The positive coherent vectors are dense in the positive L^p functions.*

Proof. This is a general result depending only on the weak denseness of $\mathfrak{A}$ in $\mathscr{M}$. It is sufficient to prove $\{\mathfrak{A}\Omega_0\}$ is $\|\ \|_p$-dense in L^∞ for each p. Let $M \in \mathscr{M}$ and let $\tilde{M}$ be the corresponding L^∞ function. By the Kaplansky density theorem, we can find $A_\alpha \in \mathfrak{A}$ with $A_\alpha \to M$ strongly and $\| A_\alpha \| \leqslant \| M \|$ for all α. Thus $(A_\alpha - M)^{2n} \to 0$ strongly for any n. In particular,

$$\| \tilde{A}_\alpha - \tilde{M} \|_{2n}^{2n} = \langle \Omega_0 , (\tilde{A}_\alpha - \tilde{M})^{2n}\, \Omega_0 \rangle \to 0.$$

As a result, $\{\mathfrak{A}\Omega_0\}$ is L^{2n} dense in L^∞ for any n integral which is enough to conclude L^p denseness for any $p < \infty$. Since $|\psi(F)|$ is a coherent vector whenever $\psi(F)$ is, the positivity statement is true. Q.E.D.

Before stating our next result, it is useful to consider the object

$$\psi_A(F) = \int dh\, F(h) \exp[i\Phi(Ah)], \tag{3.9}$$

where A is a linear transformation from V to another finite-dimensional subspace, W, of $\mathscr{H}$. By a change of variable, $\psi_A(F)$ is a coherent vector (built from a function on W) and since being of positive type is a notion invariant under change of variable $\psi_A(F) \geqslant 0$ if F is of positive type. Let us make a simple computation:

LEMMA 3.3.

$$\langle \Omega_0 , \psi_A(F)\rangle = \int dh\, F(h) \exp(-\tfrac{1}{2} \| Ah \|^2). \tag{3.10a}$$

If we consider the form $h \to \| Ah \|^2$ *and diagonalize it so it is expressed in some basis as*

$$\begin{pmatrix} a_1{}^2 & & \\ & \ddots & \\ & & a_n{}^2 \end{pmatrix}$$

and if the form $h \to \| A^{-1}h \|^2$ *is defined by*

$$\begin{pmatrix} a_1^{-2} & & \\ & \ddots & \\ & & a_n^{-2} \end{pmatrix}$$

and if $\det_V A = \prod_{i=1}^n a_i$, *then*

$$\langle \Omega_0 , \psi_A(F)\rangle = (\det_V A)^{-1}(2\pi)^{-n/2} \int du\, \hat{F}(u) \exp(-\tfrac{1}{2} \| A^{-1}u \|^2). \tag{3.10b}$$

Proof. Equation (3.10a) follows from the formula:

$$\langle \Omega_0 , \exp[i\Phi(g)]\, \Omega_0\rangle = \exp -1/2 \| g \|^2. \tag{3.11}$$

To prove (3.11), we remark that since $[a^*(g), a(g)] = -\| g \|^2$ holds on a common domain of analytic vectors, we have

$$\exp[i\Phi(g)] = \exp[ia^*(g)] \exp[ia(g)] \exp[-\tfrac{1}{2} \| g \|^2]. \tag{3.12}$$

Using $a(g)\, \Omega_0 = 0$, (3.11) follows. (Alternately, $\langle \Omega_0 , [\Phi(g)]^n\, \Omega_0\rangle$ can be explicitly calculated.) Equation (3.10b) follows from the Plancherel theorem and elementary computations of transforms of Gaussians. Q.E.D.

We are now able to explicitly compute L^p norms of coherent vectors:

THEOREM 3.4.

$$\| \psi_A(F)\|_p{}^p = \det_V A)^{-1}(2\pi)^{-n/2} \int du \, |\hat{F}(u)|^p \exp(-\tfrac{1}{2} \| A^{-1}u \|^2). \quad (3.13)$$

Proof. From (3.6b), it follows that $|\psi(F)|^p = \psi(|\hat{F}|^p)$. Thus, (3.13) follows from (3.10b) and the elementary fact $\|f\|_p^p = \langle \Omega_0, |f|^p \rangle$.
Q.E.D.

B. *Quantized Operators as Generators of Hypercontractive Semigroups*

Our goal in this section is to prove a large class of natural operators in Fock space are generators of hypercontractive semigroups. As we have mentioned before, such a result is implicit in the work of Nelson [45] and Glimm [18] and explicit in the work of Segal [53]. We wish to provide a proof which by using our coherent vector formalism uses a minimal number of cutoffs.

We again begin by reviewing a general formalism of Segal [49]. Let u be a unitary on $\mathscr{F}_1$. There is a natural induced map on $\otimes_n \mathscr{F}_1$ by $h_1 \otimes \cdots \otimes h_n \to Uh_1 \otimes \cdots \otimes Uh_n$ which leaves the symmetric tensors invariant and thus U induces a map $\Gamma(U)$ on $\mathscr{F}$ which is determined by $\Gamma(U) \upharpoonright \mathscr{F}_1 = U$ and $\Gamma(U)$ is a unitary preserving the symmetric tensor product on $\mathscr{F}$ (i.e., Γ is the natural map from automorphisms of $\mathscr{F}_1$ onto automorphisms of $\mathscr{F}$). Γ induces a natural map, $d\Gamma$ from self-adjoint operators on $\mathscr{F}_1$ to self-adjoint operators on $\mathscr{F}$ (i.e., on the Lie algebras of the unitaries) by

$$\Gamma(e^{it\omega}) = \exp[it \, d\Gamma(\omega)].$$

We will deal below only with self-adjoints $\omega : \mathscr{H} \to \mathscr{H}$. If we pass to a spectral representation where $\mathscr{H} = L^2(\mathbf{R}, d\mu)$ and $(\omega f)(k) = \omega(k) f(k)$, then $H_0 = d\Gamma(\omega)$ has the heuristic form:

$$H_0 = \int \omega(k) \, a^*(k) \, a(k) \, d\mu(k),$$

where $a^\#(k)$ is the operator valued distribution given by

$$\int h(k) \, a^\#(k) \, d\mu(k) = a^\#(h).$$

A particular simple object is $d\Gamma(1) \equiv N$ which acts as multiplication by n on $\mathscr{F}_n$: it is called "the number operator." We also remark $d\Gamma(\omega) \, \Omega_0 = 0$ for any ω and $d\Gamma(\omega) \geqslant 0$ if $\omega \geqslant 0$. For such an ω, one can explicitly compute $\exp[-t \, d\Gamma(\omega)]$ on coherent vectors:

THEOREM 3.5. *Let $\omega \geqslant 0$ be an operator on $\mathscr{H}$ and let $H_0 = d\Gamma(\omega)$. Then*:

$$e^{-tH_0}[\psi(F)] = \int dh\, F(h) \exp[-\tfrac{1}{2}\langle h, (1 - e^{-2\omega t})\, h\rangle] \exp[i\Phi(e^{-t\omega}h)]\, \Omega_0 . \quad (3.14)$$

Proof. By (3.12), $\psi(F) = \int dh\, F(h) \exp[-\frac{1}{2} \| h \|^2] \exp[ia^*(h)]\, \Omega_0$. By the fact that e^{-tH_0} preserves tensor products, we see that $e^{-tH_0}a^*(h) = a^*(e^{-t\omega}h)\, e^{-tH_0}$ on F and thus, since vectors in F are analytic vectors for $a^*(h)$, $e^{-tH_0} \exp[ia^*(h)] = \exp[ia^*(e^{-t\omega}h)]\, e^{-tH_0}$. This combined with $e^{-tH_0}\Omega_0 = \Omega_0$ implies

$$e^{-tH_0}[\psi(F)] = \int dh\, F(h) \exp[-\tfrac{1}{2} \| h \|^2] \exp[ia^*(e^{-t\omega})]\, \Omega_0 .$$

Using (3.12) in reverse, we obtain (3.14). Q.E.D.

As an immediate corollary, we recover a positivity result first proven in the infinite degree of freedom case by Glimm and Jaffe [21]:

COROLLARY 3.6. *Let $\omega \geqslant 0$ and let $H_0 = d\Gamma(\omega)$. Then e^{-tH_0} is positivity preserving on $L^1(Q)$ and thus (by Propositoin 2.1), a contraction on each $L^p(Q, d\mu)$.*

Proof. Since the positive coherent vectors in L^1 are dense in the positive L^1 functions, we need only prove $e^{-tH_0}[\psi(F)] \geqslant 0$ if $\psi(F) \geqslant 0$. By (3.8), $\psi(F)$ implies F is of positive type. Since $\omega \geqslant 0$, $\langle h, (1 - e^{-2\omega t})h\rangle$ is a positive form so that $\exp[-\frac{1}{2}, \langle h, (1 - e^{-2\omega t})h\rangle)$ is of positive type. Products of functions of positive type are of positive type so (3.14) implies $e^{-tH_0}[\psi(F)] \geqslant 0$. Q.E.D.

To prove e^{-tH_0} is hypercontractive for some class of H_0, we must look at the norm of e^{-tH_0} as a map of L^2 to L^4; our goal is obviously to prove that $\| e^{-tH_0}[\psi(F)]\|_4 \leqslant C \| \psi(F)\|_2$. Since the L^p norm is expressed in terms of $\hat{F}$, it is natural to study e^{-tH_0} as it acts on $\hat{F}$ with F lying in some fixed finite-dimensional subspace. In terms of action on F, (3.14) tells us e^{-tH_0} acts as multiplication by a Gaussian followed by a change variable. Thus, in terms of action on $\hat{F}$'s, we have convolution with a Gaussian followed by change of variable.

Explicitly, given V a finite-dimensional space, let us pick orthonormal coordinates $q_1 ,..., q_n$ which diagonalize the form $h \to \langle h, e^{-2\omega t}h\rangle$ and let $e^{-2\omega_i t}$ be the principal moments of this form. By (3.13), the $L^2(Q)$ norm of $\psi(F)$ is

$$(2\pi)^{-n/2} \int d^n q \exp(-\tfrac{1}{2}q^2)| \hat{F}(q)|^2. \quad (3.15a)$$

Suppose each $\omega_i > 0$. Then, by (3.14),

$$e^{-tH_0}[\psi(F)] = \int dh(KF)(h) \exp[i\Phi(e^{-t\omega}h)]\, \Omega_0$$

where

$$\widehat{KF}(q_1, \ldots, qn) = (2\pi)^{-n/2} \prod_{i=1}^{n} (1 - e^{-2\omega_i t})^{-1/2}$$

$$\times \int d^n q' \hat{F}(q') \prod_{i=1}^{n} \exp[-1/2(1 - e^{-2\omega_i t})^{-1}(q^i - q'^i)^2. \quad (3.15b)$$

Finally, again using (3.13), the L^4-norm of $e^{-tH_0}[\psi(F)]$ is

$$(2\pi)^{-n/2} \int d^n q' \exp(-\tfrac{1}{2}q'^2) | \widehat{KF}(e^{-\omega_1 t} q_1, \ldots)|^4. \quad (3.15c)$$

Equation (3.15) is equivalent to proving a bound on the map $P_t : L^2(\mathbf{R}^n, (2\pi)^{-n/2} e^{-q^2} d^n q) \to L^4(\mathbf{R}^n, (2\pi)^{-n/2} e^{-q^2} d^n q)$ given by

$$(P_t f)(q_1, \ldots, q_n) = (2\pi)^{-n/2} \int d^n q' \exp(-\tfrac{1}{2}q'^2) \left[\prod_{i=1}^{n} p_{\omega_i t}(q_i, q_i')\right] f(q_1', \ldots, q_n') \quad (3.16a)$$

with

$$p_s(q, q') = (1 - e^{-2s})^{-1/2} \exp\left[-\frac{1}{2} \frac{(q - e^{-s}q)^2}{(1 - e^{-2}s)} + \frac{1}{2} q'^2\right] \quad (3.16b)$$

We have thus found a Fock space proof of Mehler's formula (3.16). We remark that if one supresses the temptation of writing $e^{-1/2q^2}$ in the measure in (3.16a) thereby eliminating the final $\tfrac{1}{2}q'^2$ in (3.16b), one sees quite clearly the structure of the kernel p_s ; it is just convolution followed by a change of variable, something we have already noted is implicit in (3.14).

Since we now have essentially a finite degree of freedom problem, the general proof of boundedness from L^2 to L^4 follows from arguments of Nelson [45], Glimm [18] and Segal [53], which we briefly sketch:

(1) (Nelson [45]). Let

$$a_s^2(q) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} [p_s(q, q')]^2 e^{-1/2q'^2} dq'.$$

Let p_s be the operator on $L^2(\mathbf{R})$ generated by the kernel $p_s(q, q')$; then, by the Cauchy–Schwartz inequality, $|(p_s\psi)(q)| \leqslant a_s(q) \| \psi \|_2$, so p_s is a bounded map of $L^2(\mathbf{R}_2)$ to $L^4(\mathbf{R})$ if $(2\pi)^{-1/2} \int a_s^4(q)\, e^{-1/2q^2} dq < \infty$. By explicit computation of the integrals [45], this is so if $e^{2s} > 3$.

(2) (Glimm [18]). Following Glimm, we remark that since n kernels enter in (3.16a), we expect the nth power of any $\| \|_{2,4}$-bound to enter; thus, to get bounds uniform in n, we need to show p_s is actually a contraction. Now, we note that if the one-particle space $\mathscr{H}$ were one dimensional, p_s would be the kernel of e^{-Ns} (N = number operator) so the p_s generate a semigroup e^{-Ns}, and, in addition, since $N \leqslant 1$ on $\bigoplus_{n=1}^{\infty} \mathscr{F}_n = [\Omega_0]^1$. This can be used to prove p_s is a *contraction* from $L^2(\mathbf{R}, (2\pi)^{-\frac{1}{2}} e^{-1/2q^2} dq)$ to $L^4(\mathbf{R}, (2\pi)^{-\frac{1}{2}} e^{-1/2q^2} dq)$ for $S \geqslant G$, some universal constant; for details, see Ref. [18, p. 19]. We only remark that crude numerical estimates show $G < 2$.

