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On representations of general linear groups over principal ideal local rings of length two

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ABSTRACT

We study the irreducible complex representations of general linear groups over principal ideal local rings of length two with a fixed finite residue field. We construct a canonical correspondence between the irreducible representations of all such groups that preserves dimensions. For general linear groups of order three and four over these rings, we construct all the irreducible representations. We show that the problem of constructing all the irreducible representations of all general linear groups over these rings is not easier than the problem of constructing all the irreducible representations of all general linear groups over principal ideal local rings of arbitrary length in the function field case.

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1. Introduction

Let F be a non-Archimedean local field with ring of integers $\mathcal{O}$. Let $\wp$ be the unique maximal ideal of $\mathcal{O}$ and π be a fixed uniformizer of $\wp$. Assume that the residue field $\mathcal{O}/\wp$ is finite. The typical examples of such rings of integers are $\mathbf{Z}_p$ (the ring of p-adic integers) and $\mathbf{F}_q[[t]]$ (the ring of formal power series with coefficients over a finite field). We denote by $\mathcal{O}_\ell$ the reduction of $\mathcal{O}$ modulo $\wp^\ell$, i.e., $\mathcal{O}_\ell = \mathcal{O}/\wp^\ell$. Let Λ_k denote the set of partitions with k parts, namely, non-increasing finite sequences $(\ell_1, \ell_2, \dots, \ell_k)$ of positive integers, and let $\Lambda = \bigcup \Lambda_k$. Since $\mathcal{O}$ is a principal ideal domain with a unique maximal ideal $\wp$, every finite $\mathcal{O}$-module is of the form $\bigoplus_{i=1}^k \mathcal{O}_{\ell_i}$, where ℓ_i's can be arranged so that $\lambda = (\ell_1, \ell_2, \dots, \ell_k) \in \Lambda_k$. Let $\mathrm{M}_\lambda = \bigoplus_{i=1}^k \mathcal{O}_{\ell_i}$ and

$$\mathrm{G}_{\lambda,F} = \mathrm{Aut}_{\mathcal{O}}(\mathrm{M}_\lambda).$$

We write G_λ instead of $G_{\lambda,F}$ whenever field F is clear from the context. If $\mathrm{M}_\lambda = \mathcal{O}_\ell^n$ for some natural number n, then the group G_λ consists of invertible matrices of order n with entries in the ring $\mathcal{O}_\ell$, so we use the notation $\mathrm{GL}_n(\mathcal{O}_\ell)$ for G_λ in this case.

The representation theory of the finite groups G_λ has attracted the attention of many mathematicians. We give a brief history of this problem. Green [9] calculated the characters of the irreducible representations of $\mathrm{GL}_n(\mathcal{O}_1)$. Several authors, for instance, Frobenius [6], Rohrbach [23], Kloosterman [15,16], Tanaka [30], Nobs and Wolfart [21], Nobs [20], Kutzko [17], Nagornyĭ [18], and Stasinski [27] studied the representations of the groups $\mathrm{SL}_2(\mathcal{O}_\ell)$ and $\mathrm{GL}_2(\mathcal{O}_\ell)$. Nagornyĭ [19] obtained partial results regarding the representations of $\mathrm{GL}_3(\mathcal{O}_\ell)$ and Onn [22] constructed all the irreducible representations of the groups $\mathrm{G}_{(\ell_1,\ell_2)}$. Recently, Avni, Klopsch, Onn, and Voll [2] have announced results about the representation theory of the groups $\mathrm{SL}_3(\mathbf{Z}_p)$.

In another direction, it was observed that, being maximal compact subgroups, $\mathrm{GL}_n(\mathcal{O})$ play an important role in the representation theory of the groups $\mathrm{GL}_n(F)$. Further, every continuous representation of $\mathrm{GL}_n(\mathcal{O})$ factors through one of the natural homomorphisms $\mathrm{GL}_n(\mathcal{O}) \to \mathrm{GL}_n(\mathcal{O}_\ell)$. This brings the study of irreducible representations of groups $\mathrm{GL}_n(\mathcal{O}_\ell)$ to the forefront. Various questions regarding the complexity of the problem of determining irreducible representations of these groups have been asked. For example, Nagornyĭ [19] proved that this problem contains the matrix pair problem. Aubert, Onn, Prasad, and Stasinski [1] proved that, for $F = \mathbf{F}_q((t))$, constructing all irreducible representations of $\mathrm{GL}_n(\mathcal{O}_2)$ for all n is equivalent to constructing all irreducible representations of $G_{\lambda,\mathbf{F}_{q^m}((t))}$ for all λ and m (see also Section 6).

Motivated by Lusztig's work for finite groups of Lie type, Hill [10] partitioned all the irreducible representations of groups $\mathrm{GL}_n(\mathcal{O}_\ell)$ into geometric conjugacy classes and reduced the study of irreducible representations of $\mathrm{GL}_n(\mathcal{O}_\ell)$ to the study of its nilpotent characters. In later publications [11–13], he succeeded in constructing many irreducible representations (namely strongly-semisimple, semisimple, regular, etc.) for these groups. Following the techniques used in the representation theory of groups $\mathrm{GL}_n(\mathcal{O}_1)$ and $\mathrm{GL}_n(F)$, various notions such as cuspidality and supercuspidality were introduced for representations of $\mathrm{GL}_n(\mathcal{O})$ (for more on this see [1] and [26]), but the complete set of irreducible representations of groups $\mathrm{GL}_n(\mathcal{O}_\ell)$ for $\ell \geqslant 2$ is still unknown. From the available results, it was observed that methods of constructing irreducible representations of groups G_λ do not depend on the particular ring of integers $\mathcal{O}$, but only on the residue field. This led Onn to conjecture [22, Conjecture 1.2] that:

Conjecture 1.1. *The isomorphism type of the group algebra $\mathbb{C}[G_\lambda]$ depends only on λ and $q = |\mathcal{O}/\wp|$.*

We discuss the method of constructing complex irreducible representations of the groups $\mathrm{GL}_n(\mathcal{O}_2)$ with the help of Clifford theory and reduce this problem to constructing irreducible representations of certain subgroups of $\mathrm{GL}_n(\mathcal{O}_1)$. This enables us to give an affirmative answer to the above conjecture for $\mathrm{GL}_n(\mathcal{O}_2)$. The groups $\mathrm{GL}_n(\mathcal{O}_2)$, for distinct rings of integers $\mathcal{O}$, are not necessarily isomorphic, even when the residue fields are isomorphic. For example, for a natural number n and a prime p, the group $\mathrm{GL}_n(\mathbf{F}_p[[t]]/t^2)$ is a semi-direct product of the groups $\mathrm{M}_n(\mathbf{F}_p)$ and $\mathrm{GL}_n(\mathbf{F}_p)$, but on the other hand $\mathrm{GL}_n(\mathbf{Z}_p/p^2\mathbf{Z}_p)$ is not unless $n = 1$ or $(n, p) = (2, 2), (2, 3)$ or $(3, 2)$ (Sah [24, p. 22], Ginosar [8]). Our main emphasis is on proving that all of their irreducible representations can be constructed in a uniform way. We also succeed in showing that representation theory of groups $G_{\lambda,\mathbf{F}_{q^m}((t))}$ plays a vital role in representation theory of groups $\mathrm{GL}_n(\mathcal{O}_2)$ for any $\mathcal{O}$, in the sense that if we know irreducible representations of the groups $G_{\lambda,\mathbf{F}_{p^m}((t))}$ for all positive integers m, we can determine all the representations of $\mathrm{GL}_n(\mathcal{O}_2)$.

More precisely, let F and F′ be local fields with rings of integers $\mathcal{O}$ and $\mathcal{O}'$, respectively, such that their residue fields are finite and isomorphic (with a fixed isomorphism). Let $\wp$ and $\wp'$ be the maximal ideals of $\mathcal{O}$ and $\mathcal{O}'$ respectively. As described earlier, $\mathcal{O}_2$ and $\mathcal{O}'_2$ denote the rings $\mathcal{O}/\wp^2$ and $\mathcal{O}'/\wp'^2$, respectively. We prove:

Theorem 1.2 *(Main Theorem). There exists a canonical bijection between the irreducible representations of* $\mathrm{GL}_n(\mathcal{O}_2)$ *and those of* $\mathrm{GL}_n(\mathcal{O}'_2)$*, which preserves dimensions.*

Definition 1.3 *(Representation zeta polynomial).* Let G be a finite group. The representation zeta polynomial of G is the polynomial

$$R_G(\mathcal{D}) = \sum_{\rho \in \mathrm{Irr}\, G} \mathcal{D}^{\dim \rho} \in \mathbb{Z}[\mathcal{D}].$$

In view of the above definition, Theorem 1.2 implies that:

Corollary 1.4. *The representation zeta polynomials of* $\mathrm{GL}_n(\mathcal{O}_2)$ *and* $\mathrm{GL}_n(\mathcal{O}'_2)$ *are equal.*

In other words, the representation zeta polynomial depends on the ring only through the order of its residue field.

Concerning the complexity of the problem of constructing irreducible representations of groups $\mathrm{GL}_n(\mathcal{O}_2)$, we obtain the following generalization of [1, Theorem 6.1].

Theorem 1.5. *Let $\mathcal{O}$ be the ring of integers of a non-Archimedean local field F, such that its residue field has cardinality q. Then the problem of constructing irreducible representations of the following groups are equivalent*:

(1) $\mathrm{GL}_n(\mathcal{O}_2)$ *for all* $n \in \mathbf{N}$.
(2) $G_{\lambda,E}$ *for all partitions λ and all unramified extensions E of* $\mathbf{F}_q((t))$.

We construct all the irreducible representations of $\mathrm{GL}_2(\mathcal{O}_2)$, $\mathrm{GL}_3(\mathcal{O}_2)$, and $\mathrm{GL}_4(\mathcal{O}_2)$. As mentioned earlier, the representation theory of $\mathrm{GL}_2(\mathcal{O}_2)$ is already known. Partial results regarding the representations of $\mathrm{GL}_3(\mathcal{O}_2)$ have been obtained by Nagornyĭ [19], but the representation theory of $\mathrm{GL}_4(\mathcal{O}_2)$ seems completely novel. We find that:

Theorem 1.6. *The number and dimensions of irreducible representations of groups* $\mathrm{GL}_3(\mathcal{O}_2)$ *and* $\mathrm{GL}_4(\mathcal{O}_2)$ *are polynomials in* $\mathbb{Q}[q]$.

This theorem proves the strong version of Onn's conjecture [22, Conjecture 1.3] for the groups $\mathrm{GL}_3(\mathcal{O}_2)$ and $\mathrm{GL}_4(\mathcal{O}_2)$.

1.1. Organization of the article

In Section 2, we set up the basic notation that we use throughout the article and discuss the action of the group $\mathrm{GL}_n(\mathcal{O}_2)$ on the characters of its normal subgroup $K = \mathrm{Ker}(\mathrm{GL}_n(\mathcal{O}_2) \mapsto \mathrm{GL}_n(\mathcal{O}_1))$. We also state the results of Clifford theory, which we use later to prove Theorem 1.2.

