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Sign-changing solutions for some fourth-order nonlinear elliptic problems ☆

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Abstract

In this paper, we consider the existence and multiplicity of sign-changing solutions for some fourth-order nonlinear elliptic problems and some existence and multiple are obtained. The weak solutions are sought by means of sign-changing critical theorems.

Keywords: Sign-changing solutions; Critical point; Elliptic problems

1. Introduction

Let Ω be a bounded open set in R^n with smooth boundary. The purpose of this paper is to investigate the existence and multiplicity of sign-changing solutions to the fourth-order nonlinear elliptic boundary value problems

$$\begin{cases} \Delta^2 u + c\Delta u = f(x,u) & \text{in } \Omega, \\ u|_{\partial\Omega} = \Delta u|_{\partial\Omega} = 0 \end{cases} \tag{1.1}$$

where Δ^2 denotes the biharmonic operator, $c \in R$ and $f : \Omega \times R \to R$ is a Caratheodory function with subcritical growth: $|f(x,t)| \leqslant C(1+|t|^{s-1})$, $\forall x \in \Omega$, $\forall t \in R$, $s \in (2, 2^*)$ $(N \geqslant 3)$, $s \in (2, +\infty)$ $(N \leqslant 2)$.

In problem (1.1), let $f(x,u) = b[(u+1)^+ - 1]$, then we get the following Dirichlet problem:

$$\begin{cases} \Delta^2 u + c\Delta u = b\big[(u+1)^+ - 1\big] & \text{in } \Omega, \\ u|_{\partial\Omega} = \Delta u|_{\partial\Omega} = 0 \end{cases} \tag{1.2}$$

where $u^+ = \max\{u, 0\}$ and $b \in R$.

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Thus fourth-order problems have been studied by many authors, in [1] Lazer and McKenna have pointed out that this type of nonlinearity furnishes a model to study travelling waves in suspension bridges. Since then more general nonlinear fourth-order elliptic boundary value problems have been studied. There are many results about problems (1.1) and (1.2). We refer the reader to [2,3] for some references along this line.

For problem (1.2), Lazer and McKenna [2] proved the existence of $2k-1$ solutions when $N=1$ and $b>\lambda_k(\lambda_k-c)$ by the global bifurcation method. In [5], Tarantello found a negative solution when $b\geqslant\lambda_1(\lambda_1-c)$ by a degree argument. For problem (1.1) when $f(x,u)=bg(x,u)$, Micheletti and Pistoia [3,4] proved that there exist two or three solutions for a more general nonlinearity g by variational method. Zhang [6] proved the existence of solutions for a more general nonlinearity $f(x,u)$ under some weak assumptions. Zhang and Li [7] proved the existence of multiple nontrivial solutions by means of Morse theory and local linking. But the existence and multiple of sign-changing solutions for (1.1) have not been studied.

In this paper, we study the existence and multiple of sign-changing solutions for problem (1.1). The results include the existence of four sign-changing solutions or infinitely many sign-changing solutions for (1.1) which are different from the references [1–7]. All these results are new.

The plan of the following sections are as follows. In Section 2 we give some notations and preliminaries. In Section 3 we give some results. Section 4 is devoted to the proofs of these results.

2. Preliminaries and statements

Let Ω be a bounded open set in R^n with smooth boundary and $f:\Omega\times R\to R$ is a Caratheodory function with subcritical growth: $|f(x,t)|\leqslant C(1+|t|^{s-1})$, where $s\in(2,2^*)$ $(N\geqslant 3)$, $s\in(2,+\infty)$ $(N\leqslant 2)$ for all $x\in\Omega$ and $t\in R$. From now on, letter C is indiscriminately used to denote various positive constants. Let λ_k $(k=1,2,\ldots)$ denote the eigenvalue and φ_k $(k=1,2,\ldots)$ the corresponding eigenfunctions of the eigenvalue problem

$$\begin{cases}\Delta u+\lambda u=0 & \text{in } \Omega,\\ u|_{\partial\Omega}=0\end{cases} \tag{2.1}$$

where each eigenvalue λ_k is repeated as often as multiplicity recall that $0<\lambda_1<\lambda_2\leqslant\lambda_3\leqslant\cdots$, $\lambda_k\to\infty$. Then φ_1 is positive (or negative) and eigenfunctions associated to λ_i $(i\geqslant 2)$ is sign-changing. By reference [8], the eigenvalue problem

$$\begin{cases}\Delta^2 u+c\Delta u=\mu u & \text{in } \Omega,\\ u|_{\partial\Omega}=\Delta u|_{\partial\Omega}=0\end{cases} \tag{2.2}$$

has infinitely many eigenvalues

$$\mu_k=\lambda_k(\lambda_k-c),\quad k=1,2,\ldots,$$

and corresponding eigenfunctions $\varphi_k(x)$.

We will always assume $c<\lambda_1$. Let V denote the Hilbert space $H^2(\Omega)\cap H_0^1(\Omega)$ equipped with the inner product

$$\langle u,v\rangle=\int_\Omega[\Delta u\Delta v-\nabla u\nabla v]\,dx. \tag{2.3}$$

Then we may denote an element u of V as

$$u=\sum_{k=1}^\infty a_k\varphi_k,\qquad \sum_{k=1}^\infty a_k^2<\infty,$$

φ_k and φ_l $(k\neq l)$ is orthogonal base for V. We denote by $\|u\|_p$ the norm in $L^p(\Omega)$ and by $\|u\|$ the norm in V is given by

$$\|u\|^2=\langle u,u\rangle.$$

Let V' denote the dual of V and $\langle,\rangle$ be the duality pairing between V' and V. Let X_k denote the eigenspace associated to μ_k, then $V=\overline{\bigoplus_{j\in N}X_j}$. Let $V_k=X_1\oplus\cdots\oplus X_k$, $B_R(0)=\{u\in V,\ \|u\|<R\}$.

Definition 2.1. E is Hilbert space, $G \in C^1(E, R)$. G satisfies w-PS condition on V if $\{u_n\} \in E$ and $G(u_n)$ is bounded, $G'(u_n) \to 0$, we have either $\{u_n\}$ is bounded and has a convergent subsequence or $\|G'(u_n)\|\|u_n\| \to \infty$.

Definition 2.2. We say that $u \in V$ is the solution of problem (1.1) if the identity

$$\int_\Omega [\Delta u \Delta v - c\nabla u \nabla v]\,dx = \int_\Omega f(x,u)v\,dx \tag{2.4}$$

holds for any $v \in V$.

Definition 2.3. u is the solution of (1.1): if $u \in \{u \in E\colon u(x) \geqslant 0,\ u \neq 0\}$, then u is positive solution of (1.1); if $u \in \{u \in E\colon u(x) \leqslant 0,\ u \neq 0\}$, then u is negative solution of (1.1); if $u \in \{u \in E\colon \operatorname{meas}\{x \in \Omega\colon u(x) > 0\} > 0$, $\operatorname{meas}\{x \in \Omega\colon u(x) < 0\} > 0\}$, then u is sign-changing solution of (1.1).

Assume H is Banach space, $\Phi = \{\Gamma(\cdot,\cdot) \in C([0,1] \times E, E)\}$, where $\Gamma(\cdot,\cdot)$ satisfies

(a) $\Gamma(0,\cdot) = \mathrm{id}$;
(b) $\forall t \in [0,1)$, $\Gamma(t,\cdot)$ is a homeomorphism of E onto itself, $(t,x) \mapsto \Gamma(t,\cdot)^{-1}(x)$ is continuous on $[0,1) \times E$;
(c) there exists $x_0 \in H$ such that $\Gamma(1,x) = x_0$ for each $x \in H$ and $\Gamma(t,x) \to x_0$ as $t \to 1$ uniformly on bounded subsets of H.

Definition 2.4. (See [10, p. 21].) A subset A of H is linked (with respect to Φ) to a subset B of H if $A \cap B = \emptyset$, for every $\Gamma \in \Phi$, there is $t \in [0,1]$ such that $\Gamma(t,A) \cap B \neq \emptyset$.

In this paper, we need the following four propositions.

Proposition 2.1. *(See [11, Theorem 3.2].) Assume H is Hilbert space, f satisfies* PS *condition on H and $f'(u)$ has the expression $f'(u) = u - Au$. D_1 and D_2 are open convex subset of H, $D_1 \cap D_2 \neq \emptyset$, $A(\partial D_1) \subset D_1$, $A(\partial D_2) \subset D_2$. If there exists a path $h : [0,1] \to H$ such that*

$$h(0) \in D_1 \setminus D_2, \qquad h(1) \in D_2 \setminus D_1$$

and

$$\inf_{u \in \overline{D_1 \cap D_2}} f(u) > \sup_{t \in [0,1]} f\big(h(t)\big),$$

then f has at least four critical points: $u_1 \in D_1 \cap D_2$, $u_2 \in D_1 \setminus \overline{D_2}$, $u_3 \in D_2 \setminus \overline{D_1}$, $u_4 \in H \setminus (\overline{D_1} \cup \overline{D_2})$.