(3) (Segal [53]). Finally, we must show P_t defined by (3.16a) is a contraction from $L^2(\mathbf{R}^n, (2\pi)^{-\frac{1}{2}} e^{-1/2q^2} dq)$ to $L^4(\mathbf{R}^n, (2\pi)^{-\frac{1}{2}} e^{-1/2q^2} dq)$. This fact, which can be proven by direct computation[9], follows from a beautiful lemma of Segal which is so nice we cannot overcome the temptation presenting it to the reader:

LEMMA 3.7 (Segal [53]). *Let (M, μ) and (S, ν) be two measure spaces and let $T : L^p(M) \to L^r(M)$ and $K : L^p(S) \to L^r(S)$ be given by kernels $T(m, m')$, $K(s, s')$. Let $T \otimes K$ be defined by $[(T \otimes K) f](m, s) = \int T(m, m') K(s, s') f(m', s') \, d\mu(m') \, d\nu(s')$. If T is nonnegative (as a function on $M \times M$), then $\| T \otimes K \|_{p,r} \leqslant \| T \|_{p,r} \| K \|_{p,r}$.*

Proof. Since T is nonnegative, for any Banach space whatever $T_B : L^p(M; B) \to L^r(M; B)$ given by

$$(T_B f)(m) = \int T(m, m') f(m') \, d\mu(m')$$

is bounded by $\| T \|_{p,r}$; in particular, consider

$$T_{L^p(S)} : L^p(M, L^p(S)) \equiv L^p(M \times S) \to L^r(M, L^p(S)).$$

K induces a map $\tilde{K} : L^r(M, L^p(S)) \to L^r(M, L^r(S)) \equiv L^r(M \times S)$ which only acts on the fibers, so it has bound $\| K \|_{p,s}$. Since $T \otimes K = \tilde{K} T_L p_{(S)}$, the lemma is proven. Q.E.D.

We are thus able to conclude that as long as the numbers $\omega_i T$ are uniformly larger than G, e^{-TH_0} is a contraction from $L^2(Q)$ to $L^4(Q)$. This will happen if $\omega \geqslant cI$ for some $c > 0$; we have thus proven the major result of this section:

[9] J. Glimm, private communication.

THEOREM 3.8 (Glimm [18], Segal [53]). *Let* $H_0 = d\Gamma(\omega)$ *with* $\omega \geqslant cI$ (*for some* $c > 0$). *Then* e^{-tH_0} *is a contraction from* $L^2(Q, d\mu)$ *to* $L^4(Q, d\mu)$ *if* $t \geqslant Gc^{-1}$. *In particular,* e^{-tH_0} *is a hypercontractive semigroup in this case.*

As a simple application of this result, we consider the relation between finite particle vectors and the L^p spaces:

THEOREM 3.9. *Let* $\psi \in \mathscr{F}_n$. *Then* $\psi \in L^p$ *for any* $p < \infty$ *and explicitally*:

$$\| \psi \|_{2^{m+1}} \leqslant e^{+nmG} \| \psi \|_2 . \tag{3.17}$$

The finite particle vectors F are dense in each L^p ($p < \infty$); *in fact, for any* $H_0 = d\Gamma(\omega)$ ($\omega \geqslant cI$ *for some* $c > 0$), *the vectors in F which are analytic vectors for* H_0 *are dense in each* L^p ($p < \infty$).

Proof. By the proof of Proposition 2.2, e^{-mGN} is a contraction from L^2 to $L^{2^{m+1}}$ which proves (3.17). To prove the last statement, fix p and note that for T sufficiently large, e^{-TH_0} is a contraction from L^2 to L^p, so $\{e^{-TH_0}\psi \mid \psi \in F\}$ is a set of vectors in F which are analytic vectors for H_0. Moreover, since F is dense in L^2 and e^{-TH_0} has no kernel in L^q ($q^{-1} + p^{-1} = 1$) this last set is dense in L^p.

Remarks. (1) If $\psi \in \bigoplus_{m=1}^{n} \mathscr{F}_m$, we still have (3.17). Since $\| \psi \|_{2^{m+1}} \leqslant \| e^{+NmG}\psi \|_2 \leqslant e^{+nmG} \| \psi \|_2$.

(2) By using the Stein interpolation theorem, (3.17) holds for all m. Since they are unbounded operators affiliated with $\mathscr{M}$ we think of the fields as unbounded multiplication operators. Since $\langle \Omega_0 , [\Phi(h)]^{2n} \Omega_0 \rangle < \infty$, we should have $\Phi(h) \in L^p$ for all $p < \infty$. In fact, we can be pedantic and prove:

PROPOSITION 3.10. *The operator* $\Phi(h)$ (*for any* $h \in \mathscr{H}$) *is an unbounded multiplication operator on* $L^2(Q)$; *explicitly*

$$\| \Phi(h) \|_{2n}^{2n} = \frac{(2n)!}{2^n n!} \| h \|^{2n}.$$

Proof. Let P_n be the spectral projection for $\Phi(h)$ associated with the interval $[-n, n]$. Since $\Omega_0 \in D^m(\Phi(h))$ for any m, $(\Phi(h)P_n)^m\Omega_0 \to (\Phi(h))^m\Omega_0$ in norm. Now $\Phi(h)P_n \in \mathscr{M}$ so $\Phi(h)P_n$ is a multiplication operator by a function $\widetilde{\Phi(h)P_n}$. By the norm convergence of $(\Phi(h)P_n)^m\Omega_0$ we conclude that $\widetilde{\Phi(h)P_n}$ is L^{2m} Cauchy for any m so for any $p < \infty$, $\widetilde{\Phi(h)P_n} \to \widetilde{\Phi(h)}$ in L^p for some function $\widetilde{\Phi(h)}$.

Since any $\psi \in F$ is in all L^p, $\psi \in D(\widetilde{\Phi(h)})$. Thus

$$\widetilde{\Phi(h)}\,\psi = \lim_n \widetilde{\Phi(h)}\,P_n\,\psi = \lim_n \Phi(h)\,P_n\psi = \Phi(h)\,\psi,$$

since $\psi \in D(\Phi(h))$ also. Thus $\Phi(h) = \widetilde{\Phi(h)}$ on F. Since F is a core for $\Phi(h)$ and $\widetilde{\Phi(h)}$ is self-adjoint, $\Phi(h) = \widetilde{\Phi(h)}$. Q.E.D.

Finite particle vectors are still analytic (although no longer entire) vectors for $[\Phi(h)]^2$; but for $n > 2$ they are no longer even analytic for $[\Phi(h)]^n$. Nonetheless, we have:

THEOREM 3.11. *Let A be a polynomial in $\Phi(h_1),\ldots,\Phi(h_n)$ (which is well defined as a multiplication operator in all L^p, $p < \infty$). Then F is a core for A; in fact, if $H_0 = d\Gamma(\omega)$ ($\omega \geqslant cI$, $c > 0$), the vectors in F which are analytic for H_0 are a core for A.*

Remark. It is not hard to show $A \leqslant c(N+1)^m$ for some m, so that $C^\infty(N) \subset D(A)$ and that $C^\infty(N)$ is invariant under A (by the above, it is a core). $C^\infty(H_0) \subset D(A)$ will hold in general, but $C^\infty(H_0)$ will *not* be invariant in general.

This last theorem follows from the elementary lemma.

LEMMA 3.12. *Let (M, μ) be a finite measure space. Let $V \in L^p$ ($p > 2$). If $\mathscr{D}$ is dense in L^q with $p^{-1} + q^{-1} = 1/2$, then $\mathscr{D}$ is a core for V.*

Proof. By Hölder's inequality, $D(\overline{V/\mathscr{D}}) \supset L^q \supset L^\infty$. Since L^∞ is clearly a core for V, the lemma is proven. Q.E.D.

C. *Wick Polynomials in Two-Dimensional Space–Time*

We now turn to specific interactions in a specific Fock space. As the one-particle space $\mathscr{H}$, we take $L^2(\mathbf{R}, dk)$ (we think of k as a "momentum space variable" conjugate to the space part of a two-dimensional space–time). We view $h \to a(h)$ and ($h \in \mathscr{H}$ is real-valued) $h \to a^*(h)$ as operator valued distributions [64] which we write as a (k) and $a^*(k)$. We then define the operator valued distribution[10]:

$$\varphi(x) = (2\pi)^{-1/2}(\sqrt{2})^{-1} \int e^{-ikx}[a^*(k) + a(-k)]\,\omega(k)^{-1/2}\,dk, \tag{3.18a}$$

with

$$\omega(k) = (k^2 + m^2)^{1/2}, \tag{3.18b}$$

m some fixed constant.[11]

[10] We form the maximal Abelian algebra $\mathscr{M}$ and the coherent vectors $\psi(F)$ using smeared x-space fields, $\varphi(h) = \int h(x)\,\varphi(x)\,dx$.

[11] m is the mass of the "bare", i.e., noninteracting, particle. Equation (3.18b) is just the relatavistic energy momentum relation.

Besides the normalization factors and the $\omega(k)^{-1/2}$ (which we discuss below) $\varphi(x)$ differs from $\Phi(k)$ in an additional crucial aspact: Φ is chosen so it is self-adjoint when smeared with a real-valued k-space function, while φ is chosen to be self-adjoint when smeared with a function real in x-space; this is the reason for the $a(-k)$ in (3.18a).

Let ω be the one-particle operator which is multiplication by $\omega(k)$ and let:

$$H_0 = d\Gamma(\omega) = \int a^*(k)\, a(k)\, \omega(k)\, dk. \tag{3.18c}$$

By our discussion in Section 3B, H_0 is the generator of a hypercontractive semigroup since $\omega \geqslant mI$.

Parenthetically, we remark that the $\omega(k)^{-1/2}$ in (3.18a) is chosen for a simple reason: If one defines

$$\varphi(x, t) = e^{iH_0t}\varphi(x)\, e^{-iH_0t} = (4\pi)^{-1/2} \int e^{-ikx+it\omega(k)}[a^*(k) + a(-k)]\, \omega(k)^{-1/2}\, dk,$$

φ is made Lorentz invariant by the factor $\omega(k)^{-1/2}$ in the sense that $\mathscr{F}$ supports a unitary representation $\Lambda \to U(\Lambda)$ of the Lorentz group obeying $U(\Lambda)\, \varphi(x, t)\, U(\Lambda)^{-1} = \varphi(\Lambda(x, t))$.

To get Lorentz invariant interactions, it is natural to try to work with objects like $\varphi^n(x)$. Unfortunately, powers of distributions are often meaningless and $\varphi^n(x)$ is no exception. There is, however, a simple modification of $\varphi^n(x)$ which is well-defined. If one writes out say $\varphi^2(x, t)$ formally in terms of (3.18a), smears with some function of x and t tries to apply it to the vacuum, one finds that the only misbehaved term is the aa^* term; for $\varphi^n(x, t)$ all $a^* \cdots a^*\, a \cdots a$ terms are well behaved. This suggests one define an object: $\varphi^n(x, t)$: by ad hoc changing[12] any $a^\#(k_1) \cdots a^\#(k_n)$ into a term with all a^*'s on the left and a's on the right. Such a definition was proposed by Wick [61] and Gårding and Wightman [64] have shown $\varphi^n(x, t)$: so defined for an m-dimensional space–time is an operator-valued distribution smeared over space *and time*. What makes two-dimensional space–time special is that for the free field of mass m [(3.18a)], the Wick product is well-defined when only smeared in space [29, 51], explicitly, one defines:

$$\int g(x) : \varphi^n(x) : dx = (2\pi)^{-n/2}\, 2^{-n/2} \sum_{j=0}^{n} \binom{n}{j} \int a^*(k_1) \cdots a^*(k_j)$$
$$\times\, a(-k_{j+1}) \cdots a(-k_n)\, \hat{g}\left(\sum_{i=1}^{n} k_i\right) \prod_{i=1}^{n} \omega(k_i)^{-1/2}\, dk_i\,. \tag{3.19}$$

[12] We adopt the symbol $a^\#$ to stand for a or a^* in the same way $\pm$ stands for $+$ or $-$.

A priori, this is only a formal definition but one has:

LEMMA 3.13. *If $W(k_1, ..., k_n) \in L^2(\mathbf{R}^n)$, then for any j,*

$$\int w(k_1, ..., k_n)\, a^*(k_1) \cdots a^*(k_j)\, a(-k_{j+1}) \cdots a(-k_n)\, dk_1 \cdots ak_n$$

is well-defined applied to any finite particle vector. Its adjoint is an extension of $\int W(k_1, ..., k_n)\, a^*(-k_n) \cdots a(k_1)\, dk_1 \cdots dk_n$ defined on finite particle vectors. If $W_n \to W$ in $L^2(\mathbf{R}^n)$ and $\psi \in F$, then

$$\int W_m(k_1, ..., k_n)\, a^*(k_1) \cdots a(-k_n)\, \psi \to \int W(k_1, ..., k_n)\, a^*(k_1) \cdots a(-k_n)\, \psi$$

in norm.

Proof. An elementary computation; see Ref. [4] or [20].

LEMMA 3.14. *If $g \in L^2(\mathbf{R})$, then $\hat{g}(\sum_{i=1}^n k_i) \prod_{i=1}^n \omega(k_i)^{-1/2} \in L^2(\mathbf{R}^n)$.*

Proof.

$$\prod_{i=1}^n \omega(k_i)^{-1/2} \leqslant \sum_{j=1}^n \prod_{i \neq j} [\omega(k_i)^{-1/2}]^{n/n-1}$$

and

$$\int dk_1 \cdots dk_n \prod_{i \neq n} [\omega(k_i)^{-1}]^{n/n-1}\, |\hat{g}(\Sigma k_i)|^2 < \infty. \qquad \text{Q.E.D.}$$

Notice that the fact that dk_i is a one-dimensional integral (i.e., that space–time is two-dimensional) is critical here and also that the L^2 norm of $\hat{g}\pi\omega(K_i)^{-1/2}$ is bounded by $C \| g \|_2$. We thus see [29] that the formal object defined by (3.19) is a symmetric operator on the finite particle operators. Our main goal in this section is to prove the following result ((a) has been proven by different means by Lanford and by Jaffe and Doplicher [62] and by Segal [51]; (b) and (c) are implicit in Glimm [18] and Segal [53]; a weak form of (d) appears in [20]):

THEOREM 3.15. *Let $P(X) = \sum_{n=0}^N a_n X^n$ be a polynomial which is bounded below (i.e., N is even and $a_N > 0$) and let $g \in L^1 \cap L^2(\mathbf{R})$, $g \geqslant 0$. Let V be defined by:*

$$V = \int g(x) : P(\varphi(x)) : dx = \sum_{n=0}^N a_n \int g(x) : \varphi^n(x) : dx.$$

Then:

(a) *V is essentially self-adjoint on F (and on $C^\infty(H_0) \cap F$).*

(b) *The closure of V, which we also call V, is a multiplication operator by a function $V \in L^p(Q, d\mu)$ for any $p < \infty$.*

(c) *For any $t > 0$, $e^{-tV} \in L^1(Q, d\mu)$.*

(d) *If g has support in a set $(-a, a)$, then e^{iVt} is in $\mathscr{M}_{(-a,a)}$, where $\mathscr{M}(-a, a)$ is the weak closure of the bounded functions of the fields smeared with functions f with support in $(-a, a)$.*

The idea of the proof is the following: With any function $h \in \mathscr{S}$, Schwarz's space of functions of fast decrease, we associate a cutoff field $\phi_h(x) = \int h(x - y)\, \phi(y)\, dy$ and a cutoff interaction $V_h = \int dx\, g(x) :P(\phi_h(x)):$. We will see $:P(\phi_h(x)):$ is a polynomial in the smeared field $\phi_h(x)$, so $V_h \in L^p(Q, d\mu)$ for each p (since $g \in L^1$). We will also see that the V_h are L^p-Cauchy as $h(x) \to \delta(x)$, so there is a function $\tilde{V} \in L^p$ all $p < \infty$ with $V_h \to \tilde{V}$ in $\| \ \|_p$. Then, as in Proposition 3.10, V is essentially self-adjoint and its closure is $\tilde{V}$. Finally, we will prove (c) following Nelson's cutoff argument [45] by showing each V_h is bounded below and using explicit estimates on $\| V - V_h \|_p$.

