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Uniform Distribution of Integral Points on 3-Dimensional Spheres via Modular Forms

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The problem of the asymptotic distribution of integral points on a sequence of expanding spheres $x^2 + y^2 + z^2 = m_i$ [first penetrated by Linnik with his ergodic method] is shown to be attackable by analytic methods, especially via the theory of modular forms. Success depends on a new estimate for the Fourier coeff. of cusp forms of half integral dimension on the theta group, which we obtain after improving upon previously known estimates of the associated Kloostermann sums.

1. INTRODUCTION

Let Δ be a Jordan measurable region on the 3-dimensional unit sphere. Let $r(m; \Delta)$ be the number of solutions (x, y, z) of

$$x^2 + y^2 + z^2 = m \qquad \text{with} \quad x, y, z \in Z \text{ (the set of rational integers)} \tag{1.1}$$

and $(x, y, z)/m^{1/2}$ contained in Δ. It might be expected that

$$r(m_j\,; \Delta) \sim \frac{m(\Delta)}{4\pi}\, r(m_j) \qquad \text{as} \quad j \to \infty, \tag{1.2}$$

where $r(m) = r_3(m)$ denotes the number of (unconstrained) solutions of (1.1) and $m(\Delta)$ the area of Δ, if the sequence $\{m_j\,; j = 1, 2,...,\}$ is chosen so that $r(m_j) \to \infty$. Indeed, Malyshev [6] and Pommerenke [7] have established the analogue of (1.2) for integral positive quadratic forms in more than three variables. Thus lattice points on n-dimensional ellipsoids (centered at the origin) are asymptotically uniformly distributed for $n \geqslant 5$, and also for $n = 4$, (but then only undei a restriction, corresponding to $r(m_j) \geqslant cm_j$).

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The proofs use traditional methods from the analytic theory of quadratic forms, the theory of modular functions and the circle method.

The situation for positive ternary forms is less clear. Linnik ([4], chap. IV) with his "ergodic method" has claimed the first major result, which states that (1.2) does hold for certain $\{m_j\}$ and convex Δ. Still, it remained an interesting open question whether or not (1.2) say, is attackable with methods from the analytic theory of quadratic forms; see for instance [4, p. 4] or [7, p. 229]. Up to now analytic methods have not yielded a result on this (ternary) problem. It is our primary goal in this paper to show that a proof of the validity of (1.2) is accessible via the theory of modular forms and classical analytic number theory, at least for some sequences $\{m_j\}$.

Our approach is based on the method originated by Hecke and employed by Pommerenke [7], namely a reduction to spherical theta functions. The crucial ingredient for making this approach work, in the present 3-dimensional case, is a new estimate for the Fourier coefficients of cusp forms of half integral dimension on the theta group. In Section 7 (Theorem 3) we give such an estimate, based on an improved estimate (Theorem 2 in 6) of certain Kloostermann sums. Thereafter we show that this estimate is sharp enough to establish (1.2) for certain rare sequences $\{m_j\}$, (Theorem 4 in Section 8).

It might be of interest that we derive the formula for $r_3(m)$, (Theorem 1, Section 2). Such a formula was first obtained by Gauss [3]. Analytic derivations have been given in more recent times by several authors and are contained in [1], [2], [5], [11], for instance. Still, a finished "explicit" formula for $r_3(m)$ is hardly available in the literature. We include our version of a proof also because our result (Theorem 4) on (1.2) depends very strongly on the arithmetic nature of $r_3(m)$ as exhibited in (2.14). Finally, our proof incorporates some simplifications not given in earlier proofs.

2. The Three Squares Formula

The generating function for $r_3(n)$ is

$$\sum_{n=0}^{\infty} r_3(n)\, e^{\pi i n \tau} = \Theta^3(\tau), \qquad \text{where} \quad \Theta(\tau) = \sum_{n=-\infty}^{+\infty} e^{\pi i \tau n^2}$$

and $\operatorname{Im} \tau > 0$. Well known properties of the theta function imply that Θ^3 belongs to $\mathcal{M}_{3/2}$, the space of entire modular forms of dimension $-3/2$ with multipliers $V = V_3$, on the theta group Γ_Θ: The theta group consists of all transformations $\tau \to M\tau = (a\tau + b)/(c\tau + d)$ with

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \text{ or } \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \bmod 2,$$

$$a, b, c, d \in Z \quad \text{and} \quad \det M = 1,$$

(and has a fundamental domain described by $|\operatorname{Re}\tau| \leqslant 1 \leqslant |\tau|$); while

$$V_3(M) = \gamma^{-3}(-d, c) \quad \text{for} \quad c > 0$$

with

$$\gamma(a, c) = \frac{i^{1/2}}{2(c)^{1/2}} \sum_{r=1}^{2c} e^{\pi i r^2 a/c} \quad \text{for} \quad (a, c) = 1. \tag{2.1}$$

The Gauss sum $\gamma(a, c)$, $(c > 0, (a, c) = 1)$ can be evaluated in terms of the Jacobi symbol

$$\gamma(a, c) = \begin{cases} \left(\frac{a}{c}\right) e^{\pi i(2-c)/4}, & \text{if} \quad a \not\equiv c \equiv 1 \bmod 2, \\ 0, & \text{if} \quad ac \equiv 1 \bmod 2, \\ \left(\frac{c}{a}\right) e^{\pi i(a+1)/4}, & \text{if} \quad a \not\equiv c \equiv 0 \bmod 2. \end{cases} \tag{2.2}$$

It obeys the reciprocity law

$$L(\rho) = (i\rho)^{1/2} \cdot L(-1/\rho) \qquad \left(\operatorname{arc}(i\rho)^{1/2} = \frac{\pi}{4} \operatorname{sgn} \rho\right) \tag{2.3}$$

with $L(\rho) = c^{-1/2}\gamma(a, c) = L(\rho + 2)$, if $\rho = a/c$.

Following Hecke we consider the Dirichlet–Eisenstein series with convergence generating factor

$$\Psi(\tau; s) = y^{s/2}\left(1 + \sum_{\rho \in Q} \gamma^3(a, c)(c\tau - a)^{-3/2} \, | c\tau - a |^{-s}\right), \tag{2.4}$$

where $\rho = a/c$ runs over all rational numbers $(c > 0, (a, c) = 1)$, $y = \operatorname{Im} \tau > 0$ and $\sigma = \operatorname{Re} s > 1/2$ for absolute convergence. Direct calculations using (2.3) yield (for $\sigma > 1/2$)

$$\Psi(-1/\tau; s) = (-i\tau)^{3/2}\Psi(\tau; s), \; \Psi(\tau + 2; s) = \Psi(\tau; s); \tag{2.5}$$

hence (as for Θ^3)

$$\Psi(M\tau; s) = V_3(M)(c\tau + d)^{3/2}\Psi(\tau; s), \qquad \text{if} \quad M \in \Gamma_\Theta; \tag{2.5$'$}$$

i.e., Ψ transforms like a function in $\mathscr{M}_{3/2}$.