Proposition 2.2. *(See [8, Theorem 2.1].) Let E be a Hilbert space with inner product $\langle,\rangle$ and norm $\|.\|$. Assume that E has an orthogonal decomposition $E = N \oplus M$ with $\dim N < \infty$. Let $G \in C^1(E,R)$ and the gradient G' be of the form*

$$G'(u) = u - J'(u)$$

where $J' : E \to E$ is a continuous operator. Let P denote a closed convex positive cone of E; $D_0^{(i)}$ be an open convex subset of E, $i = 1,2$, $S = E \setminus W$, $W = D_0^{(1)} \cup D_0^{(2)}$. Assume

(H_1) $J'(D_0^{(i)}) \subset D_0^{(i)}$, $i = 1,2$.
(H_2) *If $D_0^{(1)} \cap D_0^{(2)} = \emptyset$, then either $D_0^{(1)} = \emptyset$ or $D_0^{(2)} = \emptyset$.*
(H_3) *There exist $\delta > 0$ and $z_0 \in N$ with $\|z_0\| = 1$ such that*

$$B := \big\{u \in M\colon \|u\| \geqslant \delta\big\} \cup \big\{sz_0 + v\colon v \in M,\ s \geqslant 0,\ \|sz_0 + v\| = \delta\big\} \subset S.$$

Let G maps bounded sets to bounded sets and satisfies w-PS *and*

$$b_0 = \inf_M G \neq -\infty, \qquad a_0 = \sup_N G \neq +\infty.$$

Then G has a critical point in S with critical value $\geqslant \inf_B G$.

Proposition 2.3. *(See [9, Corollary 2.1].) Assume E is a Hilbert space with inner product $\langle,\rangle$ and the corresponding norm $\|.\|$, $G \in C^1(E,R)$ and $G(u) = \frac{1}{2}\|u\|^2 - J(u)$, $u \in E$, where $J \in C^1(E,R)$ maps bounded sets to bounded sets. $G_\lambda(u) = \frac{\lambda}{2}\|u\|^2 - J(u)$, $\lambda \in \Lambda = (\frac{1}{2}, 1)$. P denote a closed convex cone of E. Assume*:

(A$_1$) *There exists $\mu_0 > 0$ such that $\operatorname{dist}(J'(u), \pm P) \leqslant \frac{1}{5}\operatorname{dist}(u, \pm P)$ for all $u \in E$ with $\operatorname{dist}(u, \pm P) < \mu_0$.*
(A$_2$) *$\pm D_0 = \{u \in E\colon \operatorname{dist}(u, \pm P) < \mu_0\}$, $D = D_0 \cup (-D_0)$, $S = E \setminus D$, let A be a bounded subset of E and link a subset B of E, $B \subset S$ and*

$$a_0(\lambda) = \sup_A G_\lambda \leqslant b_0(\lambda) = \inf_B G_\lambda, \quad \forall \lambda \in \Lambda.$$

J' is compact, then for almost all $\lambda \in \Lambda$, G_λ has a sign-changing critical point in S.

Proposition 2.4. *(See [9, Theorem 3.1].) Assume E is a Hilbert space with inner product $\langle,\rangle$ and the corresponding norm $\|.\|$, $E = \bigoplus_{j \in N} X_j$ with $\dim X_j < \infty$ for any $j \in N$, where N denotes the set of all positive integers. $G \in C^1(E,R)$ and $G(u) = \frac{1}{2}\|u\|^2 - J(u)$, where $J \in C^1(E,R)$ maps bounded sets to bounded sets, $G_\lambda(u) = \frac{\lambda}{2}\|u\|^2 - J(u)$, $\lambda \in \Lambda = (\frac{1}{2}, 1)$. P denotes a closed convex of E,*

$$\pm D_0 = \{u \in E\colon \operatorname{dist}(u, \pm P) < \mu_0\}, \qquad D = D_0 \cup (-D_0), \qquad S = E \setminus D,$$

$$E_k = \bigoplus_{j=1}^{k} X_j, \qquad Z_k = \overline{\bigoplus_{j=k}^{\infty} X_j}, \qquad B_k = \{u \in E_k\colon \|u\| \leqslant \rho_k\}, \qquad N_k = \{u \in Z_k\colon \|u\| = r_k\}$$

where $\rho_k > r_k > 0$. For $k \geqslant 2$, assume

$$\Gamma_k = \{\gamma \in C([0,1] \times B_k, E)\colon \gamma(t,u) \text{ is odd in } u \text{ and } \gamma(t,\cdot)|_{\partial B_k} = \mathrm{id} \text{ for each } t \in [0,1],$$
$$\gamma(t, D) \subset D \text{ for all } t \in [0,1]\}$$
$$a_k(\lambda) = \max_{\partial B_k} G_\lambda, \qquad b_k(\lambda) = \inf_{N_k} G_\lambda, \qquad c_k(\lambda) = \inf_{\gamma \in \Gamma_k} \max_{\gamma([0,1],B_k) \cap S} G_\lambda.$$

If (A$_1$) *and the following* (A$_3$) *hold*:

(A$_3$) *$a_k(\lambda) < b_k(\lambda)$ for any $\lambda \in \Lambda$, $N_k \subset S$,*

G_λ is even for any $\lambda \in \Lambda$, then for almost all $\lambda \in \Lambda$, there is a sequence $\{u_m\}$ depending on λ such that

$$\sup_m \|u_m\| < \infty, \quad u_m \in S, \qquad G'_\lambda(u_m) \to 0, \qquad G_\lambda(u_m) \to c_k(\lambda) \in \Big[b_k(\lambda), \max_{u \in B_k} G(u)\Big].$$

In particular, if J' is compact, then for almost all $\lambda \in \Lambda$, G_λ has a sign-changing critical point $u_\lambda \in S$ and $G_\lambda(u_\lambda) \in [b_k(\lambda), \max_{u \in B_k} G(u)]$.

The solutions of (1.1) are corresponding to the critical points of the following C^1-functional:

$$G(u) = \frac{1}{2}\|u\|^2 - \int_\Omega F(x,u)\,dx = \frac{1}{2}\|u\|^2 - J(u)$$

where $F(x,t) = \int_0^t f(x,s)\,ds$. The gradient of G at u is given by

$$G'(u) = u - J'(u).$$

Then $\langle J'(u), v\rangle = \int_\Omega f(x,u)v\,dx$, $\forall v \in V$,

$$|f(x,t)| \leqslant C(1 + |t|^{s-1}), \quad \forall x \in \Omega,\ \forall t \in R,$$

when $N \geqslant 3$, $s \in (2, 2^*)$, when $N \leqslant 2$, $s \in (2, +\infty)$, by [12, Theorem 6.3.2], $G \in C^1(E,R)$ and J' is compact.

3. Main results

Suppose

(g_1) $f \in C(\overline{\Omega} \times R, R)$;
(g_2) there exists $\eta > 2$ such that $\forall x \in \Omega$, $\forall t \in R$,

$$0 \leqslant \eta F(x,t) \leqslant f(x,t)t.$$

Moreover $f(x,t) = o(|t|)$ as $t \to 0$ uniformly in $x \in \Omega$.

It is easy seen that (g_1) and (g_2) hold for nonlinearity of the form

$$f(x,t) = \frac{1}{|x|+1}|t|^{p-2}t$$

where $p \in (2, 2^*)$ $(N \geqslant 3)$, $p \in (2, +\infty)$ $(N \leqslant 2)$.

(h_1) $\mu_l t^2 - W_1(x) \leqslant 2F(x,t) \leqslant \mu_{l+1}t^2 + W_2(x)$, a.e. $x \in \Omega$, $t \in R$, where $W_1, W_2 \in L^1(\Omega)$, $l \geqslant 2$.

This assumption implies the following double resonance case:

$$\mu_l \leqslant \liminf_{|t|\to\infty} \frac{2F(x,t)}{t^2} \leqslant \limsup_{|t|\to\infty} \frac{2F(x,t)}{t^2} \leqslant \mu_{l+1}, \quad \text{a.e. } x \in \Omega,$$

as well as jumping and oscillating between μ_l, μ_{l+1}. Furthermore, if we assume

(h_2) $f(x,t), t \geqslant 0$, for a.e. $x \in \Omega$, $t \in R$; $f(x,t) = o(|t|)$ as $|t| \to 0$ uniformly for $x \in \Omega$,

then we have

Theorem 3.1. *Assume* (g_1) *and* (g_2) *hold, then* (1.1) *has four solutions*: *one naught solution, one positive solution, one negative solution and one sign-changing solution.*

Theorem 3.2. *Assume* (g_2) *and* (h_1) *hold, then* (1.1) *has at least a sign-changing solution.*

Remark. f has subcritical growth: $|f(x,t)| \leqslant C(1 + |t|^{s-1})$, $\forall x \in \Omega$, $\forall t \in R$, $s \in (2, 2^*)$ $(N \geqslant 3)$, $s \in (2, +\infty)$ $(N \leqslant 2)$, but by (g_2) F is superquadratic because $\eta > 2$. It is easy seen that this subcritical condition and (g_2) hold for nonlinearity of the form

$$f(x,t) = |t|^{p-2}t$$

where $p \in (2, 2^*)$ $(N \geqslant 3)$, $p \in (2, +\infty)$ $(N \leqslant 2)$.

