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The Gelfand–Graev Representation of U(3, q)

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In this paper we explicitly calculate the irreducible representations of the endomorphism algebra of the Gelfand–Graev representation of the unitary group U(3, q). In addition, we compute the structure constants of this endomorhphism algebra. © 1997 Academic Press

INTRODUCTION

Let GL(3, $\overline{\mathbf{F}}_q$) denote the general linear group of invertible 3 by 3 matrices over $\overline{\mathbf{F}}_q$. Let $F: \mathrm{GL}(3, \overline{\mathbf{F}}_q) \to \mathrm{GL}(3, \overline{\mathbf{F}}_q)$ denote the homomorphism defined by $F(a_{ij}) = ((a_{ij}^q)^t)^{-1}$. Then F is a Frobenius map and the group of fixed points of F is the unitary group U(3, q). In this paper we will examine the structure and representations of the Hecke algebra H of the Gelfand–Graev representation of U(3, q). In this case, the Gelfand–Graev representation is the induced representation of any nontrivial linear representation of the maximal unipotent subgroup U of U(3, q) to U(3, q). The center of GL(3, $\overline{\mathbf{F}}_q$) is connected, thus there is only one Gelfand–Graev representation Γ of U (3, q) [3, p. 519], and Γ is independent of the choice of the nontrivial linear representation of U.

After a discussion of some preliminary results in Section 1, Section 2 contains the calculations that explicitly give the irreducible representations of H. D. Surowski [8] has previously calculated some of these irreducible representations in the case of SU(3, q) instead of U(3, q). The techniques used in Section 2 are different than Surowski's (most notably the use of Curtis's theorem to be discussed below), and this section provides a

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complete list of the irreducible representations. The structure of the Hecke algebra is further examined in Section 3 and the structure constants of the algebra are explicitly calculated.

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1. PRELIMINARY RESULTS AND BACKGROUND INFORMATION

1.1. *Gelfand–Graev Representations*

Let $\tilde{G}$ be a connected reductive algebraic group defined over a finite field $\mathbf{F}_q$. Given a Frobenius endomorphism F, let $G = \tilde{G}^F$, the fixed points of F. The Gelfand–Graev characters of G are constructed in the following way (see, for example [5, Chap. 7; 3, pp. 518–519]). Given an F-stable Borel subgroup $\tilde{B}$ of $\tilde{G}$ and an F-stable maximal torus $\tilde{T}_0$ contained in $\tilde{B}$, we have the root system Φ of $\tilde{G}$ with respect to $\tilde{T}_0$. Let Π denote the set of simple roots in Φ corresponding to $\tilde{B}$. Let $\tilde{U}$ be the unipotent radical of $\tilde{B}$ and let $\hat{U}$ be the subgroup of $\tilde{U}$ generated by the root subgroups corresponding to the nonsimple roots of Φ. Also let $\{U_\alpha\}_{\alpha \in \Pi}$ be the set of simple root subgroups. Then for each F-orbit i on Π let

$$U_i = \prod_{\alpha \in i} U_\alpha .$$

Let ψ be a linear character of $U = \tilde{U}^F$ which is trivial on $\hat{U}^F$. Then ψ is called nondegenerate if ψ is nontrivial when it is restricted to the F-fixed points of any of the U_i. Given a nondegenerate character ψ of U, let $\Gamma = \mathrm{Ind}_U^G(\psi)$. Then Γ is called a Gelfand–Graev character of G.

1.2. *Properties of Gelfand–Graev Representations*

The Gelfand–Graev characters Γ are multiplicity free [7, Theorem 49]. Thus, in principle, it might be feasible to decompose the Gelfand–Graev representation into irreducible representations. In the case that $Z(\tilde{G})$ is connected there is only one distinct Gelfand–Graev representation [3, p. 519]. Thus, in this case the choice of nondegenerate linear representation of U will not affect the results. Let $\tilde{T}_0$ denote a maximally split F-stable maximal torus of $\tilde{G}$. A maximal torus of G is defined to be a subgroup of the form $\tilde{T}^F$ where $\tilde{T}$ is an F-stable maximal torus of $\tilde{G}$. Now the G-conjugacy classes of F-stable maximal tori of $\tilde{G}$ are parametrized by the F-conjugacy classes of $N_{\tilde{G}}(\tilde{T}_0)/\tilde{T}_0$ where the F-conjugate of x by g is defined to be $gxF(g)^{-1}$ (see, for example, [1, Propositions 3.3.2 and 3.3.3]). Also, given an F-conjugacy class $[x]$, T_x is conjugate (in $\tilde{G}$) to a maximal torus of G where

$$T_x = \left\{ A \in \tilde{T}_0 \mid xAx^{-1} = F(A) \right\}.$$

Conversely, all the maximal tori of G are conjugate (in $\tilde{G}$) to T_x for some x.

Given an F-stable maximal torus $\tilde{T}$ of $\tilde{G}$ let $R^G_{T,\theta}$ denote the Deligne–Lusztig generalized character, where θ is an irreducible character of the torus $T = \tilde{T}^F$ (see, for example, [1, Chap. 7]). Let Q^G_T denote the Green function, which is defined for all unipotent elements $u \in G$ by $Q^G_T(u) = R^G_{T,\theta}(u)$ (see, for example, [1, p. 212]).

Given a pair $(\tilde{T}, \theta)$ there exists a unique irreducible character $\chi_{T,\theta}$ of G such that $\langle \chi_{T,\theta}, \Gamma \rangle \neq 0$ and $\langle \chi_{T,\theta}, R^G_{T,\theta} \rangle \neq 0$. Also any irreducible character χ of G such that $\langle \chi, \Gamma \rangle \neq 0$ coincides with a $\chi_{T,\theta}$ for some pair $(\tilde{T}, \theta)$ (see, for example, [3, Theorem 2.1]). The pairs $(\tilde{T}, \theta)$ are partitioned into geometric conjugacy classes (see, for example, [1, Sect. 4.1]). In fact, each Gelfand–Graev character of G is equal to

$$\sum_{\substack{(\tilde{T}, \theta) \in \kappa \\ \text{mod } G}} \frac{\epsilon_{\tilde{G}} \epsilon_{\tilde{T}} R^G_{T,\theta}}{\left(R^G_{T,\theta}, R^G_{T,\theta} \right)}$$

for some geometric conjugacy class κ of pairs $(\tilde{T}, \theta)$ where $\epsilon_{\tilde{G}} = (-1)^{\text{rel. rank } \tilde{G}}$ (see, for example, [1, Proposition 8.4.7]). In particular when χ is cuspidal, $\langle \chi, \Gamma \rangle = \langle \chi, R^G_{T,\theta} \rangle = 1$.

1.3. *The Hecke Algebra*

Given $G = \tilde{G}^F$, the maximal unipotent subgroup U of G and a nondegenerate linear character ψ of U, the Hecke algebra H is constructed in the following way. (This can be done more generally, see [4, Sect. 11].) Let e denote the central primitive idempotent in $\mathbf{C}U$ corresponding to ψ. That is,

$$e = |U|^{-1} \sum_{u \in U} \psi(u^{-1})u.$$

Then $H = e\mathbf{C}Ge$.

Let N be a set of double coset representatives of U, U in G. Let $\operatorname{ind}(n) = |UnU|/|U| = |U : {}^nU \cap U|$ for $n \in N$. Let $J = \{n \in N \mid {}^n\psi = \psi$ on ${}^nU \cap U\}$. Let $c_n = \operatorname{ind}(n)ene$. Then $\{c_n\}_{n \in J}$ is a basis for H called the standard basis of H [4, Proposition 11.30]. There is a bijection from the set of irreducible characters χ of G such that $\langle \chi, \Gamma \rangle \neq 0$ to the set of all irreducible characters of H. This bijection is given by restriction from $\mathbf{C}G$ to H [3, Proposition 2.2]. Also the primitive central idempotents of H are $\{e\epsilon\}$ where ϵ is a primitive central idempotent of $\mathbf{C}G$ associated with a χ such that $\langle \chi, \Gamma \rangle \neq 0$. Since Γ is multiplicity free, the Hecke algebra H is commutative [3, Proposition 2.2]. Thus these idempotents are actually primitive (non-central) idempotents. Thus they give us the simple module $\mathbf{C}Ge\epsilon$ which affords χ [4, Corollary 11.27].

1.4. *Curtis's Theorem*

As mentioned above, the set of irreducible characters of H is in bijection with the set of irreducible characters χ of G such that $\langle \chi, \Gamma \rangle \neq 0$. Namely, given a pair $(\tilde{T}, \theta)$ there exists a unique irreducible character $\chi_{T,\theta}$ of G such that $\langle \chi_{T,\theta}, \Gamma \rangle \neq 0$ and $\langle \chi_{T,\theta}, R^G_{T,\theta} \rangle \neq 0$. The restriction of $\chi_{T,\theta}$ to H is the corresponding irreducible character $f_{T,\theta}$ of H. Thus we can index the characters of H by the pairs $(\tilde{T}, \theta)$. In addition, $f_{T,\theta} = f_{T',\theta'}$ if and only if the pairs $(\tilde{T}, \theta)$ and $(\tilde{T}', \theta')$ are geometrically conjugate [4, Theorem 3.1]. So each irreducible representation $f_{T,\theta}$ of H corresponds to the unique irreducible character $\chi_{T,\theta}$ of G which occurs as a common constituent of both $R^G_{T,\theta}$ and Γ.

Curtis's theorem is as follows [3, Theorem 4.2]:

THEOREM 1.1. *Let the pair $(\tilde{T}, \theta)$, the Gelfand–Graev representation $\Gamma = \operatorname{Ind}_U^G(\psi)$, and the Hecke algebra H be given. Let $\bar{\theta}$ denote the extension of θ to $\mathbf{C}T$. Also let x_s and x_u denote the semisimple and unipotent parts of $x \in G$. Then*:

(i) *There exists a unique homomorphism $f_T : H \to \mathbf{C}T$, independent of θ, which has the property that each character $f_{T,\theta} : H \to \mathbf{C}$ can be factored as $f_{T,\theta} = \bar{\theta} \cdot f_T$.*

(ii) *$f_T(c_n) = \sum_{t \in T} f_T(c_n)(t)t$ where c_n is an element in the standard basis of H described above and the coefficients $f_T(c_n)(t)$ are given by*

$$f_T(c_n)(t) = \frac{\operatorname{ind}(n)}{\langle Q_T^G, \Gamma \rangle |U|\,|C_G(t)|} \sum_{\substack{g \in G,\, u \in U \\ (gung^{-1})_s = t}} \psi(u^{-1}) Q_T^{C_G(t)}\big((gung^{-1})_u\big). \tag{1.2}$$

Note that Curtis's theorem says that we have the following commutative diagram:

$$\begin{array}{ccc} H & \xrightarrow{f_T} & \mathbf{C}T \\ & \searrow^{f_{T,\theta}} & \downarrow{\scriptstyle \bar{\theta}} \\ & & C \end{array}$$

Thus if we first find the homomorphisms f_T for each maximal torus T of G and then compose these with the irreducible characters of T we will get all the irreducible characters of H.

2. THE GELFAND–GRAEV REPRESENTATION OF G

2.1. *Notation*

As described in the Introduction let $F : \mathrm{GL}(3, \overline{\mathbf{F}}_q) \to \mathrm{GL}(3, \overline{\mathbf{F}}_q)$ denote the twisted Frobenius map, defined by $F(a_{ij}) = ((a_{ij}^q)^t)^{-1}$. From now on $\mathrm{GL}(3, \overline{\mathbf{F}}_q)$ will be denoted by $\tilde{G}$. Instead of using the fixed point group $\tilde{G}^F$ it will be convenient to take the unitary group G given by conjugating $\tilde{G}^F$ by

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Now let $\tilde{U}$ denote the unipotent subgroup of $\tilde{G}$ which consists of upper unitriangular matrices. That is,

$$\tilde{U} = \left\{ \begin{pmatrix} 1 & t & u \\ 0 & 1 & v \\ 0 & 0 & 1 \end{pmatrix} \middle| \, t, u, v \in \overline{\mathbf{F}}_q \right\}.$$

Let U be the subgroup of G given by conjugating $\tilde{U}^F$ by

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Then

$$U = \left\{ \left| \begin{pmatrix} 1 & t & u \\ 0 & 1 & t^q \\ 0 & 0 & 1 \end{pmatrix} \right| \, t, u \in \mathbf{F}_{q^2} \text{ and } t^{q+1} = u + u^q \right\}.$$

Let $[U, U]$ denote the commutator subgroup of U. Since $U/[U, U]$ is isomorphic to the additive group $\mathbf{F}_{q^2}$, there is a correspondence between the irreducible linear characters ψ of U and the additive characters χ of $\mathbf{F}_{q^2}$. This correspondence is given by

$$\psi\begin{pmatrix} 1 & t & u \\ 0 & 1 & t^q \\ 0 & 0 & 1 \end{pmatrix} = \chi(t).$$

Choose any nontrivial irreducible linear character ψ of U. Let $\Gamma = \operatorname{Ind}_U^G(\psi)$. Then Γ is independent of the choice of ψ and Γ is the Gelfand–Graev character of G (see [3; 5, Chap. 14]). Let e denote the idempotent

$$e = \frac{1}{|U|} \sum_{u \in U} \psi(u^{-1})u.$$

Then let H be the Hecke algebra $e(\mathbf{C}G)e$.