We now continue Ψ analytically as function of s (for any fixed τ with $y > 0$) to the half plane $\sigma > -1/4$. Applying the Poisson summation formula yields

$$\begin{aligned} y^{-s/2}\Psi(\tau; s) - 1 &= \sum_{\rho \bmod 2} L^3(\rho)\, c^{-s} \sum_{n=-\infty}^{+\infty} (\tau - \rho + 2n)^{-3/2} \, | \tau - \rho + 2n |^{-s} \\ &= \sum_n q(n; s)\, I(n; \tau, s), \qquad (\sigma > 1/2), \end{aligned} \tag{2.6}$$

where

$$I(n;\tau,s)=\tfrac{1}{2}\int_{-\infty}^{+\infty} e^{-\pi int}(\tau+t)^{-(3+s)/2}\,(\bar\tau+t)^{-s/2}\,dt$$

and

$$q(n;s)=\sum_{\rho \bmod 2} L^3(\rho)\,c^{-s}e^{-\pi i\rho n},\qquad (\rho=a/c,\ c>0,\ (a,c)=1)$$
$$=\sum_{\rho \bmod 2}(-1)^c\,\bar L(\rho)\,c^{-s-1}e^{-\pi i\rho n},\qquad (\sigma>1/2)$$

since $\gamma^4(a,c)=(-1)^c$ for $ac\not\equiv 1 \bmod 2$, by (2.2). Clearly the integral is holomorphic in s for $\sigma>-1/2$. Suitably shifting the line of integration implies that, for $y=\operatorname{Im}\tau>0$,

$$I(n;\tau,s)=0(e^{-\pi|n|y/2})\quad\text{as}\quad n\to\pm\infty,\quad\text{uniformly for}\quad \sigma\geqslant -1/4. \tag{2.7}$$

Also

$$I(n;\tau,0)=\begin{cases}0, & \text{if}\quad n\leqslant 0,\\ i^{-3/2}\,2\pi(n)^{1/2}\,e^{\pi in\tau}, & \text{if}\quad n>0.\end{cases} \tag{2.8}$$

Furthermore, by definition of $L(\rho)$ in (2.3) and exchange of summation with the Gauss sum defined in (2.1) we obtain

$$q(n;s)\sum_{\substack{k>0\\ \text{odd}}} k^{-s-1}=\sum_{\substack{c>0\\ \text{even}}} c^{-s-1}\sum_{\substack{a=1\\ \text{odd}}}^{2c}\bar L(\rho)\,e^{-\pi in\rho}-\sum_{\substack{c>0\\ \text{odd}}} c^{-s-1}\sum_{a=1}^{2c}\bar L(\rho)\,e^{-\pi in\rho}$$
$$=\frac{1}{i^{1/2}}\left\{\sum_{0<c\equiv O(2)} c^{-s-1}N(-n,2c)-\sum_{c>0} c^{-s-1}N(-n,c)\right\} \tag{2.9}$$

with $N(n,c)=\sum_{h=1,h^2\equiv n \bmod c}^{c} 1$, since $N(n,2c)=N(n,c)$ for odd c. The last series have Euler products

$$\sum_{c>0} N(n,c)\,c^{-s}=\prod_p\sum_{k=0}^{\infty} N(n,p^k)\,p^{-ks},\qquad (\sigma>2)$$

and

$$\sum_{0<c\equiv O(2)} N(n,2c)\,c^{-s}=2^s\sum_{k=2}^{\infty} N(n,2^k)\,2^{-ks}\cdot\prod_{p>2}\sum_{k=0}^{\infty} N(n,p^k)\,p^{-ks}.$$

For every integer $n\neq 0$ let $d(n)$ be the discriminant of the number field $Q(n)^{1/2}$, ($d(n)=1$, iff. $n=k^2>0$ with integer k) and for every prime p let

$m = m(p, n)$ be the largest integer such that p^{2m} divides $4n/d(n)$. Then, from the known enumeration of solutions of quadratic congruences mod p^k,

$$\sum_{k=0}^{\infty} N(n, p^k)\, p^{-ks}$$

$$= \sum_{k=0}^{2m} p^{[k/2]-ks} + p^{m(1-2s)-s}\left[1 + \left(\frac{d}{p}\right)\right]\left[1 + \left(\frac{d}{p}\right)^2 \sum_{k=1}^{\infty} p^{-ks}\right]$$

$$= (1 + p^{-s})\left(1 - \left(\frac{d}{p}\right) p^{-s}\right)^{-1} h(n; p, s), \qquad (p > 2,\ n \neq 0)$$

where $m = m(p, n)$, $d = d(n)$ and

$$h(n; p, s) = p^{m(1-2s)} + \left(1 - \left(\frac{d}{p}\right) p^{-s}\right) \sum_{k=0}^{m-1} p^{k(1-2s)}.$$

For $p = 2$ and $n \neq 0$ set $m(n) = m(2, n) - (d(n)/2)^2$. Then

$$\sum_{k=1}^{\infty} N(n, 2^k)\, 2^{-ks} = \sum_{k=1}^{2m(n)} 2^{[k/2]-ks} + 2^{m(n)(1-2s)-s}$$

$$\times \left\{1 + 2\left(\frac{d}{2}\right)^2 \left(2^{-s} + \left[1 + \left(\frac{d}{2}\right)\right] \sum_{k=2}^{\infty} 2^{-ks}\right)\right\}$$

$$= 2^{-s}\left(1 - \left(\frac{d}{2}\right) 2^{-s}\right)^{-1} h(n; s)$$

with $d = d(n)$ and

$$h(n; s) = 1 + \left(\frac{d}{2}\right) + \left[2^{1-s} - \left(\frac{d}{2}\right)\right] \sum_{k=0}^{2m-1} 2^{[k/2]-ks}, \qquad m = m(2, n).$$

(Here $(d/2)$ is Kronecker's extension of the Jacobi symbol.)

Finally, since $N(0, p^k) = p^{[k/2]}$,

$$\sum_{k=0}^{\infty} N(0, p^k)\, p^{-ks} = \frac{1 + p^{-s}}{1 - p^{1-2s}}, \qquad (p > 2)$$

$$\sum_{k=1}^{\infty} N(0, 2^k)\, 2^{-ks} = \frac{2^{-s} + 2^{1-2s}}{1 - 2^{1-2s}}.$$

From these relations and (2.9) we obtain, setting $w = s + 1$,

$$q(0; s)\, i^{1/2} = \frac{(1 - 2^{-w})^{-1}}{\zeta(w)} \left\{(1 - 2^{-w}) \frac{1 + 2^{1-w}}{1 - 2^{1-2w}} - 2\right\} \prod_{p>2} \frac{1 + p^{-w}}{1 - p^{1-2w}}$$

$$= \frac{2^{1-2w} + 2^{-w} - 1}{1 - 2^{-2w}} \frac{\zeta(2w - 1)}{\zeta(2w)},$$

and

$$q(n; s)\, i^{1/2} = \frac{(1-2^{-w})^{-1}}{\zeta(w)} \left\{(1-2^{-w}) \frac{h(-n; w)}{1-(d/2)\, 2^{-w}} - 2\right\}$$
$$\times \prod_{p>2} \frac{(1+p^{-w})\, h(-n; p, w)}{1-(d/p)\, p^{-w}}$$
$$= \frac{L(w; d)}{\zeta(2w)} \left\{\frac{(1-2^{-w})\, h(-n; w) - 2 + (d/2)\, 2^{1-w}}{1-2^{-2w}}\right\}$$
$$\times \prod_{2<p|(4n/d)} h(-n; p, w) \tag{2.10}$$

for $n \neq 0$ and with $d = d(-n)$ (!), where $\zeta(w)$ is the Riemann zeta function and

$$L(w; d) = \sum_{n>0} \left(\frac{d}{n}\right) n^{-w}, \qquad (\mathrm{Re}\, w > 1). \tag{2.11}$$