In Section 3, we briefly review the similarity classes of $\mathrm{M}_n(\mathbf{F}_q)$, and in Section 4, we discuss the centralizer algebras of matrices, namely the set of matrices that commute with a given matrix. For any matrix $A \in \mathrm{M}_n(\mathbf{F}_q)$ in Jordan canonical form, we describe its centralizer explicitly in $\mathrm{M}_n(\mathbf{F}_q)$ and in $\mathrm{GL}_n(\mathbf{F}_q)$.

In Section 5, we present the proof of Theorem 1.2. For this, we prove that all the characters of the subgroup K can be extended to its stabilizer in $\mathrm{GL}_n(\mathcal{O}_2)$.

Section 6 is devoted to applications of Theorem 1.2. We express a relation between the representation zeta polynomial of $\mathrm{GL}_n(\mathcal{O}_2)$ and that of centralizers in $\mathrm{GL}_n(\mathcal{O}_1)$. Then we describe the representation zeta polynomials of $\mathrm{GL}_2(\mathcal{O}_2), \mathrm{GL}_3(\mathcal{O}_2)$, $G_{(2,1,1)}$, and $\mathrm{GL}_4(\mathcal{O}_2)$. In particular, we prove Theorem 1.6.

2. Notations and Clifford theory

In this section, we set up the basic notation that we use throughout the article. We state and apply the main results of Clifford theory to our case and state Proposition 2.2, which is an important step toward the proof of Theorem 1.2.

In this article, by "character", we mean a one dimensional representation, unless stated otherwise. For any group G, we denote by $\mathrm{Irr}(G)$ the set of its irreducible representations, and for any abelian group A, we denote by $\hat{A}$ the set of its characters.

Let $\kappa : \mathrm{GL}_n(\mathcal{O}_2) \to \mathrm{GL}_n(\mathcal{O}_1)$ be the natural quotient map and $K = \mathrm{Ker}(\kappa)$. Then $A \mapsto I + \pi A$ induces an isomorphism $\mathrm{M}_n(\mathcal{O}_1) \xrightarrow{\sim} K$. Fix a non-trivial additive character $\psi : \mathcal{O}_1 \to \mathbf{C}^*$ and for any $A \in \mathrm{M}_n(\mathcal{O}_1)$ define $\psi_A : K \to \mathbf{C}^*$ by

$$\psi_A(I + \pi X) = \psi\big(\mathrm{Tr}(AX)\big).$$

Then $A \mapsto \psi_A$ gives an isomorphism $\mathrm{M}_n(\mathcal{O}_1) \xrightarrow{\sim} \hat{K}$. The group $\mathrm{GL}_n(\mathcal{O}_2)$ acts on $\mathrm{M}_n(\mathcal{O}_1)$ by conjugation via its quotient $\mathrm{GL}_n(\mathcal{O}_1)$, and therefore on $\hat{K}$: for $\alpha \in \mathrm{GL}_n(\mathcal{O}_2)$ and $\psi_A \in \hat{K}$, we have

$$\begin{aligned}\psi_A^{\alpha}(I + \pi X) &= \psi_A\big(I + \pi\alpha X\alpha^{-1}\big)\\ &= \psi\big(\mathrm{Tr}\big(A\kappa(\alpha)X\kappa(\alpha)^{-1}\big)\big)\\ &= \psi\big(\mathrm{Tr}\big(\kappa(\alpha)^{-1}A\kappa(\alpha)X\big)\big)\\ &= \psi_{\kappa(\alpha)^{-1}A\kappa(\alpha)}(I + \pi X). \end{aligned} \tag{2.1}$$

Thus the action of $\mathrm{GL}_n(\mathcal{O}_2)$ on the characters of K transforms to its conjugation (inverse) action on elements of $\mathrm{M}_n(\mathcal{O}_1)$.

We shall use the following results of Clifford theory.

Theorem 2.1. *Let G be a finite group and N be a normal subgroup. For any irreducible representation ρ of N, let $T(\rho) = \{g \in G \mid \rho^g = \rho\}$ denote the stabilizer of ρ. Then the following hold:*

(1) *If π is an irreducible representation of G such that $\langle \pi|_N, \rho\rangle \neq 0$, then $\pi|_N = e(\bigoplus_{\rho' \in \Omega} \rho')$ where Ω is an orbit of irreducible representations of N under the action of G, and e is a positive integer.*
(2) *Suppose that ρ is an irreducible representation of N. Let*

$$A = \big\{\theta \in \mathrm{Irr}\big(T(\rho)\big) \bigm| \big\langle Res_N^{T(\rho)}\,\theta, \rho\big\rangle \neq 0\big\}$$

and

$$B = \{\pi \in \operatorname{Irr} G \mid \langle Res_N^G \pi, \rho \rangle \neq 0\}.$$

Then

$$\theta \to Ind_{T(\rho)}^G(\theta)$$

is a bijection of A onto B.

(3) *Let H be a subgroup of G containing N, and suppose that ρ is an irreducible representation of N, which has an extension $\tilde{\rho}$ to H (i.e., $\tilde{\rho}|_N = \rho$). Then the representations $\chi \otimes \tilde{\rho}$ for $\chi \in \operatorname{Irr}(H/N)$ are irreducible, distinct for distinct χ, and*

$$Ind_N^H(\rho) = \bigoplus_{\chi \in \operatorname{Irr}(H/N)} \chi \otimes \tilde{\rho}.$$

For proofs of the above, see, for example, 6.2, 6.11, and 6.17, respectively, in Isaacs [14]. Applying the above results to the group $G = \mathrm{GL}_n(\mathcal{O}_2)$ and normal subgroup $N = K$, we see that the following proposition plays an important role in the proof of Theorem 1.2.

Proposition 2.2. *For a given $A \in \mathrm{M}_n(\mathcal{O}_1)$, there exists a character χ of $T(\psi_A)$ such that $\chi|_K = \psi_A$ (such a character χ is called an extension of ψ_A).*

Its proof will be given in Section 5.

3. Primary decomposition and Jordan canonical form

In this section, we describe the primary decomposition of matrices under the action of conjugation. We also discuss the Jordan canonical form for those matrices whose characteristic polynomials split over $\mathbf{F}_q$.

Let f be an irreducible polynomial with coefficients in $\mathbf{F}_q$.

Definition 3.1 *(f-Primary matrix).* A matrix with entries in $\mathbf{F}_q$ is f-primary if its characteristic polynomial is a power of f.

Notation. If A_i's for $1 \leqslant i \leqslant l$ are matrices, then we denote by $\bigoplus_i A_i$ the block diagonal matrix

$$\begin{bmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_l \end{bmatrix}.$$

In the same spirit, if $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_l$ are the sets of matrices then $\bigoplus_{i=1}^l \mathcal{A}_i$ denotes the set of matrices $\{\bigoplus_{i=1}^l A_i \mid A_i \in \mathcal{A}_i\}$. The next two theorems are easy consequences of structure theorems for $\mathbf{F}_q[t]$-modules. For proofs of these see Bourbaki [4, A.VII.31].

Theorem 3.2 *(Primary decomposition). Every matrix $A \in \mathrm{M}_n(\mathbf{F}_q)$ is similar to a matrix of the form*

$$\bigoplus_f A_f,$$

where A_f is an f-primary matrix, and the sum is over the irreducible factors of the characteristic polynomial of A. Moreover, for every f, the similarity class of A_f is uniquely determined by the similarity class of A.

Definition 3.3 *(Split matrix).* A matrix over $\mathcal{O}_1$ is called split if its characteristic polynomial splits over $\mathbf{F}_q$.

Definition 3.4 *(Elementary Jordan blocks).* For a natural number n and an element $a \in \mathbf{F}_q$, an elementary Jordan block $J_n(a)$ is the matrix

$$\begin{bmatrix} a & 1 & 0 & 0 & \cdots & 0 \\ 0 & a & 1 & 0 & \cdots & 0 \\ 0 & 0 & a & 1 & & \\ \vdots & \vdots & & \ddots & \ddots & \vdots \\ & & & & a & 1 \\ 0 & 0 & \cdots & & 0 & a \end{bmatrix}_{n\times n}.$$

Theorem 3.5 *(Jordan canonical form for split matrices). Every split matrix $A \in \mathrm{M}_n(\mathbf{F}_q)$, up to the rearrangement of the a_i's, is similar to a unique matrix of the form*

$$\bigoplus_i J_{\lambda(a_i)}(a_i),$$

where $\lambda(a) = (\lambda_1(a), \lambda_2(a), \ldots, \lambda_k(a))$ is a partition and

$$J_{\lambda(a_i)}(a_i) = \begin{bmatrix} J_{\lambda_1(a_i)}(a_i) & 0 & \cdots & 0 \\ 0 & J_{\lambda_2(a_i)}(a_i) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & J_{\lambda_k(a_i)}(a_i) \end{bmatrix}$$

and each $J_{\lambda_j(a_i)}(a_i)$ is an elementary Jordan block with eigenvalue a_i.

4. Centralizers

Let R be a commutative ring with unity. In this section, we determine centralizers (see Definition 4.1) of certain matrices in $\mathrm{M}_n(R)$ and $\mathrm{GL}_n(R)$. We also relate the groups $G_{\lambda,\mathbf{F}_{q^m}((t))}$ with centralizers in $\mathrm{GL}_n(\mathbf{F}_q)$.

Definition 4.1 *(Centralizer of an element).* Let L be a semi-group under multiplication and l be an element of L. Assume that T is a subset of L. Then *centralizer* of l in T, $Z_T(l)$, is the set of elements of T that commute with l, i.e.,

$$Z_T(l) = \{t \in T \mid tl = lt\}.$$

Remark 4.2. If T is a group, then $Z_T(l)$ is a subgroup of T.

Definition 4.3 *(Principal nilpotent matrix).* A square matrix is called principal nilpotent if it is of the form

$$N_n = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & \cdots & & 0 & 1 \\ 0 & \cdots & & 0 & 0 \end{bmatrix}_{n\times n}.$$

In the following section, we use the notation N_n for the principal nilpotent matrix of order n.

Let $n_1, n_2, \ldots, n_l$ be a sequence of natural numbers, such that $n = n_1 + n_2 + \cdots + n_l$. Let $A = \bigoplus_{i=1}^{l} N_{n_i}$. In Lemmas 4.4–4.9 we describe the centralizer algebras $Z_{\mathrm{M}_n(R)}(A)$. The proofs of these lemmas involve only simple matrix multiplications, so we leave these for the reader.