Theorem 3.3. *Assume* (h_1) *and* (h_2) *hold. Moreover if*

(h_3) $\mu_l < L = \liminf_{|t|\to\infty} \frac{f(x,t)}{t} \leqslant \limsup_{|t|\to\infty} \frac{f(x,t)}{t} \leqslant \mu_{l+1}$ *a.e.* $x \in \Omega$;
(h_4) *there exists* $\alpha > 0$ *such that*

$$\lim_{|t|\to+\infty} \frac{f(x,t) - 2F(x,t)}{|t|^\alpha} = \beta(x) \quad \text{a.e. } x \in \Omega,$$

where $\int_\Omega \beta(x)|w(x)|^\alpha\,dx > 0$ *on the set* $\{w \in X_{l+1}\colon \|w\| = 1\}$, *then* (1.1) *has at least one sign-changing solution.*

Suppose

$$\lim_{t\to+\infty} \frac{f(x,t)}{t} = b_+(x), \qquad \lim_{t\to-\infty} \frac{f(x,t)}{t} = b_-(x),$$

uniformly for $x \in \Omega$. For $k \geqslant 2$,

(a_1) there is a constant $F_0 > \mu_k$ such that

$$4F(x,t) \leqslant F_0 t^2 \quad \text{for all } x \in \Omega,\ t \in R.$$

(a_2) $\forall (x,t) \in \Omega \times R$, $2F(x,t) \geqslant \mu_{k-1} t^2 - W_0(x)$, where $F(x,t) = \int_0^t f(x,s)\,ds$, $0 < \int_\Omega W_0(x)\,dx < \infty$.

Choose μ_l such that

$$\mu_l \geqslant \frac{64\mu_k^2}{\mu_k(\mu_k - \mu_{k-1})} F_0, \tag{3.1}$$

then exists positive constant C_{l-1} such that $\|u\|_\infty \leqslant C_{l-1}\|u\|$ $u \in V_{l-1}$.

(a_3) $2F(x,t) \leqslant \frac{\mu_k + \mu_{k-1}}{4} t^2$, for all $x \in \Omega$ and $|t| \leqslant r_0$, where

$$r_0 > C_{l-1}\left(\frac{48\mu_k}{\mu_k - \mu_{k-1}} \int_\Omega W_0(x)\,dx\right)^{\frac{1}{2}}.$$

(a_4) $H(x,t) = f(x,t)t - 2F(x,t) > 0$ for all $x \in \Omega$ and $t \neq 0$, $H(x,t)$ is convex in t.

Theorem 3.4. *Assume* (a_1)–(a_4) *and* (h_2) *hold and* $\mu_k < b_\pm(x)$ *for all* $\forall x \in \Omega$, *then* (1.1) *has one sign-changing solution.*

Theorem 3.5. *Assume* (h_2) *and*

(b_1) $\liminf_{|t|\to\infty} \frac{f(x,t)}{t} = \infty$ *uniformly for* $x \in \Omega$.
(b_2) $f(x,t)$ *is odd in* t.
(b_3) $\frac{f(x,t)}{t}$ *is nondecreasing in* $t > 0$, (1.1) *has infinitely many sign-changing solution.*

It is easy seen that (b_1)–(b_3) *and* (h_2) *hold for nonlinearity of the form*

$$f(x,t) = |t|^{p-2}t$$

where $p \in (2, 2^*)$ $(N \geqslant 3)$, $p \in (2, +\infty)$ $(N \leqslant 2)$.

4. Proof of theorems

For $\mu_{0>0}$, assume

$$D_0(\mu_0) = \{u \in V\colon \operatorname{dist}(u,P) < \mu_0\},$$
$$-D_0(\mu_0) = \{u \in V\colon \operatorname{dist}(u,-P) < \mu_0\},$$
$$P = \{u \in V\colon u(x) \geqslant 0 \text{ a.e. } x \in \Omega\}.$$

Lemma 4.1. *Assume* (g_2) *holds, then* G *satisfies* PS *condition.*

Proof. Assume $\{u_n\} \subset V$, $|G(u_n)| \leqslant C$, $G'(u_n) \to 0$. It suffices to prove that $\{u_n\}$ is bounded. By (g_2)

$$\eta C + \|G'(u_n)\|\|u_n\| \geqslant \eta G(u_n) - \langle G'(u_n), u_n\rangle \geqslant \frac{\eta - 2}{2}\|u_n\|^2,$$

thus $\{u_n\}$ is bounded. □

Lemma 4.2. *Assume* (g_2) *holds, then there exists* $\epsilon_0 > 0$ *such that*

(i) $J'(\partial D_0(\epsilon_0)) \subset D_0(\epsilon_0)$, *and if* $u \in D_0(\epsilon_0)$ *is the solution of* (1.1), *then* $u \in P$;
(ii) $J'(\partial(-D_0(\epsilon_0))) \subset -D_0(\epsilon_0)$, *and if* $u \in -D_0(\epsilon_0)$ *is the solution of* (1.1), *then* $u \in -P$.

Proof. Let $u^{\pm} = \max\{\pm u, 0\}$. $\forall u \in V$, by the definition of V and the Sobolev embedding theorem, if $s \in (2, 2^*)$, there exists $C_s > 0$ such that

$$\|u^+\|_s \leqslant \inf_{w\in(-P)} \|u-w\|_s \leqslant C_s \inf_{w\in(-P)} \|u-w\| = C_s \operatorname{dist}(u, -P). \tag{4.1}$$

By $|f(x,t)| \leqslant C(1+|t|^{s-1})$ and (g_2): $\forall \epsilon > 0$, there exists $C_\epsilon > 0$, such that

$$f(x,t)t \leqslant \epsilon t^2 + C_\epsilon |t|^s, \quad \forall x \in \Omega,\ \forall t \in R. \tag{4.2}$$

Assume $v = J'(u)$. Then by (4.1) and (4.2), for ϵ small enough,

$$\begin{aligned}
\operatorname{dist}(v, -P)\|v^+\| &\leqslant \|v^+\|^2 \\
&= \langle v, v^+\rangle \\
&\leqslant \int_\Omega f(x, u^+)v^+\,dx \\
&\leqslant \int_\Omega \left(\epsilon|u^+| + C_\epsilon |u^+|^{s-1}\right)|v^+|\,dx \\
&\leqslant \left(\frac{1}{2}\operatorname{dist}(u, -P) + C \operatorname{dist}(u, -P)^{s-1}\right)\|v^+\|.
\end{aligned}$$

That is,

$$\operatorname{dist}\big(J'(u), -P\big) \leqslant \frac{1}{2}\operatorname{dist}(u, -P) + C\big(\operatorname{dist}(u, -P)^{s-1}\big). \tag{4.3}$$

So there exists $\epsilon_0 > 0$ such that $\operatorname{dist}(J'(u), -P) \leqslant \frac{3}{4}\epsilon_0$ for every $u \in \partial(-D_0(\epsilon_0))$. Thus $J'(\partial(-D_0(\epsilon_0))) \subset -D_0(\epsilon_0)$. If $u \in D_0(\epsilon_0)$ is the solution of (1.1), then $G'(u) = u - J'(u) = 0$, that is, $J'(u) = u$. By (4.3), $u \in -P$, (i) holds. (ii) can be proved analogously. □

Lemma 4.3. *Assume* (g_2) *holds, then*

$$\inf_{\overline{D_0(\epsilon)}\cap\overline{-D_0(\epsilon)}} G(u) = d_0 > -\infty.$$

Proof. By (g_2), (4.2) and Holder inequality

$$\begin{aligned}
G(u) &\geqslant -\int_\Omega F\big(x, u(x)\big)\,dx \\
&\geqslant -\frac{1}{\eta}\int_\Omega f\big(x, u(x)\big)u(x)\,dx \\
&\geqslant -\frac{C}{\eta}\big(\|u\|_2^2 + \|u\|_p^p\big).
\end{aligned}$$

According to (4.1), $\|u^+\|_s \leqslant C_s \operatorname{dist}(u, -P) \leqslant C_s\epsilon_0$, $\|u^-\|_s \leqslant C_s \operatorname{dist}(u, P) \leqslant C_s\epsilon_0$, so

$$\inf_{\overline{D_0(\epsilon)}\cap\overline{-D_0(\epsilon)}} G(u) = d_0 > -\infty. \qquad \square$$