It follows that $q(n; s)$ is kolomorphic for $\sigma > -1/2$ (and all n), since $h(-n; 1) = 2$, when $d(-n) = 1$. Also, $q(n; s) = 0(|n|)$ as $n \to \pm\infty$, uniformly for Re $s \geqslant -1/4$. This and (2.7) imply that the last series in (2.6) converges uniformly for $\sigma \geqslant -1/4$ and Im $\tau \geqslant \delta > 0$. Thus $\Psi(\tau; s)$ is a holomorphic function of s in $\sigma > -1/4$ (at least), and by (2.6) and (2.8)

$$\Psi(\tau; 0) = 1 - 2\pi(i)^{1/2} \sum_{n>0} q(n; 0)\, n^{1/2} e^{\pi i \tau n}, \tag{2.12}$$

which is a holomorphic function of τ in Im $\tau > 0$.

The functional equations (2.5) hold also for $f(\tau) = \Psi(\tau; 0)$, by analytic continuation. They imply

$$f\left(\frac{-1}{\tau+2}\right) = f(\tau) \left(\frac{\tau+1}{i}\right)^{3/2} \left(\frac{\tau'-1}{i}\right)^{3/2} i^{-3/2}, \qquad \text{if} \quad \tau' = \frac{-1}{\tau+1},$$

and then for the function

$$F(\tau') = f(\tau)(-i\tau')^{-3/2} e^{-3\pi i \tau'/4} = F(\tau'-1).$$

Hence $F(\tau')$ has a convergent Fourier expansion in Im $\tau' > 0$ and thus

$$\Psi(\tau; 0) = f(\tau) = (-i\tau')^{3/2} \sum_{n=-\infty}^{+\infty} a_n e^{2\pi i (n+3/8)\tau'}, \qquad \left(\tau' = \frac{-1}{\tau+1}\right). \tag{2.13}$$

Since $q(n; 0) = O(n)$ (at most) as $n \to \infty$, (2.12) implies

$$|f(\tau)| = O(y^{-5/2}) \quad \text{as} \quad y = \mathrm{Im}\, \tau \to +0 \quad (\text{uniformly in Re}\, \tau),$$

and thus in (2.13) $a_n = 0$ for $n < 0$. Thus $f(\tau)$ has a zero (of order 3/8 at least) at $\tau = -1$. Hence $f(\tau)$ is in $\mathscr{M}_{3/2}$.

Since this space has dimension one (or, since $f(\tau)\,\Theta^{-3}(\tau)$ is invariant under Γ_Θ and bounded in its fundamental domain and hence constant) it follows with (2.12)

THEOREM 1. (Maass). $\Psi(\tau;\ 0) = \Theta^3(\tau)$. *Comparing Fourier coefficients it follows, using* (2.10) *for* $w = 1$ *and Dirichlet's class number formula, since* $\Pi p^{m(p,-n)} = |(4n/d)|^{1/2}$, *that*

$$r_3(n) = \frac{24h(d)}{w(d)}\left[1 - \left(\frac{d}{2}\right)\right] \prod_{2<p|N} \left\{p^{n(p)} + \frac{p^{n(p)} - 1}{p - 1}\left[1 - \left(\frac{d}{p}\right)\right]\right\}, \qquad (2.14)$$

where $d = d(-n)$ *is the discriminant of the imaginary quadratic field* $Q((-n)^{1/2})$, $(n > 0)$; $h(d)$ *its (ideal) class number,* $w(d)$ *the number of its units*; (d/q) *the quadratic character* mod $|d|$, $n = n_0N^2$ *with squarefree* n_0 *and* $N = \Pi p^{n(p)}$ *in unique prime factorization. (Clearly* $m(p, -n) = n(p)$ *for odd* p; *also* $d = d(-n_0)$, *and* $r_3(n) \neq 0$, *iff* $n_0 \not\equiv -1$ mod 8.) *For completeness we list that*

$$2h(d)/w(d) = \left[2 - \left(\frac{d}{2}\right)\right]^{-1} \sum_{m=1}^{|d/2|} \left(\frac{d}{m}\right), \qquad (d < 0). \qquad (2.14)'$$

3. A SUFFICIENT CONDITION FOR UNIFORM DISTRIBUTION

In what follows all vectors will be written as columns, the $'$ will denote transposition and $|x| = (x'x)^{1/2}$, if $x \in R^3$. Let Δ denote a Jordan measurable set on the unit sphere S of E^3, $m(\Delta)$ its area and $I(x;\ \Delta)$ its characteristic function $(x \in S)$.

Let $\epsilon > 0$. There exists a continuous function $I^*(x;\ \Delta)$ on S (depending on ϵ) such that $0 \leqslant I^*(x;\ \Delta) \leqslant 1$, $I^*(x;\ \Delta) = 1$ for $x \in \Delta$, and $I^*(x;\ \Delta) = 0$ for $x \in S - \Delta^*$, where Δ^* is a Jordan measurable open set on S with $\Delta \subset \Delta^*$ and $m(\Delta^*) - m(\Delta) < \epsilon\pi$. Let $p_k^l(x)$ (degree k, $|l| \leqslant k$) denote in E^3 harmonic homogeneous polynomials forming a complete orthonormal system on S. They span the space of spherical harmonics, which is dense in the space of continuous functions on S. Consequently there exist real c_{kl} and an integer K (all depending on ϵ and Δ) such that

$$0 \leqslant \sum_{\substack{k=0 \\ |l| \leqslant k}}^{K} c_{kl}\, p_k^l(x) - I^*(x;\ \Delta) < \frac{\epsilon}{4}, \qquad (x \in S). \qquad (3.1)$$

Then, by orthonormality and choice of Δ^*

$$c_{00} - \frac{m(\Delta)}{4\pi} < \frac{\epsilon}{2}.$$

Now, by definition of I^* and (3.1)

$$r(m; \Delta) = \sum_{n'n=m} I(m^{-1/2}n; \Delta) \leqslant \sum_{k=0}^{K} m^{-k/2} \sum_{|l|\leqslant k} c_{kl} \sum_{n'n=m} p_k{}^l(n)$$

$$< \left(\frac{m(\Delta)}{4\pi} + \frac{\epsilon}{2}\right) r(m) + \sum_{k=1}^{K}, \tag{3.2}$$

summing over $n \in Z^3$. Next we use the following

LEMMA 1. *Let $p_k(x)$ $(x \in R^3)$ be a harmonic homogeneous polynomial of degree k (in the coordinates of x). There exist finitely many isotropic vectors $\xi_h \in C^3$; (i.e., with $\xi_h'\xi_h = 0$), such that*

$$p_k(x) = \sum_h (\xi_h'x)^k.$$

A proof is in [7], p. 248. Assuming now that there exists an increasing sequence $\{m_j\}$ of natural numbers satisfying the conditions

$$\sum_{n'n=m_j} (\xi'n)^k \overset{!}{=} o(m_j^{k/2} r(m_j)) \quad \text{as} \quad j \to \infty \qquad (k \geqslant 2) \tag{3.3}$$

for all isotropic $\xi \in C^3$ and all even natural numbers k, then by (3.2) and lemma 1, for $j > N(\epsilon, \Delta)$ say,

$$r(m_j\,; \Delta) < \left(\frac{m(\Delta)}{4\pi} + \epsilon\right) r(m_j). \tag{3.4}$$

Clearly (3.4) holds also with Δ replaced by its complement on S. Thus the lower estimate corresponding to (3.4) holds also. Therefore the condition (3.3) implies (1.2); i.e. asymptotic uniform distribution of lattice points on the corresponding sequence of spheres in R^3.