From now on, the diagonal matrix

$$\begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}$$

will be denoted by $[a, b, c]$. In addition, the element

$$\begin{pmatrix} 0 & 0 & a \\ 0 & b & 0 \\ c & 0 & 0 \end{pmatrix}$$

will be denoted by $[\widehat{a, b, c}]$. Also an element

$$\begin{pmatrix} 1 & t & u \\ 0 & 1 & t^q \\ 0 & 0 & 1 \end{pmatrix} \in U$$

will be denoted by $[t, u]$.

The following is a summary of the above notation:

$$\tilde{G} = \mathrm{GL}\left(3, \overline{\mathbf{F}}_q\right)$$

$$G = \mathrm{U}(3, q) = \left[1, \widehat{-1}, 1\right] \tilde{G}^F \left[1, \widehat{-1}, 1\right]$$

$$U = \left\{ [t, u] \mid u, t \in \mathbf{F}_{q^2} \text{ and } t^{q+1} = u + u^q \right\}$$

$$\psi = \text{nontrivial linear character of } U$$

$$\chi = \text{additive character of } \mathbf{F}_{q^2} \text{ such that } \psi([t, u]) = \chi(t)$$

$$\Gamma = \operatorname{Ind}_U^G(\psi)$$

$$e = (1/|U|) \sum_{u \in U} \psi(u^{-1})u$$

$$H = e(\mathbf{C}G)e.$$

We recall Curtis's theorem, Theorem 1.1. The main result of this section will be the computations of the unique homomorphisms $f_{T_i}: H \to \mathbf{C}T_i$, for each maximal torus T_i of G.

2.2. *The Maximal Tori of G*

Let $\tilde{T}$ denote the maximally split F-stable maximal torus of $\tilde{G}$:

$$\tilde{T} = \left\{[t, u, v] \mid t, u, v \in \overline{\mathbf{F}}_q^*\right\}.$$

As discussed in the Introduction, the G-conjugacy classes of F-stable maximal tori of $\tilde{G}$ are parametrized by the F-conjugacy classes of $N_{\tilde{G}}(\tilde{T})/\tilde{T}$ (see, for example, [5, Proposition 3.23]). Also $N_{\tilde{G}}(\tilde{T})/\tilde{T} = W(\tilde{T})$, the Weyl group, which in this case is isomorphic to S_3, the symmetric group on three elements. Let, e.g., (12) denote the element of S_3 corresponding to the permutation matrix

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

which will also be denoted by (12), and similarly for the other permutation matrices.

Now since $S_3 = \langle (12), (23) \rangle$, the twisted Frobenius map F can be thought of as the automorphism of S_3 that takes (12) to (23) and takes (23) to (12). Let $[x]$ denote the F-conjugacy class of x. Then

$$[e] = \{e, (123), (132)\}$$

$$[(12)] = \{(12), (23)\}$$

$$[(13)] = \{(13)\}.$$

Thus there are three classes of maximal tori in G. Each of these maximal tori is conjugate (in $\tilde{G}$) to T_x for some F-conjugacy class $[x]$ where

$$T_x = \left\{A \in \tilde{T} \mid xAx^{-1} = F(A)\right\}.$$

Thus the three maximal tori are

$$T_e = \{[t, u, v] \mid [t, u, v] = [v^{-q}, u^{-q}, t^{-q}] \text{ and } t, u, v \neq 0\},$$
$$T_{(12)} = \{[t, u, v] \mid [u, t, v] = [v^{-q}, u^{-q}, t^{-q}] \text{ and } t, u, v \neq 0\},$$

and

$$T_{(13)} = \{[t, u, v] \mid [v, u, t] = [v^{-q}, u^{-q}, t^{-q}] \text{ and } t, u, v \neq 0\}.$$

From now on denote T_e by T_0, $T_{(12)}$ by T_1, and $T_{(13)}$ by T_2.
The above conditions on T_0, T_1, and T_2 simplify to

$$T_0 = \left\{[t, u, t^{-q}] \mid t^{q^2-1} = 1, u^{q+1} = 1\right\}$$
$$T_1 = \left\{[t, t^2, t^{-q}] \mid t^{q^3+1} = 1\right\}$$
$$T_2 = \left\{[t, u, v] \mid t^{q+1} = 1, u^{q+1} = 1, \text{ and } v^{q+1} = 1\right\}.$$

Notice that T_1 is not a subgroup of G but there exists a subgroup $\hat{T}_1$ of G such that $\hat{T}_1$ is conjugate to T_1 in $\tilde{G}$. Since T_1 is diagonal, it will be convenient to work with the group T_1 instead of $\hat{T}_1$. Notice that $|T_0| = (q+1)(q^2-1)$, $|T_1| = q^3 + 1$ and $|T_2| = (q+1)^3$. Also notice that T_0 is the maximally split torus in G.

2.3. *The Hecke Algebra*

Let $s = [1, \widehat{-1}, 1]$. Then $N_G(T_0) = \{A \mid A \in T_0 \text{ or } A \in T_0 s\}$, where $N_G(T_0)$ denotes the normalizer of T_0 in G. We have the Bruhat decomposition

$$G = \bigcup_{w=1,s} UT_0wU.$$

But then a basis of the Hecke algebra is given by

$$\Big\{c_j = \operatorname{ind}(x_j)ex_je \mid x_j \in T_0 \cup T_0s \text{ and}$$
$$\psi(y) = \psi\left(x_jyx_j^{-1}\right) \text{ for all } y \in U \cap x_jUx_j^{-1}\Big\}$$

(see, for example, [4, Proposition 11.30]). Here $\operatorname{ind}(x) = |U : {}^xU \cap U| =$ the number of left cosets of U in the double coset UxU. The following are calculations that determine this basis explicitly.

Fix a $y \in U$, i.e., $y = [v, w]$ for some $v, w \in \mathbf{F}_{q^2}$ and $v^{q+1} = w + w^q$. First let $x \in T_0s$. That is, $x = [t, \widehat{-u}, t^{-q}]$ for some t, u such that $t^{q^2-1} = 1$

and $u^{q+1} = 1$. Then

$$xyx^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ -ut^{-1}v^q & 1 & 0 \\ wt^{-q-1} & -u^{-1}vt^{-q} & 1 \end{pmatrix}.$$

Thus $xyx^{-1} \in U$ if and only if $ut^{-1}v^q = 0$, $wt^{-q-1} = 0$, and $u^{-1}vt^{-q} = 0$. But $t, u \neq 0$ so that $v = 0$ and $w = 0$. Thus ${}^xU \cap U = I$ when $x \in T_0 s$. Thus $\operatorname{ind}(x) = |U| = q^3$ when $x \in T_0 s$.

Now let $x \in T_0$. So $x = [t, u, t^{-q}]$ and $xUx^{-1} = U$. But note that the condition that $\psi(y) = \psi(xyx^{-1})$ for all $y \in U \cap xUx^{-1}$ implies that

$$\psi([v, w]) = \psi([tu^{-1}v, t^{q+1}w]).$$

But this holds if and only if $tu^{-1}v = v$ for all $v \in \mathbf{F}_{q^2}$. This holds if and only if $t = u$ which causes $t^{-q} = u^{-q} = u$ since $u^{q+1} = 1$. Thus we must have

$$x \in \{[t, t, t] \mid t^{q+1} = 1\}.$$

But then $xyx^{-1} = y$. Also for x of this type ${}^xU \cap U = U$ and $\operatorname{ind}(x) = 1$.

Thus if

$$c_{u,t} = \operatorname{ind}\left(\left[t, \widehat{-u}, t^{-q}\right]\right)e\left[t, \widehat{-u}, t^{-q}\right]e = q^3 e\left[t, \widehat{-u}, t^{-q}\right]e$$

and

$$c_u = \operatorname{ind}([u, u, u])e[u, u, u]e = [u, u, u]e,$$

then the set

$$\left\{c_{u,t}, c_u \mid t, u \in \mathbf{F}_{q^2}^* \text{ and } u^{q+1} = 1\right\}$$

is a basis for the Hecke algebra H. Note that the number of elements in this basis is $(q+1) + (q^2-1)(q+1) = q^2(q+1)$.

2.4. *Calculations of $C_G(t)$, $t \in T_i$, and $\langle Q_{T_i}^G, \Gamma\rangle$*

In order to apply Curtis's theorem we need to know $C_G(t)$ for t in a maximal torus T_i of G. Also we will need the values of $\langle Q_{T_i}^G, \Gamma\rangle$ for each of the maximal tori.

For all three tori if $t \in T_i$ and the entries on the main diagonal are all distinct then $C_G(t) = T_i$. Also, if the entries on the main diagonal are all the same, then clearly $C_G(t) = G$.

Fix a $t \in T_0$ so that $t = [t_1, t_2, t_1^{-q}]$ for some fixed t_1, t_2 such that $t_1^{q^2-1} = 1$ and $t_2^{q+1} = 1$. Suppose that $t_1 = t_1^{-q}$ and $t_1 \neq t_2$ (so $t_1^{-q} \neq t_2$). Then

$$C_G(t) = \left\{\begin{pmatrix} a & 0 & c \\ 0 & e & 0 \\ g & 0 & i \end{pmatrix}\right\}.$$

Let

$$H_2 = \left\{ \begin{pmatrix} a & 0 & c \\ 0 & 1 & 0 \\ g & 0 & i \end{pmatrix} \right\}.$$

Then $H_2 \cong U(2, q)$ under the isomorphism

$$\beta : \left\{ \begin{pmatrix} a & 0 & c \\ 0 & 1 & 0 \\ g & 0 & i \end{pmatrix} \right\} \to \left\{ \begin{pmatrix} a & c \\ g & i \end{pmatrix} \right\}.$$

Also let

$$S_2 = \{[1, w, 1] \mid w^{q+1} = 1\}.$$

Then $S_2 \cong S$ where S is the subgroup of $\mathbf{F}_{q^2}^*$ consisting of elements x such that $x^{q+1} = 1$. Thus, in this case, $C_G(t) = H_2 \times S_2 \cong U(2, q) \times S$. So $|C_G(t)| = |U(2, q)|\,|S| = q(q+1)(q^2-1)(q+1) = q(q+1)^3(q-1)$.

Note that the case $t_1 = t_2$ and $t_1 \neq t_1^{-q}$ (so $t_1^{-q} \neq t_2$) cannot occur. Also the case $t_2 = t_1^{-q}$ and $t_1 \neq t_2$ (so $t_1^{-q} \neq t_1$) cannot occur.

Now let $t \in T_1$ so that $t = [t_1, t_1^{q^2}, t_1^{-q}]$ for some fixed t_1 such that $t_1^{q^3+1} = 1$. Note that it is not possible that exactly two of the entries on the main diagonal of t are the same.

Now fix a $t \in T_2$ so that $t = [t_1, t_2, t_3]$ for some fixed t_1, t_2, t_3 such that $t_i^{q+1} = 1$. Suppose that $t_1 = t_2$ and $t_2 \neq t_3$ (so $t_1 \neq t_3$). Then

$$C_G(t) = \left\{ \begin{pmatrix} a & b & 0 \\ d & e & 0 \\ 0 & 0 & i \end{pmatrix} \right\}.$$

Let

$$H_1 = \left\{ \begin{pmatrix} a & b & 0 \\ d & e & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\}.$$

Then $H_1 \cong U(2, q)$ under the isomorphism

$$\alpha : \begin{pmatrix} a & b & 0 \\ d & e & 0 \\ 0 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} a & b \\ d & e \end{pmatrix}.$$

Also let S_1 be the torus

$$\{[1, 1, w] \mid w^{q+1} = 1\}.$$

Then S_1 is isomorphic to the multiplicative subgroup S of $\mathbf{F}_{q^2}^*$ consisting of elements x such that $x^{q+1} = 1$. So in this case $C_G(t) = H_1 \times S_1 \cong U(2, q) \times S$.

Now suppose that $t_1 = t_3$ and $t_2 \neq t_3$ (so $t_1 \neq t_2$). Then

$$C_G(t) = \left\{ \begin{pmatrix} a & 0 & c \\ 0 & e & 0 \\ g & 0 & i \end{pmatrix} \right\}.$$

So again $C_G(t) = H_2 \times S_2 \cong U(2, q) \times S$.

Now suppose that $t_2 = t_3$ and $t_1 \neq t_2$ (so $t_1 \neq t_3$). Then

$$C_G(t) = \left\{ \begin{pmatrix} a & 0 & 0 \\ 0 & e & f \\ 0 & h & i \end{pmatrix} \right\}.$$

So now let

$$H_3 = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & e & f \\ 0 & h & i \end{pmatrix} \right\}$$

and

$$S_3 = \left\{ [w, 1, 1] \mid w^{q+1} = 1 \right\}.$$

Then in this case $C_G(t) = H_3 \times S_3 \cong U(2, q) \times S$.

Notice that when $t \in T_2$ has exactly two entries on the diagonal the same, $|C_G(t)| = |U(2, q)|\,|S| = q(q + 1)(q^2 - 1)(q + 1) = q(q + 1)^3 (q - 1)$.

