Independence measures of arithmetic functions

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A B S T R A C T

The notion of algebraic dependence in the ring of arithmetic functions with addition and Dirichlet product is considered. Measures for algebraic independence are derived.

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1. Introduction

Denote by \((\mathcal{A}, +, \ast)\) the unique factorization domain of arithmetic functions equipped with addition and convolution (or Dirichlet product) defined by

\[(f + g)(n) := f(n) + g(n),\]
\[(f \ast g)(n) = \sum_{ij=n} f(i)g(j) \quad (f, g \in \mathcal{A}, \ n \in \mathbb{N}).\]

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and write $f^{*i} = f \ast \cdots \ast f$, where the right-hand expression is a convolution of $i \in \mathbb{N}$ terms. The convolution identity, $I$, is defined by $I(1) = 1$ and $I(n) = 0$ for all $n > 1$. It is well known [17, Chapter 4] that $(\mathcal{A}, +, \ast)$ is isomorphic to $(\mathcal{D}, +, \cdot)$, where

$$ \mathcal{D} := \left\{ D(s) := \sum_{n=1}^{\infty} \frac{f(n)}{n^s} \right\} $$

is the ring of formal Dirichlet series equipped with addition and multiplication, through the isomorphism $f \leftrightarrow D$; the addition in both domains is the customary addition while the multiplication of formal Dirichlet series corresponds to the convolution of the appropriate arithmetic functions appearing as coefficients of the two formal Dirichlet series. For $f \in \mathcal{A}$, its valuation ([17, Chapter 4], [16]) is defined as

$$ |f| := \frac{1}{O(f)}, $$

where $O(f)$ is the least integer $n$ for which $f(n) \neq 0$. Correspondingly, for a formal Dirichlet series $D(s) := \sum_{n \geq 1} f(n)/n^s$, its valuation is defined as

$$ |D| = |f|, $$

where the same valuation symbols are used for convenience sake. With such valuation, the isomorphism $(\mathcal{A}, +, \ast) \leftrightarrow (\mathcal{D}, +, \cdot)$ is indeed an isometry. Because of this isometry, we often refer to each domain interchangeably.

A set of arithmetic functions $f_1, \ldots, f_r$ is said to be algebraically dependent over $\mathbb{C}$ or $\mathbb{C}$-algebraically dependent if there exists

$$ P[X_1, \ldots, X_r] := \sum_{i_1, \ldots, i_r} a_{i_1, \ldots, i_r} X_1^{i_1} \cdots X_r^{i_r} \in \mathbb{C}[X_1, \ldots, X_r] \setminus \{0\} $$

such that

$$ \sum_{i_1, \ldots, i_r} a_{i_1, \ldots, i_r} f_1^{*i_1} \ast \cdots \ast f_r^{*i_r} \equiv 0, $$

and is said to be $\mathbb{C}$-algebraically independent otherwise. If the polynomial $P$ is homogeneous of degree one in each variable, we say that $f_1, \ldots, f_r$ are $\mathbb{C}$-linearly dependent and $\mathbb{C}$-linearly independent otherwise. The first investigation of dependence of arithmetic functions was due to Carlitz [3] in 1952. Popken [9] in 1962 considered the problem of algebraic dependence in a more general setting of functions defined over a unique factorization semigroup with values in a ring. His main results give necessary and sufficient conditions for algebraic dependence by analyzing the Taylor expansion of the polynomial defining the dependence. In subsequent papers [10–12], he made applications to Dirichlet series and multiplicative functions. In the direction of Dirichlet series, Popken [13] gave a measure of the so-called differential transcendence of certain Dirichlet series closely connected to the Riemann zeta function, $\zeta$ [8]. More recent works can be found in [18], where algebraic independence of Dirichlet series and transcendence over $\mathbb{C}[\zeta]$ are considered. The works of Popken mentioned above were simplified and sharpened in [6].

In the present work, our main objectives are first to derive some algebraic independence criteria and then to prove general quantitative results about measure of such independence of arithmetic functions which simultaneously implies corresponding results for formal Dirichlet series. We also apply our results to a number of interesting cases in particular to the formal Fibonacci and Lucas zeta series.
To do so, we require certain related concepts which we briefly recall now. A derivation \( d \) \([16,17]\) over \( \mathcal{A} \) is a map \( \mathcal{A} \to \mathcal{A} \) satisfying

\[
d(f \ast g) = df \ast g + f \ast dg,
\]

\[
d(c_1 f + c_2 g) = c_1 df + c_2 dg,
\]

where \( f, g \in \mathcal{A} \) and \( c_1, c_2 \in \mathbb{C} \). Derivations of higher orders are defined in the usual manner. Two typical examples of derivation are

- the \( p \)-basic derivation, \( p \) prime, defined by

\[
(d_p f)(n) = f(np)\nu_p(np) \quad (n \in \mathbb{N}),
\]

where \( \nu_p(m) \) denotes the exponent of the highest power of \( p \) dividing \( m \),

- the log-derivation defined by

\[
(d_L f)(n) = f(n)\log n \quad (n \in \mathbb{N}).
\]

Although, there are arithmetic sequences \( f(n) \) for which the corresponding Dirichlet series \( D(s) := \sum_n f(n)/n^s \) are divergent, through the isometry between \( \mathcal{A} \) and \( \mathcal{D} \), it is legitimate to define the formal derivation \( \tilde{d} \) of (formal) Dirichlet series via the derivation \( d \) of the associated arithmetic function as

\[
\tilde{d}D(s) = \sum_{n=1}^{\infty} \frac{df(n)}{n^s}.
\]

Thus, the formal differentiation of the formal Dirichlet series, \( D(s) \), with respect to the variable \( s \), i.e.,

\[
D'(s) = \sum_{n=1}^{\infty} \frac{-f(n)\log n}{n^s},
\]

corresponds to the (negative) log-derivation \(-d_L\) of the associated arithmetic function \( f \), and the \( p \)-basic derivation \( d_p \) over \( \mathcal{A} \) corresponds to the formal \( p \)-basic derivation \( \tilde{d}_p \) over \( \mathcal{D} \) defined by

\[
\tilde{d}_p D(s) = \sum_{n=1}^{\infty} \frac{(dp f)(n)}{n^s}.
\]

For convenience, in the sequel we use the same derivation symbol \( d \) for both the domains \( \mathcal{A} \) and \( \mathcal{D} \). Our investigation concerning Dirichlet series will be formal throughout, noting that should the Dirichlet series involved converge, the results so obtained are then valid (analytically) and coincide with results proved for convergent Dirichlet series in the domain of convergence.

2. Some criteria

To state some preliminary results, we need another notion. For \( f \in \mathcal{A}, f(1) > 0 \), the Rearick logarithmic operator of \( f \) (or logarithm of \( f \) \([14,15,7]\)), denoted by \( \text{Log} f \in \mathcal{A} \), is defined via

\[
(\text{Log} f)(1) = \log f(1),
\]

\[
(\text{Log} f)(n) = \frac{1}{\log n} \sum_{k|n} f(k) f^{-1} \left( \frac{n}{k} \right) \log k = \frac{1}{\log n} (d_L f \ast f^{-1})(n) \quad (n > 1).
\]
where \(d_L\) denotes the log-derivation. For \(h \in \mathcal{A}\), the Rearick exponential \(\text{Exp} h\) is defined as the unique element \(f \in \mathcal{A}\), \(f(1) > 0\) such that \(h = \log f\).