Lemma 4.4. *Let a and b be two elements of R such that $a - b$ is a unit in R. Assume that A and B are two upper triangular matrices such that all the diagonal entries of A are equal to a and those of B are equal to b. Then there does not exist any non-zero matrix X over R such that $XA = BX$.*

Lemma 4.5. *Let $a_1, a_2, \ldots, a_l$ be elements of R such that for all $i \neq j$, $a_i - a_j$ is invertible in R. Let $A = \bigoplus_{i=1}^{l} A_i$ be a square matrix of order n, where A_i's are upper triangular matrices of order n_i. Assume that all diagonal entries of A_i are equal to a_i. Then,*

$$Z_{\mathrm{GL}_n(R)}(A) = \bigoplus_{i=1}^{l} Z_{\mathrm{GL}_{n_i}(R)}(A_i).$$

Definition 4.6 *(Upper Toeplitz matrix).* A square matrix of order n is called Upper Toeplitz if it is of the form

$$\begin{bmatrix} a_1 & a_2 & \cdots & a_{n-1} & a_n \\ 0 & a_1 & a_2 & & a_{n-1} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ & \cdots & & a_1 & a_2 \\ 0 & \cdots & 0 & 0 & a_1 \end{bmatrix}_{n\times n}.$$

Lemma 4.7. *Assume that N_n and N_m are principal nilpotent matrices of order n and m, respectively. Then consider the matrices X over R such that $XN_m = N_nX$ are of the form*

$$X = \begin{cases} \begin{bmatrix} 0_{n\times m-n} & T_{n\times n} \end{bmatrix} & \text{if } n \leqslant m, \\ \begin{bmatrix} T_{m\times m} \\ 0_{n-m\times m} \end{bmatrix} & \text{if } n \geqslant m, \end{cases}$$

where $T_{s\times s}$, for a natural number s, is an upper Toeplitz matrix of order s over the ring R.

This lemma motivates the following definition of rectangular upper Toeplitz matrix.

Definition 4.8 *(Rectangular upper Toeplitz matrix).* A matrix of order $n \times m$ over a ring R is called a rectangular upper Toeplitz if it is of the form

$$\begin{bmatrix} 0_{n\times(m-n)} & T_{n\times n} \end{bmatrix} \quad \text{if } n \leqslant m \quad \text{or} \quad \begin{bmatrix} T_{m\times m} \\ 0_{(n-m)\times m} \end{bmatrix} \quad \text{if } n \geqslant m,$$

where $T_{s\times s}$, for a natural number s, is the upper Toeplitz matrix of order s.

Lemma 4.9. *Let $n_1, n_2, \dots, n_l$ be a sequence of natural numbers such that $n = n_1 + n_2 + \cdots + n_l$. Let $A = \bigoplus_{i=1}^{l} N_{n_i}$. Then the centralizer, $Z_{\mathrm{M}_n(R)}(A)$ of A in $\mathrm{M}_n(R)$ consists of matrices of the form*

$$\begin{bmatrix} T_{n_1\times n_1} & T_{n_1\times n_2} & \cdots & T_{n_1\times n_l} \\ T_{n_2\times n_1} & T_{n_2\times n_2} & \cdots & T_{n_2\times n_l} \\ \vdots & \vdots & & \vdots \\ T_{n_l\times n_1} & T_{n_l\times n_2} & \cdots & T_{n_l\times n_l} \end{bmatrix},$$

where $T_{n_i\times n_j}$ for all i, j are rectangular upper Toeplitz matrices.

Definition 4.10 *(Block upper Toeplitz matrix).* Let $T_{n_i\times n_j}$ be a rectangular upper Toeplitz matrix of order $n_i \times n_j$ over the ring R. A matrix of the form

$$\begin{bmatrix} T_{n_1\times n_1} & T_{n_1\times n_2} & \cdots & T_{n_1\times n_l} \\ T_{n_2\times n_1} & T_{n_2\times n_2} & \cdots & T_{n_2\times n_l} \\ \vdots & \vdots & & \vdots \\ T_{n_l\times n_1} & T_{n_l\times n_2} & \cdots & T_{n_l\times n_l} \end{bmatrix}$$

is called a block upper Toeplitz matrix of order $(n_1, n_2, \dots, n_l)$ over the ring R.

In the following lemma, we relate the groups of automorphisms $\mathrm{G}_{\lambda,\mathbf{F}_q((t))}$ with the centralizers in $\mathrm{GL}_n(\mathbf{F}_q)$.

Lemma 4.11. *Let $A = \bigoplus_{i=1}^{k} N_{\lambda_i}$ and $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k)$ be a partition. Then the following groups are isomorphic*:

(1) *The group of automorphisms, $\mathrm{G}_{\lambda,\mathbf{F}_q((t))} = \mathrm{Aut}_{\mathcal{O}}(\mathcal{O}_{\lambda_1} \oplus \mathcal{O}_{\lambda_2} \oplus \cdots \oplus \mathcal{O}_{\lambda_k})$.*
(2) *The centralizer $Z_{\mathrm{GL}_n(\mathbf{F}_q)}(A)$.*
(3) *The set of invertible block upper Toeplitz matrices of order $(\lambda_1, \lambda_2, \dots, \lambda_k)$ over $\mathbf{F}_q$.*

Proof. Lemma 4.9 implies that groups (2) and (3) are actually equal. We prove isomorphism between (1) and (3). Every $f \in \mathrm{G}_\lambda$ can be thought as an invertible matrix of the form

$$\begin{bmatrix} f_{11} & f_{12} & \cdots & f_{1k} \\ f_{21} & f_{22} & \cdots & f_{2k} \\ \vdots & \vdots & & \vdots \\ f_{k1} & f_{k2} & \cdots & f_{kk} \end{bmatrix}$$

with each $f_{ij} \in \mathrm{End}_{\mathcal{O}}(\mathcal{O}_{\lambda_i}, \mathcal{O}_{\lambda_j})$. Hence it is sufficient to find an isomorphism between $\mathrm{End}_{\mathcal{O}}(\mathcal{O}_{\lambda_i}, \mathcal{O}_{\lambda_j})$ and rectangular Toeplitz matrices of order $\lambda_i \times \lambda_j$ over $\mathbf{F}_q$, which takes composition to matrix multiplication. We prove it only for $\lambda_i = \lambda_1$ and $\lambda_j = \lambda_2$ with $\lambda_1 \geqslant \lambda_2$, but the rest of the parts can be proved similarly.

Let $\mathcal{T}_{\lambda_1,\lambda_2}$ be the set of rectangular upper Toeplitz matrices of order $\lambda_1 \times \lambda_2$ over the field $\mathbf{F}_q$. Define a map $\mathrm{End}_{\mathcal{O}}(\mathcal{O}_{\lambda_1}, \mathcal{O}_{\lambda_2}) \to \mathcal{T}_{\lambda_1,\lambda_2}$ by

$$f \mapsto \left[\, 0_{(\lambda_1-\lambda_2)\times\lambda_2} \;\middle|\; A \,\right]_{\lambda_1\times\lambda_2},$$

where

$$A = \begin{bmatrix} a_1 & a_2 & \cdots & a_{\lambda_2-1} & a_{\lambda_2} \\ 0 & a_1 & a_2 & & a_{\lambda_2-1} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ & \cdots & & a_1 & a_2 \\ 0 & \cdots & 0 & 0 & a_1 \end{bmatrix}_{\lambda_2 \times \lambda_2}$$

and elements $a_1, a_2, \dots, a_{\lambda_2}$ are determined by the expression $f(1) = a_1 + a_2\pi + \cdots + a_{\lambda_2}\pi^{\lambda_2 - 1}$. It is straightforward to see that this map gives the required isomorphism. □

5. Proof of the Main Theorem

In this section, we present proofs of Proposition 2.2 and Theorem 1.2.

Recall that for any character $\rho \in \hat{K}$, $T(\rho) = \{g \in \mathrm{GL}_n(\mathcal{O}_2) \mid \rho^g = \rho\}$ is the stabilizer of ρ in $\mathrm{GL}_n(\mathcal{O}_2)$. By (2.1), for each $\psi_A \in \hat{K}$,

$$T(\psi_A) = \kappa^{-1}\big(Z_{\mathrm{GL}_n(\mathcal{O}_1)}(A)\big). \tag{5.1}$$

Fix a section $s : \mathcal{O}_1 \to \mathcal{O}_2$ of the natural quotient map $\mathcal{O}_2 \to \mathcal{O}_1$, such that $s(0) = 0$ and $s(1) = 1$. By extending s entry-wise, we obtain a map $\mathfrak{s} : \mathrm{M}_n(\mathcal{O}_1) \to \mathrm{M}_n(\mathcal{O}_2)$. Observe that the restriction of $\mathfrak{s}$ to $\mathrm{GL}_n(\mathcal{O}_1)$ defines a section of κ. For every matrix A in $\mathrm{M}_n(\mathcal{O}_1)$, $Z_{\mathrm{GL}_n(\mathcal{O}_2)}(\mathfrak{s}(A))$ is the centralizer of $\mathfrak{s}(A)$ in $\mathrm{GL}_n(\mathcal{O}_2)$.

Lemma 5.1. *Assume that A is a split matrix and is in its Jordan canonical form, then*

$$Z_{\mathrm{GL}_n(\mathcal{O}_1)}(A) = \kappa\big(Z_{\mathrm{GL}_n(\mathcal{O}_2)}\big(\mathfrak{s}(A)\big)\big).$$

Proof. Let $\alpha = \kappa(t)$ for some $t \in Z_{\mathrm{GL}_n(\mathcal{O}_2)}(\mathfrak{s}(A))$. Then by definition, t satisfies $t\mathfrak{s}(A) = \mathfrak{s}(A)t$, which along with the fact that κ is a homomorphism, implies $\kappa(t)A = A\kappa(t)$. Hence $\alpha = \kappa(t) \in Z_{\mathrm{GL}_n(\mathcal{O}_1)}(A)$. This proves $\kappa(Z_{\mathrm{GL}_n(\mathcal{O}_2)}(\mathfrak{s}(A))) \subseteq Z_{\mathrm{GL}_n(\mathcal{O}_1)}(A)$. For the reverse inclusion, because A is a split matrix, by Theorems 3.2 and 3.5,

$$A = \bigoplus_{i=1}^{l} A_i,$$

where each A_i is a split primary matrix and is of the form

$$A_i = \bigoplus_{j=1}^{l_i} J_{\lambda_{ij}}(a_i)$$

with a_i's being distinct elements of the field $\mathcal{O}_1$. By using $s(0) = 0$ and $s(1) = 1$, we obtain $\mathfrak{s}(A_i)$, for all i is an upper triangular matrix with all diagonal entries equal to $s(a_i)$ and $\mathfrak{s}(A) = \bigoplus_{i=1}^{l}(\mathfrak{s}(A_i))$. Further for all $i \neq j$, $a_i \neq a_j$ implies that $s(a_i) - s(a_j)$ are invertible elements of the ring $\mathcal{O}_2$. Therefore, by Lemma 4.5,

$$Z_{\mathrm{GL}_n(\mathcal{O}_2)}\big(\mathfrak{s}(A)\big) = \bigoplus_{i=1}^{l} Z_{\mathrm{GL}_{n_i}(\mathcal{O}_2)}\big(\mathfrak{s}(A_i)\big). \tag{5.2}$$

Because we also have

$$Z_{\mathrm{GL}_n(\mathcal{O}_1)}(A) = \bigoplus_{i=1}^{l} Z_{\mathrm{GL}_{n_i}(\mathcal{O}_1)}(A_i),$$

it is sufficient to prove $Z_{\mathrm{GL}_n(\mathcal{O}_1)}(A) \subseteq \kappa(Z_{\mathrm{GL}_n(\mathcal{O}_2)}(\mathfrak{s}(A)))$ when A is a split primary.