LEMMA 2.1. *We have that*

$$\langle Q_{T_0}^G, \Gamma \rangle = 1$$

$$\langle Q_{T_1}^G, \Gamma \rangle = -1$$

$$\langle Q_{T_2}^G, \Gamma \rangle = -1.$$

Proof. For any maximal torus T we have, by definition, $Q_T^G(u) = R_{T,\theta}^G(u)$ for $u \in U$. But also, when θ is in general position, $R_{T,\theta}^G = \pm\chi$ for some irreducible character χ of G. (The sign is chosen so that $R_{T,\theta}^G(u) = 1$ when u is in the regular unipotent conjugacy class.) Thus if we choose the character θ_i of T_i appropriately we get that $\langle Q_{T_i}^G, \Gamma \rangle = \langle R_{T_i, \theta_i}^G, \Gamma \rangle = \pm 1$. The correct sign can be determined by examining the character table for G in, for example, [6]. ∎

2.5. *Characteristic Equations of Elements of T_0*

Recall, as mentioned in the Introduction, given an F-stable maximal torus $\tilde{T}$ and a character θ of $T = \tilde{T}^F$ there exists a unique homomorphism $f_T : H \to \mathbf{C}T$, independent of θ, which has the property that each homomorphism $f_{T,\theta} : H \to \mathbf{C}$ can be factored as $f_{T,\theta} = \bar{\theta} f_T$. In addition, $f_T(c_n) = \sum_{t \in T} f_T(c_n)(t)t$, where c_n is an element in the basis of H given above, and

$$f_T(c_n)(t) = \frac{\operatorname{ind}(n)}{\langle Q_T^G, \Gamma \rangle |U|\,|C_G(t)|} \sum_{\substack{g \in G \\ u \in U \\ (gung^{-1})_s = t}} \psi(u^{-1}) Q_T^{C_G(t)}\big((gung^{-1})_u\big)$$

[3, Theorem 4.2]. Here n is the element in $Z \cup T_0 w$ such that $c_n = \operatorname{ind}(n) ene$, $Z = \{[u, u, u] \mid u^{q+1} = 1\}$, and $w = [1, \widehat{-1}, 1]$.

In order to explicitly compute these homomorphisms, it remains to determine for which $g \in G$ and $u \in U$ we have that $(gung^{-1})_s = t$, for a fixed $t \in T$.

Note. Since t is semisimple, t and un have the same characteristic equation if and only if there exists a g such that $(gung^{-1})_s = t$. So it is enough to find conditions on u and n so that un and t have the same characteristic equation. In this section we will determine under what conditions un and t have the same characteristic equation when $t \in T_0$.

The following lemma is a general result about all three maximal tori of G.

LEMMA 2.2. *Let t be an element of a maximal torus T_i. Also let $u \in U$ and let $n = n_\lambda = [\lambda, \lambda, \lambda]$. Then $\{u \in U \mid (gun_\lambda g^{-1})_s = t\} = U$ if all the diagonal entries of t are equal to λ, otherwise $\{u \in U \mid (gun_\lambda g^{-1})_s = t\} = \varnothing$.*

Proof. It is clear that un and t have the same characteristic equation if and only if $t = n$. For this t, $(gung^{-1})_s = (gug^{-1}gng^{-1})_s = gng^{-1} = n = t$ for all $u \in U$ and for all $g \in G$. ∎

In what follows let D denote the set of diagonal matrices in $\tilde{G}$.

LEMMA 2.3. *Fix an element z in D, so $z = [a, b, c]$ for some $a, b, c \in \overline{\mathbf{F}}_q^*$. Again let u be an element of U, so $u = [r, s]$ for some r and s such that $r^{q+1} = s + s^q$. But now let $n \in T_0 w$, so that*

$$n = n_{\lambda, n} = \left[\,\mu, \widehat{-\lambda}, \mu^{-q}\right],$$

for some λ, μ *such that* $\lambda^{q+1} = 1$. *Then un and z have the same characteristic equation if and only if the following three conditions occur*:

$$s = (a + b + c + \lambda)\mu^q \tag{2.4}$$

$$r^{q+1} = ((ab + bc + ac)\mu^q + \mu + s\lambda)\lambda^{-1} \tag{2.5}$$

$$\mu^{q-1} = \lambda a^{-1} b^{-1} c^{-1}. \tag{2.6}$$

Proof. Note that un and z will have the same characteristic equation if and only if

$$\begin{aligned} \det(xI - un) &= (x-a)(x-b)(x-c) \\ &= x^3 - (a+b+c)x^2 + (ab+bc+ac)x - abc. \end{aligned}$$

Now

$$\begin{aligned} \det(xI - un) = x^3 &- (s\mu^{-q} - \lambda)x^2 \\ &-(\mu^{-q+1} + s\mu^{-q}\lambda - r^{q+1}\mu^{-q}\lambda)x - \lambda\mu^{-q+1}. \end{aligned}$$

Equating coefficients gives the three equations

$$s\mu^{-q} - \lambda = a + b + c \tag{2.7}$$

$$\mu^{-q}(\mu + s\lambda - r^{q+1}\lambda) = -(ab + bc + ac) \tag{2.8}$$

$$\lambda\mu^{-q+1} = abc. \tag{2.9}$$

These are the same equations as (2.4)–(2.6) above. ∎

COROLLARY 2.10. *Fix an element t in* T_0, *so* $t = [t_1, t_2, t_1^{-q}]$ *for some* $t_1, t_2 \in \mathbf{F}_{q^2}^*$ *with* $t_2^{q+1} = 1$. *Using the same notation as in the preceding lemma, un and t have the same characteristic equation if and only if the following two conditions occur*:

$$s = (t_1 + t_1^{-q} + t_2 + \lambda)\mu^q \tag{2.11}$$

$$\mu^{q-1} = \lambda t_1^{q-1} t_2^{-1}. \tag{2.12}$$

Proof. Substituting $t_1 = a$, $t_2 = b$, and $t_1^{-q} = c$ immediately gives (2.11) and (2.12) from (2.4) and (2.6) of the preceding lemma. That is, (2.4)–(2.6) of the above lemma become

$$s = (t_1 + t_2 + t_1^{-q} + \lambda)\mu^q \tag{2.13}$$

$$r^{q+1} = \left((t_1 t_2 + t_2 t_1^{-q} + t_1^{-q+1})\mu^q + \mu + s\lambda\right)\lambda^{-1} \tag{2.14}$$

$$\mu^{q-1} = \lambda t_1^{q-1} t_2^{-1}. \tag{2.15}$$

Thus it remains to show that (2.14) is implied by (2.13) and (2.15). Now (2.14) requires that $r^{q+1} = \mu^q\lambda^{-1}(t_1t_2 + t_2t_1^{-q} + t_1^{-q+1}) + \mu\lambda^{-1} + s$. By the defining conditions on U we already have the requirement $r^{q+1} = s + s^q$. Thus, from (2.13),

$$r^{q+1} = s + s^q = s + \left(t_1^q + t_2^q + t_1^{-q^2} + \lambda^q\right)\mu^{q^2}$$
$$= s + \left(t_1^q + t_2^{-1} + t_1^{-1} + \lambda^{-1}\right)\mu,$$

since $x^{q^2} = x$ for all $x \in \mathbf{F}_{q^2}^*$ and since $t_2^{q+1} = \lambda^{q+1} = 1$. Thus by (2.15),

$$r^{q+1} = \mu^q\lambda^{-1}t_1t_2 + \mu^q\lambda^{-1}t_2t_1^{-q} + \mu^q\lambda^{-1}t_1^{-q+1} + \mu\lambda^{-1} + s$$

as required. Thus (2.13) and (2.15) imply (2.14). ∎

The following lemma and proof are a modification of the lemma and proof in [2, p. 500].

LEMMA 2.16. *Let z, u and $n_{\lambda,\mu}$ be as in Lemma* 2.3. *Suppose that $un_{\lambda,\mu}$ and z have the same characteristic equation. Then $r = 0$ if and only if $-\lambda$ is an eigenvalue of z.*

Proof. Using (2.5) of Lemma 2.3, we have that $r^{q+1} = 0$ if and only if $((ab + ac + bc)\mu^q + \mu + s\lambda)\lambda^{-1} = 0$. Thus, using (2.4) and (2.6), we have that $r^{q+1} = 0$ if and only if $0 = -(ab + ac + bc)\lambda - abc - \lambda^2(a + b + c + \lambda)$. But this is true if and only if $0 = -\lambda^3 - (a + b + c)\lambda^2 - (ab + ac + bc)\lambda - abc$. But as noted above the characteristic equation of z is $x^3 - (a + b + c)x^2 + (ab + ac + bc)x - abc$. Thus $-\lambda$ is a root of the characteristic equation for z if and only if $r = 0$. ∎

COROLLARY 2.17. *Let t be an element of a maximal torus T_i. Also let $u \in U$, and $n_{\lambda,\mu} \in T_0w$. Also suppose that $un_{\lambda,\mu}$ and t have the same characteristic equation. Then $r = 0$ if and only if $-\lambda$ is an eigenvalue of t.*

Proof. This is immediate from Lemma 2.16. ∎

In the remainder of this section we will distinguish the results between when $t \in T_0$ has a characteristic equation with three distinct roots ($t_1 \neq t_2, t_2 \neq t_1^{-q}, t_1 \neq t_1^{-q}$), two equal roots ($t_1 = t_1^{-q}$ but $t_2 \neq t_1$), and three equal roots ($t_1 = t_1^{-q} = t_2$). These cases will be denoted by (t_1, t_2, t_1^{-q}), (t_1, t_2, t_1), and (t_1, t_1, t_1), respectively. (Note that the cases $t_2 = t_1$, $t_1 \neq t_1^{-q}$ and $t_2 = t_1^{-q}$, $t_2 \neq t_1$ are not possible, as mentioned in Subsection 2.4.) Note that there are $q + 1$ elements t in T_0 in case (t_1, t_1, t_1), $q(q + 1)$ elements t in case (t_1, t_2, t_1), and thus $(q^2 - 1)(q + 1) - (q + 1) - q(q + 1) = (q + 1)^2(q - 2)$ elements t in case (t_1, t_2, t_1^{-q}).

Note that the notation (a, b, c) means that a, b, and c are distinct.

PROPOSITION 2.18. *Let $t = [t_1, t_2, t_1^{-q}] \in T_0$ be fixed. Then the number of $u \in U$ and $n \in T_0 w$ such that un and t have the same characteristic equation is the following*:

(1) *If t is of type* $(t_1, t_2, t_1^{-q}) : ((q-1) + q(q-1)(q+1))$.
(2) *If t is of type* $(t_1, t_2, t_1) : (2(q-1) + (q-1)^2(q+1))$.
(3) *If t is of type* $(t_1, t_1, t_1) : ((q-1) + q(q-1)(q+1))$.

Proof. As above, we use the notation $n = n_{\lambda, \mu}$. Now for all three types of t there are $q+1$ choices for λ and by (2.12), there are $q-1$ choices for μ. Also by (2.11), s is completely determined by μ, λ, and t and by Corollary 2.17 if $-\lambda$ is chosen to be an eigenvalue (i.e., one of t_1, t_2, t_1^{-q}) then $r = 0$. Also if $-\lambda$ is not an eigenvalue, then $r^{q+1} = s + s^q \neq 0$. So if $-\lambda$ is not an eigenvalue there are $q+1$ choices for r. Thus suppose t is of type (t_1, t_2, t_1^{-q}). Then if λ is chosen so that $-\lambda$ is an eigenvalue, then $\lambda = -t_2$, since $t_1 \neq t_1^{-q}$. Thus when $-\lambda$ is an eigenvalue, there is 1 choice for λ, $q-1$ choices for μ, 1 choice for s, and 1 choice for r. Also if $-\lambda$ is not an eigenvalue, there are $q+1-1 = q$ choices for λ, $q-1$ choices for μ, 1 choice for s, and $q+1$ choices for r. This gives (1). Very similar reasoning gives results (2) and (3). The only differences are that in (2), if $-\lambda$ is an eigenvalue then it is either t_1 or t_2 and in (3), if $-\lambda$ is an eigenvalue then it is t_1. ∎

The above proposition is a modification of a similar proposition for $GL(3, q)$ proved by Carter in [2, Proposition 5].

2.6. *Characteristic Equations of Elements of T_1 and T_2*

Recall that T_1 denotes the maximal torus of elements of the form $t = [t_1, t_1^{q^2}, t_1^{-q}]$ such that $t_1^{q^3+1} = 1$. Thus there are two types of elements in T_1, either $t_1 \neq t_1^{q^2}$, $t_1 \neq t_1^{-q}$, and $t_1^{q^2} \neq t_1^{-q}$ or $t_1 = t_1^{-q} = t_1^{q^2}$. (The condition that $t_1^{q^3+1} = 1$ forces all three to be equal if any two are equal.) These two types will be denoted by $(t_1, t_1^{q^2}, t_1^{-q})$ and (t_1, t_1, t_1), respectively. Notice that there are $q+1$ elements t of type (t_1, t_1, t_1) and $q^3 + 1 - (q+1) = q(q^2-1)$ elements t of type $(t_1, t_1^{q^2}, t_1^{-q})$.