LEMMA 3.16. *Let h_n be any sequence of functions in $L^2(\mathbf{R})$ so that* (i) *$\| \hat{h}_n \|_\infty$ is bounded,* (ii) *$h_n(k) \to 1$ pointwise a.e. Let $P(x)$ be any polynomial and let $g \in L^2(\mathbf{R}) \cap L^1(\mathbf{R})$. Let*

$$V = \int g(x) : P(\phi(x)) : dx$$

and

$$V_n = \int g(x) : P(\phi_{h_n}(x)) : dx.$$

Then:

(a) *If ψ is any finite particle vector, $V_n\psi \to V\psi$ in $\| \ \|$.*

(b) *The V_n are multiplication operators in L^p and are Cauchy in each $L^p(Q)$ space and in fact*

$$\| V_n - V_m \| 2^{k+1} \leqslant e^{kdG} \| V_n - V_m \|_2 , \tag{3.20}$$

where d is the degree of P.

Proof. The effect of changing $\int :\phi^n(x): g(x)\, dx$ into $\int :\phi_h{}^n(x): g(x)\, dx$ in the definition (3.19) is to replace the kernel $\hat{g}(\sum_{i=1}^n k_i) \prod_{i=1}^n \omega(k_i)^{-1/2}$

with $\hat{g}(\sum_{i=1}^{n} k_i) \prod_{i=1}^{n} \omega(k_i)^{-1/2} \hat{h}(k_i)$. Under the assumptions on the h_n's, the kernels converge in $L^2(\mathbf{R}^n)$ so (a) follows from Lemma 3.15. We will see below [Eq. (3.21)] that $:\phi_h{}^n(x):$ is a polynomial in $\phi(x)$ so it is a multiplication operator in each L^p. Since $g \in L^1$, $V_n \in L^p$. We know $V_n\Omega_0 \to V\Omega_0$ in $\| \ \|$, so $V_n\Omega_0$ is $\| \ \|$-Cauchy, i.e., $V_n(Q)$ is $L^2(Q)$-Cauchy. Moreover, from (3.19), $V_n\Omega_0 \in \bigoplus_{i=1}^{d} \mathscr{F}_i$ so (3.20) follows from (3.17). That $\| V_n - V_m \|_p \to 0$ follows now from the $L^2(Q)$. Cauchy condition. Q.E.D.

Corollary 3.17. *Let P, g, and V be as in Lemma* 3.16. *Then* (a), (b), *and* (d) *of Theorem* 3.15 *follow.*

Proof. The V_n converge in all $L^p(Q)$ norm to some function $\tilde{V}$ ($p < \infty$). By (a) of Lemma 3.16, $\tilde{V}\psi = \lim V_n\psi = V\psi$ for any $\psi \in F$ and by Theorem 3.9, $\tilde{V}$ is essentially self-adjoint on F. Finally, to conclude (d) of Theorem 3.15, pick h_n so that $h_n(x-y)$ has support in $(-a, a)$ whenever $y \in \sup g$. Then $e^{iV_nt} \in \mathscr{M}(-a, a)$. Since $V_n \to V$ strongly on a common core, $e^{iV_nt} \to e^{iVt}$ strongly by the Trotter–Kato theorem. Thus $e^{iVt} \in \mathscr{M}_{(-a,a)}$. Q.E.D.

Finally, we need to prove $e^{-Vt} \in L^1(Q)$ and this requires explicit lower bounds on the V_n's which in turn require explicit formulas for $:\varphi_h{}^n(x):$ in terms of $\varphi_h{}^n(x)$. An expression for $\varphi(x_1) \cdots \varphi_h(x_n)$ in terms of $:\varphi_{i_1}(x_{i_1}) \cdots \varphi_{i_m}(x_{i_m}):$ is due to Wick [61] (and is known universally as Wick's theorem). One can invert Wick's theorem to express the Wick products in terms of ordinary products. This has been done explicitly by Cainiello [7]. (See also Eg. (3.24) of Ref. [64]). If one then smears with h and looks for $:\varphi_h{}^m(x):$, one finds

$$:\varphi_h{}^m(x): = \sum_{r=0}^{[m/2]} (-1)^r \frac{m!\, 2^{-r}}{(m-2r)!\, r!} \| \varphi(h)\, \Omega_0 \|^{2r} [\varphi_n(x)]^{m-2r}. \tag{3.21}$$

Equation (3.21) and elementary algebra, then implies:

Proposition 3.18. *Let $P(X)$ be a polynomial of degree d which is bounded below. Then, there are constants c_1 and c_2 so that*

$$:P(\varphi_h(x)): \geqslant -c_1 - c_2 \| \varphi(h)\, \Omega_0 \|^d. \tag{3.22}$$

In particular, if h_K is the function whose Fourier transform is the characteristic function for $\{k \mid |k| \leqslant K\}$ and $V_{K,g} = \int g(x) :P(\varphi_{h_K}(x)): dx$ for $g \in L^1 \cap L^2$, $g \geqslant 0$, then for K big (independent of g):

$$V_{K,g} \geqslant \| g \|_1 [-C_3 (\log K)^{d/2}]. \tag{3.23}$$

Proof. Equation (3.22) is arithmetic. To prove (3.23), we compute

$$\| \varphi(h_k)\, \Omega_0 \|^2 = (4\pi)^{-1} \| \Phi(\hat{h}_K \omega(k)^{-1/2}) \|^2$$
$$= (4\pi)^{-1} \int_{|k|>K} \omega(k)^{-1}\, dk \leqslant C(\log K). \qquad \text{Q.E.D.}$$

To apply Nelson's method we also need to know $\| V - V_k \|\, 2^{m^2}$:

LEMMA 3.19. *Under the assumptions of Proposition* 3.18,

$$\| V - V_k \|_{2^{m+1}}^{2^{m+p}} \leqslant (C_4 \| g \|_2{}^2\, K^{-\alpha} e^{mdG})^{2m} \tag{3.24}$$

for some $\alpha > 0$ *and* C_4 *independent of* g.

Proof. By (3.20), we only need prove the formula when $m = 0$. Since V is a linear combination of objects of the form $\int g(x) :\varphi^n(x): dx$, we need only consider that case. By (3.19),

$$\left\| \int dx\, g(x)[:\phi^n(x): - :\phi^n_{h_K}(x):] \right\|_2$$
$$= C \int \text{some } |k_i| \geqslant K \left| \hat{g}\left(\sum k_i\right)\right|^2 \prod_{i=1}^{n} \omega(k_i)^{-1}\, dk_i$$
$$\leqslant C^1 K^{-\alpha} \int \left| \hat{g}\left(\sum K_i\right)\right|^2 \prod_{i=1}^{n} \omega(K_i)^{-1+\alpha}\, dk_i\,.$$

By taking α small, this last integral is finite and bounded by $C'' \| g \|_2$ (as in the proof of Lemma 3.14).

Remark. Rosen [47] has shown that (3.24) with $K^{-\alpha}$ replaced $K^{-1}(\log K)^d$ is true. All one needs is some $\alpha > 0$.

We can now prove the result implicit in Nelson [45]:

THEOREM 3.20. *Let* P, g *be as in Theorem* 3.15 *and let*

$$V_g = \int g(x) :P(\phi(x)): dx.$$

Then for any $t > 0$, *and any* $q < \infty$, $e^{-tV} \in L^q(Q, d\mu)$. *Moreover* (*for fixed* P), $\| e^{-tV} \|_q$ *has a bound only dependent on* t, q, $\| g \|_1$, *and* $\| g \|_2$.

Proof (Nelson [45]). We only sketch the proof; for details, see Ref. [45] or [18]. The idea is simple; for any $X > 0$, pick K so that $X = \| g \|_1[-C_3(\log K)^{d/2}]$.

Then, by (3.23),

$$\mu\{q \mid V(q) \leqslant -X - 1\} \leqslant \mu\{q \mid |V(q) - V_k(q)| \geqslant 1\} \leqslant \| V - V_k \|_{2^m}^{2^m}.$$

For fixed K, one picks m to minimize the bound on $\| V - V_k \|_{2^m}^{2^m}$ given by (3.24). One then has sufficient control on $\mu\{q \mid V(q) \geqslant -X - 1\}$ to prove $\int e^{-Vtp}\, d\mu < \infty$. Q.E.D.

4. Spatially Cutoff Two-Dimensional Bose Field Theories

In this section, we wish to discuss in detail some aspects of the applications to $:P(\varphi):$ field theories in two-dimensional space–time. Many of the results we describe have been obtained for $(\varphi^4)_2$ theories by Glimm and Jaffe [20–22]. We describe our results explicitly for two reasons. First, we wish to emphasize which results depend merely on discussing $H_0 + V$ with e^{-tH_0} hypercontractive and $V \in L^p$ (some $p > 2$), $e^{-tV} \in L^1$ (all $t > 0$), which results depend on the additional property that $H_0 = d\Gamma(\omega)$ with $\omega \geqslant cI$ (some $c > 0$) and which results seem to make essential use of higher order estimates and which are thereby not accessable by our methods. We also wish to begin a study of coupling constant analyticity.

A. *Lower Bounds and Self-Adjointness*

By combining Corollary 2.14 with Theorems 3.8 and 3.15, we immediately find:

Theorem 4.1. *Let $\varphi(x)$ be a two-dimensional free field of mass m. Let $V = \int g(x) :P(\varphi(x)): dx$, where $g \in L^1 \cap L^2$; $g \geqslant 0$ and $P(X)$ is a polynomial which is bounded below. Let $N_\tau = d\Gamma(\omega^\tau)$, where ω is the operator of multiplication by $\omega(k)$ [(3.18b)]. Then $N_\tau + V$ is bounded below for any $\tau \geqslant 0$.*

Remark. Such $N_\tau + V$ estimates have already been noted by several authors, e.g., Jaffe [31].

In terms of the Glimm constant G one can explicitally write (for $\lambda > 0$):

$$E_{\lambda g} \equiv \inf[\mathrm{spec}(H_0 + \lambda V)] \geqslant -\frac{m}{2G} \log\langle \Omega_0, e^{-2GV\lambda/m}\Omega_0\rangle. \tag{4.1}$$

This should be compared with the formal perturbation theory result [6]:

$$E_{\lambda g} = -\log \left\langle \Omega_0, T\left[\exp - \int_{-\infty}^{\infty} V(t)\right] \Omega_0 \right\rangle. \tag{4.2}$$

We note the formal similarity and remark that (4.1) has a formal perturbation expansion in terms of connected diagrams. Since $\langle \Omega_0, e^{aV}\Omega_0 \rangle (a > 0)$ is not finite, this formal perturbation series diverges; this fact illuminates the known fact that the formal perturbation series coming from (4.2) diverges [30].

Our essential self-adjointness result (Theorem 2.16) implies:

THEOREM 4.2. *Let V be as in Theorem 4.1 and let $H_0 = d\Gamma(\omega)$. Then $H_0 + V$ is essentially self-adjoint on $D(H_0) \cap D(V)$. Moreover, the following sets are cores for H*[13]:

(a) $\mathcal{A}(H) \cap C^\infty(V)$;

(b) $\mathcal{A}(H) \cap \sum_{p<\infty} L^p(Q, d\mu)$;

(c) $C^\infty(N) \cap C^\infty(V) \cap D(H)$;

(d) $F \cap D(H_0)$;

(e) $C^\infty(N) \cap D(H_0)$.

Note. So far as we can determine, one cannot show $D(H_0^2) \cap D(H)$ is a core for H without proving some higher order estimate. It is likely that without some smoothness assumptions on V (i.e., in the abstract setting of Section 2), one cannot prove such a result[14]. In the $(\varphi^4)_2$ case Glimm–Jaffe [20] and in the $(\varphi^{2n})_2$ case Rosen [48] have proven $C^\infty(H_0)$ is a core for $H_0 + H_I(g)$, by utilization of higher order estimates.

We remark to the reader that essential self-adjointness enters crucially into one removal of the spatial cutoff as follows: One expects influence in the theory to propagate only at the speed of light, so Guenin [26] suggested $[\exp i(H_0 + H_I(g))t]\, A[\exp - i(H_0 + H_I(g))t]$ shouldn't depend on what g is outside the interval[15] $[a - |t|, b + |t|]$, if A is in the Von Neumann algebra $\mathcal{U}_{(a,b)}$ generated by the spectral projections of the fields and conjugate momentum smeared with functions having support in (a, b). Segal [50] then showed that Guenin's suggestion could be proven by using the Trotter product formula for $\exp i(H_0 + H_I(g))t$ and the fact that $\exp iH_I(g)t \in \mathcal{M}_{(a,b)}$ if g has support in (a, b). Essential self-adjointness is then crucial since it is needed to justify the Trotter product formula. We do not present the details of the Guenin-Segal proof since the discussion in Jaffe–Glimm [20] cannot be improved upon; we commend it to the

[13] We use $\mathcal{A}(H)$ to denote the analytic vectors for H.

[14] However, without any smoothness assumption on V, $\mathcal{A}(H_0) \cap C^\infty(V)$ is dense if V is in all L^p, $p < \infty$.