We start with some simple results.

**Proposition 2.1.** Let \(f \in \mathcal{A} \setminus \{0\}\).

1. Then \(f\) is \(\ast\)-algebraic over \(\mathbb{C}\) if and only if \(f = cI\) for some constant \(c \in \mathbb{C}\).

2. Assuming \(f(1) > 0\), then \(f\) and \(\log f\) are \(\mathbb{C}\)-algebraically dependent if and only if \(f = cI\) for some constant \(c \in \mathbb{C}\).

3. Assuming \(f(1) \in \mathbb{R}\), then \(f\) and \(\text{Exp} f\) are \(\mathbb{C}\)-algebraically dependent if and only if \(f = cI\) for some constant \(c \in \mathbb{C}\).

**Proof.** We give only a proof for assertion 1 as those for the other two assertions are similar.

The sufficiency part is trivial. To prove the necessity part, assume that \(f\) satisfies an algebraic equation of the form

\[a_k f^*k + \cdots + a_1 f + a_0 I = 0,\]

with least degree \(k \geq 1\) and \(a_k \neq 0\). Taking the log-derivation, we get

\[(ka_k f^*k-1 + \cdots + a_1 I) \ast d_L f = 0.\]

By the minimality of \(k\), we must have \(d_L f = 0\) which is the result. \(\Box\)

Shapiro–Sparer’s criterion for \(\mathbb{C}\)-algebraic dependence of arithmetic functions in [18] states that:

**Theorem 2.2.** Let \(f_1, \ldots, f_t \in \mathcal{A}\) and \(p_1, \ldots, p_t\) be distinct primes with corresponding \(p\)-basic derivations \(d_1 := d_{p_1}, \ldots, d_t := d_{p_t}\). If the Jacobian relative to \(d_1, \ldots, d_t\)

\[J := J(f_1, \ldots, f_t; d_1, \ldots, d_t) = \begin{vmatrix} d_1 f_1 & \cdots & d_t f_1 \\ \vdots & \ddots & \vdots \\ d_t f_1 & \cdots & d_t f_t \end{vmatrix} \neq 0,\]

where the multiplication in the determinant expansion is interpreted as convolution \(\ast\), then \(f_1, \ldots, f_t\) are \(\mathbb{C}\)-algebraically independent.

Evaluating the Jacobian at \(n \in \mathbb{N}\) in Theorem 2.2, we get

\[J(n) = \sum_{(i)} e_{(i)} (d_{i_1} f_{i_1} \ast \cdots \ast d_{i_t} f_{i_t})(n),\]

where the sum is taken over all possible permutations \((i) = (i_1, \ldots, i_t)\) of \((1, \ldots, t)\) with

\[e_{(i)} = \begin{cases} 1 & \text{if } (i) \text{ is an even permutation}, \\ -1 & \text{otherwise}. \end{cases}\]

Consequently, writing \(v_1, \ldots, v_t\) for \(v_{p_1}, \ldots, v_{p_t}\), respectively, we have
\[
J(n) = \sum_{i} e(i) \sum_{k_1 \cdots k_t = n} d_i f_i(k_1) \cdots d_t f_t(k_t)
\]

\[
= \sum_{k_1 \cdots k_t = n} \sum_{i} e(i) f_i(k_1 p_1) \cdots f_i(k_t p_t) v_i(k_1 p_1) \cdots v_i(k_t p_t)
\]

\[
= \sum_{k_1 \cdots k_t = n} v_i(k_1 p_1) \cdots v_i(k_t p_t) \begin{vmatrix} f_1(k_1 p_1) & \cdots & f_1(k_t p_t) \\ \vdots & \ddots & \vdots \\ f_t(k_1 p_1) & \cdots & f_t(k_t p_t) \end{vmatrix},
\]

which yields

**Corollary 2.3.** Let \( f_1, \ldots, f_t \) be arithmetic functions and \( p_1, \ldots, p_t \) be distinct primes with corresponding \( p \)-basic derivations \( d_1 := d_{p_1}, \ldots, d_t := d_{p_t} \) and corresponding \( p \)-exponent functions \( v_1 := v_{p_1}, \ldots, v_t := v_{p_t} \). If there exists \( n \in \mathbb{N} \) such that

\[
\sum_{k_1 \cdots k_t = n} v_i(k_1 p_1) \cdots v_i(k_t p_t) \begin{vmatrix} f_1(k_1 p_1) & \cdots & f_1(k_t p_t) \\ \vdots & \ddots & \vdots \\ f_t(k_1 p_1) & \cdots & f_t(k_t p_t) \end{vmatrix} \neq 0,
\]

then \( f_1, \ldots, f_t \) are \( \mathbb{C} \)-algebraically independent.

Specializing the values of \( n \), we deduce the following simple tests of algebraic independence.

**Test I.** The simplest test is obtained by taking \( n = 1 \) in Corollary 2.3. If

\[
\begin{vmatrix} f_1(p_1) & \cdots & f_1(p_t) \\ \vdots & \ddots & \vdots \\ f_t(p_1) & \cdots & f_t(p_t) \end{vmatrix} \neq 0,
\]

then \( f_1, \ldots, f_t \) are \( \mathbb{C} \)-algebraically independent.

An immediate consequence of Test I is the following convenient test of algebraic independence.

**Corollary 2.4.** Let \( f_1, \ldots, f_t \in \mathcal{A} \). If there are \( t \) distinct primes \( p_1, \ldots, p_t \) such that the set of vectors \( \{(f_1(p_i), \ldots, f_t(p_i)): i = 1, \ldots, t\} \) is \( \mathbb{C} \)-linearly independent, then \( f_1, \ldots, f_t \) are \( \mathbb{C} \)-algebraically independent.

**Test II.** Taking \( n = p_1 \), if

\[
0 \neq 2 \begin{vmatrix} f_1(p_1^2) & f_1(p_2) & \cdots & f_1(p_t) \\ \vdots & \ddots & \vdots & \vdots \\ f_t(p_1^2) & f_t(p_2) & \cdots & f_t(p_t) \end{vmatrix} + \begin{vmatrix} f_1(p_1 p_2) & f_1(p_1) & \cdots & f_1(p_t) \\ \vdots & \ddots & \vdots & \vdots \\ f_t(p_1 p_2) & f_t(p_1) & \cdots & f_t(p_t) \end{vmatrix} + \cdots
\]

\[
+ \begin{vmatrix} f_1(p_1) & f_1(p_2) & \cdots & f_1(p_1 p_t) \\ \vdots & \ddots & \vdots & \vdots \\ f_t(p_1) & f_t(p_2) & \cdots & f_t(p_1 p_t) \end{vmatrix},
\]

then \( f_1, \ldots, f_t \) are \( \mathbb{C} \)-algebraically independent.
Test III. Taking \( n = q \), prime distinct from \( p_1, \ldots, p_t \), if

\[
\begin{vmatrix}
0 & f_1(qp_1) & f_1(p_2) & \cdots & f_1(p_t) \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
f_t(qp_1) & f_t(p_2) & \cdots & f_t(p_t) & \vdots \\
f_1(p_1) & f_1(p_2) & \cdots & f_1(qp_t) & \vdots \\
f_t(p_1) & f_t(p_2) & \cdots & f_t(qp_t) & \vdots
\end{vmatrix}
\]

then \( f_1, \ldots, f_t \) are \( \mathbb{C} \)-algebraically independent.