Proof of Theorem 3.1. The $f(x,t)$ of Theorem 3.1 satisfies the condition of [12, Theorem 7.4.2], so as the same of (7.4.14) of [12, Theorem 7.4.2], there are two positive constants M_1 and M_2 such that $\forall t \in R$, $\forall x \in \Omega$,

$$F(x,t) \geqslant M_1|t|^\eta - M_2.$$

For any finitely dimensional subspace V_0 of V, we have, $\forall v \in V_0$, there exists $M_3 > 0$ such that

$$\begin{aligned} G(u) &= \frac{1}{2}\|u\|^2 - \int_\Omega F(x,u)\,dx \\ &\leqslant \frac{1}{2}\|u\|^2 - M_1\|u\|_\eta^\eta + M_2|\Omega| \\ &\leqslant \frac{1}{2}\|u\|^2 - M_3\|u\|^\eta + M_2|\Omega| \end{aligned}$$

where $|\Omega|$ denote the measure of Ω. Since $\eta < 2$, by Young inequality, there are two positive number M_4 and M_5 such that

$$G(u) \leqslant -M_4\|u\|^2 + M_5, \quad \forall v \in V_0. \tag{4.4}$$

Since $\varphi_2 \in V$ is sign-changing, that is, $\varphi_2^+ \neq 0, \varphi_2^- \neq 0$. It is clear that φ_2^+ and φ_2^- are linearly independent. Let $V_0 = \{t\varphi_2^+ + s\varphi_2^-:\ t \geqslant 0,\ s \geqslant 0\}$, then V_0 is the finitely dimensional subspace of V. Define a path $h : [0,1] \mapsto V$,

$$h(t) = t\frac{R_0}{\|\varphi_2^+\|}\varphi_2^+ + (1-t)\frac{R_0}{\|\varphi_2^-\|}\varphi_2^-$$

where $R_0 = \max\{\frac{d_0 - 2M_5 - 1}{-M_4}, 1\}$, then by (4.4)

$$\begin{aligned} G\big(h(t)\big) &= G\left(t\frac{R_0}{\|\varphi_2^+\|}\varphi_2^+\right) + G\left((1-t)\frac{R_0}{\|\varphi_2^-\|}\varphi_2^-\right) \\ &\leqslant -M_4R_0 + 2M_5 \\ &\leqslant d_0 - 1. \end{aligned} \tag{4.5}$$

So

$$\inf_{u \in \overline{D_0(\epsilon_0)} \cap \overline{-D_0(\epsilon_0)}} f(u) > \sup_{t \in [0,1]} f\big(h(t)\big).$$

Obviously, $h(0) \in -D_0(\epsilon_0), h(1) \in D_0(\epsilon_0)$, thus $h(0) \in -D_0(\epsilon_0) \setminus D_0(\epsilon_0)$. If not, $h(0) \in -D_0(\epsilon_0) \cap D_0(\epsilon_0)$, by Lemma 4.3, $G(h(0)) \geqslant d_0$. This is a contradiction. Analogously, $h(1) \in D_0(\epsilon_0) \setminus -D_0(\epsilon_0)$. Moreover, $0 \in -D_0(\epsilon_0) \cap \overline{D_0(\epsilon_0)}$, by Lemmas 4.1, 4.2 and Proposition 2.1, (1.1) has four solutions: $u_1 \in D_0(\epsilon_0) \cap (-D_0(\epsilon_0))$, $u_2 \in D_0(\epsilon_0) \setminus \overline{-D_0(\epsilon_0)}$, $u_3 \in (-D_0(\epsilon_0)) \setminus \overline{D_0(\epsilon_0)}$, $u_4 \in H \setminus (\overline{D_0(\epsilon_0)} \cup \overline{-D_0(\epsilon_0)})$. That is, u_1 is naught solution, u_2 is positive solution, u_3 is negative solution and u_4 is sign-changing solution. □

We prove Theorems 3.2 and 3.3 by Proposition 2.2. First let $N = X_1 \oplus X_2 \oplus \cdots \oplus X_l$ $(l \geqslant 2)$, $M = \overline{\bigoplus_{i=l+1}^\infty X_i}$, then $V = N \oplus M$. We take $z_0 \in X_l$, $\|z_0\| = 1$, and define

$$B = \big\{u \in M:\ \|u\| \geqslant \delta\big\} \cup \big\{u = sz_0 + v:\ v \in M,\ s \geqslant 0,\ \|u\| = \delta\big\}.$$

Then each element of B is sign-changing.

Lemma 4.4. $\operatorname{dist}(B, -P \cup P) = d_1 > 0$.

Proof. B and $-P \cup P$ are two closed subsets of V. Note that $B \cap (-P \cup P) = \emptyset$ and V is normal space, the conclusion is readily to be shown. □

Lemma 4.5. *Assume* (h_2) *holds, then there exists* $\mu_0 \in (0, d_1)$ *such that* $J'(\pm D_0(\mu_0)) \subset \pm D_0(\mu_0)$.

Proof. Let $u^{\pm} = \max\{\pm u, 0\}$. $\forall w \in -P$, we have $w(x) \leqslant 0$, so $-w(x) \geqslant 0$. Hence, $\forall w \in -P$, $s \in (2, 2^*)$,

$$\begin{aligned}\|u^+\|_s^s &= \int_\Omega |u^+|^s\,dx \\ &= \int_{u(x)\geqslant 0} |u^+|^s\,dx + \int_{u(x)<0} |u^+|^s\,dx \\ &= \int_{u(x)\geqslant 0} |u^+|^s\,dx \\ &\leqslant \int_{u(x)\geqslant 0} |u^+ - w|^s\,dx + \int_{u(x)<0} |-u^- - w|^s\,dx \\ &= \int_\Omega |u-w|^s\,dx.\end{aligned}$$

Therefore $\|u^+\|_s \leqslant \inf_{w\in(-P)} \|u-w\|_s$. Moreover, by the definition of V and Sobolev embedding theorem, when $s \in (2, 2^*)$, the embedding $V \hookrightarrow L^t(\Omega)$ is continuous. So for all $u \in V$, if $s \in (2, 2^*)$, there exists $C_s > 0$ such that

$$\|u^+\|_s \leqslant \inf_{w\in(-P)} \|u-w\|_s \leqslant C_s \inf_{w\in(-P)} \|u-w\| = C_s\,\mathrm{dist}(u, -P). \tag{4.6}$$

By (h_2), $\forall \epsilon > 0$, there exists $C_\epsilon > 0$, such that

$$f(x,t)t \leqslant \epsilon t^2 + C_\epsilon |t|^s, \quad x \in \Omega,\ t \in R. \tag{4.7}$$

Assume $v = J'(u)$. By (4.6) and (4.7), for ϵ small enough,

$$\begin{aligned}\mathrm{dist}(v, -P)\|v^+\| &\leqslant \|v^+\|^2 \\ &= \langle v, v^+\rangle \\ &\leqslant \int_\Omega f(x, u^+)v^+\,dx \\ &\leqslant \int_\Omega \left(\epsilon|u^+| + C_\epsilon|u^+|^{s-1}\right)|v^+|\,dx \\ &\leqslant \left(\frac{1}{2}\mathrm{dist}(u,-P) + C\,\mathrm{dist}(u,-P)^{s-1}\right)\|v^+\|.\end{aligned}$$

That is, $\mathrm{dist}(J'(u), -P) \leqslant \frac{1}{2}\mathrm{dist}(u,-P) + C(\mathrm{dist}(u,-P)^{s-1})$, there is $\mu_0 < d_1$ (cf. Lemma 4.4) such that $\mathrm{dist}(J'(u), -P) \leqslant \frac{3}{4}\mu_0$ for every $u \in -D_0(\mu_0)$. In a similar way, $\mathrm{dist}(J'(u), P) \leqslant \frac{3}{4}\mu_0$ for every $u \in D_0(\mu_0)$. The conclusion follows. □

Lemma 4.6. *Under the assumptions of Theorem* 3.2*, G satisfies the* w-PS *condition.*

Proof. Assume $\{u_n\} \subset V$ such that $|G(u_n)| \leqslant C$ and $G'(u_n) \to 0$. Without loss of generality, we suppose that $\{\|G'(u_n)\|\|u_n\|\}$ is bounded. It suffices to prove that $\{u_n\}$ is bounded. By (g_2),

$$\eta C + \|G'(u_n)\|\|u_n\| \geqslant \eta G(u_n) - \langle G'(u_n), u_n\rangle \geqslant \frac{\eta-2}{2}\|u_n\|^2.$$

Thus $\|u_n\|$ is bounded. □

Lemma 4.7. *Under the assumptions of Theorem* 3.3*, G satisfies the* w-PS *condition.*