[15] We have set c, the speed of light, equal to 1.

reader interested in the details. We however note the final result in a slightly stronger form suggested to us by J. Glimm[16]:

THEOREM 4.3. *Let $g_n(x)$ be positive, piecewise continuous functions in $L^1 \cap L^2$ so that $g_n(x) \to \lambda$ uniformly on compacts. Let:*

$$\alpha_{n,t}(A) = [\exp it(H_0 + H_I(g))]\, A[\exp - it(H_0 + H_I(g))]$$

with

$$H_I(g) = \int g(x) :P(\varphi(x)): dx,$$

where P is a polynomial which is bounded below. Then,

$$\lim_{n\to\infty} \alpha_{n,t}(A) \equiv \alpha_t(A)$$

exists for any A in $\mathscr{U}$, the norm closure of $\bigcup \mathscr{U}_{(a,b)}$ and is independent of the sequence g_n chosen.

Thus, one can use $g_n(x) = \lambda \exp(-x^2/n)$ for example and so let $g \to \lambda$ by a family with an analytic interpolation. By our analytic perturbation theory (Section 2D) and the analytic perturbation theory for eigenvalues [35], one obtains a family of approximate vacuum states w_κ (see Section 4B) analytic in a neighborhood of the real axis, with the "thermodynamic limit" obtained by $\kappa \to \infty$.

As a final subject in this section, we see what we can say about domains for the fields. Given a function f in $L^2(\mathbf{R})$, we define

$$\varphi(f, t) = e^{iHt}\varphi(f)\, e^{-iHt}. \tag{4.3}$$

Let $\epsilon(H)$ be the set of entire vectors for H. For the sharp time fields (4.3), one has:

THEOREM 4.4. *Let $H = H_0 + V$ as in Theorem 4.1. Then, for any $f \in L^2$, any fixed t and any n, $\epsilon(H) \subset D(\varphi(f, t)^n)$ and is a core for $\varphi(f, t)^n$.*

Proof. Since e^{iHt} leaves $\epsilon(H)$ invariant, we need only prove the result for $t = 0$. Since $\varphi(f)^n$ is an operator in L^p, all $p < \infty$, we need only prove $\epsilon(H)$ is dense in some L^p, $p > 2$. But $\epsilon(H)$ is dense in L^2 and $e^{-tH}[\epsilon(H)] = \epsilon(H)$. Since e^{-tH} takes L^2 into $L^{2+\epsilon}$, we see $\epsilon(H)$ is dense in $L^{2+\epsilon}$. Q.E.D.

[16] Private communication.

Remark. We could just as well deal with vectors of compact energy as with the larger set $\epsilon(H)$.

In space-times of dimensions bigger than 3, there are strong indications that the sharp time smeared fields won't be well-defined but that fields smeared in space and time will be well-defined. It is thus of interest to consider such objects even in the two-dimensional case. Given $f \in \mathscr{S}(\mathbf{R}^2)$, we define

$$\varphi(f) = \int dt\, \varphi(f_t, t), \tag{4.4}$$

where f_t is the function $x \to f(x, t)$ in $\mathscr{S}(\mathbf{R})$.

It is not hard to see that $\varphi(f)$ is defined on $\epsilon(H)$ but a proof of essential self-adjointness requires bounds on the fields discussed in Refs. [21] and [48].

B. *The Vacuum Vector for the Spatially Cutoff Theory*

Our first goal in this section will be to prove that a class of operators $H_0 + V$ have ground states by a method which is essentially an abstraction of the method used by Jaffe and Glimm in the $(\varphi^4)_2$ case [21]. We then present a simplified proof of the nondegeneracy of this ground state whose basic principle is again due to Jaffe and Glimm [21], but which uses crucially the irreducibility isolated by Segal [52]. We finally discuss, somewhat speculatively certain properties of this vacuum vector[17] and of the limiting state. Our basic existence theorem is:

THEOREM 4.5. *Let $L^2(Q, d\mu)$ be the Q-space associated with some Fock space. Let ω be a one-particle operator with $\omega \geqslant cI$ and let $H_0 = d\Gamma(\omega)$. Suppose V is a multiplication operator with $V \in L^p$ some $p > 2$ and $e^{-Vt} \in L^1$, all $t > 0$. Then $H_0 + V$ has discrete spectrum in $(E, E + c)$, where $E = \inf \sigma(H_0 + V)$. In particular, $H_0 + V$ has a ground state (eigenvector of eigenvalue E).*

Remark. This result depends critically on the combinatorial structure of Fock space and is surely not a result true in some abstract hypercontractive setting.

The idea is simple. One approximates $H_0 + V$ with operators, $H_{0,n} + V_n$ and uses the theory of Section 2C to prove $(H_{0,n} + V_n + \lambda)^{-1} \to (H_0 + V + \lambda)^{-1}$ in norm if λ is large. Each $H_{0,n} + V_n$ has discrete spectrum in $[E_n, E_n + c)$ and so the theorem follows from:

[17] Segal [53] has announced the preparation of a paper on the existence and uniqueness of the vacuum vector.

LEMMA 4.6. *If* $(A_n + \lambda)^{-1} \to (A + \lambda)^{-1}$ *in norm and* A_n *has discrete spectrum in* $[E_n, E_n + c)$ *with* $E_n = \inf \sigma(A_n)$, *then* A *has discrete spectrum in* $[E, E + c)$.

Proof. For the details, see Lemma 5.4 of Ref. [23]. The idea is simple: since

$$E_n \to E, \qquad (A_n - E_n + \lambda)^{-1} \to (A - E + \lambda)^{-1}$$

for $\lambda > 0$, so we can suppose $E_n = E = 0$. For an operator $B \geqslant 0$ to have discrete spectrum in $[0, c)$, it is necessary and sufficient that $f(B)$ be compact for any function f which is continuous with support in $(-\infty, c)$. By continuity of the functional calculus, $f(A_n - E_n) \to f(A - E)$ in norm so discreteness of $A_n - E_n$ in $[0, c)$ implies discreteness of $A - E$ in $[0, c)$. Q.E.D.

Given ω as in the theorem, realize the one particle space $\mathscr{F}_1$ as $L^2(\mathbf{R}, \mu)$ so that $(\omega f)(k) = \omega(k) f(k)$ with μ a finite Borel measure. For each n, we define a countable Borel partition, $\mathscr{P}_n$, of $\mathbf{R}$ inductively obeying:

(i) $\mathscr{P}_{n+1}$ is a refinement of $\mathscr{P}_n$;

(ii) For any set $A \in \mathscr{P}_n$, $\max(\omega \restriction A) - \min(\omega \restriction A) < 2^{-n}$;

(iii) Each $A \in \mathscr{P}_n$ is a subset of $(m2^{-n}, (m + 1) 2^{-n}]$ for some $m = -2^n n, -2^n n + 1, \ldots, n2^n$ or it is a subset of $(-\infty, -n]$ or (n, ∞).

(iv) For any K, and any n, there are only finitely many sets $A \in \mathscr{P}_n$ with $\min(\omega \restriction A) < K$.

It is not hard to construct such partitions. (iii) is demanded so that Proposition 4.7 (a) below will hold, (ii) so that Proposition 4.7 (b) will hold and (iv) so that the approximate Hamiltonians $H_{0,n}$ will have compact resolvents on certain invariant subspaces.

Let $\mathscr{H}_n$ be the set of functions which is constant on each $A \in \mathscr{P}_n$ and let P_n be the projection onto $\mathscr{H}_n$. Finally, define $H_{0,n} = d\Gamma(\omega_n)$, where ω_n is the function $P_n\omega$, i.e., the function whose constant value on $A \in \mathscr{P}_n$ is $\mu(A)^{-1} \int_A \omega(k)\, d\mu(k)$. We remark that $\omega_n \geqslant cI$.

PROPOSITION 4.7. (a) $P_n \to 1$ *strongly in* $\mathscr{F}_1$;

(b) $\|(N + 1)^{-1/2}(H_0 - H_{0,n})(N + I)^{-1/2}\| \to 0$ *as* $n \to \infty$.

Proof. (a) We need only prove $\bigcup \operatorname{ran} P_n$ is dense since $\{P_n\}$ is a monotone increasing family of projections. By condition (iii), the characteristic function of any rational dyadic interval is in some $\operatorname{ran} P_n$ so the union is dense.

(b) By condition (ii), $\| \omega_n - \omega \| < 2^{-n}$ so $-2^{-n}(N+1) \leqslant d\Gamma(\omega_n) - d\Gamma(\omega) \leqslant 2^{-n}(N+1)$ which implies (b). Q.E.D.

We can now apply the theory of Section 2C:

PROPOSITION 4.8. (a) *Suppose* $W \in L^p(Q, d\mu)$ *for some* $p > 2$ *is real with* $e^{-tW} \in L^1$ *for all* $t > 0$. *Then for* λ *sufficiently negative* $\|(H_0 + W - \lambda)^{-1} - (H_{0,n} + W - \lambda)^{-1}\| \to 0$. If $\mathscr{B}$ *is a set of* L^p *functions, with* $\{\| W \|_p \mid W \in \mathscr{B}\}$ *and* $\{\| e^{-tW} \|_1 \mid W \in \mathscr{B}\}$ *are bounded, the convergence is uniform in* W *for* $W \in \mathscr{B}$.

(b) *Suppose* $W_n \to W$ *in* $L^p(Q, d\mu)$ *for some* $p > 2$ *with* $\{\| e^{-tW_n} \|_1\}$ *bounded for each* f. *Then for* λ *sufficiently negative,*

$$\|(H_0 + W - \lambda)^{-1} - (H_{0,n} + W_n - \lambda)^{-1}\| \to 0.$$

Proof. (a) We must only show

$$\|(H_0 + W - \lambda)^{-1}(H_0 - H_{0,n})(H_{0,n} + W - \lambda)^{-1}\| \to 0.$$

Since the $e^{-tH_{0,n}}$ are hypercontractive with constants independent of n, the $H_{0,n} + W$ are uniformly bounded below, so we need only show $\|(H_0 + W - \lambda)^{-1/2}(H_0 - H_{0,n})(H_{0,n} + W - \lambda)^{-1/2}\| \to 0$. By Proposition 4.11, this follows if we can show $\|(N+1)^{1/2}(H_{0,n} + W - \lambda)^{-1/2}\|$ is uniformly bounded in n or equivalently that

$$(H_{0,n} + W - \lambda) \geqslant C(N+1) \tag{4.5}$$

for some C independent on n. But for C sufficiently small $H_{0,n} - C(N+1)$ are a family of generators of hypercontractive semigroups with uniform constants. Thus, an inequality of form (4.5) holds, so the first part of (a) is proven. Clearly, all estimates only depend on $\| W \|_p$ and $\| e^{-tW} \|_1$ so the rest of (a) follows.

(b) By (a), $\|(H_0 + W_m - \lambda)^{-1} - (H_{0,n} + W_m - \lambda)^{-1}\| \to 0$ as $n \to \infty$ uniformly in m and by Theorem 2.21,

$$\|(H_0 + W - \lambda)^{-1} - (H_0 + W_n - \lambda)^{-1}\| \to 0$$

so the result is proven. Q.E.D.

All that remains to prove Theorem 4.9 is to show we can find $V_n \to V$ in some L^p $(p > 2)$ with $\| e^{-tV_n} \|_1$ bounded in such a way that $H_{0,n} + V_n$ has discrete spectrum in $[E_n, E_n + c)$.

Let $\mathscr{F}^{(n)}$ be the Fock space built on $\mathscr{H}_n$, i.e.,

$$\mathscr{F}^{(n)} = \mathbf{C} \oplus \mathscr{H}_n \oplus \mathscr{H}_n \otimes_S \mathscr{H}_n \oplus \cdots$$

and let $\mathscr{F}^{(n,\perp)}$ be the Fock space built on $\mathscr{H}_n^{\perp}$. Since $\mathscr{H}_n \oplus \mathscr{H}_n^{\perp} = \mathscr{F}_1$, we have

$$\mathscr{F} = \mathscr{F}^{(n)} \otimes \mathscr{F}^{(n,\perp)},$$

$\mathscr{F}^{(n)}$ and $\mathscr{F}^{(n,\perp)}$ as Fock space have Q-space realizations,

$$\mathscr{F}^{(n)} = L^2(Q^{(n)}, d\mu_n) \quad \text{and} \quad \mathscr{F}^{(n,\perp)} = L^2(Q'^{(n)}, d\nu_n).$$

Thus since

$$\Omega_{\mathscr{F}} = \Omega_{\mathscr{F}^{(n)}} \otimes \Omega_{\mathscr{F}_{(n,\perp)}}$$

and

$$L^2(Q, d\mu) = L^2(Q^{(n)}, d\mu_n) \otimes L^2(Q'^{(n)}, d\nu_n) = L^2(Q^{(n)} \times Q'^{(n)}, d\mu_n \otimes d\nu_n),$$

we have $Q = Q^{(n)} \times Q'^{(n)}$ and $d\mu = d\mu_n \otimes d\nu_n$. Given $W \in L^p(Q, d\mu)$, define $\pi_n W \in L^p(Q, d\mu)$ by:

$$(\pi_n W)(q^{(n)}, q'^{(n)}) = \int d\nu_n(\tilde{q}')\, W(q^{(n)}, \tilde{q}'^{(n)}) \tag{4.6}$$

(i.e., $\pi_n W$ only depends on q, not q'; $\pi_n W$ is obtained by averaging W in the Q'-direction).

PROPOSITION 4.9. (a) *π_n is a contraction on each L^p.*

(b) *If $e^{-V} \in L^1$, then $e^{-\pi_n V} \in L^1$ and*

$$\| e^{-\pi_n V} \|_1 \leqslant \| e^{-V} \|_1 .$$

(c) *If $V \in L^p$ for $p < \infty$, then $\pi_n V \to V$ in L^p.*

Proof. (a) π_n is clearly positivity preserving, self-adjoint on L^2 with $\pi_n 1 = 1$. Thus, (a) follows from Proposition 2.1.

(b) Since the geometric mean of a quantity is smaller than the arithmetic mean:

$$e^{-\int d\nu W(q,\tilde{q}')} \leqslant \int d\nu(\tilde{q}')\, e^{-W(q,\tilde{q}')},$$

we have

$$\| e^{-\pi W} \|_1 = \int d\mu_n\, e^{-\int d\nu_n W(q,q')} \leqslant \int d\mu_n\, d\nu_n\, e^{-W} = \| e^{-W} \|_1 .$$

(c) Since the π_n are all contractions, we need only prove $\pi_n V \to V$ on a dense set of V's. Since $\pi_m \pi_n = \pi_n \pi_m = \pi_n$ for $m > n$

(by (i) of the properties of $\mathscr{P}$). $\pi_m V \to V$ if $V \in \operatorname{rad} \pi_n$ for any n. Let $\mathscr{M}_n$ be the algebra of bounded functions of smeared fields smeared with functions real and in $\mathscr{H}_n$. Since $P_n \to 1$, $\cup \mathscr{M}_n$ is weakly dense in $\mathscr{M}$ and thus L^p dense in $L^p(Q)$ ($p < \infty$) as in Theorem 3.2. Since $W \in \mathscr{M}_n$ implies $\pi_n W = W$, $\cup \mathscr{M}_n \subset \cup \operatorname{ran} \Gamma_n(L^p)$ so $\pi_m V \to V$ on a dense subset. Q.E.D.