Test IV. Taking \( n = p_t^2 \), if

\[
\begin{vmatrix}
0 & 3 & f_1(p_3^2) & f_1(p_2^2) & \cdots & f_1(p_t^2) \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
f_t(p_3^2) & f_t(p_2^2) & \cdots & f_t(p_t^2) & \vdots & \vdots \\
f_1(p_1) & f_1(p_2) & \cdots & f_1(p_t) & \vdots & \vdots \\
f_t(p_1) & f_t(p_2) & \cdots & f_t(p_t) & \vdots & \vdots \\
+ 2 & f_1(p_1^2) & f_1(p_1p_2) & \cdots & f_1(p_t) & \vdots & \vdots \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
f_t(p_1^2) & f_t(p_1p_2) & \cdots & f_t(p_t) & \vdots & \vdots & \vdots \\
f_1(p_1) & f_1(p_1p_2) & f_1(p_1p_3) & \cdots & f_1(p_t) & \vdots & \vdots \\
f_t(p_1) & f_t(p_1p_2) & f_t(p_1p_3) & \cdots & f_t(p_t) & \vdots & \vdots \\
+ & f_1(p_1) & f_1(p_2) & \cdots & f_1(p_{t-2}) & f_1(p_{t-1}p_t) & f_1(p_t) \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
f_t(p_1) & f_t(p_2) & \cdots & f_t(p_{t-2}) & f_t(p_{t-1}p_t) & f_t(p_t)
\end{vmatrix}
\]

then \( f_1, \ldots, f_t \) are \( \mathbb{C} \)-algebraically independent.

Let us now look at some examples. Let \( \{F_n\}_{n \geq 1} \) be the sequence of Fibonacci numbers defined by

\[
F_1 = F_2 = 1, \quad F_{n+2} = F_{n+1} + F_n \quad (n \in \mathbb{N}).
\]

The six formal Fibonacci zeta series are defined as

\[
\mathcal{F}_+^{+}(s) := \sum_{n=1}^{\infty} \frac{1}{F_n^s} = \sum_{n=1}^{\infty} \frac{f_+^{+}(n)}{n^s}, \quad \mathcal{F}_+^{-}(s) := \sum_{n=1}^{\infty} \frac{1}{F_{2n}^s} = \sum_{n=1}^{\infty} \frac{f_+^{-}(n)}{n^s},
\]

\[
\mathcal{F}_0^{+}(s) := \sum_{n=1}^{\infty} \frac{1}{F_{3n-2}^s} = \sum_{n=1}^{\infty} \frac{f_0^{+}(n)}{n^s}, \quad \mathcal{F}_0^{-}(s) := \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{F_n^s} = \sum_{n=1}^{\infty} \frac{f_0^{-}(n)}{n^s},
\]

\[
\mathcal{F}_e^{+}(s) := \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{F_{2n}^s} = \sum_{n=1}^{\infty} \frac{f_e^{+}(n)}{n^s}, \quad \mathcal{F}_e^{-}(s) := \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{F_{2n-1}^s} = \sum_{n=1}^{\infty} \frac{f_e^{-}(n)}{n^s}.
\]
Let \( \{L_n\}_{n \geq 1} \) be the sequence of Lucas numbers defined by

\[
L_1 = 1, \quad L_2 = 3, \quad L_{n+2} = L_{n+1} + L_n \quad (n \in \mathbb{N}).
\]

The six formal Lucas zeta series are defined as

\[
\mathcal{L}_+^+ (s) := \sum_{n=1}^{\infty} \frac{1}{L_n^s} = \sum_{n=1}^{\infty} \frac{\ell^+ (n)}{n^s}, \quad \mathcal{L}_-^+ (s) := \sum_{n=1}^{\infty} \frac{1}{L_{2n}^s} = \sum_{n=1}^{\infty} \frac{\ell^+ (n)}{n^s},
\]

\[
\mathcal{L}_0^+ (s) := \sum_{n=1}^{\infty} \frac{1}{L_{2n-1}^s} = \sum_{n=1}^{\infty} \frac{\ell_0^+ (n)}{n^s}, \quad \mathcal{L}_-^- (s) := \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{L_n^s} = \sum_{n=1}^{\infty} \frac{\ell^- (n)}{n^s},
\]

\[
\mathcal{L}_0^- (s) := \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{L_{2n-1}^s} = \sum_{n=1}^{\infty} \frac{\ell_0^- (n)}{n^s}.
\]

These twelve (formal) Fibonacci and Lucas zeta series were considered in [5] in order to prove that they are hypertranscendental. We now establish some of their dependence relations.

**Proposition 2.5.**

1. Three functions in each of the following sets of arithmetic functions are \( \mathbb{C} \)-algebraically independent:

\[
\{ f^+, f_e^+, f_o^+ \}, \{ f^+, f_e^+, f_o^- \}, \{ f^+, f_o^+, f_e^- \}, \{ f^+, f_o^+ \}, \{ f^+, f_e^+ \}, \{ f^+, f_o^- \}, \{ f^+, f_e^- \}, \{ f^+, f_o^- \}, \{ f^+, f_e^- \}.
\]

2. We have \( f^+ = f_e^+ + f_o^+ \), \( f^+ = 2f_o^- - f^- \), \( f^- = f_o^+ - f_e^- \), \( f^+ = f^- + 2f_e^+ \), i.e., three functions in each of the following sets are \( \mathbb{C} \)-linearly dependent

\[
\{ f^+, f_e^+, f_o^+ \}, \{ f^+, f_e^+, f_o^- \}, \{ f^+, f_o^+ \}, \{ f^+, f_e^+ \}, \{ f^+, f_e^- \}, \{ f^+, f_o^- \}, \{ f^+, f_e^- \}.
\]

**Proof.** The results of assertion 2 are clear, so we need only check those in assertion 1. We only provide two of them using different tests (Tests I and III).