PROPOSITION 2.19. *Fix an element t in T_1, so $t = [t_1, t_1^{q^2}, t_1^{-q}]$ for some t_1 such that $t_1^{q^3+1} = 1$. Using the same notation for u and $n_{\lambda, \mu}$ as in the previous section, $un_{\lambda, \mu}$ and t have the same characteristic equation if and only if the following two conditions occur*:

$$s = \left(t_1 + t_1^{q^2} + t_1^{-q} + \lambda\right)\mu^q \tag{2.20}$$

$$\mu^{q-1} = \lambda t_1^{-q^2+q-1}. \tag{2.21}$$

Proof. Substituting $t_1 = a$, $t_1^{q^2} = b$, and $t_1^{-q} = c$ immediately gives (2.20) and (2.21) from (2.4) and (2.6) of Lemma 2.3. That is, (2.4)–(2.6) of Lemma 2.3 become

$$s = \left(t_1 + t_1^{q^2} + t_1^{-q} + \lambda\right)\mu^q \tag{2.22}$$

$$r^{q+1} = \left(\left(t_1^{q^2+1} + t_1^{q^2-q} + t_1^{-q+1}\right)\mu^q + \mu + s\lambda\right)\lambda^{-1} \tag{2.23}$$

$$\mu^{q-1} = \lambda t_1^{-q^2+q-1}. \tag{2.24}$$

Thus it remains to show that (2.23) is implied by (2.22) and (2.24). Now (2.23) requires that

$$r^{q+1} = \mu^q\lambda^{-1}\left(t_1^{q^2+1} + t_1^{q^2-q} + t_1^{-q+1}\right) + \mu\lambda^{-1} + s.$$

From (2.22) we have that

$$\begin{aligned} r^{q+1} &= s + s^q = s + \left(t_1^q + t_1^{q^3} + t_1^{-q^2} + \lambda^q\right)\mu^{q^2} \\ &= s + \left(t_1^q + t_1^{-1} + t_1^{-q^2} + \lambda^{-1}\right)\mu, \end{aligned}$$

since $t_1^{q^3+1} = \lambda^{q+1} = 1$. Then using (2.24) we get that

$$r^{q+1} = \mu^q\lambda^{-1}t_1^{q^2+1} + \mu^q\lambda^{-1}t_1^{q^2-q} + \mu^q\lambda^{-1}t_1^{-q+1} + \mu\lambda^{-1} + s$$

as required. Thus (2.22) and (2.24) imply (2.23). ∎

PROPOSITION 2.25. *Let $t = [t_1, t_1^{q^2}, t_1^{-q}] \in T_1$ be fixed. Also let $u \in U$ so that $u = [r, s]$ for some r and s such that $r^{q+1} = s + s^q$. Then the number of u in U and $n_{\lambda,\mu} \in T_0 w$ such that $un_{\lambda,\mu}$ and t have the same characteristic equation is the following*:

(1) *If t is of type $(t_1, t_1^{q^2}, t_1^{-q})$: $(q+1)^2(q-1)$.*

(2) *If t is of type (t_1, t_1, t_1) : $((q-1) + q(q-1)(q+1))$.*

Proof. If t is of type $(t_1, t_1^{q^2}, t_1^{-q})$ then $-\lambda$ cannot be chosen to be an eigenvalue since $t_1 \neq t_1^{-q}$. Thus there are $q+1$ choices for λ, $q-1$ choices for μ, 1 choice for s, and $q+1$ choices for r when t is of this type by Proposition 2.19. Whereas if t is of type (t_1, t_1, t_1) we can choose $-\lambda = t_1$ leaving $q-1$ choices for μ and 1 choice for each of r and s since r must be zero by Corollary 2.17. Or we can choose $-\lambda \neq t_1$ of which there are q choices and then there are $q-1$ choices for μ, 1 choice for s, and $q+1$ choices for r. ∎

Now consider the maximal torus T_2. Recall that T_2 is the maximal torus whose elements t are of the form $t = [t_1, t_2, t_3]$ where $t_i^{q+1} = 1$. Thus there are five types of elements in $T_2:(t_1, t_2, t_3)$, (t_1, t_1, t_3), (t_1, t_2, t_1), (t_1, t_2, t_2), and (t_1, t_1, t_1). Note that there are $q(q+1)(q-1)$ elements t in T_2 of type (t_1, t_2, t_3), $q(q+1)$ elements t in T_2 in each of the types $(t_1, t_1, t_3), (t_1, t_2, t_1), (t_1, t_2, t_2)$, and $q+1$ elements t in T_2 of type (t_1, t_1, t_1).

PROPOSITION 2.26. *Fix an element t in T_2, so $t = [t_1, t_2, t_3]$ for some $t_1, t_2, t_3 \in \mathbf{F}_{q^2}^*$ with $t_i^{q+1} = 1$. Then $un_{\lambda, \mu}$ and t have the same characteristic equation if and only if the following two conditions occur*:

$$s = (t_1 + t_2 + t_3 + \lambda)\mu^q \tag{2.27}$$

$$\mu^{q-1} = \lambda t_1^{-1} t_2^{-1} t_3^{-1}. \tag{2.28}$$

Proof. Substituting $t_1 = a$, $t_2 = b$, and $t_3 = c$ immediately gives (2.27) and (2.28) from (2.4) and (2.6) of Lemma 2.3. That is, (2.4)–(2.6) of Lemma 2.3 become

$$s = (t_1 + t_2 + t_3 + \lambda)\mu^q \tag{2.29}$$

$$r^{q+1} = ((t_1 t_2 + t_2 t_3 + t_1 t_3)\mu^q + \mu + s\lambda)\lambda^{-1} \tag{2.30}$$

$$\mu^{q-1} = \lambda t_1^{-1} t_2^{-1} t_3^{-1}. \tag{2.31}$$

Thus it remains to show that (2.29) and (2.31) imply (2.30). Now (2.30) requires that

$$r^{q+1} = \mu^q \lambda^{-1}(t_1 t_2 + t_2 t_3 + t_1 t_3) + \mu\lambda^{-1} + s.$$

From (2.29) we have that

$$\begin{aligned} r^{q+1} &= s + s^q = s + (t_1^q + t_2^q + t_3^q + \lambda^q)\mu^{q^2} \\ &= s + (t_1^{-1} + t_2^{-1} + t_3^{-1} + \lambda^{-1})\mu, \end{aligned}$$

since $x^{q^2} = x$ for all $x \in \mathbf{F}_{q^2}^*$ and since $t_i^{q+1} = \lambda^{q+1} = 1$. Thus by (2.31) we have that

$$r^{q+1} = \mu^q\lambda^{-1} t_1 t_2 + \mu^q \lambda^{-1} t_2 t_3 + \mu^q \lambda^{-1} t_1 t_3 + \mu\lambda^{-1} + s. \quad \blacksquare$$

PROPOSITION 2.32. *Let $t = [t_1, t_2, t_3] \in T_2$ be fixed. Also let $u \in U$ so that $u = [r, s]$ for some r and s such that $r^{q+1} = s + s^q$. Then the number of u in U and $n_{\lambda, \mu} \in T_0 w$ such that $un_{\lambda, \mu}$ and t have the same characteristic equation is the following*:

(1) *If t is of type (t_1, t_2, t_3)* : $(3(q-1) + (q-2)(q-1)(q+1))$.

(2) *If t is of type (t_1, t_1, t_3), (t_1, t_2, t_1), or (t_1, t_2, t_2)* : $(2(q-1) + (q-1)^2(q+1))$.

(3) *If t is of type (t_1, t_1, t_1)* : $((q-1) + q(q-1)(q+1))$.

Proof. This proof is completely analogous to the proofs of Propositions 2.18 and 2.25. Note that since each of t_1, t_2, and t_3 are such that $t_i^{q+1} = 1$, if $-\lambda$ is an eigenvalue of t it can be any of the t_i. ∎

2.7. *The Homomorphism* $f_{T_0}: H \to \mathbf{C}T_0$

Let θ be an irreducible character of a maximal torus, T_i, of G. Then, as previously discussed, there exists a unique homomorphism $f_{T_i}: H \to \mathbf{C}T_i$, independent of θ, which has the property that each homomorphism $f_{T_i,\theta}: H \to \mathbf{C}$ can be factored as $f_{T_i,\theta} = \bar{\theta} f_{T_i}$. Also,

$$f_{T_i}(c) = \sum_{t \in T_i} f_{T_i}(c)(t)t.$$

One more lemma is needed before stating the results of the calculations of the coefficients $f_{T_i}(c)(t)$. The proof of the following lemma is clear and thus omitted.

LEMMA 2.33. *Let u and n be fixed as in Subsections* 2.5 *and* 2.6. *Also let $t \in T_i$ be fixed. Suppose there exists a $g \in G$ such that $(gung^{-1})_s = t$. Then $|C_G((un)_s)| = |C_G(t)|$. Also $(hunh^{-1})_s = t$ if and only if $h = gx^{-1}$ for some $x \in C_G((un)_s)$.*

In the remainder of this paper, the notation $N(x)$ and $T(x)$ will denote the norm and trace of $x \in \mathbf{F}_{q^2}$ over $\mathbf{F}_q$, respectively. Recall the basis

$$\left\{c_{\lambda,\mu}, c_\lambda \mid \lambda, \mu \in \mathbf{F}_{q^2}^* \text{ and } \lambda^{q+1} = 1\right\}$$

of the Hecke algebra described in Subsection 2.3. Here

$$c_{\lambda,\mu} = q^3 e\left[\mu, \widehat{-\lambda}, \mu^{-q}\right] e$$

and

$$c_\lambda = [\lambda, \lambda, \lambda] e.$$

Call $c_{\lambda,\mu}$ a basis element of the first type and c_λ a basis element of the second type.

The next theorem gives the value of the coefficient $f_{T_i}(c_\lambda)(t)$ for $i = 0, 1$, or 2 when c_λ is a basis element of the second type.

THEOREM 2.34. *Let t be a fixed element of T_i, for $i = 0, 1$, or 2. Let c_λ be a fixed element of the basis of the Hecke algebra of the second type. Then*

$$f_{T_i}(c_\lambda)(t) = \begin{cases} 0, & \text{if } t \text{ is not of type } (\lambda, \lambda, \lambda) \\ 1, & \text{if } t \text{ is of type } (\lambda, \lambda, \lambda). \end{cases}$$

Proof. Within this proof c_λ will be denoted by just c and T_i will be denoted by just T. As before, let $n_\lambda = [\lambda, \lambda, \lambda]$, so that $c = n_\lambda e$. Fix an irreducible character, θ of T. Let χ be the unique irreducible character of G such that $\langle \chi, R^G_{T,\theta} \rangle \neq 0$ and $\langle \chi, \Gamma \rangle \neq 0$ (see Subsection 1.2). The character table for G shows that $\chi(n_\lambda) = \theta(n_\lambda)\chi(I)$. Thus

$$\begin{aligned} \chi(n_\lambda e) &= |U|^{-1} \sum_{u \in U} \psi(u^{-1})\chi(n_\lambda u) \\ &= \theta(n_\lambda)\chi(e) \\ &= \theta(n_\lambda). \end{aligned}$$

Thus $f_{T,\theta}(c) = \theta(n_\lambda)$. But also, by Curtis's theorem,

$$f_{T,\theta}(c) = \sum_{t \in T} a_t \theta(t)$$

for some coefficients a_t. (That is, Curtis's theorem writes $f_{T,\theta}(c)$ as $\sum_{t \in T} f_T(c)(t)\theta(t)$.) Thus

$$\sum_{t \in T} a_t \theta(t) = \theta(n_\lambda).$$

But this is true for all irreducible characters θ of T. Thus $a_{n_\lambda} = 1$ and $a_t = 0$ for $t \neq n_\lambda$. ∎

THEOREM 2.35. *Again let $t = [t_1, t_2, t_1^{-q}]$ be a fixed element of T_0. But now let $c_{\lambda,\mu}$ be a fixed element of the basis of the Hecke algebra of the first type. Suppose that $-\lambda$ is an eigenvalue of t. Then*

$$f_{T_0}(c_{\lambda,\mu})(t) = \begin{cases} q+1 & \text{if } t \text{ is of type } (-\lambda, t_2, -\lambda) \\ 1 & \text{if } t \text{ is not of type } (-\lambda, t_2, -\lambda), \end{cases}$$

if $\mu^{q-1} = \lambda t_1^{q-1} t_2^{-1}$. Otherwise, $f_{T_0}(c_{\lambda,\mu})(t) = 0$.