The proof of Theorem 4.5 is thus reduced to showing $H_{0,n} + V_n$ has compact spectrum in $[E_n, E_n + C)$ with $V_n = \pi_n V$. Let us define ${}^{\perp}\mathscr{F}^{(n)}$ by

$$\mathscr{F}^{(n,\perp)} = \mathbf{C} \oplus^{\perp} \mathscr{F}^{(n)}.$$

We can identify $\mathscr{F}^{(n)}$ with a subspace of $\mathscr{F}$ by

$$\mathscr{F}^{(n)} \equiv \mathscr{F}^{(n)} \otimes \mathbf{C} \subset \mathscr{F}^{(n)} \otimes \mathscr{F}^{(n,\perp)}.$$

Under this identification, $[\mathscr{F}^{(n)}]^{\perp} = \mathscr{F}^{(n)} \otimes {}^{\perp}\mathscr{F}^{(n)}$. We would like to consider the action of $H_{0,n}$ and V_n under this breakup:

PROPOSITION 4.10. (a) *$H_{0,n}$ leaves $\mathscr{F}^{(n)}$ and $\mathscr{F}^{(n)\perp}$ invariant and*

$$\begin{aligned} H_{0,n} \upharpoonright \mathscr{F}^{(n)\perp} &\equiv H_{0,n} \upharpoonright (\mathscr{F}^{(n)} \otimes^{\perp} \mathscr{F}^{(n)}) \\ &= (H_{0,n} \upharpoonright \mathscr{F}^{(n)}) \otimes 1 + 1 \otimes (H_{0,n} \upharpoonright^{\perp} \mathscr{F}^{(n)}). \end{aligned}$$

(b) *V_n leaves $\mathscr{F}^{(n)}$ and $\mathscr{F}^{(n)\perp}$ invariant and*

$$V_n \upharpoonright (\mathscr{F}^{(n)\perp}) = V_n \upharpoonright (\mathscr{F}^{(n)} \otimes^{\perp} \mathscr{F}^{(n)}) = (V_n \upharpoonright \mathscr{F}^{(n)}) \otimes 1.$$

(c) *$H_n = H_{0,n} + V_n$ leaves $\mathscr{F}^{(n)}$ and $\mathscr{F}^{(n)\perp}$ invariant and*

$$H_n \upharpoonright \mathscr{F}^{(n)\perp} \geqslant E_n + c,$$

where $E_n = \inf \operatorname{spec}(H_n)$.

Proof. (a) Since ω_n leaves $\mathscr{H}_n$ and $\mathscr{H}_n^{\perp}$ invariant, $H_{0,n} = d\Gamma(\omega_n)$ leaves $\mathscr{F}^{(n)}$ and $\mathscr{F}^{(n,\perp)}$ invariant. By self-adjointness, $\mathscr{F}^{(n)\perp}$ is left invariant. Since Γ preserves $\otimes$, $d\Gamma(\omega)$ acts on $\mathscr{F}^{(n)} \otimes \mathscr{F}^{(n,\perp)}$ as $H_{0,n} \otimes 1 + 1 \otimes H_{0,n}$ (by Lebnitz' rule).

(b) Since V_n is independent of $q'^{(n)}$ and since $f \otimes g$ in $\mathscr{F}^{(n)} \otimes \mathscr{F}^{(n,\perp)}$ is $(f \otimes g)(q, q') = f(q)\, g(q')$, we have

$$\begin{aligned} [V_n(f \otimes g)][q, q') &= V_n(q) f(q)\, g(q') \\ &= [V_n f \otimes g](q, q'). \end{aligned}$$

(c) $H_n \upharpoonright \mathscr{F}^{(n)\perp} = (H_n \upharpoonright \mathscr{F}^{(n)}) \otimes 1 + 1 \otimes (H_{0,n} \upharpoonright {}^{\perp}\mathscr{F}^{(n)})$ by (a) and (b). We have $H_n \upharpoonright \mathscr{F}^{(n)} \geqslant E_n$ and $H_{0,n} \upharpoonright {}^{\perp}\mathscr{F}^{(n)} \geqslant c$ since $\omega_n \geqslant cI$ and ${}^{\perp}\mathscr{F}^{(n)}$ is the orthogonal complement of $\mathbf{C}$ in $\mathscr{F}^{(n,1)}$. Thus $H_n \upharpoonright \mathscr{F}_n \geqslant c + E_n$. Q.E.D.

We now complete the proof of Theorem 4.9 by proving $H_n \upharpoonright \mathscr{F}^{(n)}$ has compact resolvant:

PROPOSITION 4.11. (a) *$H_{0,n} \upharpoonright \mathscr{F}^{(n)}$ has compact resolvent;*
(b) *$H_n \upharpoonright \mathscr{F}^{(n)}$ has compact resolvent.*

Proof. (a) If A is a one-particle operator with eigenvalues $\lambda_1, \lambda_2, \ldots$ with $\lambda_n \to \infty$ and each $\lambda_n \geqslant c > 0$, then $d\Gamma(A)$ has eigenvalues, 0, $\{\lambda_n\}_{n=1}^{\infty}$; $\{\lambda_n + \lambda_m\}_{n \leqslant m=1}^{\infty} \cdots$, i.e., has discrete spectrum. Since $\omega_n \upharpoonright \mathscr{H}_n$ has compact resolvent (by (iv) of the properties of $\mathscr{P}$), $H_{0,n} \upharpoonright \mathscr{F}^{(n)}$ has compact resolvent.

(b) The Weyl min–max principle says that if A is bounded below and self-adjoint, then

$$\mu_n(A) = \max_{\phi_1, \ldots \phi_{n-1}} \Big[\min_{\substack{\psi[\phi_1, \ldots, \phi_{n-1}]^{\perp} \cap D(A) \\ \|\psi\|=1}} \langle \psi, A\psi \rangle \Big]$$

has the property that either $\mu_n(A)$ is the n-th eigenvalue from the bottom of the spectrum of A or it is the bottom of the essential spectrum and in the latter case, $\mu_n(A) = \mu_{n+1}(A) = \cdots$. It follows that A has compact resolvent if and only if $\mu_n(A) \to \infty$ as $n \to \infty$. Write $V = V_+ - V_-$ with $V_+, V_- \geqslant 0$. Since $1/2(H_0 - V_-)$ is bounded below, $|\langle \psi, V_-\psi \rangle| \leqslant 1/2\langle \psi, H_0\psi \rangle + b\langle \psi, \psi \rangle$ so:

$$\langle \psi, (H_0 + V)\psi \rangle \geqslant \langle \psi, H_0\psi \rangle - \langle \psi, V - \psi \rangle$$
$$\geqslant 1/2\langle \psi, H_0\psi \rangle - b\langle \psi, \psi \rangle.$$

Thus,

$$\mu_n(H_{0,n} + V_n \upharpoonright \mathscr{F}^{(n)}) \geqslant 1/2\mu_n(H_{0,n} \upharpoonright \mathscr{F}^{(n)}) - b,$$

so $\mu_n(H_{0,n} + V_n \upharpoonright \mathscr{F}^{(n)}) \to \infty$ by (a). This proves $H_n \upharpoonright \mathscr{F}^{(n)}$ has compact resolvant. Q.E.D.

We have thereby completed the proof of Theorem 4.9. We would emphasize once more that this proof is essentially an abstraction of that used by Jaffe and Glimm in the $(\varphi^4)_2$ case.

We next turn to the question of uniqueness of the vacuum[17]; our proof depends crucially on the irreducibility isolated by Segal [52] and a comment by him in Ref. [52]; it is basically merely a technical improvement of the Glimm–Jaffe proof of vacuum uniqueness [21], a proof whose basis is the classical theorems of Frobenius [17], Jentsch [34] and Perron [46]. We first prove a general result of the Frobenius type.

THEOREM 4.12. *Let (M, μ) be a measure space. Let H be a semi-bounded self-adjoint operator on $L^2(M, \mu)$; $H \geqslant c$ with two properties:*

(a) *e^{-tH} is positivity preserving, i.e., if $f \geqslant 0$ (pointwise), then $e^{-tH} \geqslant 0$.*

(b) *$\{e^{-tH}\}_{t>0} \cup L^\infty(M)$ form an irreducible set.*

Then $\dim\{\psi \mid H\psi = c\psi\} \leqslant 1$ and if $H\psi = c\psi$, then ψ is a cyclic vector for $L^\infty(M)$.

Proof.[18] We first note that if $H\psi = c\psi$, then $H \mid \psi \mid = c \mid \psi \mid$ (where $\mid \psi \mid (x) = \mid \psi(x) \mid$), for

$$\langle \mid \psi \mid \rangle, e^{-tH} \mid \psi \mid \rangle \geqslant \langle \psi, e^{-tH}\psi \rangle \text{ (by (a))}$$
$$= e^{-tc} = \| e^{-tH} \|,$$

so $e^{-tH} \mid \psi \mid = \| e^{-tH} \| \mid \psi \mid$.

We next show if $H\psi = c\psi$, then $\psi(x) = 0$, a.e. or $\psi(x) = 0$ a.e., for let $S = \{f \mid f(x)\, \psi(x) = 0 \text{ a.e.}\}$. We must show $S = \{0\}$ or $S = L^2(M)$, so by (b) it is enough to show S is left invariant by e^{-tH} and $L^\infty(M)$ since S is a closed subspace. S is clearly left invariant by $L^\infty(M)$. Let $S_+ = \{f \in S \mid f \geqslant 0\}$. Since $S = S_+ - S_+$, it is enough to show S_+ is left invariant by e^{-tH}. Let $f \in S_+$. Then

$$\langle e^{-tH}f, \mid \psi \mid \rangle = \langle f, e^{-tH} \mid \psi \mid \rangle = e^{-tc}\langle f, \mid \psi \mid \rangle = 0.$$

Since $e^{-tH}f$ and $\mid \psi \mid$ are positive, we conclude $e^{-tH}f \in S_+$. Thus, we conclude $\psi \neq 0$ a.e. or $\psi = 0$ a.e.

Finally, we note that since $\mid \psi \mid$ is an eigenvector, so is $\mid \psi \mid - \psi$. Thus $\mid \psi \mid - \psi$ is 0 a.e. or $= 0$ a.e., i.e., ψ is positive a.e. or negative a.e. It thus follows that two orthogonal eigenvectors are impossible and that ψ is cyclic for $L^\infty(M)$. Q.E.D.

To apply this to the field theory case, we need the following:

THEOREM 4.13. *Let $H_0 = d\Gamma(\omega)$; $\omega \geqslant cI$. Let $V \in L^p(Q)$, some*

[18] One can use our techniques to prove e^{-tH} is strictly ergodic and so fall back on the proof of Jaffe–Glimm [21].

$p > 0$; $e^{-Vt} \in L^1$, all $t > 0$. Let $H = H_0 + V$, the closure of the operator sum. Then:

(a) (Jaffe–Glimm [21]) *e^{-tH} is positivity preserving on $L^2(Q, d\mu)$*;

(b) (Segal [52]) *$\mathcal{M} \cup \{e^{-tH}\}_{t\geqslant 0}$ is an irreducible family.*

Proof. (a) By Theorem 2.21 and the Trotter–Kato theorem, if V_n are bounded L^p approximations of V,

$$\exp(-tH) = s\text{-}\lim_{n\to\infty} \exp[-t(H_0 + V_n)].$$

By the Trotter product formula:

$$e^{-tH} = s\text{-}\lim_{n\to\infty} \lim_{m\to\infty} (e^{-tH_0/m} e^{-tV_n/m})^m.$$

Since e^{-sH_0} and e^{-sV_n} are all positivity preserving, so is e^{-tH}.

(b) As in (a), applying Theorem 2.21 to $W_n = V - V_n$,

$$e^{-tH_0} = s\text{-}\lim_{n\to\infty} \lim_{m\to\infty} (e^{-tH/m} e^{+tV_n/m})^m,$$

so $[\mathcal{M} \cup \{e^{-tH}\}]'' \supset [\mathcal{M} \cup \{e^{-tH_0}\}]''$. Let $A \in [\mathcal{M} \cup \{e^{-tH_0}\}]'$. Then $A \in \mathcal{M}'$ so $A \in \mathcal{M}$ since $\mathcal{M}$ is maximal Abelian. Since A commutes with e^{-tH_0}, we see $e^{-tH_0}(A\Omega_0) = A\Omega_0$, so $A\Omega_0$ is an eigenfunction of e^{-tH_0} with eigenvalue 1, i.e., $A\Omega_0 = c\Omega_0$ which implies $A = c1$. Thus $[\mathcal{M} \cup \{e^{-tH}\}]'' \supset \{c1\}'$ is all bounded operators. Q.E.D.

COROLLARY 4.14. *The ground state in Theorem 4.5 is unique and is (a.e.) strictly positive Q-space function.*

Let us denote this "vacuum" state as Ω_H. We next turn to some of its simple Q-space properties:

THEOREM 4.15. *Let H_0 be the generator of a hypercontractive semigroup and let $V \in L^p$ (some $p > 2$); $e^{-Vt} \in L^1$ all $t > 0$. Let ϕ be an entire vector for $H = H_0 + V$. Then $\phi \in L^q$, all $q < \infty$; in particular, any eigenfunction of H (e.g., Ω_H in the context of Theorem 4.5) is in every L^q, $q < \infty$.*

Proof. For any t, $\phi = e^{-tH}\psi_t$ for some $\psi_t \in L^2$. Since $\operatorname{rad} e^{-tH} \subset L^q$, if t is big enough, the result follows. Q.E.D.

One immediate consequence of the L^p properties of the vacuum is:

COROLLARY 4.16. *Let H_0 be the generator of a hypercontractive* *semigroup and let* $V \in L^p$, *some* $p > 2$; $e^{-Vt} \in L^1$ *all* $t > 0$. *Let* ϕ *be an entire vector for* $H = H_0 + V$. *Then* $\phi \in D(H_0)$. *In particular,* $\Omega_H \in D(H_0)$.