By Test III, we have

\[
\begin{bmatrix}
f^+(2 \times 11) & f^+(3 = F_4) & f^+(5 = F_5) \\
f_e^+(2 \times 11) & f_e^+(3 = F_4) & f_e^+(5 = F_5) \\
f_o^-(2 \times 11) & f_o^-(3 = F_4) & f_o^-(5 = F_5)
\end{bmatrix}
\begin{bmatrix}
f^+(2) & f^+(3) & f^+(5) \\
f_e^+(2) & f_e^+(3) & f_e^+(5) \\
f_o^-(2) & f_o^-(3) & f_o^-(5)
\end{bmatrix}
= \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}
\]

\[
\begin{bmatrix}
f^+(2) & f^+(3) & f^+(5 \times 11 = F_{10}) \\
f_e^+(2) & f_e^+(3) & f_e^+(5 \times 11 = F_{10}) \\
f_o^-(2) & f_o^-(3) & f_o^-(5 \times 11 = F_{10})
\end{bmatrix}
= \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}
\]

\[
\begin{bmatrix}
f^+(2) & f^+(3) & f^+(5 \times 11 = F_{10}) \\
f_e^+(2) & f_e^+(3) & f_e^+(5 \times 11 = F_{10}) \\
f_o^-(2) & f_o^-(3) & f_o^-(5 \times 11 = F_{10})
\end{bmatrix}
= \begin{bmatrix} 0 & 1 & 1 \end{bmatrix}
\]

i.e. \( f^+, f_e^+, f_o^- \) are \( \mathbb{C} \)-algebraically independent.
By Test I, we have
\[
\begin{vmatrix}
  f^+(2 = F_3) & f^+(3 = F_4) & f^+(5 = F_5) \\
  f^+_e(2 = F_3) & f^+_e(3 = F_4) & f^+_e(5 = F_5) \\
  f^-_o(2 = F_3) & f^-_o(3 = F_4) & f^-_o(5 = F_5)
\end{vmatrix} = \begin{vmatrix}
  1 & 1 & 1 \\
  0 & 1 & 0 \\
  -1 & 0 & 1
\end{vmatrix} = 2 \neq 0,
\]
i.e. \( f^+, f^+_e, f^-_o \) are \( \mathbb{C} \)-algebraically independent. \( \Box \)

The situation for formal Lucas zeta series is much the same and we merely state the result.

**Proposition 2.6.**

1. Three functions in each of the following sets of arithmetic functions are \( \mathbb{C} \)-algebraically independent:

\[
\{ \ell^+, \ell^+_e, \ell^+_o \}, \{ \ell^+, \ell^+_o, \ell^-_o \}, \{ \ell^+, \ell^-_o, \ell^-_e \}, \{ \ell^+, \ell^+_e, \ell^-_e \}, \{ \ell^+, \ell^-_e, \ell^-_o \}, \{ \ell^+, \ell^-_e, \ell^-_o \};
\]

\[
\{ \ell^+_e, \ell^+_o, \ell^-_o \}, \{ \ell^+_e, \ell^+_o, \ell^-_e \}, \{ \ell^+_e, \ell^-_o, \ell^-_e \}, \{ \ell^+_e, \ell^-_o, \ell^-_o \}, \{ \ell^+_e, \ell^+_o, \ell^-_e \}, \{ \ell^+_e, \ell^-_o, \ell^-_o \};
\]

\[
\{ \ell^-, \ell^-_o, \ell^-_o \}.
\]

2. We have \( \ell^+ = \ell^+_e + \ell^+_o, \ell^- = 2\ell^+_e - \ell^-_e, \ell^- = \ell^+_o - \ell^-_e, \ell^+ = \ell^- + 2\ell^+_o \), i.e., three functions in each of the following sets are \( \mathbb{C} \)-linearly dependent

\[
\{ \ell^+, \ell^+_e, \ell^+_o \}, \{ \ell^+, \ell^+_o, \ell^- \}, \{ \ell^-, \ell^+_o, \ell^- \}, \{ \ell^+, \ell^+_e, \ell^- \}, \{ \ell^-, \ell^+_e, \ell^- \}.
\]

3. Three functions with at least one from each of the two sets \( \{ f^+, f^+_e, f^-_o, f^-_e, f^-_o \} \) and \( \{ \ell^+, \ell^+_e, \ell^+_o, \ell^-, \ell^-_e, \ell^-_o \} \) are \( \mathbb{C} \)-algebraically independent.

**3. Measure of algebraic independence**

We start with an auxiliary result whose proof resembles that of [6, Theorem 2].

**Lemma 3.1.** Let \( f_1, \ldots, f_r \in \mathcal{A} \) and \( P(X_1, \ldots, X_r) \in \mathbb{C}[X_1, \ldots, X_r] \setminus \{0\} \). For \( t = 1, \ldots, r \), define the following formal Dirichlet series

\[
D_t(s) = \sum_{n \geq 1} \frac{f_t(n)}{n^s}, \quad P(D_1, \ldots, D_r) = \sum_{n \geq 1} \frac{F(n)}{n^s}, \quad \frac{\partial P}{\partial X_t}(D_1, \ldots, D_r) = \sum_{n \geq 1} \frac{F_t(n)}{n^s}.
\]

Then for each \( n \in \mathbb{N} \) and for each prime \( p \), we have

\[
F(pn) v_p(pn) = \sum_{j=1}^{r} \sum_{k|n} f_j(pk) F_j \left( \frac{n}{k} \right) v_p(pk), \quad (3.1)
\]

\[
F(n) \log n = \sum_{j=1}^{r} \sum_{k|n} f_j(k) F_j \left( \frac{n}{k} \right) \log k, \quad (3.2)
\]

where the Dirichlet series and their operations are considered formally.
Proof. For each prime $p$, let $d = d_p$ be its $p$-basic derivation. Through the isometry $A \leftrightarrow D$, the correspondence of $p$-basic derivation in both domains and the fact that a product of formal Dirichlet series is isomorphic to a convolution of arithmetic functions, we formally have

$$
\sum_{n \geq 1} \frac{F(np) \nu_p(np)}{n^s} = \sum_{n \geq 1} \frac{dF(n)}{n^s} = dP(D_1, \ldots, D_r) = \sum_{j=1}^r dD_j \cdot \frac{\partial P}{\partial X_j} (D_1, \ldots, D_r)
$$

$$
= \sum_{j=1}^r \left( \sum_{n \geq 1} \frac{df_j(n)}{n^s} \right) \left( \sum_{n \geq 1} \frac{F_j(n)}{n^s} \right) = \sum_{j=1}^r \sum_{n \geq 1} \sum_{k|n} \frac{df_j(k) F_j(n/k)}{n^s}
$$

$$
= \sum_{n \geq 1} \sum_{j=1}^r \sum_{k|n} \frac{f_j(pk) F_j(n/k) \nu_p(pk)}{n^s}.
$$

(3.3)

Analytically, Eq. (3.3) is true only if the two Dirichlet series on the left-hand side converge absolutely, and this might not be the case for certain sequences $f_j, F_j \in A$. However, the above proof is treated formally in the sense that it holds true for formal Dirichlet series and formal operations.