Proof. In this proof T_0 will be denoted by T. First suppose t is of type (t_1, t_1, t_1). Then (1.2) becomes

$$\begin{aligned} f_T(c)(t) &= \frac{\operatorname{ind}(n)}{\langle Q^G_T, \Gamma \rangle |U|\,|C_G(t)|} \sum_{\substack{g \in G \\ u \in U \\ (gung^{-1})_s = t}} \psi(u^{-1}) Q_T^{C_G(t)}\big((gung^{-1})_u\big) \\ &= q^3 \cdot q^{-3} |G|^{-1} \sum_{\substack{g \in G \\ u \in U \\ (gung^{-1})_s = t}} \psi(u^{-1}) Q^G_T\big((gung^{-1})_u\big). \end{aligned}$$

But we also have by Lemma 2.16 that the $(1,2)$ entry of the matrix u must be zero in order for u to satisfy the condition $(gung^{-1})_s = t$ since $-\lambda$ is an eigenvalue of t. Thus $\psi(u^{-1}) = \chi(0) = 1$ for every u in the above sum. Also by Corollary 2.10 the only u that meets the conditions of this sum is $[0,(3t_1 + \lambda)\mu^q]$. For this u we must have that $(gung^{-1})_s = t$ for all $g \in G$, by Lemma 2.33. Thus

$$f_T(c)(t) = |G|^{-1} \sum_{g\in G} Q_T^G\big((gung^{-1})_u\big),$$

where $u = [0,(3t_1 + \lambda)\mu^q]$. First we will show that $gung^{-1}$ is not semisimple for any $g \in G$ such that $(gung^{-1})_s = t$. So suppose that $gung^{-1}$ is semisimple for some g. Then $gung^{-1} = t$. But t is in the center of G. So this implies $un = t$. But this is not possible. Thus $gung^{-1}$ is not semisimple. But this implies $(gung^{-1})_u \neq I$. So $Q_T^G((gung^{-1})_u) = 1$. Thus

$$f_T(c)(t) = |G|^{-1} \sum_{g\in G} Q_T^G\big((gung^{-1})_u\big) = 1.$$

This proves the theorem when t is of type (t_1, t_1, t_1).

Now suppose t is of type (t_1, t_2, t_1^{-q}). Then

$$\begin{aligned} f_T(c)(t) &= \frac{\operatorname{ind}(n)}{\langle Q_T^G, \Gamma\rangle |U|\,|C_G(t)|} \sum_{\substack{g\in G\\ u\in U\\ (gung^{-1})_s = t}} \psi(u^{-1}) Q_T^{C_G(t)}\big((gung^{-1})_u\big) \\ &= q^3 \cdot q^{-3}|T|^{-1} \sum_{\substack{g\in G\\ (gung^{-1})_s = t}} \psi(u^{-1}) \end{aligned}$$

since now $C_G(t) = T$ by Subsection 2.4, u is completely determined by the same argument as above, and $Q_T^T(x) = 1$ for all x. But, as above $\psi(u^{-1}) = \chi(0) = 1$. Thus

$$f_T(c)(t) = |T|^{-1} \sum_{\substack{g\in G\\ (gung^{-1})_s = t}} 1.$$

This in turn implies

$$f_T(c)(t) = |T|^{-1} \sum_{x\in C_G((un)_s)} 1 = 1,$$

by Lemma 2.33. This proves the theorem when t is of type (t_1, t_2, t_1^{-q}).

Now suppose t is of type (t_1, t_2, t_1). We still have that $\psi(u^{-1}) = 1$ but now $C_G(t) = H_2 \times S_2$ which has order $q(q+1)^3(q-1)$ by Subsection 2.4. Thus, in this case,

$$f_T(c)(t) = \frac{\operatorname{ind}(n)}{\langle Q_T^G, \Gamma\rangle |U|\,|C_G(t)|} \sum_{\substack{g\in G\\ u\in U\\ (gung^{-1})_s = t}} \psi(u^{-1}) Q_T^{C_G(t)}\big((gung^{-1})_u\big)$$

$$= \big(q(q+1)^3(q-1)\big)^{-1} \sum_{\substack{g\in G\\ u\in U\\ (gung^{-1})_s = t}} Q_T^{H_2}\big((gung^{-1})_u\big).$$

(Here $(gung^{-1})_u \in C_G(t)$ so it is a matrix of the form $[0, x]$, by Subsection 2.4.) Now

$$Q_T^{H_2}\big((gung^{-1})_u\big) = q + 1$$

if and only if $x = 0$. But this is true if and only if $gung^{-1}$ is semisimple. Suppose that $gung^{-1}$ is semisimple for some g in the above sum. Then $gung^{-1} = t$ and thus $gun = tg$. That is,

$$\begin{pmatrix} as\mu^{-q} + c\mu^{-q} & -b\lambda & a\mu \\ ds\mu^{-q} + f\mu^{-q} & -e\lambda & d\mu \\ hs\mu^{-q} + j\mu^{-q} & -i\lambda & h\mu \end{pmatrix} = \begin{pmatrix} at_1 & bt_1 & ct_1 \\ dt_2 & et_2 & ft_2 \\ ht_1 & it_1 & jt_1 \end{pmatrix},$$

$$\text{where } g = \begin{pmatrix} a & b & c \\ d & e & f \\ h & i & j \end{pmatrix}.$$

But $-\lambda$ is an eigenvalue, so first suppose that $-\lambda = t_2$ and thus $-\lambda \neq t_1$, since t is of type (t_1, t_2, t_1). A comparison of the second columns of these two matrices gives $b = i = 0$. Then by comparing the first and third columns we get that both matrices must be equal to

$$\begin{pmatrix} at_1 & 0 & a\mu \\ dt_2 & et_2 & d\mu \\ ht_1 & 0 & h\mu \end{pmatrix}.$$

But this matrix has determinant zero, a contradiction. Thus $-\lambda \neq t_2$. Thus if $gung^{-1}$ is semisimple then $-\lambda = t_1$ and thus $-\lambda \neq t_2$. By Subsection 2.5, $s = (2t_1 + t_2 + \lambda)\mu^q = (t_1 + t_2)\mu^q$. Now let

$$g = \begin{pmatrix} 1 & 0 & t_1^{-1}\mu \\ 1 & 0 & t_2^{-1}\mu \\ 0 & 1 & 0 \end{pmatrix}.$$

Note that $\det(g)$ is not zero since $t_2 \neq t_1$. Now we will show that $gung^{-1} = t$. Note that

$$gun = \begin{pmatrix} t_1 + t_2 + t_1^{-1}\mu^{-q+1} & 0 & \mu \\ t_1 + t_2 + t_2^{-1}\mu^{-q+1} & 0 & \mu \\ 0 & -\lambda & 0 \end{pmatrix}$$

$$= \begin{pmatrix} t_1 + t_2 + t_1^{-1}(-t_1t_2) & 0 & \mu \\ t_1 + t_2 + t_2^{-1}(-t_1t_2) & 0 & \mu \\ 0 & t_1 & 0 \end{pmatrix},$$

since

$$\mu^{-q+1} = \lambda^{-1}t_1^2t_2 = -t_1^{-1}t_1^2t_2 = -t_1t_2$$

by Subsection 2.5 and since $-\lambda = t_1$. Thus

$$gun = \begin{pmatrix} t_1 & 0 & \mu \\ t_2 & 0 & \mu \\ 0 & t_1 & 0 \end{pmatrix}.$$

Also

$$tg = \begin{pmatrix} t_1 & 0 & \mu \\ t_2 & 0 & \mu \\ 0 & t_1 & 0 \end{pmatrix}.$$

Thus $gun = tg$ as claimed. Thus when $-\lambda = t_1$ there exists a g such that $gung^{-1}$ is semisimple. But then $g(un)_s g^{-1} = (gung^{-1})_s = gung^{-1}$. Thus $(un)_s = un$. So un is semisimple. This implies $Q_T^{H_2}(gung^{-1}) = q + 1$ for all g in the above sum if t is of type $(-\lambda, t_2, -\lambda)$, otherwise $Q_T^{H_2}(gung^{-1}) = 1$. Thus

$$f_T(c)(t) = \left(q(q+1)^3(q-1)\right)^{-1} \sum_{\substack{g \in G \\ (gung^{-1})_s = t}} Q_T^{H_2}\left((gung^{-1})_u\right)$$

$$= \left(q(q+1)^3(q-1)\right)^{-1} \sum_{\substack{g \in G \\ (gung^{-1})_s = t}} 1$$

if t is of type (t_1, t_2, t_1) but not of type $(-\lambda, t_2, -\lambda)$ and

$$f_T(c)(t) = \left(q(q+1)^3(q-1)\right)^{-1} \sum_{\substack{g\in G \\ (gung^{-1})_s = t}} (q+1)$$

if t is of type $(-\lambda, t_2, -\lambda)$. But by Lemma 2.33 these sums are over a set of size $|C_G((un)_s)| = q(q+1)^3(q-1)$. Thus $f_T(c)(t) = 1$ if t is of type (t_1, t_2, t_1) but not of type $(-\lambda, t_2, -\lambda)$ and $f_T(c)(t) = q+1$ if t is of type $(-\lambda, t_2, -\lambda)$. ∎

We will now prove a similar result where we instead assume that the $-\lambda$ entry of $n_{\lambda,\mu} = n$ is not an eigenvalue of the fixed $t \in T_0$.

THEOREM 2.36. *Let $t = [t_1, t_2, t_1^{-q}]$ be a fixed element of T_0. Let $c_{\lambda,\mu}$ be a fixed element of the basis of the Hecke algebra of the first type. Suppose that $-\lambda$ is not an eigenvalue of t. Let $s = (t_1 + t_2 + t_1^{-q} + \lambda)\mu^q$. Then*

$$f_T(c_{\lambda,\mu})(t) = \sum_{\substack{r\in \mathbf{F}_{q^2} \\ N(r) = T(s)}} \psi([r,s])^{-1} = \sum_{\substack{r\in \mathbf{F}_{q^2} \\ N(r) = T(s)}} \chi(-r),$$

if $\mu^{q-1} = \lambda t_1^{q-1} t_2^{-1}$. Otherwise, $f_T(c_{\lambda,\mu})(t) = 0$.

Proof. Again consider (1.2),

$$f_T(c)(t) = \frac{\operatorname{ind}(n)}{\langle Q_T^G, \Gamma\rangle |U|\,|C_G(t)|} \sum_{\substack{g\in G \\ u\in U \\ (gung^{-1})_s = t}} \psi(u^{-1}) Q_T^{C_G(t)}\left((gung^{-1})_u\right).$$

The proof of the previous theorem almost works here again except for the places where we used that $-\lambda$ is an eigenvalue. This was used in two different places. One was in determining the values of the Green functions on $(gung^{-1})_u$ and the other was to conclude that $\psi(u^{-1})$ was 1. First consider the first of these two. If t is of type (t_1, t_2, t_1^{-q}) then $Q_{T_0}^{C_G(t)}(x) = Q_{T_0}^{T_0}(x) = 1$ for all x. Thus when t is of type (t_1, t_2, t_1^{-q}) the Green function in (1.2) will always take the value 1. So suppose that t is of type (t_1, t_2, t_1) or (t_1, t_1, t_1) and that g is included in (1.2) with $gung^{-1}$ semisimple. Then similar to the above argument we must have $gun = tg$. That is,

$$\begin{pmatrix} as\mu^{-q} + br^q\mu^{-q} + c\mu^{-q} & -ar\lambda - b\lambda & a\mu \\ ds\mu^{-q} + er^q\mu^{-q} + f\mu^{-q} & -dr\lambda - e\lambda & d\mu \\ hs\mu^{-q} + ir^q\mu^{-q} + j\mu^{-q} & -hr\lambda - i\lambda & h\mu \end{pmatrix} = \begin{pmatrix} at_1 & bt_1 & ct_1 \\ dt_2 & et_2 & ft_2 \\ ht_1 & it_1 & jt_1 \end{pmatrix},$$

where

$$g = \begin{pmatrix} a & b & c \\ d & e & f \\ h & i & j \end{pmatrix}$$

and we allow $t_1 = t_2$. A comparison of the last columns of these two matrices shows that

$$j = h\mu t_1^{-1} \qquad \text{and} \qquad c = a\mu t_1^{-1}.$$

A comparison of the second columns shows that

$$-hr\lambda = it_1 + i\lambda = i(t_1 + \lambda) \qquad \text{and} \qquad -ar\lambda = bt_1 + b\lambda = b(t_1 + \lambda).$$

But $-\lambda$ is not an eigenvalue of t so $t_1 + \lambda \neq 0$. Thus $i = -hr\lambda(t_1 + \lambda)^{-1}$ and $b = -ar\lambda(t_1 + \lambda)^{-1}$. Substituting these expressions for b, c, i, and j into the matrix on the right hand side we get

$$\begin{pmatrix} at_1 & -art_1\lambda(t_1 + \lambda)^{-1} & a\mu \\ dt_2 & et_2 & ft_2 \\ ht_1 & -hrt_1\lambda(t_1 + \lambda)^{-1} & h\mu \end{pmatrix}.$$

But this matrix is singular since the first and third rows are multiples of each other. This is a contradiction. So $gung^{-1}$ is not semisimple. Thus the values of the Green functions $Q_T^{H_2}((gung^{-1})_u)$, $Q_T^G((gung^{-1})_u)$, and $Q_T^T((gung^{-1})_u)$ will always be 1.