Split primary case: Now we assume that A is split primary and is in its Jordan canonical form. Theorems 3.2 and 3.5 give that $A = aI_n + (\bigoplus_{i=1}^{t} N_{n_i})$ for some $a \in \mathcal{O}_1$. Let $\alpha \in Z_{\mathrm{GL}_n(\mathcal{O}_1)}(A)$. By Lemma 4.11, α is an invertible block Toeplitz matrix of order $(n_1, n_2, \ldots, n_t)$ over the ring $\mathcal{O}_1$. Our choice of section ensures that $\mathfrak{s}(A) = s(a)I_n + (\bigoplus_{i=1}^{t} N_{n_i})$, and $\mathfrak{s}(\alpha)$ is an invertible block Toeplitz matrix of order $(n_1, n_2, \ldots, n_t)$ over the ring $\mathcal{O}_2$. However, then by Lemma 4.9, $\mathfrak{s}(\alpha) \in Z_{\mathrm{GL}_n(\mathcal{O}_2)}(\mathfrak{s}(A))$. Hence $\alpha = \kappa(\mathfrak{s}(\alpha)) \in \kappa(Z_{\mathrm{GL}_n(\mathcal{O}_2)}(\mathfrak{s}(A)))$. □

From the proof of the above lemma we obtain:

Corollary 5.2. *If A is a split matrix and is in its Jordan canonical form, then $\alpha \in Z_{\mathrm{GL}_n(\mathcal{O}_1)}(A)$ if and only if $\mathfrak{s}(\alpha) \in Z_{\mathrm{GL}_n(\mathcal{O}_2)}(\mathfrak{s}(A))$.*

Corollary 5.3. *If A is a split matrix and is in its Jordan canonical form, then $T(\psi_A) = K Z_{\mathrm{GL}_n(\mathcal{O}_2)}(\mathfrak{s}(A))$.*

Proof. The inclusion $T(\psi_A) \subseteq K Z_{\mathrm{GL}_n(\mathcal{O}_2)}(\mathfrak{s}(A))$ follows from (5.1) and Lemma 5.1. □

Lemma 5.4. *Let G be a finite group with two subgroups N and M, such that N is normal in G and $G = N.M$. If ψ_1 and ψ_2 are one dimensional representations of N and M, respectively, such that $\psi_1(mnm^{-1}) = \psi_1(n)$ for all $m \in M$, $n \in N$ and $\psi_1|_{N\cap M} = \psi_2|_{N\cap M}$, then $\psi_1.\psi_2$ defined by $\psi_1.\psi_2(n.m) := \psi_1(n)\psi_2(m)$ is the unique one dimensional representation of G extending both ψ_1 and ψ_2*

Proof. We simply prove that $\psi_1.\psi_2$ is well defined; the rest of the proof is straightforward. Suppose $nm = n'm'$, where $n, n' \in N$ and $m, m' \in M$. Then $n'^{-1}n = m'm^{-1} \in M \cap N$:

$$\begin{aligned}
\psi_1.\psi_2(nm) &= \psi_1(n)\psi_2(m) \\
&= \psi_1\big(n'n'^{-1}n\big).\psi_2\big(mm'^{-1}m'\big) \\
&= \psi_1\big(n'\big).\psi_1\big(n'^{-1}nmm'^{-1}\big).\psi_2\big(m'\big) \\
&= \psi_1\big(n'\big).\psi_2\big(m'\big) \\
&= \psi_1\psi_2\big(n'm'\big). \qquad \square
\end{aligned}$$

Proof of Proposition 2.2. It follows from (2.1) that orbits of the action of $\mathrm{GL}_n(\mathcal{O}_2)$ on K are the same as orbits of $\mathrm{M}_n(\mathcal{O}_1)$ under the action of $\mathrm{GL}_n(\mathcal{O}_1)$, namely the similarity classes. It is easy to see that, if we can extend the character ψ_A from K to $T(\psi_A)$, then we can extend any $\psi_{A'}$ in the orbit of ψ_A under the action of $\mathrm{GL}_n(\mathcal{O}_2)$ on $T(\psi_A)$. Therefore to prove the proposition, it is enough to choose a representative A of each similarity class of $\mathrm{M}_n(\mathcal{O}_1)$ and to extend the corresponding character ψ_A from K to $T(\psi_A)$. We prove existence of this extension in three steps:

Step 1 *(A is split primary).* Let A be a split primary matrix with unique eigenvalue $a \in \mathcal{O}_1$. Replace A with a matrix in its similarity class of the form $\bigoplus_{i=1}^{l} J_{\lambda_i}(a)$, where each $J_{\lambda_i}(a)$ is an elementary Jordan block.

We define a character $\psi_a : \mathcal{O}_1 \to \mathbb{C}^*$ by $\psi_a(x) = \psi(ax)$. The map $x \mapsto 1 + \pi x$ gives an isomorphism from $\mathcal{O}_1$ onto the subgroup $1 + \pi\mathcal{O}_1$ of the multiplicative group $\mathcal{O}_2^*$. Choose $\chi \in \hat{\mathcal{O}}_2^*$ such that $\chi(1 + \pi x) = \psi_a(x)$ for all $x \in \mathcal{O}_1$. Define a character $\tilde{\chi} : Z_{\mathrm{GL}_n(\mathcal{O}_2)}(\mathfrak{s}(A)) \to \mathbb{C}^*$ by $\tilde{\chi}(x) = \chi(\det(x))$.

Lemma 5.5. *The character $\tilde{\chi}$ of $Z_{\mathrm{GL}_n(\mathcal{O}_2)}(\mathfrak{s}(A))$ satisfies*

$$\tilde{\chi}|_{K\cap Z_{\mathrm{GL}_n(\mathcal{O}_2)}(\mathfrak{s}(A))} = \psi_A|_{K\cap Z_{\mathrm{GL}_n(\mathcal{O}_2)}(\mathfrak{s}(A))}.$$

Proof. By Lemma 4.11, $K \cap Z_{\mathrm{GL}_n(\mathcal{O}_2)}(\mathfrak{s}(A)) = I + \pi Z_{\mathrm{M}_n(\mathcal{O}_1)}(A)$. If $X = (x_{ij}) \in Z_{\mathrm{M}_n(\mathcal{O}_1)}(A)$, then by Lemma 4.9, X is a block upper Toeplitz matrix. Therefore, $\mathrm{Tr}(AX) = a(x_{11} + x_{22} + \cdots + x_{nn})$. We have

$$\begin{aligned}\psi_A(I+\pi X) &= \psi\big(\mathrm{Tr}(AX)\big)\\ &= \psi\big(a(x_{11}+x_{22}+\cdots+x_{nn})\big)\\ &= \chi\big(\det(I+\pi X)\big) = \tilde{\chi}(I+\pi X). \qquad \square\end{aligned}$$

Applying Lemma 5.4 to the group $T(\psi_A)$ with its subgroups K and $Z_{\mathrm{GL}_n(\mathcal{O}_2)}(\mathfrak{s}(A))$, and characters $\psi_1 = \psi_A$ and $\psi_2 = \tilde{\chi}$, we obtain that the character $\psi_A.\tilde{\chi}$ is an extension of ψ_A from K to $T(\psi_A)$.

Step 2 *(A is split).* Let A be a split matrix with distinct eigenvalues $a_1, a_2, \ldots, a_l$. Then by Theorem 3.2, A can be written as $\bigoplus_{i=1}^{l} A_i$, where each A_i is a split primary matrix, say of order n_i, and has a unique eigenvalue a_i. We may assume each A_i in its Jordan canonical form. Then by (5.2),

$$K \cap Z_{\mathrm{GL}_n(\mathcal{O}_2)}\big(\mathfrak{s}(A)\big) = \bigoplus_{i=1}^{l}\big(K \cap Z_{\mathrm{GL}_{n_i}(\mathcal{O}_2)}\big(\mathfrak{s}(A_i)\big)\big).$$

As in Step 1, define the characters $\tilde{\chi}_i$ of $Z_{\mathrm{GL}_{n_i}(\mathcal{O}_2)}(\mathfrak{s}(A_i))$ such that

$$\tilde{\chi}_i|_{K\cap Z_{\mathrm{GL}_{n_i}(\mathcal{O}_2)}(\mathfrak{s}(A_i))} = \psi_{A_i}|_{K\cap Z_{\mathrm{GL}_{n_i}(\mathcal{O}_2)}(\mathfrak{s}(A_i))}.$$

Then the character $\tilde{\chi} = \tilde{\chi}_1 \times \tilde{\chi}_2 \times \cdots \times \tilde{\chi}_l$ is a character of $Z_{\mathrm{GL}_n(\mathcal{O}_2)}(\mathfrak{s}(A))$, such that

$$\tilde{\chi}|_{K\cap Z_{\mathrm{GL}_n(\mathcal{O}_2)}(\mathfrak{s}(A))} = \psi_A|_{K\cap Z_{\mathrm{GL}_n(\mathcal{O}_2)}(\mathfrak{s}(A))}.$$

Again by Lemma 5.4, $\psi_A.\tilde{\chi}$ is an extension of ψ_A from K to $T(\psi_A)$.

Step 3 *(General case).* Let $\tilde{\mathcal{O}}_1$ be a splitting field for the characteristic polynomial of A, and let $\tilde{\mathcal{O}}_2$ be the corresponding unramified extension of $\mathcal{O}_2$. Let $\tilde{K} = \mathrm{Ker}(\mathrm{GL}_n(\tilde{\mathcal{O}}_2) \to \mathrm{GL}_n(\tilde{\mathcal{O}}_1))$ under the natural quotient map, and let $\tilde{\psi} : \tilde{\mathcal{O}}_1 \to \mathbb{C}^*$ be a character such that $\tilde{\psi}|_{\mathcal{O}_1} = \psi$. Then $\tilde{\psi}_A : \tilde{K} \to \mathbb{C}^*$, defined by

$$\tilde{\psi}_A(I+\pi X) = \tilde{\psi}\big(\mathrm{Tr}(AX)\big)$$

is a character of $\tilde{K}$. Let $\tilde{T}(\psi_A)$ be the stabilizer of $\tilde{\psi}_A$ in $\mathrm{GL}_n(\tilde{\mathcal{O}}_2)$. Because A splits over $\tilde{\mathcal{O}}_1$, by Step 2, there exists a character $\tilde{\chi} : \tilde{T}(\psi_A) \to \mathbb{C}^*$, such that

$$\tilde{\chi}|_{\tilde{K}} = \tilde{\psi}_A.$$

Define a character $\chi : T(\psi_A) \to \mathbb{C}^*$ by $\chi = \tilde{\chi}|_{T(\psi_A)}$. Then χ is an extension of ψ_A to $T(\psi_A)$. This completes the proof of Proposition 2.2. $\square$

Fix an extension χ_A of ψ_A from K to $T(\psi_A)$, and let $\mathcal{S}$ denote the set of similarity classes of $\mathrm{M}_n(\mathcal{O}_1)$. By (5.1), the groups $T(\psi_A)/K$ and $Z_{\mathrm{GL}_n(\mathcal{O}_1)}(A)$ are isomorphic. Therefore, by Clifford theory, there exists a bijection between the sets

$$\coprod_{A\in\mathcal{S}}\bigl\{\mathrm{Irr}\bigl(Z_{\mathrm{GL}_n(\mathcal{O}_1)}(A)\bigr)\bigr\}\longleftrightarrow \mathrm{Irr}\bigl(\mathrm{GL}_n(\mathcal{O}_2)\bigr), \tag{5.3}$$

given by,

$$\phi\mapsto Ind_{T(\psi_A)}^{\mathrm{GL}_n(\mathcal{O}_2)}(\chi_A\otimes\phi). \tag{5.4}$$

As $[\mathrm{GL}_n(\mathcal{O}_2):T(\psi_A)]=[\mathrm{GL}_n(\mathcal{O}_1):Z_{\mathrm{GL}_n(\mathcal{O}_1)}(A)]$, this already proves that there exists a dimension preserving bijection between the sets $\mathrm{Irr}(\mathrm{GL}_n(\mathcal{O}_2))$ and $\mathrm{Irr}(\mathrm{GL}_n(\mathcal{O}'_2))$. To prove that this bijection is canonical we need to do little more work.