The relation (3.1) follows from equating the terms with $n \geq 2$. The relation (3.2) follows in the same manner by taking log-derivation and equating the terms with $n \geq 2$. □

Our main result reads:

**Theorem 3.2.** Let $P(X_1, \ldots, X_r) \in \mathbb{C}[X_1, \ldots, X_r] \setminus \{0\}$ be of total degree $\text{deg} \, P = g$. For $t = 1, \ldots, r$, define the following formal Dirichlet series

$$
D_t(s) = \sum_{n \geq 1} \frac{f_t(n)}{n^s}, \quad P(D_1, \ldots, D_r) = \sum_{n \geq 1} \frac{F(n)}{n^s}, \quad \frac{\partial P}{\partial X_t}(D_1, \ldots, D_r) = \sum_{n \geq 1} \frac{F_t(n)}{n^s}.
$$

Let $\{p_1 < p_2 < p_3 < \cdots < p_r\}$ be a set of primes. If the set of vectors $\{(f_1(p_i)), \ldots, f_r(p_i))\}$: $i = 1, \ldots, r$ is linearly independent over $\mathbb{C}$, then

$$
|P(D_1, \ldots, D_r)| \geq p_r^{-g},
$$

where the Dirichlet series, their derivatives and operations are considered formally.

Proof. If $\text{deg} \, P = 0$, then clearly $|P(D_1, \ldots, D_r)| = 1$. If $\text{deg} \, P = 1$, then

$$
P(X_1, \ldots, X_r) = a_0 I + a_1 X_1 + \cdots + a_r X_r,
$$

where all the coefficients $a_j$ ($j = 1, \ldots, r$) do not vanish simultaneously. Equating coefficients, we get

$$
F(p_j) = a_1 f_1(p_j) + \cdots + a_r f_r(p_j).
$$

Since the set of vectors $\{(f_1(p_j)), \ldots, f_r(p_j)): j = 1, \ldots, r\}$ is linearly independent over $\mathbb{C}$, then at least one of the values $F(p_1), \ldots, F(p_r)$ must be non-zero, which renders

$$
|P(D_1, \ldots, D_r)| \geq p_r^{-1}.
$$

Now proceed by induction on $\text{deg} \, P$. Let $P$ be of total degree $g + 1 \geq 2$, and assume that the assertion has already been proved for polynomials of degree $\leq g$. Consider the polynomials $\partial P/\partial X_t$ ($t =$
1, \ldots, r), which are all of degree \leq g. Unless each \( \partial P / \partial X_t \) vanishes identically, then by induction we have

\[
\left| \frac{\partial P}{\partial X_t}(D_1, \ldots, D_r) \right| \geq p_r^{-g},
\]

which implies that not all of the \( p_r^g \) vectors

\[
\{(F_1(1), \ldots, F_r(1)), (F_1(2), \ldots, F_r(2)), \ldots, (F_1(p_r^g), \ldots, F_r(p_r^g))\}
\]

(3.4)
can be a zero vector. Let \((F_1(m), \ldots, F_r(m))\) be the first non-zero vector in the sequence (3.4) so that

\( (F_1(d), \ldots, F_r(d)) = (0, \ldots, 0) \) for \( d = 1, 2, \ldots, m - 1 \).

By the minimality of \( m \) and the result of Lemma 3.1, we get

\[
F(pm)v(pm) = f_1(p)F_1(m) + \cdots + f_r(p)F_r(m).
\]

Since the set \{(f_1(p_j), \ldots, f_r(p_j)): \( j = 1, \ldots, r \)} is linearly independent over \( \mathbb{C} \), among the \( r \) values of \( F(p_1m), \ldots, F(p_r m) \) at least one must be non-zero. This yields

\[
|P(D_1, \ldots, D_r)| \geq (mp_r)^{-1} \geq \left( \frac{g+1}{p_r} \right)^{-1}. \quad \Box
\]

As a simple example, we make quantitative one of the algebraic independence results of Proposition 2.5. Taking the first three primes \( 2 = F_3, 3 = F_4, 5 = F_5 \). As seen in the proof of Proposition 2.5, part 1, the set

\[
\{(f^+(2), f_e^+(2), f_o^-(2)), (f^+(3), f_e^+(3), f_o^-(3)), (f^+(5), f_e^+(5), f_o^-(5))\}
\]
is \( \mathbb{C} \)-linearly independent. By Theorem 3.2, we have

\[
|P(\mathcal{F}^+, \mathcal{F}_e^+, \mathcal{F}_o^-)| \geq 5^{-g},
\]

for any \( P(X_1, X_2, X_3) \in \mathbb{C}[X_1, X_2, X_3] \setminus \{0\} \) of total degree \( g \).

For a more complex example, let us note that the four Lucas zeta functions \( \ell^+, \ell^-, \ell_e^-, \ell_o^- \) are algebraically independent over \( \mathbb{C} \) because by Test I, we have

\[
\begin{bmatrix}
\ell^+(3 = L_2) & \ell^+(7 = L_4) & \ell^+(11 = L_5) & \ell^+(29 = L_7) \\
\ell^-(3 = L_2) & \ell^-(7 = L_4) & \ell^-(11 = L_5) & \ell^-(29 = L_7) \\
\ell_e^-(3 = L_2) & \ell_e^-(7 = L_4) & \ell_e^-(11 = L_5) & \ell_e^-(29 = L_7) \\
\ell_o^-(3 = L_2) & \ell_o^-(7 = L_4) & \ell_o^-(11 = L_5) & \ell_o^-(29 = L_7)
\end{bmatrix}
= \begin{bmatrix}
1 & 1 & 1 & 1 \\
-1 & -1 & 1 & 1 \\
1 & -1 & 0 & 0 \\
0 & 0 & 1 & -1
\end{bmatrix}
= -8 \neq 0.
\]

By Theorem 3.2, we have

\[
|P(\mathcal{L}^+, \mathcal{L}^-, \mathcal{L}_e^-, \mathcal{L}_o^-)| \geq 29^{-g},
\]

for any \( P(X_1, X_2, X_3, X_4) \in \mathbb{C}[X_1, X_2, X_3, X_4] \setminus \{0\} \) of total degree \( g \).

Theorem 3.2 enables us to derive a measure of the so-called differential transcendence of formal Dirichlet series encompassing the special case of the Riemann zeta function.
Corollary 3.3. Let \( D(s) = \sum_{n \geq 1} f(n)n^{-s} \in \mathcal{D} \) and \( P(X_0, \ldots, X_r) \in \mathbb{C}[X_1, \ldots, X_r] \setminus \{0\} \) be of total degree \( g \). If there is a set of \( r+1 \) primes \( \{p_1 < \cdots < p_{r+1}\} \) such that \( f(p_i) \neq 0 \) (\( i = 1, \ldots, r+1 \)), then

\[
|P(D, D', \ldots, D^{(r)})| \geq (p_{r+1}^{g})^{-1},
\]

where the Dirichlet series, their derivatives and operations are considered formally.