4. Some Results from the Theory of Modular Forms

The sum appearing on the left side of (3.3) is the m_jth Fourier coefficient of the spherical theta function

$$\Theta(\tau; \xi, k) = \sum_{n \in Z^3} (\xi'n)^k\, e^{\pi i \tau n'n} \qquad (\operatorname{Im} \tau > 0) \tag{4.1}$$

with isotropic $\xi \in C^3$. These functions (e.g., [7], section 2) are in $C_{k+3/2}$, the space of entire cusp forms on Γ_Θ with dimension $-(k + 3/2)$, $(k > 0$ even) and multipliers V_3 (the same as for Θ^3) defined in (2.1).

We only require the following known facts about the (Hilbert) space $C_{k+3/2}$:

(1) The dimension of $C_{k+3/2}$ is finite, in particular, $\dim C_{k+3/2} = [k/4]$ for $k > 0$ and even. (Thus especially $\Theta(\tau; \xi, 2)$ is identically zero.)

(2) For every natural n and k one defines the Poincaré series for $C_{k+3/2}$ by (cf. (2.4))

$$P_n(\tau; k) = e^{\pi i n \tau} + \sum_{c>0} \sum_{(d,c)=1} e^{\pi i n M\tau} \gamma^3(-d, c)(c\tau + d)^{-k-3/2}, \quad M = \begin{pmatrix} * & * \\ c & d \end{pmatrix} \tag{4.2}$$

where for the summation (in each term) the first row of M is to be chosen so that $M \in \Gamma_\Theta$; ($\gamma(-d, c) = 0$, if $d \equiv c \bmod 2$ here). These functions are all in $C_{k+3/2}$, and the special system

$$\{P_n(\tau; k) \mid 1 \leqslant n \leqslant [k/4]\}$$

forms a (linear) basis of $C_{k+3/2}$, if $k \geqslant 4$ is even.

(3) The Fourier expansion in $\operatorname{Im} \tau > 0$ of the Poincaré series can be calculated as

$$P_n(\tau; k) = \sum_{m=1}^{\infty} A_{mn}^k e^{\pi i m \tau} \tag{4.3}$$

with

$$A_{mn}^k = \delta_{mn} + i^{-k-3/2}(m/n)^{(2k+1)/4}\, \pi \sum_{c=1}^{\infty} J_{k+1/2}\left(\frac{2\pi}{c}(mn)^{1/2}\right) c^{-1} K(m, n; c),$$

where $J_\nu(t)$ denotes the standard Bessel function of the 1. kind, δ_{mn} the Kronecker delta symbol, and

$$K(m, n; c) = \sum_{\substack{d=1 \\ (d,c)=1}}^{2c} \gamma^3(-d, c)\, e^{\pi i(na+md)/c}, \qquad \text{with } \begin{pmatrix} a & * \\ c & d \end{pmatrix} \in \Gamma_\Theta \tag{4.4}$$

in each term, is a generalized Kloostermann sum.

It is clear from these facts and (4.1) and (4.3) that the conditions (3.3) for a sequence $\{m_j\}$ will be satisfied, if that sequence satisfies

$$A_{m_j n}^k \overset{!}{=} o(m_j^{k/2} r_3(m_j)) \quad \text{as } j \to \infty, \qquad \text{for } 1 \leqslant n \leqslant [k/4], \; k \text{ even}. \tag{4.5}$$

The remainder of this paper will be devoted to establish (4.5) for certain $\{m_j\}$.

5. Estimation of the Generalized Kloostermann Sum (I)

We now derive an estimate for the Kloostermann sum $K(m, n; c)$ defined in (4.4), which will be used to estimate the A_{mn}^{k}. Such sums, but without the Gauss sums (or their powers) as coefficients, have been calculated by Salié [9], among others. The following calculations were inspired by this work, but contain several simplifications.

Case I. Let $c > 0$ be odd.

By (2.2) and substitution of $d = 2d'$, $d' \equiv fh \bmod c$, where f is a solution of $4f \equiv 1 \bmod c$, and $a = 2\bar{h}$ into (4.4) we get

$$K(m, n; c) = e\left(\frac{3}{8}(2 - c)\right) \sum_{\substack{h=1 \\ \bar{h}h \equiv 1 \bmod c}}^{c} \left(\frac{-2fh}{c}\right) e\left(\frac{mfh + n\bar{h}}{c}\right), \quad (5.1)$$

where $e(z)$ is an abbreviation for $e^{2\pi i z}$, to be used from here on. Let

$$c = c_1 \cdot \cdots \cdot c_k, \quad c_r = p_r^{a_r} \quad \text{with natural } a_r$$

and distinct odd primes p_r, $(r = 1, \ldots, k)$. Selecting integers b_r so that

$$b_1 \frac{c}{c_1} + \cdots + b_k \frac{c}{c_k} = 1,$$

and applying the direct decomposition of the ring of residue classes mod c, reduces (5.1) to

$$|K(m, n; c)| = \prod_{r=1}^{k} \left| \sum_{\substack{h_r=1 \\ \bar{h}_r h_r \equiv 1 \bmod c_r}}^{c_r} \left(\frac{h_r}{c_r}\right) e\left(\frac{mfb_r h_r + nb_r \bar{h}_r}{c_r}\right) \right|. \quad (5.2)$$

It is convenient to define the sums

$$S_n(u, v; p^a) = \sum_{h \bmod p^a}^{*} \left(\frac{h}{p}\right)^n e\left(\frac{uh + v\bar{h}}{p^a}\right), \qquad (a > 0, p > 2 \text{ prime}) \quad (5.3)$$

where the asterisk indicates that the sum is over residue classes prime to p^a and that $\bar{h}$ is to be determined from $h\bar{h} \equiv 1 \bmod p^a$. Then we have (always for odd p)

Lemma 2. (i) *If* $u + v \not\equiv uv \equiv 0 \bmod p$, *then*

$$S_n(u, v; p^a) = \begin{cases} 0, & \text{if } a \geqslant 2, \\ \left(\frac{u+v}{p}\right)\left(\left(\frac{-1}{p}\right)p\right)^{1/2}, & \text{if } a = 1 \text{ and } n = 1, \\ -1, & \text{if } a = 1 \text{ and } n = 2. \end{cases} \quad (5.4)$$

(ii) *If* $u \equiv 0 \equiv v \bmod p$, *then*

$$S_n(u, v; p^a) = \begin{cases} pS_n\left(\frac{u}{p}, \frac{v}{p}; p^{a-1}\right), & \text{if } a \geqslant 2, \\ 0, & \text{if } a = 1 \text{ and } n = 1, \\ p - 1, & \text{if } a = 1 \text{ and } n = 2. \end{cases} \tag{5.5}$$

Proof. Let $a \geqslant 2$. All $h \bmod p^a$ with $(h, p) = 1$ are given by $h = l + rp^{a-1}$ with $l = 1,..., p^{a-1} - 1$, $(l, p) = 1$ and $r = 0,..., p - 1$. Then $\bar{h} \equiv \bar{l} - r\bar{l}^2p^{a-1} \bmod p^a$ with $l\bar{l} \equiv 1 \bmod p^a$. Using this in (5.3) gives

$$S_n(u, v; p^a) = \sum_{0<l<p^{a-1}} \left(\frac{l}{p}\right)^n e\left(\frac{ul + v\bar{l}}{p^a}\right) \sum_{r=0}^{p-1} e\left(\frac{r}{p}(u - \bar{l}^2v)\right), \quad (a \geqslant 2). \tag{5.6}$$

This yields the result in both cases (i) and (ii). Now let $a = 1$. The case (i) with $n = 1$ is a well known identity for Gauss sums. The remaining three cases are trivial.