Now for the other adjustment that has to be made to the previous proof. By Corollary 2.10 in order for $gung^{-1}$ to be included in (1.2) we must have that the entry $s = (t_1 + t_2 + t_1^{-q} + \lambda)\mu^q$. This corollary also showed that there are no restrictions on r except for the condition that $r^{q+1} = s + s^q$ (i.e., $N(r) = T(s)$) which just comes from the requirement that $u \in U$. Thus

$$\begin{aligned} f_T(c)(t) &= \frac{\operatorname{ind}(n)}{\langle Q_T^G, \Gamma\rangle |U|\,|C_G(t)|} \sum_{\substack{g \in G \\ u \in U \\ (gung^{-1})_s = t}} \psi(u^{-1}) Q_T^{C_G(t)}\big((gung^{-1})_u\big) \\ &= q^3 \cdot 1 \cdot q^{-3} |C_G(t)|^{-1} \sum_{\substack{x \in C_G((un)_s) \\ u \in U \\ N(r) = T(s)}} \psi(u^{-1}) \\ &= \sum_{\substack{u \in U \\ N(r) = T(s)}} \psi(u^{-1}). \end{aligned}$$

∎

Note. In summary the coefficients $f_{T_0}(c)(t)$ are equal to:

If c is of the first type ($c = c_{\lambda,\mu}$),

$$\begin{cases} q+1, & \text{if } t_1 = -\lambda,\ t_2 \neq -\lambda, \text{ and } N(n_{\lambda,\mu}) = N(t) \\ 1, & \text{if } t_2 = -\lambda \text{ and } N(n_{\lambda,\mu}) = N(t) \\ \displaystyle\sum_{\substack{r \in \mathbf{F}_{q^2} \\ N(r) = T(s)}} \chi(-r), & \text{if } t_i \neq -\lambda \text{ and } N(n_{\lambda,\mu}) = N(t) \\ 0, & \text{if } N(n_{\lambda,\mu}) \neq N(t). \end{cases}$$

If c is of the second type ($c = c_\lambda$),

$$\begin{cases} 0, & \text{if } t \text{ is not of type } (\lambda, \lambda, \lambda) \\ 1, & \text{if } t \text{ is of type } (\lambda, \lambda, \lambda). \end{cases}$$

2.8. *The Homomorphism* $f_{T_1}\colon H \to \mathbf{C}T_1$

Now we will compute the coefficients $f_{T_i}(c)(t)$ when $i = 1$ by proving the analogues of the theorems in Subsection 2.7 when $T_i = T_1$. Note that Lemma 2.33 and Theorem 2.34 did not assume a particular maximal torus. Thus we will begin with the theorem parallel to Theorem 2.35.

THEOREM 2.37. *Let $t = [t_1, t_1^{q^2}, t_1^{-q}]$ be a fixed element of T_1. Let $c_{\lambda,\mu}$ be a fixed element of the basis of the Hecke algebra of the first type. Suppose that $-\lambda$ is an eigenvalue of t. Then $f_{T_1}(c_{\lambda,\mu})(t) = -(q+1)$ if $\mu^{q-1} = \lambda t_1^{-q^2+q-1}$. Otherwise, $f_{T_1}(c_{\lambda,\mu})(t) = 0$.*

Proof. Notice that since $-\lambda$ is an eigenvalue we must have that t is of type $(-\lambda, -\lambda, -\lambda)$. This follows from the assumptions that $\lambda^{q+1} = 1$ and $t_1^{q^3+1} = 1$. For if $t_1 = -\lambda$ then $t_1^{q+1} = 1$ and so $t_1 = t_1^{-q}$. But, as mentioned in Subsection 2.4, if any two of the diagonal elements of t are equal then all three are equal. Thus assuming $t_1 = -\lambda$ forces t to be of type $(-\lambda, -\lambda, -\lambda)$. Similarly, if we assume $t_1^{q^2} = -\lambda$, then $t_1^{q^3+q^2} = 1$. Thus $t_1^{-1+q^2} = 1$ and thus $t_1^{q^2} = t_1$, again forcing t to be of type $(-\lambda, -\lambda, -\lambda)$. If we assume $t_1^{-q} = -\lambda$, then $t_1^{-q^2-q} = 1$. Thus $t_1^{-q} = t_1^{q^2}$. So we again get that t is of type $(-\lambda, -\lambda, -\lambda)$. Now, as in the first part of the proof of Theorem 2.35 (with the adjustment $\langle Q^G_{T_1}, \Gamma\rangle = -1$ instead of 1), (1.2) becomes

$$f_{T_1}(c)(t) = -|G|^{-1} \sum_{g \in G} Q^G_{T_1}\big((gung^{-1})_u\big),$$

where $u = [0, (-2\lambda)\mu^q]$. Also, as the proof of Theorem 2.35 shows, $gung^{-1}$ is not semisimple. Note that un is conjugate (in $\tilde{G}$) to

$$x = \begin{pmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 0 \\ 0 & 0 & -\lambda \end{pmatrix}.$$

In fact, let

$$g = \begin{pmatrix} -\lambda^{-1} & 0 & 0 \\ 1 & 0 & -\mu\lambda^{-1} \\ 1 & 1 & -\mu\lambda^{-1} \end{pmatrix}.$$

But $\mu^{q-1} = \lambda t_1^{-3} = -\lambda^{-2}$. Thus

$$gun = \begin{pmatrix} 2 & 0 & -\mu\lambda^{-1} \\ -\lambda & 0 & \mu \\ -\lambda & -\lambda & \mu \end{pmatrix}.$$

Also

$$xg = \begin{pmatrix} 2 & 0 & -\mu\lambda^{-1} \\ -\lambda & 0 & \mu \\ -\lambda & -\lambda & \mu \end{pmatrix}.$$

Thus $gung^{-1} = x$. Thus $Q_{T_1}^G((gung^{-1})_u) = q + 1$. So

$$\begin{aligned} f_{T_1}(c)(t) &= -|G|^{-1} \sum_{g \in G} Q_{T_1}^G\big((gung^{-1})_u\big) \\ &= -|G|^{-1} \sum_{g \in G} (q+1) = -(q+1). \end{aligned}$$ ∎

THEOREM 2.38. *Again let $t = [t_1, t_1^{q^2}, t_1^{-q}]$ be a fixed element of T_1. Also let $c_{\lambda,\mu}$ be a fixed element of the basis of the Hecke algebra of the first type. But now suppose that $-\lambda$ is not an eigenvalue of t. Let $s = (t_1 + t_1^{q^2} + t_1^{-q} + \lambda)\mu^q$. Then*

$$f_T(c_{\lambda,\mu})(t) = -\sum_{\substack{r \in \mathbf{F}_{q^2} \\ N(r) = T(s)}} \psi([r,s])^{-1} = -\sum_{\substack{r \in \mathbf{F}_{q^2} \\ N(r) = T(s)}} \chi(-r).$$

Proof. Now the proof of Theorem 2.36 almost translates to a proof of this theorem by just changing T_0 to T_1 except that in this case the Green functions are not determined to be 1 just by knowing that $gung^{-1}$ is not semisimple. In the case $C_G(t) = T_1$ (i.e., t is of type $(t_1, t_1^{q^2}, t_1^{-q})$) we still have that $Q_{T_1}^{C_G(t)} = Q_{T_1}^{T_1}$ which is always 1. But suppose that t is of type (t_1, t_1, t_1). In this case it is necessary to show that un is also not conjugate (in $\tilde{G}$) to

$$\begin{pmatrix} t_1 & 1 & 0 \\ 0 & t_1 & 0 \\ 0 & 0 & t_1 \end{pmatrix}$$

in order to conclude that the Green function is always 1. So suppose un is conjugate to

$$x = \begin{pmatrix} t_1 & 1 & 0 \\ 0 & t_1 & 0 \\ 0 & 0 & t_1 \end{pmatrix}$$

and let

$$g = \begin{pmatrix} a & b & c \\ d & e & f \\ h & i & j \end{pmatrix}$$

be the matrix such that $gun = xg$. This equation in matrices is

$$\begin{pmatrix} as\mu^{-q} + br^q\mu^{-q} + c\mu^{-q} & -ar\lambda - b\lambda & a\mu \\ ds\mu^{-q} + er^q\mu^{-q} + f\mu^{-q} & -dr\lambda - e\lambda & d\mu \\ hs\mu^{-q} + ir^q\mu^{-q} + j\mu^{-q} & -hr\lambda - i\lambda & h\mu \end{pmatrix}$$
$$= \begin{pmatrix} at_1 + d & bt_1 + e & ct_1 + f \\ dt_1 & et_1 & ft_1 \\ ht_1 & it_1 & jt_1 \end{pmatrix}.$$

A comparison of the last columns of these two matrices shows that $j = h\mu t_1^{-1}$ and $f = d\mu t_1^{-1}$. A comparison of the second columns shows that $-dr\lambda = et_1 + e\lambda = e(t_1 + \lambda)$ and $hr\lambda = it_1 + i\lambda = i(t_1 + \lambda)$. But $-\lambda$ is not an eigenvalue of t so $t_1 + \lambda \neq 0$. So $i = -hr\lambda(t_1 + \lambda)^{-1}$ and $e = -dr\lambda(t_1 + \lambda)^{-1}$. Substituting these expressions for j, f, i, and e into the matrix on the right hand side gives

$$\begin{pmatrix} at_1 + d & bt_1 + e & ct_1 + f \\ dt_1 & -drt_1\lambda(t_1 + \lambda)^{-1} & d\mu \\ ht_1 & -hrt_1\lambda(t_1 + \lambda)^{-1} & h\mu \end{pmatrix}.$$

But the second and third rows of this matrix are multiples of each other, so it is not invertible. This is a contradiction. So un is not conjugate to x. Thus as in Theorem 2.36, the values of the Green functions in (1.2) will always be 1. The rest of the proof of Theorem 2.36 goes through here verbatim. ∎

Note. In summary the coefficients $f_{T_1}(c)(t)$ are equal to:

If c is of the first type ($c = c_{\lambda,\mu}$),

$$\begin{cases} -(q+1), & \text{if } -\lambda \text{ an eigenvalue and } N(n_{\lambda,\mu}) = N(t) \\ -\sum_{\substack{r\in \mathbf{F}_{q^2} \\ N(r)=T(s)}} \chi(-r), & \text{if } -\lambda \text{ not an eigenvalue and } N(n_{\lambda,\mu}) = N(t) \\ 0, & \text{if } N(n_{\lambda,\mu}) \neq N(t). \end{cases}$$

If c is of the second type ($c = c_\lambda$),

$$\begin{cases} 0, & \text{if } t \text{ is not of type } (\lambda,\lambda,\lambda) \\ 1, & \text{if } t \text{ is of type } (\lambda,\lambda,\lambda). \end{cases}$$

2.9. *The Homomorphism* $f_{T_2}\colon H \to \mathbf{C}T_2$

Now we will compute the coefficients $f_{T_i}(c)(t)$ when $i = 2$. Theorem 2.34 computed these coefficients when c is a basis element of the second type. The following two theorems compute these coefficients when c is of the first type.

THEOREM 2.39. *Let $t = [t_1, t_2, t_3]$ be a fixed element of T_2. Let $c_{\lambda,\mu}$ be a fixed element of the basis of the Hecke algebra of the first type. Suppose that $-\lambda$ is an eigenvalue of t. Then*

$$f_{T_2}(c_{\lambda,\mu})(t) = \begin{cases} 2q-1 & \textit{if the eigenvalue } -\lambda \textit{ occurs with multiplicity } 3 \\ -(q+1) & \textit{if the eigenvalue } -\lambda \textit{ occurs with multiplicity } 2 \\ -1 & \textit{if the eigenvalue } -\lambda \textit{ occurs with multiplicity } 1, \end{cases}$$

if $\mu^{q-1} = \lambda t_1^{-1} t_2^{-1} t_3^{-1}$. Otherwise, $f_{T_2}(c_{\lambda,\mu})(t) = 0$.

Proof. First suppose t is of type (t_1, t_1, t_1). Then, as in the first part of the proof of Theorem 2.35 (with the adjustment $\langle Q_{T_2}^G, \Gamma\rangle = -1$ instead of 1), (1.2) becomes

$$f_{T_2}(c)(t) = -|G|^{-1} \sum_{g\in G} Q_{T_2}^G\big((gung^{-1})_u\big),$$

where $u = [0, (3t_1 + \lambda)\mu^q]$. Also, as the proof of Theorem 2.35 shows, $gung^{-1}$ is not semisimple. As shown in the proof of Theorem 2.37, un is conjugate to

$$\begin{pmatrix} t_1 & 1 & 0 \\ 0 & t_1 & 0 \\ 0 & 0 & t_1 \end{pmatrix}.$$

Thus $Q_{T_2}^G((gung^{-1})_u) = -2q + 1$. So that

$$\begin{aligned} f_{T_2}(c)(t) &= -|G|^{-1} \sum_{g \in G} Q_{T_2}^G\big((gung^{-1})_u\big) \\ &= -|G|^{-1} \sum_{g \in G} (-2q + 1) = 2q - 1. \end{aligned}$$

This proves the theorem when t is of type (t_1, t_1, t_1).

Now suppose t is of type (t_1, t_2, t_3). For t of this type the proof of Theorem 2.35 translates to prove this theorem by just changing everywhere you see T_0 to T_2 and adjusting by a multiple of -1.