Proof. $\phi = e^{-tH}\psi$ with $\psi \in L^p$, all $p < \infty$. Thus, by our argument in Theorem 2.16, $\phi \in D(H_0)$.

Remarks. (1) If $V \in L^p$, all $p < \infty$, then ϕ need only be an analytic vector for H in order that Corollary 4.16 hold.

(2) For $(\Phi^4)_2$, perturbation theory suggests $\phi \notin D(H^{3/2})$ [20], so we can't much improve on $\phi \in D(H_0)$.

(3) In the $(\Phi^4)_2$ case, this result has been proven by Glimm–Jaffe [20] and in $(\Phi^{2n})_2$ by Rosen [48].

(4) On the surface, one would think of $\Omega_H \in D(H_0)$ as following only from some sort of higher order estimate (as used by Glimm–Jaffe and Rosen), but, in fact, we have used no higher order estimates.

One can also make statements about L^p convergence of the vacuums of H_n, thereby extending a result of Glimm and Jaffe [21], who prove L^2 convergence:

COROLLARY 4.17. (a) *Let* $V_n \to V$ *in* L^p (*some* $p > 2$) *with* $\| e^{-V_n t} \|_1$ *bounded for each* t *uniformly in* n. *Let* $H_0 = d\Gamma(\omega)$, $\omega \geqslant cI$. *Let* $H_n = H_0 + V_n$; $H = H_0 + V$. *Then* $\Omega_{H_n} \to \Omega_H$ *in each* L^q $(q < \infty)$.

(b) *If* $H_{0,n} = d\Gamma(\omega_n)$ *with each* $\omega_n \geqslant cI$ *and with*

$$\| (N+1)^{-1/2} (H_0 - H_{0,n}) (N+1)^{-1/2} \| \to 0$$

as $n \to \infty$ *and if* $H_n = H_{0,n} + V_n$, $H = H_0 + V$, *then* $\Omega_{H_n} \to \Omega_H$ *in each* L^q $(q < \infty)$.

Proof. In either case, $(H_n - E)^{-1} \to (H - E)^{-1}$ in norm, so by the phase condition $\langle \Omega_{H_n}, \Omega_H \rangle > 0$, we have $\Omega_{H_n} \to \Omega_H$ in L^2. By arguments in Section 2C, $e^{-H_n T} \to e^{-HT}$ converge in norm as maps from L^2 to L^q if T is large enough. Thus, $e^{-E_n T}\Omega_{H_n} \to e^{-ET}\Omega_H$ in L^q. Since $E_n \to E$, the proof is complete. Q.E.D.

Before leaving the subject of the L^p properties of the vacuums, we would like to point out that L^p estimates might be of use in studying the field theory without cutoffs. Explicitly, let

$$H(g) = H_0 + \int g(x) :P(\varphi(x)): dx$$

and let Ω_g be the vacuum of $H(g)$. Glimm and Jaffe [22] have proven there is a *sequence* of cutoffs, $g_n \to 1$, so that the states $\omega_n(A) = \langle \Omega_{g_n}, A\Omega_{g_n} \rangle$ on $\mathscr{U}$ converge to a state ω in the W^*-topology. Moreover, the representation, π, of $\mathscr{U}$ generated by ω via the G.N.S. construction is also locally Fock, i.e., $\pi \upharpoonright \mathscr{U}_{(a,b)}$ is unitarily equivalent to the Fock space representation restricted to $\mathscr{U}_{(a,b)}$ [22]. In particular, by using this unitary operator, one can define smeared fields $\phi_\pi(h)$ on $\mathscr{H}_{\text{phys}}$, the underlying Hilbert space for π. Jaffe and Glimm also prove that the G.N.S. vacuum $\Omega_{\text{phys}} \in D(\phi_\pi(h))$. For reasons related to the Wightman axioms [58], one would want $\Omega_{\text{phys}} \in D(\phi_\pi(h_1) \cdots \phi_\pi(h_k))$ for any set $h_1, \ldots, h_k$ of functions on space–time. One is also interested in the "equal" time fields, i.e., $\Omega_{\text{phys}} \in D(\phi_\pi(h_1) \cdots \phi_\pi(h_k))$, where the h_i are functions of space alone [see (4.3) and (4.4)].

We should like to make a few *speculative* remarks about the relation of L^p-estimates to this domain question. We will first obtain a result that follows from a uniform estimate on $\| \Omega_{g_n} \|_p$ ($p > 2$). We feel such an estimate is almost surely false as we shall explain, but we will conjecture that a weaker estimate holds with esssentially the same consequences. For simplicity, we first state the consequences of the stronger but probably false estimate:

Proposition 4.18. *Suppose* $\| \Omega_{g_n} \|_p$ *(in* $L^p(Q, d\mu)$*) is uniformly bounded as* $n \to \infty$ *for some* $p > 2$. *Then* $\Omega_{\text{phys}} \in D(\phi_\pi(h_1) \cdots \phi_\pi(h_m))$ *[equal time fields] for any set of functions* h_i *with*

$$\int dk\, |(\hat{h}_i(k)|^2\, \omega(k)^{-1} < \infty.$$

In fact:

$$\langle \Omega_{g_n}, \phi(h_1) \cdots \phi(h_m)\, \Omega_{g_n} \rangle \to \langle \Omega_{\text{phys}}, \phi_\pi(h_1) \cdots \phi_\pi(h_m)\, \Omega_{\text{phys}} \rangle.$$

Proof. Let P_k be the spectral projection for $\phi(h_1) \cdots \phi(h_m)$ onto the interval $(-k, k)$. It is enough to prove $\omega(P_k[\phi(h_m)^* \cdots \phi(h_1)^*]$ $[\phi(h_1) \cdots \phi(h_m)])$ is bounded independently of k. Since $\omega_n \to \omega$ on bounded operators in $\mathscr{M}$, we need only show $\omega_n(P_k[\phi(h_m)^* \cdots \phi(h_m)])$ is bounded uniformly in n and k. This follows in turn if we can show $\omega_n([\phi^*(h_m) \cdots \phi(h_m)])$ is bounded uniformly in n. But, since

$$A = \phi^*(h_m) \cdots \phi(h_m) \in L^q(Q)(q^{-1} + 2p^{-1} = 1),$$

we have $| \omega_n(A)| \leqslant \| A \|_q \| \Omega_n \|_p^2$ which provides the needed bound. The convergence of the vacuum expectation values is not very hard.

Remarks. While we cannot use this method to say anything about the unequal real time vacuum expectation values, we can say something about the "imaginary" time vacuum expectation values[19], $\omega(\phi(h_1)\, e^{iH(t_1+i\epsilon_1)} \cdots e^{iH(t_n+i\epsilon_n)}\phi(h_{n+1}))$ with $\epsilon_1, \ldots, \epsilon_n > 0$, if we can also obtain uniform estimates on $e^{-t(H(g)-E_g)}$ as maps of L^2 to $L^{2+\delta}$.

We suspect in fact that as $g \to 1$ the $\| \Omega_g \|_p$ are not bounded if $p > 2$[20]. For as $g \to 1$, more and more modes are "coupled essentially to Ω_g"; thus, we expect $\| \Omega_g \|_p$ to grow as $C^{n(g)}$ where $C > 1$ and $n(g)$ is the size of the set on which g is 1. There is in x-space a coupling of the modes associated with disjoint regions of space but it is well-known to be a coupling which falls off exponentially with the distance between regions. This suggests the following picture: Fix a, b and consider the set S of one-particle functions $h \in \mathscr{H}$ whose x-space representation has support in (a, b). As in Proposition 4.9, we have

$$L^2(Q, d\mu) = L^2(Q_s \times Q_s', d\mu_s \otimes d\nu_s),$$

where

$$\mathscr{M}_{(a,b)} = L^\infty(Q_s, d\mu_s).$$

As in Lemma 4.9, define $\pi_s : L^p(Q, d\mu) \to L^p(Q, d\mu)$ by averaging in the Q_s' direction. As n increases, we expect the additional wiggles in Ω_{g_n} should concentrate in the Q' direction because of the falloff in the coupling of the Q_s modes to the new modes introduced by increasing n. Thus we conjecture that $\pi_s(\Omega_g^2)$ will have bounded $L^{p/2}(Q_s, d\mu_s)$ norms. This would imply $\Omega_{\text{phys}} \in D(\phi_\pi(h_1) \cdots \phi_\pi(h_m))$ as long as $h_1, \ldots, h_m$ all have compact support.

As support for this picture which we conjecture we remark the following: Because of Haag's theorem Ω_n^2 cannot converge in $L^1(Q)$, but by the local Fock property proven by Glimm–Jaffe [22] $\pi_s(\Omega_n^2)$ does converge in $L^1(Q)$; some smoothing due to Q_s' averaging is already in evidence—we see no reason for there not to be enough smoothing for the $\pi_s(\Omega_n^2)$ to converge in all L^p $(p < \infty)$.

C. *Coupling Constant Analyticity*

As a final subject in our general study of spatially cutoff $(\varphi^{2n})_2$ theories, we should like to examine the analyticity in κ of the various objects associated with $H_0 + \kappa V$. After showing full cut plane

[19] These are the analogs of the continuation of the Wightman functions into the forward tube.

[20] Thus, the VanHove phenomenon, $\Omega_g \to 0$ weakly in L^2 [60, 32] which is not yet proven in $(\Phi^4)_2$, does not hold in L^p. Since Ω_g is L^2-normalized, this is not surprising.

analyticity of the resolvent we study the ground state energy and vacuum vector in the right half-plane. In particular, we will prove the asymptotic nature of various perturbation series and we will make one comment on the convergence of the Padé approximants to the ground state.

Our general analytic perturbation theory (Section 2D) implies $H_0 + \kappa V$ is an analytic family for $\operatorname{Re} \kappa > 0$. But Jaffe's study of the finite number of degree of freedom case[21] suggests one try to prove analyticity in the cut plane, $\kappa \neq$ negative real. In fact, this can be proven and, in fact, could have been proven after Glimm's lower boundedness paper [18], for:

THEOREM 4.19. *Let H_0 and V be self-adjoint operators on some Hilbert space $\mathscr{H}$ with $H_0 \geqslant 0$. Suppose $V = V_+ - V_-$ where*

(a) $V_+, V_- \geqslant 0$

(b) *V_- is a tiny form perturbation of H_0, i.e., $Q(V_-) \supset Q(H_0)$ and for all $a > 0$, there is a, b so that*

$$\langle\psi, V_-\psi\rangle \leqslant a\langle\psi, H_0\psi\rangle + b\langle\psi, \psi\rangle$$

for all $\psi \in Q(H_0)$. Define the quadratic form $t(\kappa) = H_0 + \kappa V$ on $Q(H_0) \cap Q(V_+)$ for κ in the cut plane $\kappa \neq$ negative real. Then $t(\kappa)$ is a holomorphic family of type (b) [35, p. 392]; *that is*

(i) *$t(\kappa)$ is a closed sectorial form*[22].

(iv) *The domain $Q = Q(t(\kappa))$ is independent of κ and $\langle u, t(\kappa)u\rangle$ is analytic in κ for all $u \in Q$.*

In particular, if $H_0 + \kappa V$ is essentially self-adjoint for $\kappa > 0$, then the family of closures of the positive κ operators has a resolvent analytic continuation to the cut plane.

Before proving this theorem in a series of lemmas, we note that it has the corollary:

COROLLARY 4.20. *Let V be as in Theorem 4.1 and let $H(\kappa) = H_0 + \kappa V$ be the closure of the operator sum defined for $\kappa > 0$. Then $H(\kappa)$ has a resolvent analytic continuation to the cut κ-plane.*

[21] A. Jaffe, private communication and Ref. [33]; see also Ref. [54].

[22] Our definition of sectorial is slightly more general than Kato's in that we only require the numerical range to lie in some closed sector of opening angle strictly less than π; he requires the sector to be symmetric about the positive real axis.

Proof. Take $V_+(Q) = \max(V(Q), 0)$. Condition (b) of Theorem 4.19 is a consequence of the fact that $H_0 + a^{-1}V_-$ is bounded below for any a. Q.E.D.

Let us turn to the proof of Theorem 4.4. Lemmas 4.21 and 4.22 are both stated supposing conditions (a) and (b) of the theorem hold:

LEMMA 4.21. *For any fixed* κ, $H_0 - \kappa V_-$ *has a numerical range in a symmetric translated sector* $\{z \mid |\arg(z - z_0)| < 1/2\,\theta\}$ *of arbitrarily small opening angle* θ.

Proof. Let $\| u \| = 1$. Then $\langle u, V_- u\rangle \leqslant a\langle u, H_0 u\rangle + b$, so

$$|(\operatorname{Im} \kappa)\langle u, V_- u\rangle| \leqslant \frac{a \mid \operatorname{Im} \kappa \mid}{(1 - a \mid \operatorname{Re} \kappa \mid)} [\operatorname{Re}\langle u, (H_0 - \kappa V_-)\, u\rangle] + b,$$

which proves the result since $H_0 - (\operatorname{Re} \kappa)\, V_-$ is bounded below.

LEMMA 4.22. $t(\kappa) = H_0 + \kappa V$ *defined on* $Q(H_0) \cap Q(V_+)$ *is a closed sectorial form if* κ *is not a negative real.*

Proof. $H_0 + \kappa V_-$ has numerical range in a sector of opening angle $\theta < \pi - |\arg \kappa|$. Then $(H_0 + \kappa V_-) + \kappa V_+$ is the sum of sectorial forms whose sectors generate a sector of opening angle less than π. Thus $t(\kappa)$ is sectorial. It is not hard to show that if u_n is Cauchy in the norms $|||\, a \,|||_t = |\langle u, (t(\kappa) + b) + b)u\rangle|$, it is Cauchy in the norms $|||\, u \,|||_{H_0} = \langle u, (H_0 + 1)u\rangle$ and $|||\, u \,|||_{V_+} = \langle u, (V_+ + 1)u\rangle$, so $t(\kappa)$ is closed.

Proof of Theorem 4.19. We have just proven (i), and (ii) is obvious. The final statement of the theorem follows from the theory of perturbations of type (B) [35, pp. 395–397]. Q.E.D.