If $u \equiv 0 \equiv v \bmod p^k$ with $1 \leqslant k \leqslant a - 1$, then (5.5) implies

$$S_n(u, v; p^a) = p^kS_n(up^{-k}, vp^{-k}; p^{a-k}). \tag{5.7}$$

Lemma 2 reduces calculation of the sums defined in (5.3) to the case that $(uv, p) = 1$.

LEMMA 3. *Let* $(uv/p) = -1$. *Then* $S_n(u, v; p^a) = 0$ *for* $a \geqslant 2$; *and for* $n = a = 1$.

Proof. If $a \geqslant 2$, eq. (5.6) holds again. The inner sum is different from zero only, when $p \mid u - \bar{l}^2v$; i.e., $uv \equiv (lu)^2 \bmod p$, since now $p \nmid uv$. This is impossible, since uv is not a quadratic residue mod p. If $a = 1$ and $n = 1$, the summation transformation $h \to \bar{u}v\bar{h} \bmod p$ applied in (5.3) reproduces the sum with the factor (uv/p); hence the result.

LEMMA 4. *Let* $(uv/p) = 1$ *and* w *be a solution of* $w^2 \equiv uv \bmod p^a$. *Then*

$$S_n(u, v; p^a) = \left(\frac{u}{p}\right)^n \left(\frac{w}{p}\right)^{a+n} \left(e(2wp^{-a}) + \left(\frac{-1}{p}\right)^{a+n} e(-2wp^{-a})\right) (p^{*a})^{1/2} \tag{5.8}$$

for $a \geqslant 2$; *and for* $n = a = 1$; *where* $p^* = (-1/p)p$.

Proof. Applying the transformation $h \to \bar{u}wh \bmod p^a$ to the sum in (5.3) yields

$$S_n(u, v; p^a) = \left(\frac{uw}{p}\right)^n S_n(w, w; p^a).$$

Now let $a \geqslant 2$, and set $b = a - c$ with $c = [a/2]$. Then all h mod p^a with $(h, p) = 1$ are given by $h = l + rp^b$ with $l = 1, \ldots, p^b - 1$, $(l, p) = 1$ and $r = 0, \ldots, p^c - 1$. Then $\bar{h} \equiv \bar{l} - r\bar{l}^2 p^b$ mod p^a with $l\bar{l} \equiv 1$ mod p^a, thus with (5.3)

$$S_n(w, w; p^a) = p^c \sum_{\substack{l=1 \\ l^2 \equiv 1 \bmod p^c}}^{p^b} \left(\frac{l}{p}\right)^n e\left(\frac{w(l+\bar{l})}{p^a}\right). \tag{5.9}$$

If a is even, then $b = c = a/2$ and the summation conditions allow only the values $l = 1$ and $l = p^c - 1$, with $\bar{l} = 1$ and $\bar{l} = p^a - p^c - 1$, respectively. Thus the result in (5.8) follows (with the positive square root).
If a is odd, then $c = (a-1)/2 = b - 1$ and the summation conditions in (5.9) have solutions $l = 1 + hp^c$ with $h = 0, \ldots, p - 1$, and $l = -1 + kp^c$ with $k = 1, \ldots, p$. Then $\bar{l} \equiv 1 - hp^c + h^2 p^{a-1}$, respectively $\bar{l} \equiv -1 - kp^c - k^2 p^{a-1}$ mod p^a. Thus (5.9) implies, using (2.1)

$$S_n(w, w; p^a) = p^{(a-1)/2}(-ip)^{1/2} \times \left\{e(2wp^{-a})\,\gamma(2w, p) + \left(\frac{-1}{p}\right)^n e(-2wp^{-a})\,\gamma(-2w, p)\right\}.$$

Now the result in (5.8) for odd $a > 2$ (with the square root positive or positive imaginary) follows by using (2.2). (The preceding proof is modeled on the one in [13].) The result (5.8) in the remaining case $a = n = 1$ of lemma 4 can be found in the paper of Salié [9], (54). Another, most elegant derivation is in [14].

LEMMA 5. *For the sums* $S = S_n(u, v; p^a)$ *defined in* (5.3) *we have*

(i) $|S| \leqslant [1 + (uv/p)]\, p^{a/2}$ *for* $a > 1$ *and for* $n = a = 1$, *if* $p \nmid uv$;

(ii) $|S| \leqslant \begin{cases} p^{1/2}, & \text{if } a = 1 \\ 0, & \text{if } a > 1 \end{cases}$ *and* $p \mid uv$, *but* $p \nmid (u + v)$;

(iii) $|S| \leqslant (1 + e_p)(v, p^a)\, p^{a/2}$, *if* $p \mid u$ *and* $p \mid v$; *where* $e_p = 0$ *for* $p > 3$ *and* $e_p = 1$ *for* $p = 3$.

Proof. (i) Follows from lemmas 3 and 4. (ii) follows from (5.4) in lemma 2. For the proof of (iii) set $(u, p^a) = p^b$, $(v, p^a) = p^c$. We consider three cases:

(1) Let $b \neq c$. Setting $k = \min\{b, c\}$ we have $0 < k < a$ and by (5.7) and (ii) above $|S| \leqslant p^{k+1/2} \leqslant (v, p^a)\, p^{1/2}$.

(2) Let $b = c < a$. Setting $k = c$ we have by (5.7) and (i) above

$$|S| \leqslant 2p^{k+(a-k)/2} < (1 + e_p)\, p^{k+a/2} \leqslant (1 + e_p)(v, p^a)\, p^{a/2},$$

if $a - k > 1$ or n is odd. For $a - k = 1$ and n even we have by (5.7)

$$|S| \leqslant p^k(p-1) < p^{(k+1+a)/2} \leqslant (v, p^a)\, p^{a/2}, \qquad \text{since} \quad k \geqslant 1.$$

(3) Let $b = c = a$. Setting $k = a - 1$ we have by (5.7)

$$|S| \leqslant p^k(p-1) < (v, p^a).$$

Returning to (5.2) we observe that in the product there occur sums as defined in (5.3) with $n = a (= a_r)$ only. Thus by Lemma 5

$$|K(m, n; c)| \leqslant 2(n, c)\, c^{1/2} \prod_{p \mid c} \left[1 + \left(\frac{mn}{p}\right)\right] \qquad (c > 0 \text{ odd}) \quad (5.10)$$

since $(b_r, p_r) = 1$ and f is a quadratic residue mod p_r for every $p_r \mid c$. We remark also that (5.1), (5.3) and Lemmas 3 and 4 imply especially

$$K(m, n; c) = i^{3/2} c^{1/2} \left(\frac{-n}{c}\right) \prod_{r=1}^{k} \left\{\left[1 + \left(\frac{mn}{p_r}\right)\right] \cos(4\pi h_r b_r / c_r)\right\}$$

$$= i^{3/2} c^{1/2} \left(\frac{-n}{c}\right) \sum_{\substack{h=1 \\ 4h^2 \equiv mn \bmod c}}^{c} e(2h/c), \qquad \text{if} \quad (c, mn) = 1, \quad (5.11)$$

where in the product each h_r is a solution of $4h_r^2 \equiv mn \bmod c_r$ (if solvable). This completes our study in Case I.