Proof. Assume $\{u_n\} \subset V$ such that $|G(u_n)| \leqslant C$ and $G'(u_n) \to 0$. Without loss of generality, we suppose that $\{\|G'(u_n)\| \|u_n\|\}$ is bounded. It suffices to prove that $\{u_n\}$ is bounded. If not, assume $\|u_n\| \to \infty$. For $\epsilon > 0$ small enough, since the limits if (h_3) and (h_4) are taken in pointwise sense, by Egorov theorem, we obtain an $\Omega_{l+1} \subset \Omega$ such that $\operatorname{meas}(\Omega \setminus \Omega_{l+1}) < \epsilon$ and that $\limsup_{|t|\to\infty} \frac{f(x,t)}{t} \leqslant \mu_{l+1}$ uniformly for $x \in \Omega_{l+1}$. Therefore, there exists $C_1 > 0$ such that

$$\frac{f(x,t)}{t} \leqslant \mu_{l+1} + \epsilon + \frac{C_1}{|t|}, \quad \forall t \neq 0,\ x \in \Omega_{l+1}. \tag{4.8}$$

Similarly, there exists $\Omega_l \subset \Omega$ such that $\operatorname{meas}(\Omega \setminus \Omega_l) < \epsilon$ and that $\liminf_{|t|\to\infty} \frac{f(x,t)}{t} \geqslant L > \mu_l$ uniformly for $x \in \Omega_l$. Hence, there exists $C_2 > 0$ such that

$$\frac{f(x,t)}{t} \geqslant L - \frac{C_2}{|t|}, \quad \forall t \neq 0,\ x \in \Omega_l. \tag{4.9}$$

Let $\Omega^* = \Omega_l \cap \Omega_{l+1}$, then $\operatorname{meas}(\Omega^*)^c < 2\epsilon$. Write u_n with $u_n = u_n^+ + u_n^0 + u_n^-$, where $u_n^- \in N$, $u_n^0 \in X_{l+1}$, $u_n^+ \in \overline{\bigoplus_{i=l+2}^{\infty} X_i}$. If $|u_n^+ + u_n^0| \geqslant |u_n^-|$ on Ω^*, by (4.8) we have that

$$f(x,u_n)\big(u_n^+ + u_n^0 - u_n^-\big) \leqslant (\mu_{l+1} + \epsilon)\big(u_n^+ + u_n^0\big)^2 - L\big(u_n^-\big)^2 + C_1\big|u_n^+ + u_n^0 - u_n^-\big|. \tag{4.10}$$

If $|u_n^+ + u_n^0| < |u_n^-|$ on Ω^*, by (4.9) we have that

$$f(x,u_n)\big(u_n^+ + u_n^0 - u_n^-\big) \leqslant (\mu_{l+1} + \epsilon)\big(u_n^+ + u_n^0\big)^2 - L\big(u_n^-\big)^2 + C_2\big|u_n^+ + u_n^0 - u_n^-\big|. \tag{4.11}$$

Since $\|u^-\|^2 - L\int_\Omega (u^-)^2\,dx \leqslant -\frac{L-\mu_l}{\mu_l}\|u^-\|^2 := -\gamma\|u^-\|^2$ for every $u^- \in N$. Therefore, by (4.10) and (4.11),

$$\begin{aligned}
\big\langle G'(u_n), u_n^+ + u_n^0 - u_n^-\big\rangle &= \big\|u_n^+\big\|^2 + \big\|u_n^+\big\|^0 - \big\|u_n^-\big\|^2 - \int_\Omega f(x,u_n)\big(u_n^+ + u_n^0 - u_n^-\big)\,dx \\
&\geqslant \big\|u_n^+\big\|^2 + \big\|u_n^+\big\|^0 - \big\|u_n^-\big\|^2 - \int_{\Omega^*} \big((\mu_{l+1} + \epsilon)\big(u_n^+ + u_n^0\big)^2 - L\big(u_n^-\big)^2\big)\,dx \\
&\quad - \int_{\Omega^*} (C_1 + C_2)\big|u_n^+ + u_n^0 - u_n^-\big|\,dx - \int_{\Omega\setminus\Omega^*} \big|f(x,u_n)\big|\big|u_n^+ + u_n^0 - u_n^-\big|\,dx \\
&\geqslant \big\|u_n^+\big\|^2\bigg(1 - \frac{\mu_{l+1} + \epsilon}{\mu_{l+2}}\bigg) - \epsilon\|u_0\|_2^2 + L\int_\Omega \big(u_n^-\big)^2\,dx - \big\|u_n^-\big\|^2 \\
&\quad - L\int_{\Omega\setminus\Omega^*} \big(u_n^-\big)^2 - \int_{\Omega\setminus\Omega^*} \big|f(x,u_n)\big|\big|u_n^+ + u_n^0 - u_n^-\big|\,dx \\
&\quad - \int_{\Omega^*} (C_1 + C_2)\big|u_n^+ + u_n^0 - u_n^-\big|\,dx \\
&\geqslant \big\|u_n^+\big\|^2\bigg(1 - \frac{\mu_{l+1} + \epsilon}{\mu_{l+2}}\bigg) - \epsilon\big\|u_n^0\big\|_2^2 + \gamma\big\|u_n^-\big\|^2 - \int_{\Omega^*} (C_1 + C_2)\big|u_n^+ + u_n^0 - u_n^-\big|\,dx \\
&\quad - L\int_{\Omega\setminus\Omega^*} \big(u_n^-\big)^2 - \int_{\Omega\setminus\Omega^*} \big|f(x,u_n)\big|\big|u_n^+ + u_n^0 - u_n^-\big|\,dx,
\end{aligned}$$

which implies that $\frac{u_n^+}{\|u_n\|} \to 0$, $\frac{u_n^-}{\|u_n\|} \to 0$, $\frac{u_n^0}{\|u_n\|} \to 1$, hence $\frac{u_n}{\|u_n\|} \to w$ with $w \in X_{l+1}$, $\|w\| = 1$. So

$$0 = \lim_{n\to\infty} \frac{G(u_n) - \frac{1}{2}\langle G'(u_n), u_n\rangle}{\|u_n\|^\alpha} = \lim_{n\to\infty} \frac{\int_\Omega (f(x,u_n)u_n - 2F(x,u_n))\,dx}{\|u_n\|^\alpha} \geqslant \int_\Omega \beta(x)\big|w(x)\big|^\alpha\,dx > 0.$$

This is a contradiction. The conclusion follows. $\square$

Proof of Theorems 3.2 and 3.3. Assume

$$D_0^{(1)} = D_0(\mu_0), \qquad D_0^{(2)} = -D_0(\mu_0),$$
$$W = D_0^{(1)} \cup D_0^{(2)}, \qquad S = V \setminus W.$$

By Lemma 4.4, $B \subset S$, that is, the condition (H_3) of Proposition 2.2 holds. Lemma 4.5 says that condition (H_1) of proposition is also satisfied. Since $0 \in D_0^{(1)} \cap D_0^{(2)}$, then ($H_2$) holds automatically. By Lemmas 4.6 and 4.7, G satisfies w-PS condition. Moreover, note that $\|v\|^2 \leqslant \mu_l \|v\|_2^2$ for all $v \in N$ and $\mu_{l+1}\|w\|_2^2 \leqslant \|w\|^2$ for all $w \in M$. Combining (h_1), we have that

$$G(v) \leqslant \frac{1}{2}\|v\|^2 - \frac{\mu_l}{2}\|v\|_2^2 + \frac{\int_\Omega W_1(x)\,dx}{2} \leqslant \frac{\int_\Omega W_1(x)\,dx}{2}, \quad \forall v \in N,$$

and

$$G(w) \geqslant \frac{1}{2}\|w\|^2 - \frac{\mu_{l+1}}{2}\|w\|_2^2 - \frac{\int_\Omega W_2(x)\,dx}{2} \geqslant \frac{-\int_\Omega W_2(x)\,dx}{2}, \quad \forall w \in M.$$

Therefore, we have

$$\sup_N G = a_0 < \infty, \qquad \inf_M = b_0 > -\infty.$$

Since

$$\left|f(x,t)\right| \leqslant c\left(1 + |t|^{s-1}\right), \quad \forall x \in \Omega,\ \forall t \in R,$$

G maps bounded sets to bounded sets. By Proposition 2.2, G has a critical point in S. Therefore, (1.1) has a sign-changing solution. □

Lemma 4.8. *Assume* (a_2) *holds, then* $G_\lambda(u) \leqslant \frac{1}{2}\int_\Omega W_0(x)\,dx$ *for all* $u \in V_{k-1}$, $\lambda \in \Lambda$.

