Sign-changing solutions for some fourth-order nonlinear elliptic problems

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Abstract

In this paper, we consider the existence and multiplicity of sign-changing solutions for some fourth-order nonlinear elliptic problems and some existence and multiple are obtained. The weak solutions are sought by means of sign-changing critical theorems. © 2007 Elsevier Inc. All rights reserved.

Keywords: Sign-changing solutions; Critical point; Elliptic problems

1. Introduction

Let \( \Omega \) be a bounded open set in \( \mathbb{R}^n \) with smooth boundary. The purpose of this paper is to investigate the existence and multiplicity of sign-changing solutions to the fourth-order nonlinear elliptic boundary value problems

\[
\begin{align*}
\Delta^2 u + c \Delta u &= f(x, u) \quad \text{in } \Omega, \\
{u|}_{\partial \Omega} &= {\Delta u|}_{\partial \Omega} = 0
\end{align*}
\]

where \( \Delta^2 \) denotes the biharmonic operator, \( c \in \mathbb{R} \) and \( f : \Omega \times \mathbb{R} \to \mathbb{R} \) is a Carathéodory function with subcritical growth: \(|f(x, t)| \leq C(1 + |t|^{s-1}), \forall x \in \Omega, \forall t \in \mathbb{R}, s \in (2, 2^*) (N \geq 3), s \in (2, +\infty) (N \leq 2)\).

In problem (1.1), let \( f(x, u) = b[(u + 1)^+ - 1] \), then we get the following Dirichlet problem:

\[
\begin{align*}
\Delta^2 u + c \Delta u &= b[(u + 1)^+ - 1] \quad \text{in } \Omega, \\
{u|}_{\partial \Omega} &= {\Delta u|}_{\partial \Omega} = 0
\end{align*}
\]

where \( u^+ = \max\{u, 0\} \) and \( b \in \mathbb{R} \).

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Thus fourth-order problems have been studied by many authors, in [1] Lazer and McKenna have pointed out that this type of nonlinearity furnishes a model to study travelling waves in suspension bridges. Since then more general nonlinear fourth-order elliptic boundary value problems have been studied. There are many results about problems (1.1) and (1.2). We refer the reader to [2,3] for some references along this line.

For problem (1.2), Lazer and McKenna [2] proved the existence of $2k - 1$ solutions when $N = 1$ and $b > \lambda_k (\lambda_k - c)$ by the global bifurcation method. In [5], Tarantello found a negative solution when $b \geq \lambda_1 (\lambda_1 - c)$ by a degree argument. For problem (1.1) when $f(x, u) = bg(x, u)$, Micheletti and Pistoia [3,4] proved that there exist two or three solutions for a more general nonlinearity $g$ by variational method. Zhang [6] proved the existence of solutions for a more general nonlinearity $f(x, u)$ under some weak assumptions. Zhang and Li [7] proved the existence of multiple nontrivial solutions by means of Morse theory and local linking. But the existence and multiple of sign-changing solutions for (1.1) have not been studied.

In this paper, we study the existence and multiple of sign-changing solutions for problem (1.1). The results include the existence of four sign-changing solutions or infinitely many sign-changing solutions for (1.1) which are different from the references [1–7]. All these results are new.

The plan of the following sections are as follows. In Section 2 we give some notations and preliminaries. In Section 3 we give some results. Section 4 is devoted to the proofs of these results.

2. Preliminaries and statements

Let $\Omega$ be a bounded open set in $\mathbb{R}^n$ with smooth boundary and $f : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ is a Caratheodory function with subcritical growth: $|f(x, t)| \leq C (1 + |t|^q - 1)$, where $s \in (2, 2^*) \ (N \geq 3), s \in (2, +\infty) \ (N \leq 2)$ for all $x \in \Omega$ and $t \in \mathbb{R}$. From now on, letter $C$ is indiscriminately used to denote various positive constants. Let $\lambda_k \ (k = 1, 2, \ldots)$ denote the eigenvalue and $\phi_k \ (k = 1, 2, \ldots)$ the corresponding eigenfunctions of the eigenvalue problem

$$
\begin{align*}
\Delta u + \lambda u &= 0 \quad \text{in } \Omega, \\
\partial \Omega &= 0
\end{align*}
$$

(2.1)

where each eigenvalue $\lambda_k$ is repeated as often as multiplicity recall that $0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots, \lambda_k \rightarrow \infty$. Then $\phi_1$ is positive (or negative) and eigenfunctions associated to $\lambda_i \ (i \geq 2)$ is sign-changing. By reference [8], the eigenvalue problem

$$
\begin{align*}
\Delta^2 u + c \Delta u &= \mu u \quad \text{in } \Omega, \\
\partial \Omega &= \Delta u \partial \Omega = 0
\end{align*}
$$

(2.2)

has infinitely many eigenvalues

$$
\mu_k = \lambda_k (\lambda_k - c), \quad k = 1, 2, \ldots,
$$

and corresponding eigenfunctions $\phi_k(x)$.