We have seen for λ positive, $H_0 + \lambda V$ has a unique vacuum (Theorems 4.5 and 4.14). It thus follows from the general theory of analytic families of operators and Corollary 4.20 that:

PROPOSITION 4.21. *Let* $H_0 = d\Gamma(\omega)$, $\omega \geqslant cI$ *and* $V \in L^p$, *some* $p > 2$; $e^{-Vt} \in L^1$ *all* $t > 0$. *Let* Ω_λ *be the vacuum for* $H_0 + \lambda V$ *normalized by* $\| \Omega_\lambda \| = 1$, $\Omega_\lambda \geqslant 0$. *It has an analytic continuation to a neighborhood of the real axis and the ground state energy* $E(\lambda) = \langle \Omega_\lambda, H_\lambda \Omega_\lambda\rangle / \langle \Omega_\lambda, \Omega_\lambda\rangle$ *has a continuation to a neighborhood of the real axis.*

Naturally, one is interested in the singularities of $E(\lambda)$ when one continues it to the cut plane and beyond. Even in the case of one degree of freedom, this is an interesting problem which has recently been studied in the case $p^2 + x^2 + \lambda x^4$ by one of us [54] and by Loeffel–Martin [39]. In that case, the principle analyticity results were the following:

(1) $E(\lambda)$ has a continuation without singularities to the cut λ plane [39].

(2) If $E(\lambda)$ does not develope natural boundaries, it has a continuation to a domain which is three sheeted about $\lambda = 0$, with cuts at $\lambda \neq 0$ allowed [54].

(3) On this three sheeted surface, $\lambda = 0$ is not an isolated point of analytic or an isolated singularity, but it is a limit point of singularities whose asympototic phase is $\pm 3\pi/2$ [54].[23]

(4) For any $\Theta < 3\pi/2$, there is a Λ with $E(\lambda)$ analytic in $\{\lambda \mid |\lambda| < \Lambda, |\arg\lambda| < \Theta\}$ and the Rayleigh–Schrödinger series is asymptotic[24] as $\lambda \to 0$ uniformly in the sector [54].

We are not able to extend these results to the infinite number of degrees of freedom case[25] but we can prove several weaker results. First, if zero is to be a limit point of singularities (as it almost surely is), they must have asymptotic phase at least $\pi/2$, for:

PROPOSITION 4.22. *Let H_0, V be as in Proposition 4.21. Let $\Theta < \pi/2$ be given. Then there is a Ω so that $E(\lambda)$ and Ω_λ are analytic in $\{\lambda \mid |\lambda| \Omega; |\arg\lambda| < \Theta\}$ and $E(\lambda)$ is the only eigenvalue of $H_0 + \lambda V$ near* 0.

Proof. It is enough to show $H_0 + |\lambda| e^{i\theta} V$ converges in norm resolvent sense as $|\lambda| \to 0$ for each fixed $|\theta| < \pi/2$ with this convergence uniform in $|\theta| < \Theta$. This follows from Theorem 2.22. Q.E.D.

Moreover, the Rayleigh–Schrödinger series for $E(\lambda)$ is asymptotic. We first note the lemmas:

LEMMA 4.23. *Let $\omega \geqslant cI$; $H_0 = d\Gamma(\omega)$ and suppose $0 < |\mu| < c$. Then $(H_0 - \mu)^{-1}$ is a bounded operator of L^p into L^p for any $1 < p < \infty$*

[23] If we are to believe the approximate calculation of C. Bender and T. T. Wu [2, 3] these singularities are square root branch points.

[24] It is known to diverge [2, 54].

[25] Case (2)–(4) are known for the many (i.e., finite but bigger than 1) degree of freedom case, but case (1) isn't even known in the two degree of freedom case.

and the bound is uniform on compact subsets of $\{\mu \mid 0 < |\mu| < c\} \times \{p \mid 1 < p < \infty\}$.

Proof. Write $L^p = \mathbf{C}1 \oplus \widetilde{L^p}$ where $\widetilde{L^p} = \{f \in L^p \mid \int f \, d\mu = 0\}$. $(H_0 - \mu)^{-1}$ leaves $\mathbf{C}1$ and $\widetilde{L^p}$ invariant so it is sufficient to show $(H_0 - \mu)^{-1}$ is bounded on each $\widetilde{L^p}$. Without loss of generality, suppose $p > 2$ since duality can be used for $p < 2$. Pick T so that e^{-TH_0} is a contraction of L^2 to L^p, and consider how $e^{-(t+T)H_0}$ acts on $\widetilde{L^p}$. Since $\widetilde{L^p} \subset \widetilde{L^2}$,

$$\| e^{-tH_0}\psi \|_2 \leqslant e^{-tc} \| \psi \|_2 \leqslant e^{-tc} \| \psi \|_p \qquad \text{if} \quad \psi \in \widetilde{L^p}.$$

Thus

$$\| e^{-(t+T)H_0}\psi \|_p \leqslant e^{-tc} \| \psi \|_p \qquad \text{if} \quad \psi \in \widetilde{L^p}.$$

For any μ with $\operatorname{Re} \mu < c$, $\int_0^\infty e^{\mu t} \, e^{-H_0 t}\psi \, dt$ exists if $\psi \in \widetilde{L^p}$ and it is uniformly bounded on the $\widetilde{L^p}$ unit ball. Thus $\widetilde{L^p}$ is mapped into itself by $(H_0 - \mu)^{-1}$ in a bounded manner. Q.E.D.

Remark. Let V be real and in L^q for some $q > 2$ with $e^{-tV} \in L^1$ for all t. Let $H_0 = d\Gamma(\omega)$, $\omega \geqslant cI$. We have already seen that H has a simple L^2 eigenvalue at $E(V)$ and a gap of some size $\epsilon(V)$ before the next spectral point. By an argument identical to the above proof with $\widetilde{L^p}$ replaced by $\{f \in L^p \mid \langle \Omega_V, f \rangle = 0\}$, one can show the L^p spectrum of H also has the gap; explicitly

$$\operatorname{spec}(H \restriction L^p) \cap \{z \mid \operatorname{Re} z < E(V) + \epsilon(V)\} = \{E(V)\}.$$

In fact, one can prove further that the L^p-spectrum is also purely discrete in $[E(V), E(V) + C)$ and is identical in that interval to the L^2-spectrum.

LEMMA 4.24. *Let* $H_0 = d\Gamma(\omega)$; $\omega \geqslant cI$, $V \in L^q$ (*some* $q < \infty$), $e^{-Vt} \in L^1$ *all* t. *Suppose* $\Theta < \pi/2$, p, *and* $0 < \epsilon < c$ *are given. Then, there is a* Ω *so that for* $|\lambda| \leqslant \Omega$, $|\arg \lambda| \leqslant \Theta$ *and any* μ *with* $|\mu| = \epsilon$, $(H_0 + \lambda V - \mu)^{-1}$ *is a bounded map of* L^p *into* L^p *with bound uniform on the set of* μ *and* λ *in question.*

Proof. This follows from Lemma 4.23 and the uniform L^p norm resolvant convergence of $H_0 + |\lambda| e^{i\theta} V$ to H_0 as $|\lambda| \to 0$ with $|\theta| < \Theta$ (Theorem 2.22).

THEOREM 4.25. *Let* $H_0 = d\Gamma(\omega)$; $\omega \geqslant cI$; $V \in L^p$, *all* $p < \infty$, *real,* $e^{-Vt} \in L^1$ *all* $t > 0$. *Let* $E(\lambda)$ *and* Ω_λ *be the vacuum energy and vacuum vector for* $H_0 + \lambda V$. *Then,*

(a) *The Rayleigh–Schrödinger series for* $E(\lambda)$ *is asymptotic as* $\lambda \downarrow 0$ *in any sector* $\{\lambda \mid |\arg \lambda| < \Theta < \pi/2\}$.

(b) *The Rayleigh–Schrödinger series for* Ω_λ *is asymptotic as* $\lambda \downarrow 0$ *in any sector with remainders going to zero in each* L^p ($p < \infty$).

Proof. Consider first the resolvant $(H_0 + \lambda V - \mu)^{-1}$ where $|\mu| = \epsilon$ and $|\lambda| < \Lambda$ as in Lemma 4.24. Then applied to vectors in L^∞, we have, for any fixed N:

$$(H_0 + \lambda V - \mu)^{-1} = \sum_{n=0}^{n} (-\lambda)^n [(H_0 - \mu)^{-1} V]^n (H_0 - \mu)^{-1}$$

$$+ (-\lambda)^{N+1} (H_0 + \lambda V - \mu)^{-1} [V(H_0 - \mu)^{-1}]^{N+1}. \quad (4.7)$$

If we apply both sides of this equation to Ω_0 we see that the geometric series for $(H_0 + [\lambda V - \mu])^{-1} \Omega_0$ is asymptotic in L^p with remainder terms uniformly bounded in μ. The projection onto Ω_λ is given by:

$$P_\lambda = -(2\pi i)^{-1} \int_{|\mu|=\epsilon} d\mu (H_0 + \lambda V - \mu)^{-1}, \quad (4.8)$$

so $P_\lambda \Omega_0$ has an asymptotic series. Since $\Omega_\lambda = P_\lambda \Omega_0 / \langle \Omega_0 , P_\lambda \Omega_0 \rangle^{1/2}$, Ω_λ has an asymptotic series as does $E(\lambda) = \langle \Omega_0 , (H_0 + \lambda V) \Omega_\lambda \rangle / \langle \Omega_0 , \Omega_\lambda \rangle$. But it is well-known that the series obtained by putting (4.7) into (4.8) is just the Rayleigh–Schrödinger series.

COROLLARY 4.26. *Let* $H_0 = d\Gamma((k^2 + m^2)^{1/2})$. *Let*

$$V = \int g(x){:}\, P(\phi(x)){:}\, dx,$$

where P is a polynomial which is bounded below. Then the following have asymptotic series, asymptotic uniformly in subsectors of $|\arg \lambda| < \pi/2$:

(a) $E_g(\lambda)$, *the ground state energy*;

(b) $\langle \Omega_\lambda , \phi(h_1) \cdots \phi(h_k) \Omega_\lambda \rangle$, *the equal-time vacuum expectation values*;

Proof. This follows from the asymptotic results in Theorem 4.25.

Before leaving the question of $H_0 + V$ with V fixed, we would like to make a few comments on the possibility of these perturbation series having convergent Padé approximants [1, 5]. For the one degree of freedom anharmonic oscillator Loeffel *et al* [40] have shown that the diagonal Padé approximants for the Rayleigh–Schrödinger series converge to the actual energy levels. The most difficult part of the proof [39], namely that $E(\lambda)$ is analytic in the cut plane is not even known to extend to several degrees of freedom and we have nothing to say about it extending to an infinite number of degrees of freedom case. In the anharmonic oscillator case, one easily established element of the proof is that $E(\lambda)$ is a Herglotz function—what we wish to remark is that this property cannot hold in the $(\phi^{2n})_2$ case, for:

PROPOSITION 4.27. *Let $E(\lambda)$ be the ground state energy of $H_0 + \lambda V$ with $H_0 \geqslant 0$. If*

(a) *$E(\lambda)$ is analytic in the cut plane.*

(b) *V is not bounded below as a form on $D(H_0) \cap D(V)$. Then $E(\lambda) + C\lambda$ is not a Herglotz function for any C.*

Proof. By a well-known property of Herglotz functions, if $E(\lambda) + C\lambda$ were Herglotz, it would be linearly bounded as $\lambda \to \infty$ along the positive axis; thus $E(\lambda)$ would be linearly bounded, say $|E(\lambda)| < D\lambda$. Then $\langle\psi, (H_0 + \lambda V)\psi\rangle \geqslant -D\lambda$ for any ψ with $\|\psi\| = 1$. This implies $\langle\psi, V\psi\rangle \geqslant -D$ contradicting (b). Q.E.D.

In particular, the ground state energy for

$$H_0 + \lambda \int g(x) :P(\phi(x)): dx$$

is not a Herglotz function, even if it can be proven analytic in the cut plane. It is rather simple to understand heuristically why this happens. If we take $p^2 + x^2 + \lambda x^4$ and partially "Wick-order" x^4 by subtracting $c = \langle \Omega_0, x^4 \Omega_0 \rangle$, the new energy $\tilde{E}(\lambda) = E(\lambda) - c\lambda$ is no longer Herglotz either. Of course, $\tilde{E}(\lambda) + c\lambda$ is. In the field theory case, Wick ordering involves infinite subtractions, so no finite constant can restore the Herglotz nature. One is tempted to suggest that the Pade approximants to $E(\lambda)$ still might converge, with the Rayleigh–Schrödinger coefficients, a_n, still a series of Steiljes [1] for $n \geqslant 2$, i.e., $a_n = (-1)^{n+1} \int_0^\infty x^n \, d\rho$ might hold for $n \geqslant 2$. But, unlike the one degree of freedom case $\int_0^\infty x \, d\rho$ would be infinite.[26] Because of this loss of the Herglotz property, it will be hard to prove the discontinuity across the cut is positive even if cut plane analyticity is established.

[26] The divergence coming from $x = 0$.

We would like to end this section with a brief discussion of how coupling constant analyticity of the physical vacuum in a neighborhood of the real axis might hold or fail to hold. According to the discussion of Wightman [63], there are two ways in which singularities of $\omega_\lambda^{\text{phys}}$ might develop on the real axis as $g \to 1$[27]:

(a) For $g \in L^1 \cap L^2$, there are complex singularities in Ω_λ which as $g \to 1$ "pinch" the real axis.

(b) There is some nonuniformity in the approach to the limit $\omega_{g,\lambda} \to \omega_\lambda$ as $g \to 1$.

We should like to discuss how this nonuniformity (in (b)) would have to manifest itself. Suppose there is some neighborhood N of $\lambda_0 > 0$ so that for g_n in a sequence $g_n \to 1$, $\Omega_{\lambda,n}$, the ground state of $H_0 + \lambda \int gn(x) :P(\phi(x)): dx$, is analytic in N (i.e., suppose (a) does not occur). We can, without loss, normalize Ω so $\int \Omega(Q)^2 \, d\mu(Q) = 1$. We emphasize that since Ω is complex for λ non-real, $\| \Omega \|_2 > 1$ for λ nonreal. Any nonuniformity must manifest itself by $\| \Omega \|_2 \to \infty$ for:

THEOREM 4.27. *Suppose $\Omega_{n,\lambda}$ is analytic for $\lambda \in N$ for some g_n with $g_n \to 1$. Suppose also, that $\| \Omega_{n,\lambda} \|_2$ is bounded as $n \to \infty$ uniformly on compacts of N. Then there is a subsequence $g_{n(i)}$ so that for any $\lambda \in N$,*

$$\omega_{\lambda,n(i)}(A) = \langle \overline{\Omega_{\lambda,n(i)}} \, , A\Omega_{\lambda,n(i)} \rangle$$

converges as $i \to \infty$ in the weak star topology to a limiting functional ω_λ analytic in N.