Proof. $\forall u \in V_{k-1}$, $\forall \lambda \in \Lambda$, by ($a_2$), we have that

$$\begin{aligned}
G_\lambda(u) &\leqslant \frac{1}{2}\|u\|^2 - \int_\Omega F(x,u)\,dx \\
&\leqslant \frac{1}{2}\mu_{k-1}\int_\Omega u^2\,dx - \int_\Omega F(x,u)\,dx \\
&\leqslant \frac{1}{2}\mu_{k-1}\int_\Omega u^2\,dx - \frac{1}{2}\mu_{k-1}\int_\Omega u^2\,dx + \frac{1}{2}\int_\Omega W_0(x)\,dx \\
&\leqslant \frac{1}{2}\int_\Omega W_0(x)\,dx. \qquad \square
\end{aligned}$$

Lemma 4.9. *Assume* (h_2) *holds and* $\forall x \in \Omega$, $\mu_k < b_\pm(x)$. *Then* $G_\lambda(u) \to -\infty$ *for* $u \in V_k$ *as* $\|u\| \to \infty$ *uniformly in* $\lambda \in \Lambda$.

Proof. Write $G(u)$ as

$$G(u) = \frac{1}{2}\|u\|^2 - \int_\Omega \left(\frac{1}{2}b_+(x)\left(u^+\right)^2 + \frac{1}{2}b_-(x)\left(u^-\right)^2 + P(x,u)\right)dx, \quad \forall u \in V$$

where $P(x,u) = \int_0^u p(x,t)\,dt$; $p(x,t) = f(x,t) - (b_+(x)t^+ - b_-(x)t^-)$, $t^\pm = \max\{\pm t, 0\}$. Note that $\mu_k < b_\pm(x)$ and the variational characterization of eigenvalues $\{\mu_k\}$: $\forall u \in V_k$, $\lambda \in \Lambda$, we have

$$\begin{aligned}
G_\lambda(u) &\leqslant G(u)\\
&= \frac{1}{2}\|u\|^2 - \int_\Omega P(x,u)\,dx - \frac{1}{2}\Big(\int_{b_-(x)\geqslant b_+(x)} + \int_{b_-(x)<b_+(x)}\Big)\big(b_+(x)\big(u^+\big)^2 + b_-(x)\big(u^-\big)^2\big)\,dx\\
&= \frac{1}{2}\|u\|^2 - \int_\Omega P(x,u)\,dx - \frac{1}{2}\int_{b_-(x)\geqslant b_+(x)} b_+(x)u^2\,dx - \frac{1}{2}\int_{b_-(x)\geqslant b_+(x)} \big(b_-(x)-b_+(x)\big)\big(u^-\big)^2\,dx\\
&\quad - \frac{1}{2}\int_{b_-(x)<b_+(x)} b_-(x)u^2\,dx - \frac{1}{2}\int_{b_-(x)<b_+(x)} \big(b_+(x)-b_-(x)\big)\big(u^+\big)^2\,dx\\
&\leqslant \frac{1}{2}\|u\|^2 - \int_\Omega P(x,u)\,dx - \frac{1}{2}\int_{b_-(x)\geqslant b_+(x)} b_+(x)u^2\,dx - \frac{1}{2}\int_{b_-(x)<b_+(x)} b_-(x)u^2\,dx\\
&\leqslant \frac{1}{2}\|u\|^2 - \int_\Omega P(x,u)\,dx - \frac{1}{2}\int_\Omega \min\big\{b_+(x),b_-(x)\big\}u^2\,dx\\
&\leqslant -\delta\|u\|^2 - \int_\Omega P(x,u)\,dx
\end{aligned}$$

where $\delta = \frac{\min\{b_+(x),b_-(x)\}-\mu_k}{2\mu_k}$. By ($h_2$), $\lim_{t\to\infty}\frac{p(x,t)}{t} = 0$, therefore,

$$\lim_{u\in V_k,\,\|u\|\to\infty} \frac{G_\lambda(u)}{\|u\|^2} \leqslant -\delta.$$

The conclusion follows. □

Lemma 4.10. *Assume* (a_1) *and* (a_3) *hold, then there exists* $\rho_0 > 0$ *independent of* λ *such that* $G_\lambda(u) \geqslant \frac{1}{2}\int_\Omega W_0(x)\,dx$ *for all* $u \in \overline{\bigoplus_{i=k}^\infty X_i}$ *with* $\|u\| = \rho_0$ *and all* $\lambda \in \Lambda$.

Proof. By (a_1), if $x \in \Omega$, $|t| \geqslant r_0$,

$$F(x,t) \leqslant \frac{1}{2}F_0t^2 - \frac{1}{4}F_0r_0^2. \tag{4.12}$$

$\forall u \in \overline{\bigoplus_{i=k}^\infty X_i}$, write u as $u = v + w$, where $v \in \bigoplus_{i=k}^{l-1} X_i$, $w \in \overline{\bigoplus_{i=l}^\infty X_i}$. Let

$$\begin{aligned}
\beta_0 &= \frac{\mu_k+\mu_{k-1}}{2},\\
\xi_1 &= \frac{2F_0+\mu_l}{8}w^2 + \frac{\mu_k+\beta_0}{8}v^2 - F(x,v+w).
\end{aligned} \tag{4.13}$$

If $|v+w| \leqslant r_0$, then by (a_3) and the choice of μ_l, we see that

$$\begin{aligned}
\xi_1 &\geqslant \frac{2F_0+\mu_l}{8}w^2 + \frac{\mu_k+\beta_0}{8}v^2 - \frac{1}{4}\beta_0(v+w)^2\\
&\geqslant \frac{2F_0+\mu_l-2\beta_0}{8}w^2 + \frac{\mu_k+\beta_0-2\beta_0}{8}v^2 - \frac{1}{2}\beta_0|vw|\\
&\geqslant \left(\frac{((2F_0+\mu_l-2\beta_0)(\mu_k-\beta_0))^{\frac{1}{2}}}{4} - \frac{1}{2}\beta_0\right)|vw| \geqslant 0.
\end{aligned} \tag{4.14}$$

If $|v+w| > r_0$, then by (4.12), we conclude that

$$\xi_1 \geqslant \left(\frac{\mu_l+2F_0-4F_0}{8}w^2 + \frac{\mu_k+\beta_0-4F_0}{8}v^2\right) - F_0vw + \frac{F_0r_0}{4} = \xi_2 + \xi_3 \tag{4.15}$$

where

$$\xi_2 = \frac{\mu_l - 2F_0}{16} w^2 + \frac{\mu_k - \beta_0}{8} v^2 - \frac{\beta_0 v w}{2}, \tag{4.16}$$

$$\xi_3 = \frac{\mu_l - 2F_0}{16} w^2 - \frac{2F_0 - \beta_0}{4} v^2 - \frac{(2F_0 - \beta_0) v w}{2} + \frac{F_0 r_0^2}{4}. \tag{4.17}$$

Next, we estimate ξ_2 and ξ_3. If $\frac{\mu - \beta_0}{8}|v| - \frac{\beta_0 |w|}{2} \geqslant 0$, then

$$\xi_2 \geqslant \frac{\mu_l - 2F_0}{16} w^2 + \left(\frac{\mu_k - \beta_0}{8}|v| - \frac{\beta_0 |w|}{2}\right)|v| \geqslant 0. \tag{4.18}$$

If $\frac{\mu - \beta_0}{8}|v| - \frac{\beta_0 |w|}{2} < 0$, by the choice of μ_l, we deduce that

$$\xi_2 \geqslant \left(\frac{\mu_l - 2F_0}{16} - \frac{2\beta_0^2}{\mu_k - \beta_0}\right) w^2 + \frac{\mu_k - \beta_0}{8} v^2 \geqslant 0. \tag{4.19}$$

On the other hand,

$$\xi_3 \geqslant \frac{\mu_l + 2F_0 - 4F_0}{16} w^2 - \left(F_0 - \frac{\beta_0}{2}\right)\big(|v| + |w|\big)|v| + \frac{F_0 r_0^2}{4} = \xi_4. \tag{4.20}$$

Thus

$$\begin{aligned} \xi_4 &\geqslant \frac{\mu_l - 10F_0 + 4\beta_0}{16} w^2 - \frac{3(2F_0 - \beta_0)}{4} v^2 + \frac{F_0 r_0^2}{4} \\ &\geqslant -\frac{3(2F_0 - \beta_0)}{4} v^2 + \frac{F_0 r_0^2}{4}. \end{aligned} \tag{4.21}$$