Similarly the proof when t is of type (t_1, t_2, t_1) is exactly the same as the proof of Theorem 2.35 where we showed that $gung^{-1}$ is semisimple. Thus $Q_{T_2}^{H_2} = q + 1$. Also by symmetry that proof translates into a proof of this theorem when t is of type (t_1, t_1, t_3) or of type (t_1, t_2, t_2). ∎

THEOREM 2.40. *Again let $t = [t_1, t_2, t_3]$ be a fixed element of T_2. Also let $c_{\lambda,\mu}$ be a fixed element of the basis of the Hecke algebra of the first type. But now suppose that $-\lambda$ is not an eigenvalue of t. Let $s = (t_1 + t_2 + t_3 + \lambda)\mu^q$. Then*

$$f_T(c_{\lambda,\mu})(t) = -\sum_{\substack{r \in \mathbf{F}_{q^2} \\ N(r) = T(s)}} \psi([r, s])^{-1} = -\sum_{\substack{r \in \mathbf{F}_{q^2} \\ N(r) = T(s)}} \chi(-r),$$

if $\mu^{q-1} = \lambda t_1^{-1} t_2^{-1} t_3^{-1}$. Otherwise, $f_T(c_{\lambda,\mu})(t) = 0$.

Proof. Now the proofs of Theorems 2.36 and 2.38 almost translate to a proof of this theorem by just changing T_0 or T_1 to T_2. In the case $C_G(t) = T_2$ (i.e., t is of type (t_1, t_2, t_3)) we still have that $Q_{T_1}^{C_G(t)} = Q_{T_2}^{T_2} = 1$. In the case t is of type (t_1, t_2, t_1), (t_1, t_1, t_3), or (t_1, t_2, t_2) it is enough to show $gung^{-1}$ is not semisimple to conclude $Q_{T_2}^{C_G(t)} = Q_{T_2}^{H_i} = 1$ as was done in Theorem 2.36. In Theorem 2.38 it was shown that if t is of type (t_1, t_1, t_1), the value of the Green functions in (1.2) will still always be 1. Thus as in Theorem 2.36, the values of the Green functions in (1.2) will always be 1. The rest of the proof of Theorem 2.36 goes through here verbatim. ∎

Note. In summary the coefficients $f_{T_2}(c)(t)$ are equal to:

If c is of the first type ($c = c_{\lambda,\mu}$),

$$\begin{cases} 2q-1, & \text{if } -\lambda \text{ a multiplicity 3 eigenvalue and } N(n_{\lambda,\mu}) = N(t) \\ -(q+1), & \text{if } -\lambda \text{ a multiplicity 2 eigenvalue and } N(n_{\lambda,\mu}) = N(t) \\ -1, & \text{if } -\lambda \text{ a multiplicity 1 eigenvalue and } N(n_{\lambda,\mu}) = N(t) \\ -\sum\limits_{\substack{r\in \mathbf{F}_{q^2} \\ N(r)=T(s)}} \chi(-r), & \text{if } t_i \neq -\lambda \text{ and } N(n_{\lambda,\mu}) = N(t) \\ 0, & \text{if } N(n_{\lambda,\mu}) \neq N(t). \end{cases}$$

If c is of the second type ($c = c_\lambda$),

$$\begin{cases} 0, & \text{if } t \text{ is not of type } (\lambda,\lambda,\lambda) \\ 1, & \text{if } t \text{ is of type } (\lambda,\lambda,\lambda). \end{cases}$$

The results of Subsections 2.7, 2.8, and 2.9 give us the homomorphisms $f_{T_i} : H \to \mathbf{C}T_i$. Composing these homomorphisms with the irreducible characters of T_i will give all the irreducible characters of H, by Curtis's theorem, Theorem 1.1.

3. THE STRUCTURE CONSTANTS OF THE HECKE ALGEBRA H

In this section we will continue to use the notation given in Section 2. As explained in Subsection 2.3, if

$$c_{u,t} = q^3 e\left[t, \widehat{-u}, t^{-q}\right]$$

and

$$c_u = [u,u,u]e$$

then

$$\left\{c_{u,t}, c_u \mid t,u \in \mathbf{F}_{q^2}^* \text{ and } u^{q+1} = 1\right\}$$

is a basis for H.

To simplify notation, let J be an index set of cardinality $(q+1) + (q^2-1)(q+1) = (q+1)(q^2)$. For $j \le q+1$ let

$$x_j = [u_j, u_j, u_j],$$

where u_j runs over the elements of $\mathbf{F}_{q^2}^*$ such $N(u_j) = 1$. For $j > q+1$ let

$$x_j = \left[t_j, \widehat{-u_j}, t_j^{-q}\right],$$

where again u_j runs over the elements of $\mathbf{F}_{q^2}^*$ such $N(u_j) = 1$ and t_j runs over the elements of $\mathbf{F}_{q^2}^*$. Note that $\operatorname{ind}(x_j) = 1$ for j less than or equal to $q+1$ and $\operatorname{ind}(x_j) = q^3$ for j greater than $q+1$. Now let $a_j = \operatorname{ind}(x_j)ex_je$. Then $\{a_j\}_{j\in J}$ is the same basis of H as described above.

Now given two basis elements a_i and a_j of the Hecke algebra H, their product

$$a_ia_j = \sum_{k\in J} \mu_{ijk} a_k$$

for some structure constants μ_{ijk}. The purpose of this section is to explicitly compute these constants.

In the three cases, (1) $i \le q+1$, $j \le q+1$, (2) $i \le q+1$, $j > q+1$, and (3) $i > q+1$, $j \le q+1$, this computation is trivial. First note that the Hecke algebra is commutative so that cases (2) and (3) are the same. Clearly in case (1), $a_ia_j = a_k$ where $u_iu_j = u_k$. That is, $c_{u_i}c_{u_j} = c_{u_iu_j}$. In case (2) we have that

$$a_ia_j = a_k, \qquad \text{where } x_k = \left[u_it_j, \widehat{-u_iu_j}, u_it_j^{-q}\right].$$

That is, $c_{u_i}c_{u_j,t_j} = c_{u_iu_j,u_it_j}$. Thus the only interesting case is when both i and j are greater than $q+1$. So in the remainder of this section assume that i and j are both greater than $q+1$.

The elements of the group algebra $\mathbf{C}G$ can be identified with the set of functions $f: G \to \mathbf{C}$, where $\sum_{x\in G} \alpha_x x \in \mathbf{C}G$ corresponds to the function $f: G \to \mathbf{C}$ defined by $f(x) = \alpha_x$. Using this correspondence, the structure constants

$$\mu_{ijk} = |U| \sum_{y\in D_i \cap x_k D_j^{-1}} a_i(y)a_j\left(y^{-1}x_k\right),$$

where D_i is the double coset Ux_iU and D_j^{-1} is the double coset $Ux_j^{-1}U$ (see, for example, [4, p. 280]).

First we will compute μ_{ijk} when $k \le q+1$. To do this we need to determine when $y \in D_i \cap x_k D_j^{-1}$. Now since $i > q+1$ we have that $x_i = [t_i, \widehat{-u_i}, t_i^{-q}]$. Thus $D_i = Ux_iU =$

$$\left\{ \begin{pmatrix} bt_i^{-q} & bct_i^{-q} - au_i & bdt_i^{-q} - ac^qu_i + t_i \\ a^qt_i^{-q} & a^qct_i^{-q} - u_i & a^qdt_i^{-q} - c^qu_i \\ t_i^{-q} & ct_i^{-q} & dt_i^{-q} \end{pmatrix} \;\middle|\; N(a) = T(b), N(c) = T(d) \right\}.$$

Also since $k \le q+1$ and $j > q+1$ we have that

$$x_k = [u_k, u_k, u_k]$$

and

$$x_j^{-1} = \left[t_j^q, \widehat{-u_j^{-1}}, t_j^{-1}\right].$$

Thus $x_k D_j^{-1}$ equals

$$\left\{ \begin{pmatrix} yu_k t_j^{-1} & -xu_k u_j^{-1} + yzu_k t_j^{-1} & u_k t_j^q - xz^q u_k u_j^{-1} + ywu_k t_j^{-1} \\ x^q u_k t_j^{-1} & -u_k u_j^{-1} + x^q zu_k t_j^{-1} & -z^q u_k u_j^{-1} + x^q wu_k t_j^{-1} \\ u_k t_j^{-1} & zu_k t_j^{-1} & wu_k t_j^{-1} \end{pmatrix} \;\middle|\; N(x) = T(y), N(z) = T(w) \right\}.$$

A comparison of the (3,1) entries reveals that $t_i^{-q} = u_k t_j^{-1}$ for an element to be in both D_i and $x_k D_j^{-1}$. Then using the fact that none of t_i, t_j, u_k are zero we get from comparison of the first columns and third rows that $b = y$, $a^q = x^q$, $c = z$, and $d = w$. But we already have that $a^{q+1} = b + b^q$. Thus $a^{q+1} = y + y^q = x^{q+1} = a^q x$. Thus $a = 0$ or $a = x$. But if $a = 0$ then $a^q = 0$ so that $x^q = 0$ and thus $x = 0$. So $a = x$. Thus if $u_1 x_i u_2 = x_k u_3 x_j^{-1} u_4$ for some $u_i \in U$ then $u_1 = u_3$ and $u_2 = u_4$. But x_k is central. Thus $x_i = x_k x_j^{-1}$. That is,

$$\left[t_i, \widehat{-u_i}, t_i^{-q}\right] = \left[u_k t_j^q, \widehat{-u_k u_j^{-1}}, u_k t_j^{-1}\right].$$

Thus $u_k = u_i u_j = t_i t_j^{-q} = t_i^{-q} t_j$. But $u_i u_j = t_i^{-q} t_j$ implies $u_i u_j t_i^q = t_j$. So that $u_i^{-1} u_j^{-1} t_i = t_j^q$ since $u_i^q = u_i^{-1}$, $u_j^q = u_j^{-1}$ and $t_i^{q^2} = t_i$. This implies $t_i t_j^{-q} = u_i u_j$. Thus $u_i u_j = t_i^{-q} t_j$ implies $u_i u_j = t_i t_j^{-q}$. Similarly $u_i u_j = t_i t_j^{-q}$ implies $u_i u_j = t_i^{-q} t_j$. Thus $D_i \cap x_k D_j^{-1} = \varnothing$ unless $u_k = u_i u_j = t_i^{-q} t_j$. That is, $D_i \cap x_k D_j^{-1} = \varnothing$ unless $x_k = [u_i u_j, u_i u_j, u_i u_j]$.

So from now on assume $u_k = u_i u_j = t_i^{-q} t_j$. Then by directly multiplying the matrices we see that $D_i = x_k D_j^{-1}$. Thus $D_i \cap x_k D_j^{-1} = D_i$. So $\mu_{ijk} = |U| \sum_{y \in D_i} a_i(y) a_j(y^{-1} x_k)$.

Now

$$a_i = q^3 e x_i e = q^3 |U|^{-1} \sum_{v \in U} \psi(u^{-1}) u x_i |U|^{-1} \sum_{v \in U} \psi(v^{-1}) v$$
$$= q^{-3} \sum_{u, v} \psi(u^{-1} v^{-1}) u x_i v.$$

Also if $y \in D_i$ then $y = u_y x_i v_y$ for some $u_y, v_y \in U$. Thus $a_i(y) = q^{-3}\psi(u_y^{-1}v_y^{-1})$. Also notice that $y^{-1}x_k = (u_y x_i v_y)^{-1}x_k = v_y^{-1}x_i^{-1}u_y^{-1}x_k = v_y^{-1}x_i^{-1}x_k u_y^{-1} = v_y^{-1}x_j u_y^{-1}$, since x_k is central and $x_j = x_i^{-1}x_k$ by above. Thus $a_j(y^{-1}x_k) = q^{-3}\psi(v_y u_y)$. Thus $a_i(y)a_j(y^{-1}x_k) = q^{-3}\psi(u_y^{-1}v_y^{-1})q^{-3}\psi(v_y u_y) = q^{-6}$. Thus $\mu_{ijk} = q^3\sum_{y \in D_i} q^{-6} = q^3$ when $u_k = u_i u_j = t_i^{-q}t_j$, and $\mu_{ijk} = 0$ otherwise. Let

$$\delta(r, s) = \begin{cases} 0 & \text{if } r \neq s \\ 1 & \text{if } r = s. \end{cases}$$

Then $\mu_{ijk} = \delta(u_i u_j, t_i^{-q}t_j)q^3$ only for the one index number $k \leq q+1$ such that $u_k = u_i u_j$. Otherwise $\mu_{ijk} = 0$ when $k \leq q+1$.