Let $\mathcal{O}'$ be an another ring of integers of local non-Archimedean field F', such that residue fields of both $\mathcal{O}$ and $\mathcal{O}'$ are isomorphic. We fix an isomorphism ϕ between their residue fields. From now onwards we shall assume that section $s:\mathcal{O}_1\to\mathcal{O}_2$ satisfies $s(0)=0$ and $s|_{\mathcal{O}_1^*}$ is multiplicative. The existence and uniqueness of this section is proved in, for example, Serre [25, Proposition 8]. In the sequel, this unique section will be called *the multiplicative section* of $\mathcal{O}_1$ (or of $\mathcal{O}_1^*$) (depending on the domain). Given above isomorphism ϕ:

Lemma 5.6. *There exists a canonical isomorphism between groups $\hat{\mathcal{O}}_2^*$ and $\hat{\mathcal{O}}_2'^*$.*

Proof. Let $s:\mathcal{O}_1^*\to\mathcal{O}_2^*$ and $s':\mathcal{O}_1'^*\to\mathcal{O}_2'^*$ be the multiplicative sections of $\mathcal{O}_1^*$ and $\mathcal{O}_1'^*$, respectively. Then the following exact sequences split,

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{O}_1 \arrow[r, "i"'] & \mathcal{O}_2^* \arrow[r] & \mathcal{O}_1^* \arrow[r] \arrow[l, dotted, bend left, "s"] & 1 \\ 0 \arrow[r] & \mathcal{O}'_1 \arrow[r, "i'"'] & \mathcal{O}_2'^* \arrow[r] & \mathcal{O}_1'^* \arrow[r] \arrow[l, dotted, bend right, "s'"'] & 1. \end{tikzcd}$$

The uniqueness of the sections s and s' implies the existence of a unique isomorphism $f:\mathcal{O}_2^*\to\mathcal{O}_2'^*$ such that $f\circ s=s'\circ\phi$ and $f\circ i=i'\circ\phi$. This gives a canonical isomorphism between $\hat{\mathcal{O}}_2^*$ and $\hat{\mathcal{O}}_2'^*$.

Let $K'=\mathrm{Ker}(\mathrm{GL}_n(\mathcal{O}'_2)\to\mathrm{GL}_n(\mathcal{O}'_1))$. Then

$$K\cong K'\cong\mathrm{M}_n(\mathcal{O}_1),$$

hence the set $\{\psi_A\mid A\in\mathrm{M}_n(\mathcal{O}_1)\}$ can also be thought of as the set of characters of K'. Let $T'(\psi_A)$ denote the stabilizer of character ψ_A in $\mathrm{GL}_n(\mathcal{O}'_2)$. By (5.1), groups $T'(\psi_A)/K'$ and $T(\psi_A)/K$ are canonically isomorphic. Further, to prove that there exists a canonical bijection between $\mathrm{Irr}(\mathrm{GL}_n(\mathcal{O}_2))$ and $\mathrm{Irr}(\mathrm{GL}_n(\mathcal{O}'_2))$, it is sufficient to prove that for a given $A\in\mathrm{M}_n(\mathcal{O}_1)$, and an extension $\chi_A:T(\psi_A)\to\mathbf{C}^*$ of ψ_A, there exists a canonical extension $\chi'_A:T'(\psi_A)\to\mathbf{C}^*$ of ψ_A from K' to $T'(\psi_A)$. For that, in Steps 1 and 2 of the proof of Proposition 2.2, choose the character of $\mathcal{O}_2'^*$ by using the given character of $\mathcal{O}_2^*$ and canonical isomorphism between $\hat{\mathcal{O}}_2^*$ and $\hat{\mathcal{O}}_2'^*$; the rest of the argument follows easily from this. For Step 3, any isomorphism between $\tilde{\mathcal{O}}_1$ and $\tilde{\mathcal{O}}'_1$ that extends $\phi:\mathcal{O}_1\to\mathcal{O}'_1$ is defined uniquely up to an element of the Galois group. To complete the proof for this step, we observe that, if $\tilde{\psi}$ is an extension of ψ from $\mathcal{O}_1$ to $\tilde{\mathcal{O}}_1$ and γ is an element of Galois group $\mathrm{Gal}(\tilde{\mathcal{O}}_1/\mathcal{O}_1)$ then by definition $(\tilde{\psi}\circ\gamma)_A=\tilde{\psi}_A\circ\gamma$ where on the right side element γ is thought as scalar matrix with all its diagonal entries equal to γ. Choose $\tilde{\chi}_A$ an extension of $\tilde{\psi}_A$ from $\tilde{K}$ to $\tilde{T}(\psi_A)$ and let $\tilde{\gamma}$ be a lift of γ from $\mathrm{Gal}(\tilde{\mathcal{O}}_1/\mathcal{O}_1)$ to $\mathrm{Gal}(\tilde{F}/F)$, which takes the maximal ideal of $\tilde{\mathcal{O}}$ to itself (for existence of this see [7, p. 26]), then $\tilde{\chi} \circ \tilde{\gamma}$ (again $\tilde{\gamma}$ is thought as scalar matrix with diagonal entries equal to $\tilde{\gamma}$) extends $\tilde{\psi}_A \circ \gamma$. The restrictions of $\tilde{\chi}_A$ and $\tilde{\chi}_A \circ \tilde{\gamma}$ to $T(\psi_A)$ coincide. This completes the proof of Theorem 1.2. □

Corollary 5.7. *The isomorphism type of group algebra* $\mathbf{C}[\mathrm{GL}_n(\mathcal{O}_2)]$ *depends only on the cardinality of the residue field of* $\mathcal{O}$.

This can be restated as:

Corollary 5.8. *The number and dimensions of irreducible representations of groups* $\mathrm{GL}_n(\mathcal{O}_2)$ *depends only on the residue field.*

The equivalence between number of conjugacy classes and irreducible representations further gives:

Corollary 5.9. *The number of conjugacy classes of groups* $\mathrm{GL}_n(\mathcal{O}_2)$ *depends only on the cardinality of the residue fields.*

Remark 5.10. In a forthcoming paper, we will sharpen this result by showing that the class equation of $\mathrm{GL}_n(\mathcal{O}_2)$ depends only on the cardinality of the residue field.

6. Complexity of the problem

In this section we comment on the complexity of the problem of constructing all the irreducible representations of $\mathrm{GL}_n(\mathcal{O}_2)$.

In this context, Aubert, Onn, Prasad, and Stasinski have proved the following (see [1, Theorem 6.1]):

Theorem 6.1. *Let* $F = \mathbf{F}_q((t))$ *be a local function field. Then the problems of constructing all the irreducible representations of the following are equivalent*:

(1) $G_{2^n,F}$ *for all* $n \in \mathbf{N}$.
(2) $G_{k^n,F}$ *for all* $k, n \in \mathbf{N}$.
(3) $G_{\lambda,E}$ *for all partitions* λ *and all unramified extensions* E *of* F.

The above, combined with Theorem 1.2, proves the following:

Theorem 6.2. *Let* $\mathcal{O}$ *be ring of integers of a non-Archimedean local field* F, *such that the residue field has cardinality* q. *Then the problems of constructing irreducible representations of the following groups are equivalent*:

(1) $\mathrm{GL}_n(\mathcal{O}_2)$ *for all* $n \in \mathbf{N}$.
(2) $G_{\lambda,E}$ *for all partitions* λ *and all unramified extensions* E *of* $\mathbf{F}_q((t))$.

7. Applications

In this section, we discuss a few applications of the theory developed so far. In particular, we discuss the relation between the representation zeta polynomial of $\mathrm{GL}_n(\mathcal{O}_2)$ and those of centralizers in $\mathrm{GL}_n(\mathcal{O}_1)$. We also construct all the irreducible representations of groups $\mathrm{GL}_2(\mathcal{O}_2)$, $\mathrm{GL}_3(\mathcal{O}_2)$, $\mathrm{GL}_4(\mathcal{O}_2)$ and obtain their representation zeta polynomials.

Recall the following definition from Section 1:

Definition 7.1 *(Representation zeta polynomial).* Let G be a finite group. The representation zeta polynomial of G is the polynomial:

$$R_G(\mathcal{D}) = \sum_{\rho \in \operatorname{Irr} G} \mathcal{D}^{\dim \rho} \in \mathbb{Z}[\mathcal{D}].$$

7.1. Representation zeta polynomial of $\mathrm{GL}_n(\mathcal{O}_2)$

Let $\mathcal{S}$ be the set of similarity classes of $\mathrm{M}_n(\mathcal{O}_1)$. From (5.3), it is clear that representations of centralizers play an important role in determining irreducible representations of $\mathrm{GL}_n(\mathcal{O}_2)$. Moreover, we obtain the following relation between their representation zeta polynomials,

$$R_{\mathrm{GL}_n(\mathcal{O}_2)}(\mathcal{D}) = \sum_{A \in \mathcal{S}} R_{Z_{\mathrm{M}_n(\mathcal{O}_1)}(A)}\big(\mathcal{D}^{[\mathrm{GL}_n(\mathcal{O}_1):Z_{\mathrm{GL}_n(\mathcal{O}_1)}(A)]}\big), \tag{7.1}$$

where $[\mathrm{GL}_n(\mathcal{O}_1) : Z_{\mathrm{GL}_n(\mathcal{O}_1)}(A)] = [\mathrm{GL}_n(\mathcal{O}_2) : T(\psi_A)]$ is the index of $Z_{\mathrm{GL}_n(\mathcal{O}_1)}(A)$ in $\mathrm{GL}_n(\mathcal{O}_1)$. Following Green [9], a similarity class c of $\mathrm{M}_n(\mathcal{O}_1)$ can be denoted by the symbol

$$c = \big(\ldots, f^{\nu_c(f)}, \ldots\big),$$

where f is an irreducible polynomial appearing in the characteristic polynomial of c, and $\nu_c(f)$ is the partition associated with f in the canonical form of c.