**Proof.** Formally differentiating the Dirichlet series with respect to \( s \) for \( j \in \mathbb{N} \) times, we get

\[
D^{(j)}(s) = \sum_{n \geq 1} \frac{f(n)(-\log n)^j}{n^s}.
\]

For each \( i \in \{1, \ldots, r+1\} \), since \( f(p_i)(-\log p_i)^j \neq 0 \), the determinant

\[
\begin{vmatrix}
 f(p_1) & f(p_1)(-\log p_1) & \cdots & f(p_1)(-\log p_1)^r \\
 \vdots & \vdots & & \vdots \\
 f(p_{r+1}) & f(p_{r+1})(-\log p_{r+1}) & \cdots & f(p_{r+1})(-\log p_{r+1})^r \\
 1 & (-\log p_1) & \cdots & (-\log p_1)^r & \vdots & \vdots \\
 1 & (-\log p_{r+1}) & \cdots & (-\log p_{r+1})^r & \neq 0,
\end{vmatrix}
\]

implying that the set of vectors

\[
\left\{ (f(p_1), \ldots, f(p_{r+1})), (-f(p_1)\log p_1, \ldots, -f(p_{r+1})\log p_{r+1}), \ldots, \\
(-f(p_1)(\log p_1)^r, \ldots, -f(p_{r+1})(\log p_{r+1})^r) \right\}
\]

is \( \mathbb{C} \)-linearly independent. The assertion now follows from Theorem 3.2. \( \square \)

Applying the result of Corollary 3.3 to the formal Riemann zeta series, we get a nice measure

\[
|P(\zeta(s), \zeta'(s), \ldots, \zeta^{(r)}(s))| \geq p_{r+1}^{g},
\]

for any \( P(X_0, \ldots, X_r) \in \mathbb{C}[X_1, \ldots, X_r] \setminus \{0\} \) of total degree \( g \).

The condition of linear independence at primes in Theorem 3.2 can be relaxed at the expense of an extra condition, as we show next.

**Theorem 3.4.** Let \( P(X_1, \ldots, X_r) \in \mathbb{C}[X_1, \ldots, X_r] \setminus \{0\} \) be of total degree \( g \) and let

\[
D_t(s) = \sum_{n \geq 1} \frac{f_t(n)}{n^s}, \quad P(D_1, \ldots, D_r) = \sum_{n \geq 1} \frac{F(n)}{n^s},
\]

\[
\frac{\partial P}{\partial X_t}(D_1, \ldots, D_r) = \sum_{n \geq 1} \frac{F_t(n)}{n^s} \quad (t = 1, \ldots, r).
\]

Assume that there are a set of \( r \) primes \( \{p_1 < p_2 < \cdots < p_r\} \) and a set of \( r \) positive integers \( \{n_1, \ldots, n_r\} \) such that

\[
f_t(p_i n_i) \neq 0 \quad \text{but} \quad f_t(p_i k) = 0 \quad \text{for} \ 1 \leq k < n_i \ (t = 1, \ldots, r; \ i = 1, \ldots, r).
\]
If the vectors \( \{(f_1(p_1n_1), \ldots, f_r(p_rn_1)); i = 1, \ldots, r\} \) are \( \mathbb{C} \)-linearly independent, then
\[
|P(D_1, \ldots, D_r)| \geq M_1^{-g},
\]
where \( M_1 = \max\{p_1n_1, \ldots, p_rn_r\} \), and the Dirichlet series, their derivatives together with operations are considered formally.

**Proof.** All the Dirichlet series, their derivatives and operations are formally treated here. If \( \deg P = 0 \), then \( |P(D_1, \ldots, D_r)| = 1 \). If \( \deg P = 1 \), then
\[
P(D_1, \ldots, D_r) = a_0 + \sum_{t=1}^{r} a_t D_t
\]
with not all \( a_t \)'s vanishing simultaneously. Now
\[
\sum_{n \geq 1} \frac{F(n)}{n^s} = a_0 + \sum_{n \geq 1} \sum_{t=1}^{r} a_t \frac{f_t(n)}{n^s}.
\]
Then
\[
F(n) = \sum_{t=1}^{r} a_t f_t(n) \quad (n \geq 2).
\]
Since the vectors \( \{(f_1(p_1n_1), \ldots, f_r(p_rn_1)); i = 1, \ldots, r\} \) are \( \mathbb{C} \)-linearly independent and not all \( a_t \)'s are zero, at least one of the values \( F(p_1n_1), \ldots, F(p_rn_r) \) must be non-zero yielding
\[
|P(D_1, \ldots, D_r)| \geq M_1^{-1}.
\]
Assume that \( \deg P = g + 1 \geq 2 \) and for any polynomial \( Q \) of degree \( d \leq g \), we have
\[
|Q(D_1, \ldots, D_r)| \geq M_1^{-d}.
\]
For each \( t = 1, \ldots, r \), if \( \partial P/\partial X_t = 0 \), then \( |\partial P/\partial X_t(D_1, \ldots, D_r)| = 0 \), while if \( \partial P/\partial X_t \neq 0 \), we have \( |\partial P/\partial X_t(D_1, \ldots, D_r)| \geq M_1^{-g} \). Consequently, not all of the \( M_1^g \) vectors
\[
\{(F_1(1), \ldots, F_r(1)), (F_1(2), \ldots, F_r(2)), \ldots, (F_1(M_1^g), \ldots, F_r(M_1^g))\}
\]
can be zero vector. Let \( (F_1(m), \ldots, F_r(m)) \) be the first non-zero such vector. Then for \( t = 1, \ldots, r \),
\[
1 \leq m \leq M_1^g, \quad F_t(d) = 0, \quad \text{for } 1 \leq d < m.
\]
By Lemma 3.1 and the minimality of \( m \), for each \( i = 1, \ldots, r \), we have
\[
F(p_i n_i m) v_{p_i}(p_i n_i) = v_{p_i}(p_i n_i) \sum_{t=1}^{r} f_t(p_i n_i) F_t(m)
\]
with at least one \( F_i(m) \neq 0 \). Since the vectors \( \{ (f_1(p_i n_i), \ldots, f_r(p_i n_i)); i = 1, \ldots, r \} \) are \( \mathbb{C} \)-linearly independent, at least one of the values \( F(p_i n_i m), \ldots, F(p_i n_i m) \) must be non-zero. Thus,

\[
|P(D_1, \ldots, D_r)| \geq \frac{1}{M_1 m} \geq \frac{1}{M_1^{g+1}}. 
\]

A counterpart of Corollary 3.3 is:

**Corollary 3.5.** Let \( D(s) = \sum_{n \geq 1} f(n)n^{-s} \in \mathcal{D} \) and \( P(X_0, \ldots, X_r) \in \mathbb{C}[X_1, \ldots, X_r] \setminus \{0\} \) be of total degree \( g \). If there are a set of \( r+1 \) primes \( \{p_1 < p_2 < \cdots < p_{r+1}\} \) and a set of \( r+1 \) positive integers \( \{n_1, \ldots, n_{r+1}\} \) such that

\[
f(p_i n_i) \neq 0 \quad \text{and} \quad f(p_i k) = 0 \quad \text{for } 1 \leq k < n_i (i = 1, \ldots, r+1),
\]

then

\[
|P(D, D', \ldots, D^{(r)})| \geq M_2^{-g},
\]

where \( M_2 = \max(p_1 n_1, \ldots, p_{r+1} n_{r+1}) \), and the Dirichlet series, its derivatives and operations are considered formally.