6. Estimation of the Generalized Kloostermann Sum (II)

We now study the sums defined in (4.4) in

Case II. $c > 0$ is even.
Then, using (2.2) and again $e(z) = e^{2\pi i z}$,

$$K(m, n; c) = \sum_{\substack{d=1 \\ ad \equiv 1 \bmod 2c}}^{2c} e\left(\frac{3}{8}(1-d)\right)\left(\frac{c}{d}\right) e\left(\frac{na + md}{2c}\right) \qquad (6.1)$$

Let

$$2c = 2^{a_0} c' = \prod_{r=0}^{k} c_r, \qquad c_r = p_r^{a_r} \qquad (p_0 = 2,\ c' \text{ odd})$$

with natural a_r and distinct primes p_r, ($a_0 \geqslant 2$); and

$$b_0 \frac{2c}{c_0} + \cdots + b_k \frac{2c}{c_k} = 1 \qquad (b_r \in Z).$$

Introducing (for fixed a_0 and odd c') the functions

$$g(d) = e\left(\frac{1}{8} i^{d-1}\right)\left(\frac{2}{d}\right)^{a_0} (-1)^{(d-1)(c'-1)/4} \qquad (d \text{ odd}) \tag{6.2}$$

we obtain from (6.1) using the quadratic reciprocity law, the direct decomposition of the ring of residue classes mod 2c (observing that $g(d) = g(h)$, if $d \equiv h \bmod c_0$) and (5.3)

$$\begin{aligned} |K(m, n; c)| &= \Big| \sum_{d \bmod 2c}^{*} \left(\frac{d}{c'}\right) g(d)\, e\left(\frac{md + n\bar{d}}{2c}\right)\Big| \\ &= |S_0| \prod_{r=1}^{k} |S_{a_r}(mb_r, nb_r; c_r)|, \end{aligned} \tag{6.3}$$

where $S_0 = S(mb_0, nb_0; a_0)$ with odd b_0 and

$$S(u, v; a) = \sum_{h \bmod 2^a}^{*} g(h)\, e\left(\frac{uh + v\bar{h}}{2^a}\right) \qquad (\bar{h}h \equiv 1 \bmod 2^a,\ a > 0). \tag{6.4}$$

Clearly, the factors under the product in (6.3) can be estimated by Lemma 5 again, thus similar to (5.10)

$$|K(m, n; c)| \leqslant 2\,|S_0|\,(n, c')(c')^{1/2} \prod_{p \mid c'} \left[1 + \left(\frac{mn}{p}\right)\right]. \tag{6.5}$$

It remains to estimate the sum in (6.4) with $g(h)$ defined in (6.2) having period 8 in h and $|g(h)| = 1$ or 0.

LEMMA 6. *Let $a \geqslant 4$. Then*

$$S(u, v; a) = \begin{cases} 0, & \textit{if} \quad u \equiv v + 1 \bmod 2, \\ 2S\left(\frac{u}{2}, \frac{v}{2}; a - 1\right), & \textit{if} \quad u \equiv 0 \equiv v \bmod 2. \end{cases} \tag{6.6}$$

The proof is the same as that of Lemma 2. We infer

$$S(u, v; a) = 2^k S(2^{-k}u, 2^{-k}v; a - k), \quad \text{if } u \equiv 0 \equiv v \bmod 2^k,\ 1 \leqslant k \leqslant a - 3. \tag{6.7}$$

LEMMA 7. *Let $a \geqslant 6$, $uv \equiv 1 \bmod 2$ and $uv \not\equiv 1 \bmod 8$. Then $S(u, v; a) =$* 0.

Proof. Imitating the derivation of (5.6), for $p = 2$, and since $g(h)$ in (6.4) has period 8, we get

$$S = S(u, v; a) = 2 \sum_{l=1}^{2^{a-1}} g(l)\, e\left(\frac{ul + v\bar{l}}{2^a}\right) \qquad (l\bar{l} \equiv 1 \bmod 2^a)$$

observing that in the inner sum r takes the values 0 and 1 only, and that $u - \bar{l}^2 v$ is even. Repeating this process yields

$$S = 2\left[1 + e\left(\frac{u-v}{4}\right)\right] \sum_{l=1}^{2^{a-2}} g(l)\, e\left(\frac{ul + v\bar{l}}{2^a}\right) \qquad (l\bar{l} \equiv 1 \bmod 2^a).$$

The square bracket gives zero, when $uv \equiv 3 \bmod 4$; and two, when $uv \equiv 1 \bmod 4$. In the latter case, repeating the process again (observing that $2(a-3) \geqslant a$ and $2^{a-3} \equiv 0 \bmod 8$) we obtain for S the factor

$$1 + e\left(\frac{u-v}{8}\right) = 0, \qquad \text{since now} \quad uv \equiv 5 \bmod 8.$$

LEMMA 8. *Let $a \geqslant 6$ and $uv \equiv 1 \bmod 8$. Then*

$$|S(u, v; a)| \leqslant 2^{(a+5)/2}.$$

Proof. With $b = a - c$ and $c = [a/2]$ all $h \bmod 2^a$, h odd, are given by $h = l + r2^b$ with $l = 1, \ldots, 2^b - 1$, l odd and $r = 0, \ldots, 2^c - 1$. Then $\bar{h} \equiv \bar{l} - r\bar{l}^2 2^b \bmod 2^a$ with $l\bar{l} \equiv 1 \bmod 2^a$. Thus from (6.4)

$$S = S(u, v; a) = 2^c \sum_{l=1}^{2^b} g(l)\, e\left(\frac{ul + v\bar{l}}{2^a}\right), \qquad \bar{l}^2 v \equiv u \bmod 2^c$$

since $b \geqslant 3$. If a is even, then $b = c = a/2$ and the summation conditions for l have exactly 4 solutions, since uv is quadratic residue mod 8. Thus $|S| \leqslant 2^{c+2}$. If a is odd, then $c = (a-1)/2 \geqslant 3$ and $b = c + 1$, thus the sum has 8 terms and $|S| \leqslant 2^{c+3}$.

Lemmas 6, 7 and 8 together with (6.7) and the trivial estimate (when $a \leqslant 5$) imply that for all $a > 0$

$$|S(u, v; a)| < K_0(v, 2^a)\, 2^{a/2} \quad \text{with an absolute constant } K_0. \tag{6.8}$$

With $a = a_0$, $v = nb_0$ and (6.5) we obtain finally our

THEOREM 2. *The sum defined in* (4.4) *satisfies for all natural m, n and c*

$$|K(m, n; c)| \leqslant Knc^{1/2} \prod_{2<p|c} \left[1 + \left(\frac{mn}{p}\right)\right], \tag{6.9}$$

where K is an absolute constant.