We will always assume $c < \lambda_1$. Let $V$ denote the Hilbert space $H^2(\Omega) \cap H^1_0(\Omega)$ equipped with the inner product

$$
\langle u, v \rangle = \int_{\Omega} [\Delta u \Delta v - \nabla u \nabla v] \, dx.
$$

(2.3)

Then we may denote an element $u$ of $V$ as

$$
u = \sum_{k=1}^{\infty} a_k \phi_k, \quad \sum_{k=1}^{\infty} a_k^2 < \infty,
$$

$\phi_k$ and $\phi_l \ (k \neq l)$ is orthogonal base for $V$. We denote by $\|u\|^p$ the norm in $L^p(\Omega)$ and by $\|u\|$ the norm in $V$ is given by

$$
\|u\|^2 = \langle u, u \rangle.
$$

Let $V'$ denote the dual of $V$ and $\langle , \rangle$ be the duality pairing between $V'$ and $V$. Let $X_k$ denote the eigenspace associated to $\mu_k$, then $V = \bigoplus_{j \in \mathbb{N}} X_j$. Let $V_k = X_1 \oplus \cdots \oplus X_k$, $B_R(0) = \{ u \in V, \|u\| < R \}$. 

Definition 2.1. \( E \) is Hilbert space, \( G \in C^1(E, R) \). \( G \) satisfies w-PS condition on \( V \) if \( \{u_n\} \in E \) and \( G(u_n) \) is bounded, \( G'(u_n) \to 0 \), we have either \( \{u_n\} \) is bounded and has a convergent subsequence or \( \|G'(u_n)\|\|u_n\| \to \infty \).

Definition 2.2. We say that \( u \in V \) is the solution of problem (1.1) if the identity

\[
\int_\Omega [\Delta u \Delta v - c \nabla u \nabla v] \, dx = \int_\Omega f(x, u) v \, dx
\]  

(2.4)

holds for any \( v \in V \).

Definition 2.3. \( u \) is the solution of (1.1): if \( u \in \{u \in E: u(x) \geq 0, \ u \neq 0\} \), then \( u \) is positive solution of (1.1); if \( u \in \{u \in E: u(x) \leq 0, \ u \neq 0\} \), then \( u \) is negative solution of (1.1); if \( u \in \{u \in E: \text{meas}\{x \in \Omega: \ u(x) > 0\} > 0, \ \text{meas}\{x \in \Omega: \ u(x) < 0\} > 0\} \), then \( u \) is sign-changing solution of (1.1).

Assume \( H \) is Banach space, \( \Phi = \{\Gamma(\cdot, \cdot) \in C([0, 1] \times E, E)\} \), where \( \Gamma(\cdot, \cdot) \) satisfies

(a) \( \Gamma(0, \cdot) = \text{id}; \)
(b) \( \forall t \in [0, 1], \Gamma(t, \cdot) \) is a homeomorphism of \( E \) onto itself, \( (t, x) \mapsto \Gamma(t, \cdot)^{-1}(x) \) is continuous on \([0, 1] \times E; \)
(c) there exists \( x_0 \in H \) such that \( \Gamma(1, x) = x_0 \) for each \( x \in H \) and \( \Gamma(t, x) \to x_0 \) as \( t \to 1 \) uniformly on bounded subsets of \( H \).

Definition 2.4. (See [10, p. 21].) A subset \( A \) of \( H \) is linked (with respect to \( \Phi \)) to a subset \( B \) of \( H \) if \( A \cap B = \emptyset \), for every \( \Gamma \in \Phi \), there is \( t \in [0, 1] \) such that \( \Gamma(t, A) \cap B \neq \emptyset \).

In this paper, we need the following four propositions.