Choose $\rho_0 = \frac{1}{C_{l-1}}\left(\frac{F_0 r_0^2}{3(2F_0 - \beta_0)}\right)^{\frac{1}{2}}$. If $\|u\| = \rho_0$, then $\|v\|_\infty \leqslant C_{l-1}\|v\| \leqslant C_{l-1}\|u\| \leqslant C_{l-1}\rho_0$. Hence, $\xi_4 \geqslant 0$. Therefore, by (4.13)–(4.21), $\xi_1 \geqslant 0$. Finally

$$\begin{aligned} G_\lambda(u) &= G_\lambda(v + w) \\ &\geqslant \frac{1}{4}\big(\|v\|^2 + \|w\|^2\big) - \int_\Omega F(x, v + w)\,dx \\ &\geqslant \frac{1}{8}\|v\|^2 + \frac{1}{8}\|w\|^2 + \frac{1}{8}\mu_k\|v\|_2^2 + \frac{1}{8}\mu_l\|w\|_2^2 - \int_\Omega F(x, v + w)\,dx \\ &\geqslant \frac{1}{8}\left(1 - \frac{\beta_0}{\mu_k}\right)\|v\|^2 + \frac{1}{8}\left(1 - \frac{2F_0}{\mu_l}\right)\|w\|^2 + \int_\Omega \xi_1\,dx \\ &\geqslant \frac{1}{8}\min\left\{\left(1 - \frac{\beta_0}{\mu_k}\right), \left(1 - \frac{2F_0}{\mu_l}\right)\right\}\|u\|^2 \\ &\geqslant \frac{1}{8}\left(1 - \frac{\beta_0}{\mu_k}\right)\rho_0^2 \\ &\geqslant \frac{1}{2}\int_\Omega W_0(x)\,dx. \qquad \square \end{aligned}$$

By Lemma 4.9, there exists $R > \rho_0$ such that $G_\lambda(u) \leqslant 0$ for all $u \in V_k$, $\|u\| \geqslant R$. Choose $y_0 \in X_k$, $\|y_0\| = 1$. Let $B = \overline{\bigoplus_{i=k}^\infty X_i} \cap \partial B_{\rho_0}(0)$, $A = \{u = v + s y_0\colon\ v \in V_{k-1},\ s \geqslant 0,\ \|u\| = R\} \cup (V_{k-1} \cap B_R(0))$. By the definite of A, B and link, A links B and each element of B is sign-changing. Similar to Lemma 4.4, $\mathrm{dist}(B, -P \cup P) = d_2 > 0$. In the same as that of the proof of Lemma 4.5, we have that

Lemma 4.11. *Under the assumptions of Theorem* 3.4, *then there exists* $\mu_0 \in (0, d_2)$, $\mu_0 < \frac{1}{2}$ *such that*

$$\operatorname{dist}\big(J'(u), \pm P\big) \leqslant \frac{1}{5} \operatorname{dist}(u, \pm P)$$

for $u \in V$ *and* $\operatorname{dist}(u, \pm P) < \mu_0$.

Proof of Theorem 3.4. By Lemmas 4.8–4.10, for $\lambda \in \Lambda$,

$$a_0(\lambda) = \sup_A G_\lambda \leqslant \frac{1}{2} \int_\Omega W_0(x)\,dx = b_0 \leqslant \inf_B G_\lambda.$$

Let

$$D = \big(-D_0(\mu_0)\big) \cup D_0(\mu_0), \qquad S = V \setminus D,$$

then $B \subset S$. That is, condition (A_2) of Proposition 2.3 holds. By Lemma 4.11, condition (A_1) of Proposition 2.3 also satisfied. Since

$$\big|f(x,t)\big| \leqslant c\big(1 + |t|^{s-1}\big), \quad \forall x \in \Omega,\ \forall t \in R,$$

G maps bounded sets to bounded sets. Therefore by Proposition 2.3 and [9, Theorem 2.1], for almost all $\lambda \in \Lambda$, G_λ has a sign-changing critical point $u_\lambda \in S$ such that

$$G'_\lambda(u_\lambda) = 0, \quad G_\lambda(u_\lambda) \in \Big[b_0, \sup_{(t,u)\in[0,1]\times A} G\big((1-t)u\big)\Big].$$

Then we prove $\{u_\lambda\}_{\lambda\in\Lambda}$ is bounded as follows.

Assume $\{u_\lambda\}_{\lambda\in\Lambda}$ is unbounded, then there exists $\lambda_n \in \Lambda$ such that $\|u_{\lambda_n}\| \to \infty$ for $n \to \infty$. We consider $w_{\lambda_n} = \frac{u_{\lambda_n}}{\|u_{\lambda_n}\|}$. Then, up to a subsequence, we get that

$$\begin{aligned} &w_{\lambda_n} \rightharpoonup w \quad \text{in } V,\\ &w_{\lambda_n} \to w \quad \text{in } L^t(\Omega) \text{ for } 2 \leqslant t < 2^*,\\ &w_{\lambda_n}(x) \to w(x) \quad \text{a.e. } x \in \Omega. \end{aligned}$$

If $w \neq 0$ in V, since $G'_{\lambda_n}(u_{\lambda_n}) = 0$, we have that

$$\begin{aligned} \frac{1}{2}\int_\Omega H(x, u_{\lambda_n}) &= \int_\Omega \left(\frac{1}{2} f(x, u_{\lambda_n})u_{\lambda_n} - F(x, u_{\lambda_n})\right)\\ &= G_{\lambda_n}(u_{\lambda_n}) - \frac{1}{2}\big\langle G'_{\lambda_n}(u_{\lambda_n}), u_{\lambda_n}\big\rangle\\ &= G_{\lambda_n}(u_{\lambda_n})\\ &\leqslant \sup_{(t,u)\in[0,1]\times A} G\big((1-t)u\big). \end{aligned}$$

However, by (a_4), $H(x,t) \to \infty$ as $|t| \to \infty$ for each $x \in \Omega$. Therefore

$$\int_\Omega H(x, u_{\lambda_n})\,dx \geqslant \int_{\{w(x)\neq 0\}} H(x, u_{\lambda_n})\,dx \to \infty$$

as $n \to \infty$. This is a contradiction.

If $w = 0$ in V, we define

$$G_{\lambda_n}(t_n u_{\lambda_n}) = \max_{t\in[0,1]} G_{\lambda_n}(t u_{\lambda_n}).$$

For any $c > 0$ and $\overline{w}_{\lambda_n} = \sqrt{4c}\,w_{\lambda_n}$, we have, for n large enough, that

$$G_{\lambda_n}(t_n u_{\lambda_n}) \geqslant G_{\lambda_n}(\overline{w}_{\lambda_n}) \geqslant 2c\lambda_n - \int_\Omega F(x, \overline{w}_{\lambda_n}) \geqslant \frac{c}{2},$$

which implies $\lim_{n\to\infty} G_{\lambda_n}(t_n u_{\lambda_n}) = \infty$. Evidently, $t_n \in (0,1)$, hence, we have $\langle G'_{\lambda_n}(t_n u_{\lambda_n}), t_n u_{\lambda_n}\rangle = 0$. It follows that

$$\int_\Omega \left(\frac{1}{2} f(x, t_n u_{\lambda_n}) t_n u_{\lambda_n} - F(x, t_n u_{\lambda_n})\right) dx \to \infty.$$

By the convexity of $H(x,t)$ in t, we have that

$$\int_\Omega \left(\frac{1}{2} f(x, u_{\lambda_n}) u_{\lambda_n} - F(x, u_{\lambda_n})\right) dx \geqslant \int_\Omega \left(\frac{1}{2} f(x, t_n u_{\lambda_n}) t_n u_{\lambda_n} - F(x, t_n u_{\lambda_n})\right) dx \to \infty.$$

We get a contradiction since

$$\int_\Omega \left(\frac{1}{2} f(x, u_{\lambda_n}) u_{\lambda_n} - F(x, u_{\lambda_n})\right) dx = G_{\lambda_n}(u_{\lambda_n}) \in \left[b_0, \sup_{(t,u)\in[0,1]\times A} G\big((1-t)u\big)\right].$$

Therefore, $\{u_\lambda\}_{\lambda\in\Lambda}$ is bounded.

Let $\lambda_m \to 1$ $(m\to\infty)$, since $\{u_{\lambda_m}\}$ is bounded, then, up to a subsequence, we get $u_{\lambda_m} \to u$, we will prove u is sign-changing.

Let $u^{\pm}_{\lambda_m} = \max\{\pm u_{\lambda_m}, 0\}$, then

$$\lambda_m \|u^{\pm}_{\lambda_m}\|^2 \leqslant \int_\Omega f\big(x, u^{\pm}_{\lambda_m}\big) u^{\pm}_{\lambda_m}\, dx.$$

By (h_2), there exists $C_3 > 0$ such

$$f(x,u)u \leqslant \frac{\mu_1}{4}|u|^2 + C_3|u|^s, \quad x\in\Omega,\ u\in R.$$

Note that if $\forall u \in V$, $\|u\|^2 \geqslant \sqrt{\mu_1}\|u\|_2^2$. It follows that

$$\frac{1}{2}\|u^{\pm}_{\lambda_m}\|^2 \leqslant \frac{1}{4}\|u^{\pm}_{\lambda_m}\|^2 + C_3\|u^{\pm}_{\lambda_m}\|_s^s.$$

Hence, $\|u^{\pm}_{\lambda_m}\| \geqslant C_4 > 0$, where C_4 is a constant independent of λ_m. So u is sign-changing. That is, u is the sign-changing solution of (1.1). □

We are going to prove Theorem 3.5 by applying Proposition 2.4. For $k \geqslant 2$, assume

$$E_k = \bigoplus_{j=1}^{k} X_j, \qquad Z_k = \overline{\bigoplus_{j=k}^{\infty} X_j}, \qquad B_k = \big\{u\in E_k\colon \|u\| \leqslant \rho_k\big\}, \qquad N_k = \big\{u\in Z_k\colon \|u\| = r_k\big\}$$

where $\rho_k > r_k > 0$. We need the following lemma.