It remains to consider μ_{ijk} when i, j, and k are all greater than $q+1$. In this case we have

$$D_i = Ux_iU = U\left[t_i, \widehat{-u_i}, t_i^{-q}\right]U,$$

$$D_j^{-1} = Ux_j^{-1}U = U\left[t_j^q, \widehat{-u_j^{-1}}, t_j^{-1}\right]U$$

and

$$x_k = \left[t_k, \widehat{-u_k}, t_k^{-q}\right].$$

Suppose that $y \in D_i \cap x_k D_j^{-1}$. Then $y = wx_iz = x_k u x_j^{-1} v$ for some $u, v, w, z \in U$. So $yv^{-1} = wx_iz' = x_k u x_j^{-1}$ for some $u, v, w, z' \in U$. Thus

$$[a, b]\left[t_i, \widehat{-u_i}, t_i^{-q}\right][c, d] = \left[t_k, \widehat{-u_k}, t_k^{-q}\right][r, s]\left[t_j^q, \widehat{-u_j^{-1}}, t_j^{-1}\right],$$

for some $a, b, c, d, r, s \in U$ such that $N(a) = T(b)$, $N(c) = T(d)$, and $N(r) = T(s)$. Then

$$\begin{pmatrix} bt_i^{-q} & bct_i^{-q} - au_i & bdt_i^{-q} - ac^q u_i + t_i \\ a^q t_i^{-q} & a^q ct_i^{-q} - u_i & a^q dt_i^{-q} - c^q u_i \\ t_i^{-q} & ct_i^{-q} & dt_i^{-q} \end{pmatrix}$$

$$= \begin{pmatrix} t_j^{-1}t_k & 0 & 0 \\ -r^q t_j^{-1}u_k & u_j^{-1}u_k & 0 \\ st_j^{-1}t_k^{-q} & -ru_j^{-1}t_k^{-q} & t_j^q t_k^{-q} \end{pmatrix}.$$

Thus $s = t_i^{-q}t_jt_k^q$, $b = t_i^qt_j^{-1}t_k$, and $d = t_i^qt_j^qt_k^{-q}$. Also $bct_i^{-q} - au_i = 0$ so that $au_i = bct_i^{-q} = ct_i^qt_j^{-1}t_kt_i^{-q} = ct_j^{-1}t_k$. Thus $a = ct_j^{-1}t_ku_i^{-1}$. Also $r = -ct_i^{-q}t_k^qu_j$. Substituting these values for a, b, d, r, and s in the above matrix equation gives the new equation

$$\begin{pmatrix} t_j^{-1}t_k & 0 & t_i^qt_j^{q-1}t_k^{-q+1} - c^{q+1}t_j^{-1}t_k + t_i \\ c^qt_i^{-q}t_j^{-q}t_k^qu_i & c^{q+1}t_i^{-q}t_j^{-q}t_k^qu_i - u_i & 0 \\ t_i^{-q} & ct_i^{-q} & t_j^qt_k^{-q} \end{pmatrix}$$
$$= \begin{pmatrix} t_j^{-1}t_k & 0 & 0 \\ c^qt_i^{-1}t_j^{-1}t_ku_j^{-1}u_k & u_j^{-1}u_k & 0 \\ t_i^{-q} & ct_i^{-q} & t_j^qt_k^{-q} \end{pmatrix}.$$

But $c^{q+1} = d + d^q = t_i^qt_j^qt_k^{-q} + t_it_jt_k^{-1}$. Thus $t_i^qt_j^{q-1}t_k^{-q+1} - c^{q+1}t_j^{-1}t_k + t_i = t_i^qt_j^{q-1}t_k^{-q+1} - t_i^qt_j^{q-1}t_k^{-q+1} - t_i + t_i = 0$ and $c^{q+1}t_i^{-q}t_j^{-q}t_k^qu_i - u_i = u_i + t_i^{-q+1}t_j^{-q+1}t_k^{q-1}u_i - u_i = t_i^{-q+1}t_j^{-q+1}t_k^{q-1}u_i$. So

$$\begin{pmatrix} t_j^{-1}t_k & 0 & 0 \\ c^qt_i^{-q}t_j^{-q}t_k^qu_i & t_i^{-q+1}t_j^{-q+1}t_k^{q-1}u_i & 0 \\ t_i^{-q} & ct_i^{-q} & t_j^qt_k^{-q} \end{pmatrix}$$
$$= \begin{pmatrix} t_j^{-1}t_k & 0 & 0 \\ c^qt_i^{-1}t_j^{-1}t_ku_j^{-1}u_k & u_j^{-1}u_k & 0 \\ t_i^{-q} & ct_i^{-q} & t_j^qt_k^{-q} \end{pmatrix}.$$

Thus $u_k = t_i^{-q+1}t_j^{-q+1}t_k^{q-1}u_iu_j$. Notice that now the (2,1) entries can both be simplified to $c^qt_i^{-q}t_j^{-q}t_k^qu_i$ by substituting in this expression for u_k. Also note that the requirements $N(a) = T(b)$, $N(c) = T(d)$, and $N(r) = T(s)$ are satisfied when we take the above formulas for a, b, c, d, r, and s. Thus the above shows that μ_{ijk} will be nonzero only when x_k is of the form $[t_k, -\widehat{t_i^{-q+1}t_j^{-q+1}}t_k^{q-1}u_iu_j, t_k^{-q}]$, i.e., when the u_k entry of x_k is $t_i^{-q+1}t_j^{-q+1}t_k^{q-1}u_iu_j$. So we fix an x_k of this form.

Let

$$A_c = \left[ct_j^{-1}t_ku_i^{-1}, t_i^qt_j^{-1}t_k\right] = [a, b],$$
$$B_c = \left[c, t_i^qt_j^qt_k^{-q}\right] = [c, d],$$

and

$$C_c = \left[-ct_i^{-q}t_k^q u_j, t_i^{-q}t_j t_k^q\right] = [r, s],$$

where $c \in \mathbf{F}_{q^2}$ is such that $c^{q+1} = t_i^q t_j^q t_k^{-q} + t_i t_j t_k^{-1}$.

Then the above calculations show that

$$\begin{aligned}(Ux_iU) &\cap \left(x_k U x_j^{-1} U\right) \\ &= \left\{y \mid yv^{-1} \in (Ux_iU) \cap \left(x_k U x_j^{-1}\right), v \in U\right\} \\ &= \left\{y \mid yv^{-1} \in A_c x_i B_c, v \in U, c^{q+1} = t_i^q t_j^q t_k^{-q} + t_i t_j t_k^{-1}\right\} \\ &= \left\{y \mid yv^{-1} \in x_k C_c x_j^{-1}, v \in U, c^{q+1} = t_i^q t_j^q t_k^{-q} + t_i t_j t_k^{-1}\right\}.\end{aligned}$$

So suppose that y is in the above set. So $y = A_c x_i B_c v = x_k C_c x_j^{-1} v$ for some $v \in U$ and for some c such that $c^{q+1} = t_i^q t_j^q t_k^{-q} + t_i t_j t_k^{-1}$. Now, using the notation explained earlier,

$$a_i = |U|^{-1} \sum_{u, w \in U} \psi(u^{-1}w^{-1}) u x_i w.$$

Thus $a_i(y) = |U|^{-1}\psi(A_c^{-1})\psi((B_c v)^{-1})$. Also

$$a_j = |U|^{-1} \sum_{u, w \in U} \psi(u^{-1}w^{-1}) u x_j w$$

and $y = x_k C_c x_j^{-1} v$. So that $y^{-1}x_k = v^{-1} x_j C_c^{-1}$. Thus

$$a_j\left(y^{-1}x_k\right) = a_j\left(v^{-1}x_j C_c^{-1}\right) = |U|^{-1}\psi(v)\psi(C_c).$$

Thus

$$\begin{aligned}\mu_{ijk} &= |U| \sum_{D_i \cap x_k D_j^{-1}} a_i(y) a_j\left(y^{-1}x_k\right) \\ &= |U| \sum_{v \in U} \sum_{\substack{c \\ c^{q+1} = t_i^q t_j^q t_k^{-q} + t_i t_j t_k^{-1}}} |U|^{-1}\psi\left(A_c^{-1}\right)\psi\left(v^{-1}\right) \\ &\qquad \times \psi\left(B_c^{-1}\right)|U|^{-1}\psi(v)\psi(C_c) \\ &= |U|^{-1} \sum_{v \in U} \sum_{c} \psi\left(A_c^{-1}B_c^{-1}C_c\right) \\ &= \sum_{\substack{c \\ c^{q+1} = t_i^q t_j^q t_k^{-q} + t_i t_j t_k^{-1}}} \psi\left(A_c^{-1}B_c^{-1}C_c\right).\end{aligned}$$

But $A_c^{-1}B_c^{-1}C_c = [-c(1 + t_j^{-1}t_k u_i^{-1} + t_i^{-q}t_k^q u_j), *]$, where the $*$ entry is an expression involving t_i, t_j, u_i, u_j, t_k, and c. Recall there exists an additive character χ of $\mathbf{F}_{q^2}$ such that $\psi([a,b]) = \chi(a)$. Thus

$$\mu_{ijk} = \sum_{\substack{c \\ c^{q+1}=t_i^q t_j^q t_k^{-q} + t_i t_j t_k^{-1}}} \psi\left(A_c^{-1}B_c^{-1}C_c\right)$$

$$= \sum_{\substack{c \\ c^{q+1}=t_i^q t_j^q t_k^{-q} + t_i t_j t_k^{-1}}} \chi\left(-c\left(1 + t_j^{-1}t_k u_i^{-1} + t_i^{-q}t_k^q u_j\right)\right)$$

$$= \sum_{N(x)=1} \chi\left(-a_k x\left(1 + t_j^{-1}t_k u_i^{-1} + t_i^{-q}t_k^q u_j\right)\right),$$

where in the last summation a_k is a fixed element of $\mathbf{F}_{q^2}$ such that $a_k^{q+1} = t_i^q t_j^q t_k^{-q} + t_i t_j t_k^{-1}$.

Combining all of the above calculations gives the following proposition, which describes the structure constants.

PROPOSITION 3.1. *Let $\{c_{u,t}, c_u\}$ be the basis for the Hecke Algebra described above. Then*

$$c_u c_v = c_{uv}, \tag{3.2}$$

$$c_{u,t} c_v = c_{vu,vt}, \tag{3.3}$$

$$c_{u_1,t_1} c_{u_2,t_2} = q^3 \delta(u_1 u_2, t_1^{-q} t_2) c_{u_1 u_2} + \sum_{t_k \in \mathbf{F}_{q^2}^*} \sum_{N(x)=1} \chi\left(-a_k x\left(1 + t_2^{-1} t_k u_1^{-1} + t_1^{-q} t_k^q u_2\right)\right) c_{u_k,t_k}, \tag{3.4}$$

where in (3.4), $u_k = t_1^{-q+1} t_2^{-q+1} t_k^{q-1} u_1 u_2$ *and* a_k *is fixed for a fixed* t_k *and is such that* $a_k^{q+1} = t_1^q t_2^q t_k^{-q} + t_1 t_2 t_k^{-1}$.

For a possibly more usable form of (3.4) of this proposition make the change of variable $r = t_1^{-1} t_2^{-1} t_k$ in the equation

$$c_{u_1,t_1} c_{u_2,t_2} = q^3 \delta(u_1 u_2, t_1^{-q} t_2) c_{u_1 u_2} + \sum_{t_k \in \mathbf{F}_{q^2}^*} \times \sum_{\substack{c \\ c^{q+1}=t_1^q t_2^q t_k^{-q} + t_1 t_2 t_k^{-1}}} \chi\left(-c\left(1 + t_2^{-1} t_k u_1^{-1} + t_1^{-q} t_k^q u_2\right)\right) c_{u_k,t_k}.$$

Then $u_k = r^{q-1}u_1u_2$ and (3.4) becomes

$$c_{u_1,t_1}c_{u_2,t_2} = q^3\delta(u_1u_2, t_1^{-q}t_2)c_{u_1u_2} + \sum_{r\in \mathbf{F}_{q^2}^*} \times \sum_{\substack{c\in \mathbf{F}_{q^2} \\ N(c)=T(r^{-1})}} \chi\big(-c\big(1 + rt_1u_1^{-1} + r^qt_2^qu_2\big)\big)c_{r^{q-1}u_1u_2, rt_1t_2}.$$

For another form of (3.4) we can rewrite the preceding equation in terms of matrices. Let

$$K_i = [t_i, u_i, t_i^{-q}],$$
$$R = [r, 1, r^{-q}],$$

and

$$\tilde{R} = [r, r^{q-1}, r^{-q}].$$

Also let $U_r = \{A \in U \mid \text{the (1,3) entry of } A \text{ is } r\}$.

Then $A_c = (K_1R)B_c(K_1R)^{-1}$ and $C_c = (K_2R)^qB_c^{-1}(K_2R)^{-q}$. Thus (3.4) becomes

$$c_{u_1,t_1}c_{u_2,t_2} = c_{K_1}c_{K_2} = q^3\delta(u_1u_2, t_1^{-q}t_2)c_{u_1u_2} + \sum_{r\in \mathbf{F}_{q^2}^*} \sum_{B\in U_r^{-1}} \psi\big((K_1R)B^{-1}(K_1R)^{-1}\big)\psi(B^{-1}) \times \psi\big((K_2R)^qB^{-1}(K_2R)^{-q}\big)c_{\tilde{R}K_1K_2}.$$

Thus (3.4) becomes

$$c_{K_1}c_{K_2} = q^3\delta(u_1u_2, t_1^{-q}t_2)c_{u_1u_2} + \sum_{r\in \mathbf{F}_{q^2}^*} \sum_{B\in U_r^{-1}} \psi^{K_1R}(B^{-1}) \times \psi(B^{-1})\psi^{(K_2R)^q}(B^{-1})c_{\tilde{R}K_1K_2}.$$

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