Proof. By a result of Jaffe and Glimm [22], we can pick a subsequence with ω_{n,λ_0} convergent for some fixed real λ_0. By a standard argument, we can get a subsequence converging for λ any real rational in N. Since $\| \Omega_{\lambda,n} \|_2$ is bounded on compacts, $| \omega_{\lambda,n}(A) | \leqslant \| \Omega_{\lambda,n} \|^2 \| A \|$ is also bounded on compacts. The theorem now follows from the Vitali convergence theorem. Q.E.D.

5. APPLICATIONS TO NONRELATAVISTIC QUANTUM MECHANICS

We would like to briefly discuss two results in nonrelatavistic quantum mechanics which the methods of this paper enable us to prove. First, Theorem 2.25 implies:

[27] In analogy with thermodynamics, one is tempted to call these singularities "phase transitions"; they might occur, for example, at values of the coupling constant at which bound states appear in the theory.

THEOREM 5.1. *Let V be a measurable function on $\mathbf{R}^n$ so that:*

(a) $V(q) \geqslant aq^2 - b$ *for some* a *and* b; $a > 0$;

(b) $\int | V(q)|^2[\exp(-a^{1/2}q^2)]\, d^nq < \infty$.

Then $-\Delta + V$ *is essentially self-adjoint (as an operator on* $L^2(\mathbf{R}^n, d^nq)$*) on* $D(-\Delta) \cap D(V)$.

Remark. This should be compared with the Jaffe–Carleman theorem [9, 33, 62] which requires smoothness for V but in place of (a) and (b) only demands $V \geqslant -b$.

Proof. Write $-\Delta + V = (-\Delta + aq^2) + (V - aq^2)$. $-\Delta + aq^2$ is unitary equivalent to the generator of a hypercontractive semigroup. Case (a) tells us $V - aq^2$ is bounded below and (b) tells us V is in L^2 of the relevant measure space. Q.E.D.

Our uniqueness of the vacuum proof (Theorem 4.12) immediately implies the nondegeneracy and nodelessness of the ground state of n-body Bose systems and n-body systems without statistics.

THEOREM 5.2. *Let* $H_0 = \sum_{i,j=1}^n a_{ij}\vec{P}_i \cdot \vec{P}_j$ *be the Hamiltonian of a free n-body system on* $\mathbf{R}^{3n}$, *i.e.,* a_{ij} *is a positive definite matrix. Let* $V = \sum_{i<j=1}^n V_{ij}(r_{ij}) + \sum_{i=1}^n V_i(r_i)$. *Put* $V^{(n)}(q) = V(q)$ *if* $| V(q)| < n$ *and* $V^{(n)}(q) = 0$, *otherwise. Suppose a Hamiltonian H can be defined so that*

(a) $\lim_{n\to\infty} (H_0 + V^{(n)} + \lambda)^{-1} = (H + \lambda)^{-1}$ *in norm.*

(b) $\lim_{n\to\infty} (H - V^{(n)} + \lambda)^{-1} = (H_0 + \lambda)^{-1}$ *in norm.*

Then $H_0 + V$ *has a unique a.e. positive ground state if it has a ground state. This holds true if the Hamiltonian is symmetric under certain particle interchanges and we restrict ourselves to the symmetric (Bose) subspace.*

Remarks. (1) Cases (a) and (b) hold, in particular, if each V_i, $V_{ij} \in L^2 + L^\infty$ or more generally if V_i, $V_{ij} \in R + L^\infty$ [55].

(2) This should be compared with the usual proof [12] which depends critically on smoothness assumptions on any nodes of wavefunctions.

Notes added to proofs.

(1) During the typing of this manuscript Segal has announced a result of the form of Corollary 4.14 in his talk before the International Congress of Mathematicians."

(2) Since the appearance of the original manuscript, Proposition 4.22 has been improved to $\Theta < \pi$ with consequences for summability of the perturbation series for $E(\lambda)$; see B. Simon, Borel summability of the ground-state energy in spatially cutoff $(\phi^4)_2$, *Phys. Rev. Lett.* **25** (1970), 1583–1586, and L. Rosen and B. Simon, *Trans. A.M.S.* (to appear).

(3) Recently, L. Gross has shown that the existence of a ground state (part of Theorem 4.5) follows from abstract hypercontractivity results alone, see L. Gross, "A noncommutative extension of the Perron–Frobenius theorem," *Bull. A.M.S.* **77** (1971), 343–347.

(4) The functions F used as smearing functions for coherent vectors (Egn 3-66) should be functions with $\hat{F}$ in L^∞ rather than only smooth functions.

(5) The one particle space in $\mathcal{H}$ should be those $h \in L^2(R, dk)$ with $\hat{h}$ real-valued.

(6) In the proof of Prop. 2.1 we did not show that e^{-tH_0} was a contraction on complex valued functions. This follows from the fact that it is a contraction on real valued functions and

$$\sqrt{a^2 + b^2} = c \int_0^{2\pi} | a \cos \theta + b \sin \theta | \, d\theta \text{ for suitable } c.$$

References

1. G. A. Baker, The theory and application of the Padé approximant method, *Advan. Theoret. Phys.* **1** (1965), 1–58.
2. C. Bender and T. T. Wu, Analytic structure of energy levels in a field-theory model, *Phys. Rev. Lett.* **21** (1968), 406–409.
3. C. Bender and T. T. Wu, *Phys. Rev.* **184** (1969), 1231–1260.
4. F. Berezin, "The Method of Second Quantization," Academic Press, New York, 1966.
5. D. Bessis and M. Pusterla, Unitary Padé approximants in strong coupling field theory and application to the calculation of ρ and f_0 meson Regge trajectories, *Nuovo Cimento* **54A** (1968), 243–294.
6. J. Bjorken and S. Drell, "Relatavistic Quantum Fields," McGraw-Hill, New York, 1965.
7. E. Caianiello, Theory of coupled quantized fields, *Nuovo Cimento Suppl.* **14** (1959), 177–191.
8. J. Cannon and A. Jaffe, Lorentz covariance of the $\lambda(\Phi^4)_2$ quantum field theory, *Commun. Math. Phys.* **17** (1970), 261–321.
9. T. Carleman, Sur la théorie mathématique de l'équation de Schrödinger, *Ark. Mat. Astronom. Fys.* **24B**, No. 11 (1934).
10. P. Chernoff, Note on the product formula for operator semigroups, *J. Functional Analysis* **2** (1968), 238–242.
11. J. Cook, The mathematics of second quantization, *Trans. Amer. Math. Soc.* **74** (1953), 222–245.
12. R. Courant and D. Hilbert, "Methods of Mathematical Physics," Vol I., Interscience, New York, 1953.
13. J. L. Doob, The Brownian movement and Stochastic equations, *Ann. Math.* **43** (1942), 351–369.

14. N. DUNFORD AND J. SCHWARTZ, "Linear Operators," Vol. I, Interscience, New York, 1958.
15. P. FEDERBUSH, A partially alternative derivation of a result of Nelson, *J. Math. Phys.* **10** (1969), 50–52.
16. K. O. FRIEDRICHS AND H. N. SHAPIRO, "Integration over Hilbert Space and Outer Expansions," *Proc. Natl. Acad. Sci.* **43** (1957), 336–338.
17. G. FROBENIUS, Über Matrizen mit positiven Elementen, *Sitzber Preuss. Akad. Wissen.* **1** (1908), 471–476.
18. J. GLIMM, Boson fields with nonlinear self-interaction in two dimensions, *Commun. Math. Phys.* **8** (1968), 12–25.
19. J. GLIMM AND A. JAFFE, Singular perturbations of self adjoint operators, *Commun. Pure Appl. Math.* **22** (1969), 401–414.
20. J. GLIMM AND A. JAFFE, A $\lambda\varphi^4$ quantum field theory without cutoffs, I, *Phys. Rev.* **176** (1968), 1945–1951.
21. J. GLIMM AND A. JAFFE, A $\lambda\varphi^4$ quantum field theory without cutoffs, II. The field operators and the approximate vacuum, *Ann. Math.* **91** (1970), 362–401.
22. J. GLIMM AND A. JAFFE, A $\lambda\varphi^4$ quantum field theory without cutoffs, III. The physical vacuum, *Acta Math.* **125** (1970), 203–267.
23. J. GLIMM AND A. JAFFE, The Yukawa_2 Quantum Field Theory Without Cutoffs, *Ann. Phys. New York*, in press.
24. L. GROSS, Measurable functions on Hilbert space, *Trans. Amer. Math. Soc.* **105** (1962), 372–390.
25. L. GROSS, Abstract weiner spaces, *in* "Proceedings of the Fifth Berkeley Symposium on Mathematical Statistics and Probability," Univ. of Calif. Press, Berkeley, 1968.
26. M. GUENIN, On the interaction picture, *Commun. Math. Phys.* **3** (1966), 120–132.
27. E. HILLE, A class of reciprocal functions, *Ann. Math.* **27** (1926), 427–464.
28. R. HÖEGH-KROHN, A general class of quantum fields without cutoffs in two space–time dimensions, *Commun. Math. Phys.* **21** (1971), 244–255.
29. A. JAFFE, Wick polynomials at a fixed time, *J. Math. Phys.* **7** (1966), 1251–1255.
30. A. JAFFE, Divirgence of perturbation theory for bosons, *Commun. Math. Phys.* **1** (1965), 127–149.
31. A. JAFFE, Constructing the $\lambda(\Phi^4)_2$ theory, *in* "Rendiconti della Scuola Internazionale di Fisica 'E. Fermi'—XLV Corso," pp. 120–151, Academic Press, New York, 1969.
32. A. JAFFE, Whither axiomatic field theory?, *Rev. Mod. Phys.* **41** (1969), 576–580.
33. A. JAFFE, Dynamics of a cutoff $\lambda\phi^4$ field theory, *Amer. Math. Soc. Memoir* (to appear).
34. R. JENTZSCH, Über Integralgleichungen mit positiver Kern, *J. Reine Angew. Math.* **141** (1912), 235–244.
35. T. KATO, "Perturbation Theory for Linear Operators," Springer-Verlag, New York, 1966.
36. J. KLAUDER, Continuous representation theory. I, II, III, *J. Math. Phys.* **4** (1963), 1055–1058; **4** (1963), 1058–1073; **5** (1964), 177–187.
37. J. KLAUDER AND J. MCKENNA, Continuous representation theory. IV, V, *J. Math. Phys.* **5** (1964), 878–896; **6** (1965), 68–87.
38. J. KLAUDER AND E. SUDERSHAN, "Fundamentals of Quantum Optics," Benjamin, New York, 1968.
39. J. J. LOEFFEL AND A. MARTIN, Proceedings Programme 25 Conference, May 1970, Strassbourg.

40. J. J. LOEFFEL, A. MARTIN, B. SIMON, AND A. S. WIGHTMAN, Padé approximants and the anharmonic oscillator, *Phys. Lett.* **30B** (1969), 656–658.
41. K. LÖWNER, Über Monotone Matrix Functionen, *Math. Z.* **38** (1934), 177–216.
42. E. MEHLER, Reihenentwicklungen nach Laplaceschen Funktionen hoherer Ordnung, *J. Math.* **66** (1866), 161–176.
43. E. NELSON, Analytic vectors, *Ann. Math.* **67** (1959), 572–615.
44. E. NELSON, Feynman integrals and the Schrödinger equation, *J. Math. Phys.* **5** (1964), 332–343.
45. E. NELSON, A quartic interaction in two dimensions, *in* "Mathematical Theory of Elementary Particles" (R. Goodman and I. Segal, Eds.), M.I.T. Press, Cambridge, 1966.
46. O. PERRON, Zür Theorie der Matrices, *Math. Ann.* **64** (1907), 248–263.
47. L. ROSEN, A $\lambda\Phi^{2n}$ field theory without cutoffs, *Commun. Math. Phys.* **16** (1970), 157–183.
48. L. ROSEN, The $:\varphi^{2n}:$ quantum field theory: Higher order estimates, *Commun. Pure Appl. Math.* **24** (1971), 417–457.
49. I. SEGAL, Tensor algebras over Hilbert spaces. I, *Trans. Amer. Math. Soc.* **81** (1956), 106–134.
50. I. SEGAL, Notes towards the construction of nonlinear relatavistic quantum fields. I: The hamiltonian in two space–time dimensions as the generator of a C^*-automorphism group, *Proc. Natl. Acad. Sci.* **57** (1967), 1178–1183.
51. I. SEGAL, Nonlinear functions of weak processes. I, *J. Functional Analysis* **4** (1969), 404–456.
52. I. SEGAL, Notes towards the construction of nonlinear relatavistic quantum fields. III. Properties of the C^*-dynamics for a certain class of interactions, *Bull. Amer. Math. Soc.* **75** (1969), 1390–1395.
53. I. SEGAL, Construction of nonlinear local quantum processes. I, *Ann. Math.*, to appear.
54. B. SIMON, Coupling constant analyticity for the anharmonic oscillator, *Ann. Phys. New York* **58** (1970), 76–136.
55. B. SIMON, Hamiltonians defined as quadratic forms, *Commun. Math. Phys.* **21** (1971), 192–210.
56. E. STEIN, Interpolation of linear operators, *Trans. Amer. Math. Soc.* **83** (1956), 482–492.
57. E. STEIN, Topics in harmonic analysis related to the Littlewood–Paley theory, *Ann. of Math. Study* **63** (1970), Princeton University Press.
58. R. STREATER AND A. S. WIGHTMAN, "PCT, Spin and Statistics and All That," Benjamin, New York, 1964.
59. H. TROTTER, On the product of semigroups of operators, *Proc. Amer. Math. Soc.* **10** (1959), 545–551.
60. L. VAN HOVE, *Acad. Roy. Belg. Bull. Cl. Sci.* **39** (1951), 1055.
61. G.-C. WICK, The evaluation of the collision matrix, *Phys. Rev.* **80** (1950), 268–272.
62. A. S. WIGHTMAN, Introduction to some aspects of the relatavistic dynamics of quantized fields, *in* "High Energy Electromagnetic Interactions and Field Theory" (M. Levy, Ed.), pp. 171–291, Gordon and Breach, New York, 1967.
63. A. S. WIGHTMAN, Analyticity in the Coupling Constant, *Colloq. Int. Centre Nat. Rech. Sci.* **181** (1970), 63.
64. A. S. WIGHTMAN AND L. GÅRDING, Fields as operator-valued distributions in relatavistic quantum theory, *Ark. Fys.* **28** (1964), 129–184.