Theorem 2 implies

$$| K(m, n; c)| \leqslant Knc^{1/2}2^{\omega(c)}, \tag{6.10}$$

where $\omega(c)$ is the number of prime divisors of c. This last estimate, however, is not sharp enough for our intended application, which requires the full content of (6.9). It is interesting that the proof of Theorem 2 is elementary. It does not require the well known estimate of Weil [12]; (cf. [10], Section 3). The rarely mentioned result of Salié ([9], (54)), namely (5.8) for $n = a = 1$, instead yields the crucial estimate.

7. Estimation of the Fourier Coefficients of $P_n(\tau; k)$

We now turn to the Fourier coefficients of the Poincaré series in (4.3). For the spherical Bessel function we use the familiar estimate

$$| J_{k+1/2}(x)| \leqslant C(k) \min\{x^{k+1/2}, x^{-1/2}\} \qquad (x > 0). \tag{7.1}$$

Because of Theorem 2 we introduce for natural a, c the function

$$K^*(a; c) = \prod_{2<p|c} \left[1 + \left(\frac{a}{p}\right)\right] \tag{7.2}$$

and obtain from (4.3), using (7.1) and (6.9), for the mth Fourier coefficient

$$A^k_{mn} \ll nC_1(k) \Big\{m^{k/2}n^{-(k+1)/2} \cdot \sum_{c<2\pi(a)^{1/2}} K^*(a; c) + m^{k+1/2} \cdot \sum_{c>2\pi(a)^{1/2}} K^*(a; c)\, c^{-k-1}\Big\} \tag{7.3}$$

with $a = mn > 0$. [$A \ll B$ means $| A | \leqslant$ const. B with an absolute constant.] Since $K^*(a; c) = O(c^\epsilon)$ is multiplicative in c, we introduce the Dirichlet series with Euler product

$$G(s; a) = \sum_{c=1}^{\infty} K^*(a; c)\, c^{-s} = (1 - 2^{-s})^{-1} \prod_{p>2} \frac{1 + (a/p)\, p^{-s}}{1 - p^{-s}}, \qquad (\sigma > 1). \tag{7.4}$$

Then

$$G(s; a) = L(s; d)H(s; a)\zeta(s)/\zeta(2s), \qquad d = d(a), \tag{7.5}$$

where $d(a)$ is the discriminant of $Q(a)^{1/2}$ as before, $L(s; d)$ the associated Dirichlet L-function from (2.11), $\zeta(s) = L(s; 1)$ and

$$H(s; a) = \prod_{p|2a} \left[\left(1 - \left(\frac{d}{p}\right) p^{-s}\right)(1 - p^{-2s})^{-1}\right], \qquad d = d(a). \tag{7.6}$$

LEMMA 9. *If* $\sigma \geqslant 3/4$ *and* $\epsilon > 0$, *then*

$$| H(s; a)| < C_2(\epsilon)(a/d(a))^{\epsilon}.$$

Proof. From (7.6), with $k = 8a/d(a)$

$$| H(s; a)| \leqslant \zeta(3/2) \prod_{p|k} (1 + p^{-\sigma}).$$

Now, if k has h prime divisors and p_j is the jth prime,

$$\log \prod_{p|k} (1 + p^{-\sigma}) < \sum_{j=1}^{h} p_j^{-3/4} < 4p_h^{1/4} \ll (h \log h)^{1/4} = o(\log k),$$

since $k \geqslant 2^h$. This implies the lemma.

LEMMA 10. *If* $\sigma \geqslant 3/4$ *and* $s = \sigma + it$, *then*

$$L(s; d) \ll d^{1/4}(1 + | t |)^{1/4} \qquad \textit{for} \quad d > 1,$$
$$\zeta(s) \ll (1 + | t |)^{1/4} \qquad \textit{for} \quad | s - 1 | \geqslant 1/4.$$

The proof follows from [8], p. 115.

Next, from a well-known formula (e.g., [8], p. 376) and (7.4)

$$\sum_{c<x} K^*(a; c) - \frac{1}{2\pi i} \int_{b-iT}^{b+iT} G(s; a)\, x^s \frac{ds}{s} \ll \frac{x^b}{(b-1)^2\, T} + C_3(\delta) \frac{x^{1+\delta}}{T}, \tag{7.7}$$

where $x > 0$ is half an odd integer, $1 < b < 2 < T$ and $\delta > 0$; since $G(s; 1)$ has a pole of order 2 at $s = 1$ by (7.5). Now let $d = d(a) > 1$, so that $G(s; a)$ has a simple pole at $s = 1$. Then by the residue theorem (using a rectangular contour) and (7.5)

$$\int_{b-iT}^{b+iT} G(s; a)\, x^s \frac{ds}{s} = 2\pi i \frac{L(1; d)\, H(1; a)}{\zeta(2)} x + \int_{b-iT}^{c-iT} + \int_{c-iT}^{c+iT} + \int_{c+iT}^{b+iT}$$

with $c = 3/4$. Here, by Lemmas 9 and 10

$$\int_{b\pm iT}^{c\pm iT} G(s; a)\, x^s \frac{ds}{s} \ll C_2(\epsilon)(a/d)^{\epsilon}\, d^{1/4} x^b T^{-1/2},$$

$$\int_{c-iT}^{c+iT} G(s; a)\, x^s \frac{ds}{s} \ll C_2(\epsilon)(a/d)^{\epsilon}\, d^{1/4} x^{3/4} T^{1/2}.$$

Taking $b = 1 + 1/\log x$, $T = x^{1/4}$ and $\delta = 1/8$ in (7.7) we obtain

$$\sum_{c<x} K^*(a; c) - \frac{6x}{\pi^2} H(1; a)\, L(1; d) \ll x^{7/8}(1 + C_2(\epsilon)\, a^\epsilon d^{(1/4)-\epsilon}), \quad (7.8)$$

if $d = d(a) > 1$, $\epsilon > 0$ (and $x > 16$).

Now let $q(k)$ denote the squarefree factor of the integer k.

LEMMA 11. *Let $a = mn$ with natural m, n and $q(n) \neq q(m) \ll m^{1/5}$. Then*

$$\sum_{c<x} K^*(a; c) \ll Bx \qquad \textit{with} \quad B = H(1; a)\, L(1; d) + n^{3/16} \quad (7.9)$$

and $d = d(a)$, for $x > (mn)^{1/2}$.