Let $c = (\ldots, f^{\nu_c(f)}, \ldots)$. Let d be a positive integer, and let ν be a partition other than zero. Let $r_c(d,\nu)$ be the number of f appearing in the characteristic polynomial of c with degree d and $\nu_c(f) = \nu$. Let $\rho_c(\nu)$ be the partition

$$\big\{n^{r_c(n,\nu)}, (n-1)^{r_c(n-1,\nu)}, \ldots\big\}.$$

Then two classes b and c are of the same *type* if and only if $\rho_b(\nu) = \rho_c(\nu)$ for each non-zero partition ν. By abusing notation, we shall also say that matrices of class c and d have the same type.

Let ρ_ν be a partition-valued function on the non-zero partitions ν (ρ_ν may take value zero). The condition for ρ_ν to describe a type of $\mathrm{M}_n(\mathcal{O}_1)$ is

$$\sum_\nu |\rho_\nu||\nu| = n.$$

The total number $t(n)$ of functions ρ_ν satisfying the above expression is independent of q, and so is the number of types of $\mathrm{M}_n(\mathcal{O}_1)$ (for large enough q). The following lemma (which is easy) underlines the importance of types in the calculation of the representation zeta polynomials of the groups $\mathrm{GL}_n(\mathcal{O}_2)$:

Lemma 7.2. *If matrices A and B in* $\mathrm{M}_n(\mathcal{O}_1)$ *are of the same* type, *then their centralizers are isomorphic.*

Let $\mathcal{T}$ denote the set of representatives of types of $\mathrm{M}_n(\mathcal{O}_1)$, and for each $A \in \mathcal{T}$, let n_A be the total number of similarity classes of type A. The expression (7.1) simplifies to

$$R_{\mathrm{GL}_n(\mathcal{O}_2)}(\mathcal{D}) = \sum_{A \in \mathcal{T}} n_A R_{Z_{\mathrm{M}_n(\mathcal{O}_1)}(A)}\big(D^{[\mathrm{GL}_n(\mathcal{O}_1):Z_{\mathrm{GL}_n(\mathcal{O}_1)}(A)]}\big). \tag{7.2}$$

Table 1
Group $\mathrm{GL}_2(\mathcal{O}_2)$.

Type A	Number of similarity classes of given type (n_A)	Isomorphism type of centralizer $Z_{\mathrm{GL}_2(\mathcal{O}_1)}(A)$	Index $[\mathrm{GL}_2(\mathcal{O}_1) : Z_{\mathrm{GL}_2(\mathcal{O}_1)}(A)]$
$\begin{pmatrix} \rho^a & 0 \\ 0 & \rho^a \end{pmatrix}$	q	$R_{\mathrm{GL}_2(\mathcal{O}_1)}(\mathcal{D})$	1
$\begin{pmatrix} \rho^a & 0 \\ 0 & \rho^b \end{pmatrix}$	$\frac{1}{2}q(q-1)$	$(q-1)^2\mathcal{D}$	$q(q+1)$
$\begin{pmatrix} \rho^a & 1 \\ 0 & \rho^a \end{pmatrix}$	q	$q(q-1)\mathcal{D}$	q^2-1
$\begin{pmatrix} \sigma^a & 0 \\ 0 & \sigma^{aq} \end{pmatrix}$	$\frac{1}{2}q(q-1)$	$(q^2-1)\mathcal{D}$	q^2-q

The elements ρ and σ are primitive elements of $\mathbf{F}_q$ and $\mathbf{F}_{q^2}$ respectively, such that $\rho = \sigma^{q+1}$.

Summarizing the discussion so far, to determine the irreducible representations of groups $\mathrm{GL}_n(\mathcal{O}_2)$, it is sufficient to determine the representations of the centralizers $Z_{\mathrm{GL}_n(\mathcal{O}_1)}(A)$, where A varies over the set of types of $\mathrm{M}_n(\mathcal{O}_1)$. However, determining representations of groups $Z_{\mathrm{GL}_n(\mathcal{O}_1)}(A)$ for general n is still an open problem. We discuss representations of these groups for $n=2$, $n=3$, and $n=4$.

We shall use the following theorems. For proofs of these results, see, for example, [29, Chapter 1].

Theorem 7.3. *Let $A \in \mathrm{M}_n(\mathbf{F}_q)$. Then the centralizer of A in $\mathrm{M}_n(\mathbf{F}_q)$ is the algebra $\mathbf{F}_q[A]$ if and only if the minimal polynomial of A coincides with its characteristic polynomial. Moreover, in this case, $\dim_{\mathbf{F}_q} Z_{\mathrm{M}_n(\mathbf{F}_q)}(A) = n$.*

Theorem 7.4. *Let $A \in \mathrm{M}_n(\mathbf{F}_q)$. Then its centralizer in $\mathrm{M}_n(\mathbf{F}_q)$ is a field if and only if its characteristic polynomial is irreducible over $\mathbf{F}_q$.*

7.2. *Representations of* $\mathrm{GL}_2(\mathcal{O}_2)$

The irreducible representations of groups $\mathrm{GL}_2(\mathcal{O}_2)$ have already been described by Nagornyĭ [18] and Onn [22]. Because they fall out of our discussion very easily and are used in representation theory of groups $\mathrm{GL}_4(\mathcal{O}_2)$, we add a brief description of them as well. For representation theory of groups $\mathrm{GL}_n(\mathcal{O}_1)$, we refer to Green [9] and Steinberg [28]. In Table 1, we describe types of $\mathrm{M}_2(\mathcal{O}_1)$ (set of 2×2 matrices over $\mathcal{O}_1$) with their centralizers. To determine centralizers, wherever required, we have used Theorems 7.3 and 7.4.

The representation zeta polynomial of the group $\mathrm{GL}_2(\mathcal{O}_1)$ (see Steinberg [28]) is

$$R_{\mathrm{GL}_2(\mathcal{O}_1)}(\mathcal{D}) = (q-1)\mathcal{D} + (q-1)\mathcal{D}^q + \frac{1}{2}(q-1)(q-2)\mathcal{D}^{q+1} + \frac{1}{2}q(q-1)\mathcal{D}^{q-1}.$$

Feeding all these data into 7.2, we easily obtain the representation zeta polynomial of $\mathrm{GL}_2(\mathcal{O}_2)$:

$$\begin{aligned} R_{\mathrm{GL}_2(\mathcal{O}_2)}(\mathcal{D}) = {} & qR_{\mathrm{GL}_2(\mathcal{O}_1)}(\mathcal{D}) + \frac{1}{2}q(q-1)^3\mathcal{D}^{q(q+1)} + q^2(q-1)\mathcal{D}^{q^2-1} \\ & + \frac{1}{2}q(q+1)(q-1)^2\mathcal{D}^{q^2-q}. \end{aligned} \tag{7.3}$$

7.3. *Representations of* $\mathrm{GL}_3(\mathcal{O}_2)$

Partial results regarding representations of groups $\mathrm{GL}_3(\mathcal{O}_2)$ have already been given by Nagornyĭ [19]. We complete his results for these groups.

Table 2
Group $\mathrm{GL}_3(\mathcal{O}_2)$.

Type A	Number of similarity classes of given type (n_A)	Isomorphism type of centralizer $Z_{\mathrm{GL}_3(\mathcal{O}_1)}(A)$	Index $[\mathrm{GL}_3(\mathcal{O}_1) : Z_{\mathrm{GL}_3(\mathcal{O}_1)}(A)]$
$\begin{pmatrix} \rho^a & 0 & 0 \\ 0 & \rho^a & 0 \\ 0 & 0 & \rho^a \end{pmatrix}$	q	$\mathcal{R}_{\mathrm{GL}_3(\mathcal{O}_1)}(\mathcal{D})$	1
$\begin{pmatrix} \rho^a & 0 & 0 \\ 0 & \rho^a & 0 \\ 0 & 0 & \rho^b \end{pmatrix}$	$q(q-1)$	$(q-1)\mathcal{R}_{\mathrm{GL}_2(\mathcal{O}_1)}(\mathcal{D})$	$q^2(q^2+q+1)$
$\begin{pmatrix} \rho^a & 0 & 0 \\ 0 & \rho^b & 0 \\ 0 & 0 & \rho^c \end{pmatrix}$	$\frac{1}{6}q(q-1)(q-2)$	$(q-1)^3\mathcal{D}$	$q^3(q+1)(q^2+q+1)$
$\begin{pmatrix} \rho^a & 1 & 0 \\ 0 & \rho^a & 0 \\ 0 & 0 & \rho^b \end{pmatrix}$	$q(q-1)$	$q(q-1)^2\mathcal{D}$	$q^2(q^3-1)(q+1)$
$\begin{pmatrix} \rho^a & 1 & 0 \\ 0 & \rho^a & 0 \\ 0 & 0 & \rho^a \end{pmatrix}$	q	$\mathcal{R}_{G_{(2,1)}}(\mathcal{D})$	$(q^3-1)(q+1)$
$\begin{pmatrix} \rho^a & 1 & 0 \\ 0 & \rho^a & 1 \\ 0 & 0 & \rho^a \end{pmatrix}$	q	$q^2(q-1)\mathcal{D}$	$q(q^3-1)(q^2-1)$
$\begin{pmatrix} \rho^a & 0 & 0 \\ 0 & \sigma^a & 0 \\ 0 & 0 & \sigma^{aq} \end{pmatrix}$	$\frac{1}{2}q^2(q-1)$	$(q-1)(q^2-1)\mathcal{D}$	$q^3(q^3-1)$
$\begin{pmatrix} \tau & 0 & 0 \\ 0 & \tau^{bq} & 0 \\ 0 & 0 & \tau^{bq^2} \end{pmatrix}$	$\frac{1}{3}q(q^2-1)$	$(q^3-1)\mathcal{D}$	$q^3(q-1)^2(q+1)$

The elements ρ, σ, and τ are primitive elements of $\mathbf{F}_q$, $\mathbf{F}_{q^2}$, and $\mathbf{F}_{q^3}$, respectively, such that $\rho = \sigma^{q+1} = \tau^{q^2+q+1}$.