**Proof.** Differentiating formally with respect to \( s \), we have

\[
D^{(j)}(s) = \sum_{n \geq 1} \frac{f(n)(-\log n)^j}{n^s} \quad (j \in \mathbb{N}),
\]

and since

\[
f(p_i n_i)(-\log p_i n_i)^j \neq 0, \quad f(p_i k)(-\log p_i k)^j = 0 \quad (i \in \{1, \ldots, r+1\}),
\]

we see that the determinant

\[
\begin{vmatrix}
  f(p_1 n_1) & f(p_1 n_1)(-\log p_1 n_1) & \cdots & f(p_1 n_1)(-\log p_1 n_1)^r \\
  \vdots & \vdots & \ddots & \vdots \\
  f(p_{r+1} n_{r+1}) & f(p_{r+1} n_{r+1})(-\log p_{r+1} n_{r+1}) & \cdots & f(p_{r+1} n_{r+1})(-\log p_{r+1} n_{r+1})^r \\
  f(p_1 n_1) \cdots f(p_{r+1} n_{r+1}) & 1 & \cdots & (-\log p_1 n_1)^r \\
  \vdots & \vdots & \ddots & \vdots \\
  1 & (-\log p_{r+1} n_{r+1}) & \cdots & (-\log p_{r+1} n_{r+1})^r
\end{vmatrix} \neq 0.
\]

This implies that the vectors

\[
(f(p_1 n_1), \ldots, f(p_{r+1} n_{r+1})), (-f(p_1 n_1)\log p_1 n_1, \ldots, -f(p_{r+1} n_{r+1})\log p_{r+1} n_{r+1}), \ldots,
\]

\[
(-f(p_1 n_1)(\log p_1 n_1)^r, \ldots, -f(p_{r+1} n_{r+1})(\log p_{r+1} n_{r+1})^r)
\]

are \( \mathbb{C} \)-linearly independent. By Theorem 3.4,

\[
|P(D, D', \ldots, D^{(r)})| \geq M_2^{-g}. \quad \square
\]
In connection with the Jacobian, the result of Shapiro–Sparer (Theorem 2.2 above) can be made quantitative as follows:

**Corollary 3.6.** Let \( f_1, \ldots, f_r \in \mathcal{A}, P(X_1, \ldots, X_r) \in \mathbb{C}[X_1, \ldots, X_r] \setminus \{0\} \) and

\[
D_i(s) = \sum_{n \geq 1} \frac{f_i(n)}{n^s} \quad (i = 1, \ldots, r).
\]

Assume that there are a set of \( r \) primes \( \{p_1 < p_2 < \cdots < p_r\} \) and a set of positive integers \( \{n_1, \ldots, n_r\} \) such that

\[
f_t(p_{ni}) \neq 0 \quad \text{but} \quad f_t(p_k) = 0 \quad \text{for} \quad 1 \leq k < n_i \quad (t = 1, \ldots, r; \quad i = 1, \ldots, r).
\]

If the value of the Jacobian

\[
J(f_1, \ldots, f_r; p_1, \ldots, p_r)(n_1 \cdots n_r) := \begin{vmatrix} d_{p_1} f_1 & \cdots & d_{p_1} f_r \\ \vdots & \ddots & \vdots \\ d_{p_r} f_1 & \cdots & d_{p_r} f_r \end{vmatrix} (n_1 \cdots n_r)
\]

(where the product in the expansion of the determinant is taken as the convolution) is non-zero, then

\[
|P(D_1, \ldots, D_r)| \geq M_1^{-g},
\]

where \( M_1 = \max\{p_1 n_1, \ldots, p_r n_r\} \), and the Dirichlet series, their derivatives together with operations are considered formally.

**Proof.** By the minimality of \( n_1, \ldots, n_r \), we get

\[
0 \neq J(f_1, \ldots, f_r; p_1, \ldots, p_r)(n_1 \cdots n_r) = \begin{vmatrix} d_{p_1} f_1 & \cdots & d_{p_1} f_r \\ \vdots & \ddots & \vdots \\ d_{p_r} f_1 & \cdots & d_{p_r} f_r \end{vmatrix} (n_1 \cdots n_r)
\]

\[
= \prod_{c_1 \cdots c_r = n_1 \cdots n_r} \prod_{i=1}^r v_{p_i}(p_i c_1) \prod_{i=2}^r v_{p_i}(p_i^2 c_1) \cdots v_{p_i}(p_i^r c_r) \begin{vmatrix} f_1(p_1 c_1) & \cdots & f_r(p_1 c_1) \\ \vdots & \ddots & \vdots \\ f_1(p_1 c_r) & \cdots & f_r(p_1 c_r) \end{vmatrix}
\]

\[
= \prod_{i=1}^r v_{p_i}(p_i n_i) \prod_{i=2}^r v_{p_i}(p_i^2 n_i) \cdots v_{p_i}(p_i^r n_r) \begin{vmatrix} f_1(p_1 n_1) & \cdots & f_r(p_1 n_1) \\ \vdots & \ddots & \vdots \\ f_1(p_1 n_r) & \cdots & f_r(p_1 n_r) \end{vmatrix},
\]

and so \( \det(f_t(p_i n_i))_{i,t=1}^{r,r} \neq 0 \), implying that the vectors

\[
(f_1(p_1 n_1), \ldots, f_r(p_1 n_1)), \ldots, (f_1(p_r n_r), \ldots, f_r(p_r n_r))
\]

are \( \mathbb{C} \)-linearly independent. The desired result follows at once from Theorem 3.4. \( \Box \)

Regarding linear dependence, using the notion of Wronskian, we have:
Corollary 3.7. Let \( f_1, \ldots, f_r \in \mathcal{A} \), \( D_i(s) = \sum_{n \geq 1} f_i(n)n^{-s} \) (\( i = 1, \ldots, r \)) and \( P(X_1, \ldots, X_r) = c_0 + \sum_{i=1}^r c_i X_i \in \mathbb{C}[X_1, \ldots, X_r] \setminus \{0\} \). Assume that there is a prime \( p \) and a set of positive integers \( \{n_1, \ldots, n_r\} \) such that

\[
 f_i(p^j n_i) \neq 0 \quad \text{but} \quad f_i(p^j k) = 0 \quad \text{for} \quad 1 \leq k < n_i \quad (t = 1, \ldots, r; \ i \in \mathbb{N}).
\]

If the value of the Wronskian

\[
 W(dp f_1, \ldots, dp f_r)(n_1 \cdots n_r) := \begin{vmatrix}
 d_p f_1 & \cdots & d_p f_r \\
 d_p^2 f_1 & \cdots & d_p^2 f_r \\
 \vdots & \ddots & \vdots \\
 d_p^r f_1 & \cdots & d_p^r f_r \\
\end{vmatrix}
\]

(where the product in the expansion of the determinant is taken as the convolution) is non-zero, then

\[
 |P(D_1, \ldots, D_r)| \geq M_3^{-1},
\]

where \( M_3 = \max \{pn_1, p^2 n_2, \ldots, p^n n_r\} \), and the Dirichlet series, their derivatives together with operations are considered formally.