Proof. From the assumption: $1 < d(mn) \leqslant 4q(m)q(n) \ll nm^{1/5}$. Especially, (7.8) is valid for such $a = mn$, and with $\epsilon = 1/64$ the result follows. Thus we have an estimate for the finite sum in (7.3). The infinite series there can be estimated by partial summation and (7.9), observing that $K^*(a; h) \geqslant 0$:

$$\sum_{c>x} K^*(a; c)\, c^{-k-1} = \sum_{c>x} (c^{-k-1} - (c + 1)^{-k-1}) \sum_{x<h\leqslant c} K^*(a; h)$$
$$\ll (k + 1)\, B \sum_{c>x} c^{-k-1} \ll Bx^{-k} \qquad (k > 0). \quad (7.10)$$

With $x = 2\pi(mn)^{1/2}$ in (7.9) and (7.10) we obtain from (7.3)

THEOREM 3. *If the squarefree kernels of m and n satisfy $q(n) \neq q(m) \ll m^{1/5}$, then for the mth Fourier coefficient of the Poincaré series $P_n(\tau; k) \in C_{k+3/2}$*

$$| A^k_{mn} | < C_1(k)\, n^{(5-2k)/4}(H(1; mn)\, L(1; d(mn)) + 1)\, m^{(k+1)/2}. \quad (7.11)$$

This yields (*see* (8.4))

$$| A^k_{mn} | < C(k)\, m^{(k+1)/2}(\log q(m)) \log\log(m/q(m)). \quad (7.11')$$

Theorem 3 is sufficient to establish (4.5) and thus (1.2) for some sequences $\{m_j\}$, as will be demonstrated shortly. It is also interesting that the estimate (7.11) is apparently sharp enough to exhibit some of the expected arithmetic structure of the Fourier coefficients of modular forms of half-integral dimension $-(k + 3/2)$ on the theta group. If we use the weaker estimate (6.10) instead of (6.9), then $G(s; a)$ in the preceding treatment will be replaced by $\zeta^2(s)/\zeta(2s)$, and instead of (7.11) we get

$$| A^k_{mn} | < C_1(k)\, n^{1-k/2} m^{(k+1)/2} \log 2mn \quad (7.12)$$

without restriction on m. This (though better than the usually quoted estimate $A^k_{mn} = O(m^{(k+1)/2+\epsilon})$; e.g., [10], (1.13′)) is not sharp enough for satisfying the condition (4.5).

8. Spheres with Uniform Distribution of Lattice Points

We shall here construct sequences $\{m_j\}$, for which our earlier condition (4.5) is satisfied. In order that we can apply Theorem 3 we choose m to be of the form

$$m = fg^2 \quad \text{with} \quad f > 1 \quad \text{squarefree and} \quad g > f^2, g \text{ odd}; \tag{8.1}$$

which implies $q(m) = f < m^{1/5}$. Then, from (7.6) with $a = mn > 0$ and $b = 8a/d(a)$

$$H(1; m) < \zeta(2) \prod_{p \mid b} (1 + p^{-1}) \ll (\log \log 3n) \prod_{p \mid g} (1 + p^{-1}), \tag{8.2}$$

since with $n = hk^2$, h squarefree, we have $b = 2^c g^2 k^2 (f, h)^2$, thus every prime divisor of b divides g or $2n$, and the product can be estimated similar as in lemma 9. Also, since $d(mn) \leqslant 4fn$,

$$L(1; d(mn)) \ll \log d(mn) \ll (\log 2n) \log f. \tag{8.3}$$

If we assume that $f \neq q(n)$, then by Theorem 3 and the foregoing

$$| A^k_{mn} | < C_2(k)\, m^{(k+1)/2} (\log f) \prod_{p \mid g} (1 + p^{-1}) \qquad (m = fg^2, k \geqslant 4). \tag{8.4}$$

Also, by (2.14) and (8.1), if $f \not\equiv -1 \bmod 8$,

$$r_3(m) \geqslant 4h(d)\, g \prod_{p \mid g} \left[1 + \left(1 - \left(\frac{d}{p}\right)\right) p^{-1}\right], \qquad d = d(-f). \tag{8.5}$$

Here, by the well known theorem of Siegel about the class number,

$$h(d) > c(\epsilon) f^{(1-\epsilon)/2} \qquad (\epsilon > 0)$$

hence with (8.4) and (8.1), if $q(n) \neq f \not\equiv -1 \bmod 8$ and $k \geqslant 4$

$$| A^k_{mn} |\, m^{-k/2} r_3^{-1}(m) < C(\epsilon, k) f^{\epsilon} \prod_{p \mid g} \frac{p + 1}{p + 1 - (d/p)}, \qquad d = d(-f). \tag{8.6}$$

Now we require

LEMMA 12. *Let $(q, d) = 1$ and $b > 0$ be fixed, then uniformly in d, q and x*

$$\prod_{\substack{p \leqslant x \\ p \equiv q \bmod d}} \frac{p+1}{p+2} \ll (\log x)^{-1/\phi(|d|)}, \qquad \text{if} \quad d \ll \log^b x. \tag{8.7}$$

Proof. The Siegel-Walfisz theorem implies

$$\Big| \phi(|d|) \sum_{\substack{p \leqslant x \\ p \equiv q \bmod d}} p^{-1} - \log \log x \Big| < B_0 \qquad (x > 3)$$

where $\phi(*)$ is the Euler function and B_0 a constant (depending on b). Then

$$-\log \prod_{\substack{p \leqslant x \\ p \equiv q \bmod d}} \frac{p+1}{p+2} > \sum_{\substack{p \leqslant x \\ p \equiv q \bmod d}} \frac{1}{p+2} > \frac{1}{\phi(|d|)} \log \log x - B$$

with another constant B, and the statement follows.

For the intended application in (4.5) let $k \geqslant 4$ be even, $1 \leqslant n \leqslant [k/4]$ and $\epsilon > 0$ be fixed. Let $\{f_j\}$, $(j = 1, 2, ...)$ be an increasing sequence of squarefree natural numbers with $f_j \not\equiv -1 \bmod 8$. Then $f_j \neq q(n) \leqslant k/4$ for $j > N(k)$, say. For each j let q_j be a natural number with

$$\left(\frac{d_j}{q_j}\right) = -1, \qquad \text{where} \quad d_j = d(-f_j) < 0, \tag{8.8}$$

and set

$$g_j = \prod_{2 < p \leqslant x_j, \, p \equiv q_j \bmod d_j} p \qquad \text{with} \quad x_j = \exp\{(jf_j)^{\epsilon\phi(-d_j)}\}. \tag{8.9}$$

Then $g_j \gg \exp(x_j/\phi(-d_j)) \gg f_j^2$ (amply) and $f_j \ll \log^b x_j$ with $b = 1/\epsilon$. Finally let $m_j = f_j g_j^2$, so that it has the form of m in (8.1). Setting $m = m_j$ in (8.6) lets the product there assume the same form as that in Lemma 12, by (8.9) and (8.8). Hence from (8.6) and (8.7) for $j > N(k)$

$$|A^k_{m_j n}| < C(\epsilon, k)\, j^{-\epsilon} m_j^{k/2} r_3(m_j) = o(m_j^{k/2} r_3(m_j)) \qquad \text{as} \quad j \to \infty. \tag{8.10}$$

Thus the condition (4.5) holds for this sequence $\{m_j\}$, and consequently we have proven

THEOREM 4. *Let $\{f_i\}$ be an increasing sequence of squarefree natural numbers with $f_i \not\equiv -1 \bmod 8$. There exist sequences $\{m_i\}$ with $m_i = f_i g_i^2$ $(g_i \in Z)$ such that for every Jordan measurable region Δ on the unit sphere in E^3*

$$\lim_{i \to \infty} \frac{r(m_i\,; \Delta)}{r(m_i)} = \frac{m(\Delta)}{4\pi}.$$

Thus, from the notation given in the introduction, lattice points (of Z^3) on the sequence of spheres

$$x^2 + y^2 + z^2 = m_i \qquad (i = 1, 2, \ldots)$$

(in R^3) are asymptotically uniformly distributed.

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