Lemma 4.12. *Under the assumptions of Theorem* 3.5*, there exists* $\rho_k > r_k > 0$ *independent of* λ *such that*

$$\max_{\partial B_k} G_\lambda \leqslant a_k \leqslant 0 < b_k \leqslant \inf_{N_k} G_\lambda$$

for $\lambda \in \Lambda$. *Here* a_k *and* b_k *are independent of* λ. *Moreover,* $b_k \to \infty$ *as* $(k\to\infty)$.

Proof. By (h_2), there exists $C_5 > 0$ such that $|F(x,u)| \leqslant \frac{\mu_1}{8}u^2 + C_5|u|^s$ for $x\in\Omega, u\in R$. Recall Gagliardo–Nirenberg inequality

$$\|u\|_s \leqslant c_0\|\nabla u\|_2^\alpha \|u\|_2^{1-\alpha}, \quad \forall u \in H^1\big(R^N\big),$$

where $\alpha = N(\frac{1}{2} - \frac{1}{s})$, c_0 is a constant depending on s, N. Note $\mu_k \|u\|_2^2 \leqslant \|u\|^2$ for all $u \in \overline{\bigoplus_{i=k}^{\infty} X_i}$, if $u \in \overline{\bigoplus_{i=k}^{\infty} X_i}$, $\|u\| = r_k = \frac{\mu_k^{\frac{s(1-\alpha)}{2(s-2)}}}{(16c_0^s C_5)^{\frac{1}{s-2}}}$, we have the following estimates:

$$\begin{aligned} G_\lambda(u) &\geqslant \frac{1}{4}\|u\|^2 - \frac{\mu_1}{8}\int_\Omega u^2\,dx - C_5\int_\Omega |u|^s\,dx \\ &\geqslant \frac{1}{8}\|u\|^2 - C_5 c_0^s \|\nabla u\|_2^{s\alpha}\|u\|_2^{s(1-\alpha)} \\ &\geqslant \frac{1}{8}\|u\|^2 - C_5 c_0^s \|u\|^s \mu_k^{\frac{-s(1-\alpha)}{2}} \\ &\geqslant \frac{1}{16} r_k^2 = b_k. \end{aligned}$$

Since $\dim V_k < \infty$, then by (b_1) and (h_2),

$$\frac{G_\lambda(u)}{\|u\|^2} \leqslant \frac{1}{2} - \int_\Omega \frac{F(x,u)}{\|u\|^2} \to -\infty$$

as $u \in V_k$, $\|u\| \to \infty$ uniformly for $\lambda \in \Lambda$. Then there exists $\rho_k > r_k > 0$ independent of λ such that

$$\max_{\partial B_k} G_\lambda \leqslant a_k \leqslant 0. \qquad \square$$

Proof of Theorem 3.5. Since each element of N_k $(k \geqslant 2)$ is sign-changing, there exists $\gamma_k > 0$ such that $\operatorname{dist}(N_k, -P \cup P) = \gamma_k$. Under the assumptions of Theorem 3.5, Lemma 4.11 is also true. That is, (A_1) holds. Let

$$D = \big(-D_0(\mu_0)\big) \cup D_0(\mu_0), \qquad S = V \setminus D.$$

Then $N_k \subset S$. Moreover, by Lemma 4.12, (A_3) is also satisfied and J' is compact. Thus, by Proposition 2.4, G_λ has a sign-changing critical point $u_\lambda \in S$ and $G_\lambda(u_\lambda) \in [b_k, \max_{u\in B_k} G(u)]$, an interval independent of λ. We will prove $\{u_\lambda\}_{\lambda\in\Lambda}$ is bounded.

Assume $\{u_\lambda\}_{\lambda\in\Lambda}$ is unbounded, then there exists $\lambda_n \in \Lambda$ such that $\|u_{\lambda_n}\| \to \infty$ for $n \to \infty$. We consider $w_{\lambda_n} = \frac{u_{\lambda_n}}{\|u_{\lambda_n}\|}$. Then, up to a subsequence, we get that

$$\begin{aligned} &w_{\lambda_n} \rightharpoonup w \quad \text{in } V, \\ &w_{\lambda_n} \to w \quad \text{in } L^t(\Omega) \text{ for } 2 \leqslant t < 2^*, \\ &w_{\lambda_n}(x) \to w(x) \quad \text{a.e. } x \in \Omega. \end{aligned}$$

If $w \neq 0$ in V, since $G'_{\lambda_n}(u_{\lambda_n}) = 0$, we have that

$$\int_\Omega \frac{f(x,u_{\lambda_n})u_{\lambda_n}}{\|u_{\lambda_n}\|^2}\,dx \leqslant 1.$$

On the other hand, by (h_2), (b_1) and Fatou's lemma,

$$\int_\Omega \frac{f(x,u_{\lambda_n})u_{\lambda_n}}{\|u_{\lambda_n}\|^2}\,dx = \int_{w(x)\neq 0} \big|w_{\lambda_x}(x)\big|^2 \frac{f(x,u_{\lambda_n})u_{\lambda_n}}{|u_{\lambda_n}|^2}\,dx \to \infty.$$

This is a contradiction.

If $w = 0$ in V, define

$$G_{\lambda_n}(t_n u_{\lambda_n}) = \max_{t\in[0,1]} G_{\lambda_n}(t u_{\lambda_n}).$$

For any $c > 0$ and $\overline{w}_{\lambda_n} = \sqrt{4c}\, w_{\lambda_n}$, we have, for n large enough, that

$$G_{\lambda_n}(t_n u_{\lambda_n}) \geqslant G_{\lambda_n}(\overline{w}_{\lambda_n}) \geqslant 2c\lambda_n - \int_\Omega F(x, \overline{w}_{\lambda_n}) \geqslant \frac{c}{2},$$

which implies that $\lim_{n\to\infty} G_{\lambda_n}(t_n u_{\lambda_n}) = \infty$. Evidently, $t_n \in (0,1)$, hence, we have $\langle G'_{\lambda_n}(t_n u_{\lambda_n}), t_n u_{\lambda_n}\rangle = 0$. It follows that

$$\int_\Omega \left(\frac{1}{2} f(x, t_n u_{\lambda_n}) t_n u_{\lambda_n} - F(x, t_n u_{\lambda_n})\right) dx \to \infty.$$

If condition (b$_3$) holds, $h(t) = \frac{1}{2}t^2 f(x,s)s - F(x,ts)$ is increasing in $t \in [0,1]$, hence $\frac{1}{2} f(x,s)s - F(x,s)$ is increasing in $s > 0$. Combining the oddness of f, we have that

$$\int_\Omega \left(\frac{1}{2} f(x, u_{\lambda_n}) u_{\lambda_n} - F(x, u_{\lambda_n})\right) dx \geqslant \int_\Omega \left(\frac{1}{2} f(x, t_n u_{\lambda_n}) t_n u_{\lambda_n} - F(x, t_n u_{\lambda_n})\right) dx \to \infty.$$

Therefore, we get a contradiction since

$$\int_\Omega \left(\frac{1}{2} f(x, u_{\lambda_n}) u_{\lambda_n} - F(x, u_{\lambda_n})\right) dx = G_{\lambda_n}(u_{\lambda_n}) \in \Big[b_0, \sup_{(t,u)\in[0,1]\times A} G\big((1-t)u\big)\Big].$$

Thus $\{u_\lambda\}_{\lambda\in\Lambda}$ is bounded.

Let $\lambda_m \to 1$ ($m \to \infty$), since $\{u_{\lambda_m}\}$ is bounded, then, up to a subsequence, we get $u_{\lambda_m} \to u$. In the same as that of the proof of Theorem 3.4, u is sign-changing. Hence, u is the sign-changing solution of (1.1). Since $b_k \to \infty$ ($k \to \infty$), we obtain infinitely many sign-changing solutions of (1.1). □

Remark. As far as we know, the sign-changing solutions of (1.1) have not studied. In this paper, we study the existence and multiple of sign-changing solutions for problem (1.1). The results include the existence of four sign-changing solutions or infinitely many sign-changing solutions for (1.1) which are different from the references [1–7]. All these results are new.

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