Proof. By the minimality of \( n_1, \ldots, n_r \), we get

\[
 0 \neq W(dp f_1, \ldots, dp f_r)(n_1 \cdots n_r) = \begin{vmatrix}
 d_p f_1 & \cdots & d_p f_r \\
 d_p^2 f_1 & \cdots & d_p^2 f_r \\
 \vdots & \ddots & \vdots \\
 d_p^r f_1 & \cdots & d_p^r f_r \\
\end{vmatrix} (n_1 \cdots n_r)
\]

\[
 = \sum_{c_1 \cdots c_r = n_1 \cdots n_r} \prod_{i=1}^r v_p(p c_i) \prod_{i=2}^r v_p(p^2 c_i) \cdots v_p(p^r c_r) f_1(p c_1) \cdots f_r(p c_1) \\
 f_1(p^2 c_2) \cdots f_r(p^2 c_2) \\
 \vdots \\
 f_1(p^r c_r) \cdots f_r(p^r c_r)
\]

\[
 = \prod_{i=1}^r v_p(p c_i) \prod_{i=2}^r v_p(p^2 c_i) \cdots v_p(p^r c_r) f_1(p c_1) \cdots f_r(p c_1) f_1(p^2 c_2) \cdots f_r(p^2 c_2) \\
 \vdots \\
 f_1(p^r c_r) \cdots f_r(p^r c_r)
\]

\[
 := \prod_{i=1}^r v_p(p c_i) \prod_{i=2}^r v_p(p^2 c_i) \cdots v_p(p^r c_r) \det(f_i(p^r n_i))
\]

showing that \( \det(f_i(p^r n_i)) \neq 0 \). Putting

\[
 \sum_{n \geq 1} \frac{F(n)}{n^s} := P(D_1, \ldots, D_r) = c_0 + \sum_{i=1}^r c_i D_i = c_0 + \sum_{n \geq 1} \sum_{i=1}^r \frac{c_i f_i(n)}{n^s},
\]

we get

\[
 F(n) = \sum_{i=1}^r c_i f_i(n) \quad (n \geq 2).
\]
Since $\det (f_r(p'n_i)) \neq 0$, the vectors $(f_1(p_1n_1), \ldots, f_1(p'n_r)), \ldots, (f_r(p_1n_1), \ldots, f_r(p'n_r))$ are $\mathbb{C}$-linearly independent and since the $c_i$'s do not all vanish simultaneously, at least one of the values $F(p_1n_1), \ldots, F(p'r_n)$ must be non-zero and the result follows. □

4. Other cases

It is to be observed that one of the main hypotheses in Theorems 3.2 and 3.4 is the linear independence of the set of vectors of functional values at different primes. This restricts their applicability to many interesting cases, such as the independence of the formal Riemann zeta series and the formal log zeta series. However, using direct approach, in this particular case, we have the following independence measure:

**Theorem 4.1.** Let $D_1 = \sum_{n \geq 1} \frac{f(n)}{n^s}$, $D_2 = \sum_{n \geq 1} \frac{g(n)}{n^s}$ be formal Dirichlet series. Assume that

\begin{align}
  f(1) &= f(p_1 \cdots p_r) = c_f \in \mathbb{C} \setminus \{0\} \quad (r \geq 1), \\
  g(p) &= c_g \in \mathbb{C} \setminus \{0\}, \quad g(1) = g(p_1 \cdots p_s) = 0 \quad (s \geq 2),
\end{align}

where $p$ and the $p_i$’s are distinct primes. Let $P(X, Y) = \sum_{i,j} a_{ij}X^iY^j \in \mathbb{C}[X, Y] \setminus \{0\}$ with total degree $g$ and formally put $P(D_1, D_2) := \sum_{n \geq 1} F(n)/n^s \in D$. Then there is a positive, absolute and computable constant $c$ such that

$$|P(D_1, D_2)| \geq \left\{ c^{g(g+1)} \prod_{j=2} \frac{g^j}{j \log j} \right\}^{-1},$$

where the Dirichlet series and their operations are considered formally.

**Proof.** Formally setting the product of formal Dirichlet series

$$D_1(s)^iD_2(s)^j := \sum_{n \geq 1} \frac{f_{ij}(n)}{n^s}$$

and noting that this corresponds to the convolution of associated arithmetic functions, we have

$$f_{ij}(n) = \sum_{a_1 \cdots a_i b_1 \cdots b_j = n} f(a_1) \cdots f(a_i) g(b_1) \cdots g(b_j).$$

Taking $k \geq i + j$, $n = p_1 p_2 \cdots p_k$, where $p_1 < p_2 < \cdots < p_k$ are primes and using the assumptions (4.1) and (4.2) we get

$$f_{ij}(p_1 p_2 \cdots p_k) = i^{k-j}c_f^{j} j! \binom{k}{j} c_g^j.$$

Thus,

$$F(p_1 p_2 \cdots p_k) = \sum_{i=0}^g \sum_{j=0}^g a_{ij} i^{k-j} j! \binom{k}{j}.$$
The right-hand side is an exponential polynomial in \( k \) with the maximum degree in the polynomial part and the number of frequencies both being at most \( g \). By a well-known result about the number of zeros of exponential polynomials (see e.g. the lemma in [4, Chapter 12]) the number of zeros of this exponential polynomial is at most \((g + 1)g - 1\) and so

\[
|P(D_1, D_2)| \geq \{p_1 p_2 \cdots p_{(g + 1)g}\}^{-1}.
\]

The result now follows from Chebychev's inequality (see e.g. [1, Theorem 4.7, p. 84]) that \( p_r \leq c_1 r \log r \ (r \geq 2) \) for some computable constant \( c_1 \). □

Applying Theorem 4.1 to the case of zeta and log zeta series, we have:

**Corollary 4.2.** Let \( P(X, Y) = \sum_{i,j} a_{ij} X^i Y^j \in \mathbb{C}[X, Y] \setminus \{0\} \) with total degree \( g \) and put \( P(\zeta, \log \zeta) := \sum_{n \geq 1} F(n)/n^s \). Then there is a positive, absolute and computable constant \( c \) such that

\[
|P(\zeta, \log \zeta)| \geq \left\{cg^{(g+1)} \prod_{j=2} \log j \right\}^{-1},
\]

where the zeta, log zeta series and their operations are considered formally.

**Proof.** The result follows immediately from Theorem 4.1 through the observation that [2,14]

\[
\zeta(s) = \sum_{n \geq 1} \frac{1}{n^s}, \quad \log \zeta(s) = \sum_{n \geq 1} \frac{\log A(n)}{n^s},
\]

where \( A \) is the von Mangoldt function defined by

\[
A(n) = \begin{cases} 
\log p & \text{if } n = p^j \text{ is a prime positive power}, \\
0 & \text{otherwise}.
\end{cases}
\]

**References**