In Table 2, we describe types and their corresponding centralizers for the group $\mathrm{M}_3(\mathcal{O}_1)$. The representation zeta polynomial of $\mathrm{GL}_3(\mathcal{O}_1)$ (Steinberg [28]) is

$$\begin{aligned}\mathcal{R}_{\mathrm{GL}_3(\mathcal{O}_1)}(\mathcal{D}) = {} & (q-1)\mathcal{D} + (q-1)\mathcal{D}^{q^2+q} + (q-1)\mathcal{D}^{q^3} \\ & + (q-1)(q-2)\mathcal{D}^{q^2+q+1} + (q-1)(q-2)\mathcal{D}^{q(q^2+q+1)} \\ & + \frac{1}{6}(q-1)(q-2)(q-3)\mathcal{D}^{(q+1)(q^2+q+1)} \\ & + \frac{1}{2}q(q-1)^2\mathcal{D}^{(q-1)(q^2+q+1)} \\ & + \frac{1}{3}q(q-1)(q+1)\mathcal{D}^{(q+1)(q-1)^2}. \end{aligned} \tag{7.4}$$

The irreducible representations of all the centralizers appearing in Table 2 except $G_{2,1}$ are either very easy or well known. Onn [22, Theorem 4.1] has described all the irreducible representations of groups $G_{(\ell,1)}$ for $\ell > 1$. As a consequence, we have

Lemma 7.5. *The representation zeta polynomial of the group $G_{(2,1)}$ is*

$$R_{G_{(2,1)}}(\mathcal{D}) = (q-1)^2\mathcal{D} + \left(q^2-1\right)\mathcal{D}^{q-1} + (q-1)^3\mathcal{D}^q.$$

Collecting all the pieces together, we obtain the expression for the representation zeta polynomial of $\mathrm{GL}_3(\mathcal{O}_2)$:

$$\begin{aligned}\mathcal{R}_{\mathrm{GL}_3(\mathcal{O}_2)}(\mathcal{D}) = {} & q\mathcal{R}_{\mathrm{GL}_3(\mathcal{O}_1)}(\mathcal{D}) + q(q-1)^2\mathcal{R}_{\mathrm{GL}_2(\mathcal{O}_1)}\big(\mathcal{D}^{q^2(q^2+q+1)}\big)\\ & + \frac{1}{6}q(q-2)(q-1)^4\mathcal{D}^{q^3(q+1)(q^2+q+1)}\\ & + q^2(q-1)^3\mathcal{D}^{q^2(q^3-1)(q+1)} + q\mathcal{R}_{\mathrm{G}_{(2,1)}}\big(\mathcal{D}^{(q^3-1)(q+1)}\big)\\ & + q^3(q-1)\mathcal{D}^{q(q^3-1)(q^2-1)}\\ & + \frac{1}{2}q^2(q-1)^2\big(q^2-1\big)\mathcal{D}^{q^3(q^3-1)}\\ & + \frac{1}{3}q\big(q^2-1\big)\big(q^3-1\big)\mathcal{D}^{q^3(q-1)^2(q+1)}.\end{aligned} \tag{7.5}$$

7.4. *Representations of* $\mathrm{GL}_4(\mathcal{O}_2)$

In this section we discuss representation theory of groups $\mathrm{GL}_4(\mathcal{O}_2)$.

In Table 3, we give all the data required for the representations of $\mathrm{GL}_4(\mathcal{O}_2)$. The expression for the representation zeta polynomial of $\mathrm{GL}_4(\mathcal{O}_1)$ is rather long, so we omit the details here (see Steinberg [28]). Among the other centralizers appearing in this table, only the results regarding the representations of group $G_{(2,1,1)}$ are not clear from our discussion so far. We follow a method of Uri Onn to discuss the representations of these groups.

For the proof of next Proposition (which follows from the theory of finite Heisenberg groups) we refer to Bushnell and Fröhlich [5, Proposition 8.3.3].

Proposition 7.6. *Let $1 \to N \to G \to^{\phi} G/N \to 1$ be an extension, where $V = G/N$ is an elementary finite abelian p-group so also viewed as a finite dimensional vector space over $\mathbf{F}_p$. Further, let $\chi : N \mapsto \mathbb{C}^*$ be a non-trivial character such that G stabilizes χ. Assume furthermore that $h_\chi(g_1N, g_2N) = \langle g_1N, g_2N\rangle_\chi = \chi([g_1, g_2])$ is an alternating non-degenerate bilinear form on V. Then there exists a unique irreducible representation ρ_χ of G such that $\rho_\chi|_N$ is χ-isotypic. Moreover,* $\dim(\rho_\chi)^2 = [G : N]$.

Lemma 7.7. *The representation zeta polynomial of the group $G_{(2,1,1)}$ is*

$$\mathcal{R}_{\mathrm{G}_{(2,1,1)}}(\mathcal{D}) = (q-1)^2\mathcal{R}_{\mathrm{GL}_2(\mathcal{O}_1)}\big(\mathcal{D}^{q^2}\big) + (q-1)\mathcal{R}_{(\mathcal{O}_1^2\times\mathcal{O}_1^2)\rtimes\mathrm{G}_{(1,1)}}(\mathcal{D}),$$

where

$$\begin{aligned}\mathcal{R}_{(\mathcal{O}_1^2\times\mathcal{O}_1^2)\rtimes\mathrm{G}_{(1,1)}}(\mathcal{D}) = {} & \mathcal{R}_{(1,1)}(\mathcal{D}) + 2(q-1)\mathcal{D}^{q^2-1} + (q-1)^2\mathcal{D}\big(q^2-1\big)q\\ & + (q+2)\mathcal{D}^{(q^2-1)(q-1)}.\end{aligned} \tag{7.6}$$

Proof. Using the notation in the proof of Lemma 4.11, let H be the kernel of map $\mathrm{G}_{(2,1,1)} \to \mathrm{G}_1 \times \mathrm{G}_{(1,1)}$,

$$g \mapsto \left(g_{11}(\mathrm{mod}\,\wp), \begin{pmatrix} g_{22} & g_{23}\\ g_{32} & g_{33}\end{pmatrix}\right).$$

Then

$$H = I + \begin{bmatrix} \wp & \wp & \wp\\ \mathcal{O}_1 & & \\ \mathcal{O}_1 & & \end{bmatrix}.$$

Table 3
Group $\mathrm{GL}_4(\mathcal{O}_2)$.

Type A	Number of similarity classes of given type (n_A)	Isomorphism type of centralizer $Z_{\mathrm{GL}_4(\mathcal{O}_1)}(A)$	Index $[\mathrm{GL}_4(\mathcal{O}_1) : Z_{\mathrm{GL}_4(\mathcal{O}_1)}(A)]$
$\begin{pmatrix} \rho^a & 0 & 0 & 0 \\ 0 & \rho^a & 0 & 0 \\ 0 & 0 & \rho^a & 0 \\ 0 & 0 & 0 & \rho^a \end{pmatrix}$	q	$\mathrm{GL}_4(\mathcal{O}_1)$	1
$\begin{pmatrix} \rho^a & 0 & 0 & 0 \\ 1 & \rho^a & 0 & 0 \\ 0 & 0 & \rho^a & 0 \\ 0 & 0 & 0 & \rho^a \end{pmatrix}$	q	$G_{(2,1,1)}$	$(q^2+1)(q^3-1)(q+1)$
$\begin{pmatrix} \rho^a & 0 & 0 & 0 \\ 1 & \rho^a & 0 & 0 \\ 0 & 0 & \rho^a & 0 \\ 0 & 0 & 1 & \rho^a \end{pmatrix}$	q	$G_{(2,2)}$	$q(q^4-1)(q^3-1)$
$\begin{pmatrix} \rho^a & 0 & 0 & 0 \\ 1 & \rho^a & 0 & 0 \\ 0 & 1 & \rho^a & 0 \\ 0 & 0 & 0 & \rho^a \end{pmatrix}$	q	$G_{(3,1)}$	$q^2(q^4-1)(q^3-1)(q+1)$
$\begin{pmatrix} \rho^a & 0 & 0 & 0 \\ 1 & \rho^a & 0 & 0 \\ 0 & 1 & \rho^a & 0 \\ 0 & 0 & 1 & \rho^a \end{pmatrix}$	q	$\mathcal{O}_4^*$	$q^3(q^4-1)(q^3-1)(q^2-1)$
$\begin{pmatrix} \rho^a & 0 & 0 & 0 \\ 0 & \rho^a & 0 & 0 \\ 0 & 0 & \rho^a & 0 \\ 0 & 0 & 0 & \rho^b \end{pmatrix}$	$q(q-1)$	$\mathrm{GL}_3(\mathcal{O}_1) \times \mathcal{O}_1^*$	$q^3(q+1)(q^2+1)$
$\begin{pmatrix} \rho^a & 0 & 0 & 0 \\ 1 & \rho^a & 0 & 0 \\ 0 & 0 & \rho^a & 0 \\ 0 & 0 & 0 & \rho^b \end{pmatrix}$	$q(q-1)$	$G_{(2,1)} \times \mathcal{O}_1^*$	$q^3(q^2+1)(q+1)^2(q^3-1)$
$\begin{pmatrix} \rho^a & 0 & 0 & 0 \\ 1 & \rho^a & 0 & 0 \\ 0 & 1 & \rho^a & 0 \\ 0 & 0 & 0 & \rho^b \end{pmatrix}$	$q(q-1)$	$\mathcal{O}_3^* \times \mathcal{O}_1^*$	$q^4(q^4-1)(q^3-1)(q+1)$
$\begin{pmatrix} \rho^a & 0 & 0 & 0 \\ 0 & \rho^a & 0 & 0 \\ 0 & 0 & \rho^b & 0 \\ 0 & 0 & 0 & \rho^b \end{pmatrix}$	$\frac{1}{2}q(q-1)$	$\mathrm{GL}_2(\mathcal{O}_1) \times \mathrm{GL}_2(\mathcal{O}_1)$	$q^4(q^2+1)(q^2+q+1)$
$\begin{pmatrix} \rho^a & 0 & 0 & 0 \\ 1 & \rho^a & 0 & 0 \\ 0 & 0 & \rho^b & 0 \\ 0 & 0 & 0 & \rho^b \end{pmatrix}$	$q(q-1)$	$\mathcal{O}_2^* \times \mathrm{GL}_2(\mathcal{O}_1)$	$q^4(q^2+q+1)(q^4-1)$
$\begin{pmatrix} \rho^a & 0 & 0 & 0 \\ 1 & \rho^a & 0 & 0 \\ 0 & 0 & \rho^b & 0 \\ 0 & 0 & 1 & \rho^b \end{pmatrix}$	$\frac{1}{2}q(q-1)$	$\mathcal{O}_2^* \times \mathcal{O}_2^*$	$q^4(q+1)(q^4-1)(q^3-1)$
$\begin{pmatrix} \rho^a & 0 & 0 & 0 \\ 0 & \rho^a & 0 & 0 \\ 0 & 0 & \rho^b & 0 \\ 0 & 0 & 0 & \rho^c \end{pmatrix}$	$\frac{1}{2}q(q-1)(q-2)$	$\mathrm{GL}_2(\mathcal{O}_1) \times \mathcal{O}_1^* \times \mathcal{O}_1^*$	$q^5(q+1)(q^2+1)(q^2+q+1)$

Table 3 (*continued*)

Type A	Number of similarity classes of given type (n_A)	Isomorphism type of centralizer $Z_{\mathrm{GL}_4(\mathcal{O}_1)}(A)$	Index $[\mathrm{GL}_4(\mathcal{O}_1) : Z_{\mathrm{GL}_4(\mathcal{O}_1)}(A)]$
$\begin{pmatrix} \rho^a & 0 & 0 & 0 \\ 1 & \rho^a & 0 & 0 \\ 0 & 0 & \rho^b & 0 \\ 0 & 0 & 0 & \rho^c \end{pmatrix}$	$\frac{1}{2}q(q-1)(q-2)$	$\mathcal{O}_2^* \times \mathcal{O}_1^* \times \mathcal{O}_1^*$	$q^5(q^2+1)(q+1)^2(q^3-1)$
$\begin{pmatrix} \rho^a & 0 & 0 & 0 \\ 0 & \rho^b & 0 & 0 \\ 0 & 0 & \rho^c & 0 \\ 0 & 0 & 0 & \rho^d \end{pmatrix}$	$\frac{1}{24}q(q-1)(q-2)(q-3)$	$\mathcal{O}_1^* \times \mathcal{O}_1^* \times \mathcal{O}_1^* \times \mathcal{O}_1^*$	$q^6(q^3+q^2+q+1)(q+1) \times (q^2+q+1)$
$\begin{pmatrix} \rho^a & 0 & 0 & 0 \\ 0 & \rho^a & 0 & 0 \\ 0 & 0 & \sigma^b & 0 \\ 0 & 0 & 0 & \sigma^{bq} \end{pmatrix}$	$\frac{1}{2}q^2(q-1)$	$\mathrm{GL}_2(\mathcal{O}_1) \times \mathbf{F}_{q^2}^*$	$q^5(q^2+1)(q^3-1)$
$\begin{pmatrix} \rho^a & 0 & 0 & 0 \\ 1 & \rho^a & 0 & 0 \\ 0 & 0 & \sigma^b & 0 \\ 0 & 0 & 0 & \sigma^{bq} \end{pmatrix}$	$\frac{1}{2}q^2(q-1)$	$\mathcal{O}_2^* \times \mathbf{F}_{q^2}^*$	$q^5(q^3-1)(q^4-1)$
$\begin{pmatrix} \rho^a & 0 & 0 & 0 \\ 0 & \rho^b & 0 & 0 \\ 0 & 0 & \sigma^c & 0 \\ 0 & 0 & 0 & \sigma^{cq} \end{pmatrix}$	$\frac{1}{4}q^2(q-1)^2$	$\mathcal{O}_1^* \times \mathcal{O}_1^* \times \mathbf{F}_{q^2}^*$	$q^6(q+1)(q^2+1)(q^3-1)$
$\begin{pmatrix} \sigma^a & 0 & 0 & 0 \\ 0 & \sigma^{aq} & 0 & 0 \\ 0 & 0 & \sigma^a & 0 \\ 0 & 0 & 0 & \sigma^{aq} \end{pmatrix}$	$\frac{1}{2}q(q-1)$	$\mathrm{GL}_2(\mathbf{F}_{q^2})$	$q^4(q-1)(q^3-1)$
$\begin{pmatrix} \sigma^a & 0 & 0 & 0 \\ 0 & \sigma^{aq} & 0 & 0 \\ 1 & 0 & \sigma^a & 0 \\ 0 & 1 & 0 & \sigma^{aq} \end{pmatrix}$	$\frac{1}{2}q(q-1)$	$\mathbf{F}_{q^2} \times \mathbf{F}_{q^2}^*$	$q^4(q^4-1)(q^3-1)(q-1)$
$\begin{pmatrix} \sigma^a & 0 & 0 & 0 \\ 0 & \sigma^{aq} & 0 & 0 \\ 0 & 0 & \sigma^b & 0 \\ 0 & 0 & 0 & \sigma^{bq} \end{pmatrix}$	$\frac{1}{8}q(q-1)(q^2-q-2)$	$\mathbf{F}_{q^2}^* \times \mathbf{F}_{q^2}^*$	$q^6(q^2+1)(q^3-1)(q-1)$
$\begin{pmatrix} \rho^a & 0 & 0 & 0 \\ 0 & \tau^b & 0 & 0 \\ 0 & 0 & \tau^{bq} & 0 \\ 0 & 0 & 0 & \tau^{bq^2} \end{pmatrix}$	$\frac{1}{3}q^2(q^2-1)$	$\mathcal{O}_1^* \times \mathbf{F}_{q^3}^*$	$q^6(q^4-1)(q^2-1)$
$\begin{pmatrix} \omega^a & 0 & 0 & 0 \\ 0 & \omega^a q & 0 & 0 \\ 0 & 0 & \omega^{aq^2} & 0 \\ 0 & 0 & 0 & \omega^{aq^3} \end{pmatrix}$	$\frac{1}{4}q(q^3-q)$	$\mathbf{F}_{q^4}^*$	$q^6(q-1)(q^2-1)(q^3-1)$

The elements ρ, σ, τ, and ω are primitive elements of $\mathbf{F}_q$, $\mathbf{F}_{q^2}$, $\mathbf{F}_{q^3}$, and $\mathbf{F}_{q^4}$, respectively, such that $\rho = \sigma^{q+1} = \tau^{q^2+q+1} = \omega^{q^3+q^2+q+1}$ and $\sigma = \omega^{q^2+1}$.

The center of H, i.e., $Z(H) \cong \mathcal{O}_1$. Firstly we claim that H has $q-1$ irreducible representations of dimension q^2 that lie above the non-trivial characters of $Z(H)$. We identify $Z(H)$ with its dual by $z \mapsto \psi_z(.) = \psi(\mathrm{Tr}(z.))$. H stabilizes these characters of $Z(H)$, and furthermore, each of the non-trivial characters gives rise to an alternating non-degenerate bilinear form $\langle h_1 Z(H), h_2 Z(H)\rangle_{\psi_z} = \psi(\mathrm{Tr}(z[h_1, h_2]))$ on $H/Z(H)$. Proposition 7.6 gives $q-1$ pairwise inequivalent irreducible representations of dimension $|H/Z(H)|^{1/2} = q^2$. This proves the claim. Furthermore, the group $\mathrm{G}_{2,1,1}$ stabilizes each of these representations of H. Let $\rho_\chi \in \hat{H}$ be such a representation lying over a non-trivial character $\chi \in \widehat{Z(H)}$.

We claim that the representation ρ_χ can be extended to $\mathrm{G}_{(2,1,1)}$. Let $H^i = Z(H) \times \mathcal{O}_1 \times \mathcal{O}_1$ be the pre-image in H of the maximal isotropic subgroup $\mathcal{O}_1 \times \mathcal{O}_1$ for the above bilinear form. Let χ^i be any extension of χ to H^i. Indeed, the subgroup $G_2 \times G_{(1,1)}$ stabilizes both H^i and χ^i. Let $\tilde{\chi}$ be an extension of χ to $G_2 \times G_{(1,1)}$. Then by Lemma 5.4, $\chi^i.\tilde{\chi}(a.b) = \chi^i(a)\tilde{\chi}(b)$ for all $a \in H^i$, and $b \in G_2 \times G_{(1,1)}$ is a well-defined linear character of $H^i.(G_2 \times G_{(1,1)})$. By the proof of Proposition 7.6, ρ_χ does not depend on the choice of isotropy group and the extension χ^i. Therefore, for the induced representation

$$\rho_{\chi^i} = \mathrm{ind}_{H^i(G_2 \times G_{(1,1)})}^{H.G_2 \times G_{(1,1)}}(\chi^i),$$

$\rho_\chi \leqslant \rho_{\chi^i}$, and as $\dim \rho_\chi = \dim \rho_{\chi^i} = q^2$, we conclude that ρ_{χ^i} is an extension of ρ_χ to $\mathrm{G}_{(2,1,1)}$. By the Clifford theory, it follows that all the representations of $\mathrm{G}_{(2,1,1)}$ that lie above ρ_χ are of the form $\{\rho_{\chi^i}.\phi \mid \phi \in \mathrm{G}_{(2,1,1)}/H\}$. Hence the contribution to the representation zeta polynomial of $\mathrm{G}_{(2,1,1)}$ from these representations is $(q-1)\mathcal{R}_{\mathcal{O}_1^* \times \mathrm{GL}_2(\mathcal{O}_1)}(\mathcal{D}^{q^2})$. The remaining representations correspond to representations of H whose central character is trivial, that is, representations pulled back from $((\mathcal{O}_1^2 \times \mathcal{O}_1^2) \rtimes G_{(1,1)}) \times \mathcal{O}_1^*$. The action of $G_{(1,1)}$ on $\mathcal{O}_1^2 \times \mathcal{O}_1^2$ is given by

$$\begin{pmatrix} 1 & \\ & D \end{pmatrix} \begin{pmatrix} 1 & \pi v \\ w & I \end{pmatrix} \begin{pmatrix} 1 & \\ & D^{-1} \end{pmatrix} = \begin{pmatrix} 1 & \pi v D^{-1} \\ Dw & I \end{pmatrix}.$$

After a choice of identification of $\mathcal{O}_1^2 \times \mathcal{O}_1^2$ with its dual: $\langle (\hat{v}, \hat{w}), (v, w) \rangle = \psi(v\hat{v} + w\hat{w})$, we get

$$g^{-1}(\hat{v}, \hat{w}) = \big(D^{-1}\hat{v}, \hat{w}D\big), \quad \text{where } g = \begin{pmatrix} 1 & \\ & D \end{pmatrix},$$

and the orbits and stabilizers of this action are given by

	Orbits	Stabilizers
(1)	$\left[\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \end{pmatrix}\right]$	$\mathrm{G}_{(1,1)}$
(2)	$\left[\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \mathcal{O}_1^2 \setminus 0 & 0 \end{pmatrix}\right]$	$\mathcal{O}_1 \rtimes \mathcal{O}_1^*$
(3)	$\left[\mathcal{O}_1^2 \setminus \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \end{pmatrix}\right]$	$\mathcal{O}_1 \rtimes \mathcal{O}_1^*$
(4)	$\left[\mathcal{O}_1^2 \setminus \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 & \mathcal{O}_1^* \end{pmatrix}\right]$	$\mathcal{O}_1$
(5)	$\left[\mathcal{O}_1^2 \setminus \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} u^* & \mathcal{O}_1 \end{pmatrix}\right]$, $u^* \in \mathcal{O}_1^*$	$\mathcal{O}_1^*$

Collecting all the pieces, we get the desired result. □

From the above discussion, one easily obtains the representation zeta polynomial for the group $\mathrm{GL}_4(\mathcal{O}_2)$.

Remark 7.8. The representation zeta polynomial $R_G(\mathcal{D})$ for $\mathcal{D} = 1$ gives the number of conjugacy classes of G. From above, we can easily obtain the number of conjugacy classes of groups $\mathrm{GL}_2(\mathcal{O}_2)$, $\mathrm{GL}_3(\mathcal{O}_2)$, and $\mathrm{GL}_4(\mathcal{O}_2)$. The number of conjugacy classes of $\mathrm{GL}_2(\mathcal{O}_2)$ and $\mathrm{GL}_3(\mathcal{O}_2)$ is already known; see Avni, Onn, Prasad, and Vaserstein [3].

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