Proposition 2.1. (See [11, Theorem 3.2].) Assume \( H \) is Hilbert space, \( f \) satisfies PS condition on \( H \) and \( f'(u) \) has the expression \( f'(u) = u - Au \). \( D_1 \) and \( D_2 \) are open convex subset of \( H \), \( D_1 \cap D_2 \neq \emptyset \), \( A(\partial D_1) \subset D_1 \), \( A(\partial D_2) \subset D_2 \). If there exists a path \( h : [0, 1] \to H \) such that

\[
h(0) \in D_1 \setminus D_2, \quad h(1) \in D_2 \setminus D_1
\]

and

\[
\inf_{u \in D_1 \cap D_2} f(u) > \sup_{t \in [0, 1]} f(h(t)),
\]

then \( f \) has at least four critical points: \( u_1 \in D_1 \cap D_2, u_2 \in D_1 \setminus \overline{D_2}, u_3 \in D_2 \setminus \overline{D_1}, u_4 \in H \setminus (\overline{D_1} \cup \overline{D_2}). \)

Proposition 2.2. (See [8, Theorem 2.1].) Let \( E \) be a Hilbert space with inner product \( \langle \cdot, \cdot \rangle \) and norm \( \|\cdot\| \). Assume that \( E \) has an orthogonal decomposition \( E = N \oplus M \) with \( \dim N < \infty \). Let \( G \in C^1(E, R) \) and the gradient \( G' \) be of the form

\[
G'(u) = u - J'(u)
\]

where \( J' : E \to E \) is a continuous operator. Let \( P \) denote a closed convex positive cone of \( E \); \( D_0^{(i)} \) be an open convex subset of \( E \), \( i = 1, 2 \), \( S = E \setminus W \), \( W = D_0^{(1)} \cup D_0^{(2)} \). Assume

(H1) \( J'(D_0^{(i)}) \subset D_0^{(i)}, \ i = 1, 2. \)
(H2) If \( D_0^{(1)} \cap D_0^{(2)} = \emptyset \), then either \( D_0^{(1)} = \emptyset \) or \( D_0^{(2)} = \emptyset \).
(H3) There exist \( \delta > 0 \) and \( z_0 \in N \) with \( \|z_0\| = 1 \) such that

\[
B := \{u \in M: \|u\| \geq \delta\} \cup \{sz_0 + v: v \in M, s \geq 0, \|sz_0 + v\| = \delta\} \subset S.
\]

Let \( G \) maps bounded sets to bounded sets and satisfies w-PS and

\[
b_0 = \inf_M G \neq -\infty, \quad a_0 = \sup_N G \neq +\infty.
\]

Then \( G \) has a critical point in \( S \) with critical value \( \geq \inf_B G \).
Proposition 2.3. (See [9, Corollary 2.1].) Assume E is a Hilbert space with inner product (,) and the corresponding norm ||.||. G ∈ C¹(E, R) and G(u) = ½ ||u||² − J(u), u ∈ E, where J ∈ C¹(E, R) maps bounded sets to bounded sets. G₃(u) = ½ ||u||² − J(u), λ ∈ Λ = (¼, 1). P denote a closed convex cone of E. Assume:

(A₁) There exists μ₀ > 0 such that dist(J'(u), ±P) ≤ ½ dist(u, ±P) for all u ∈ E with dist(u, ±P) < μ₀.

(A₂) ±D₀ = {u ∈ E: dist(u, ±P) < μ₀}, D = D₀ ∪ (−D₀), S = E \ D, let A be a bounded subset of E and link a subset B of E, B ⊂ S and

\[ a₀(λ) = \sup_A G_λ ≤ b₀(λ) = \inf_B G_λ, \quad \forall λ ∈ Λ. \]

\( J' \) is compact, then for almost all \( λ ∈ Λ, G_λ \) has a sign-changing critical point in S.

Proposition 2.4. (See [9, Theorem 3.1].) Assume E is a Hilbert space with inner product (,) and the corresponding norm ||.||, E = \( \bigoplus_{j ∈ N} X_j \) with \( \dim X_j < ∞ \) for any \( j ∈ N \), where \( N \) denotes the set of all positive integers. G ∈ C¹(E, R) and G(u) = ½ ||u||² − J(u), where J ∈ C¹(E, R) maps bounded sets to bounded sets, G₃(u) = ½ ||u||² − J(u), λ ∈ Λ = (½, 1). P denote a closed convex of E,

\[ ±D₀ = \{u ∈ E: \text{dist}(u, ±P) < μ₀\}, \quad D = D₀ ∪ (−D₀), \quad S = E \setminus D, \]

\( E_k = \bigoplus_{j=1}^k X_j, \quad Z_k = \bigoplus_{j=k}^∞ X_j, \quad B_k = \{u ∈ E_k: ||u|| ≤ ρ_k\}, \quad N_k = \{u ∈ Z_k: ||u|| = r_k\} \)

where \( ρ_k > r_k > 0 \). For \( k ≥ 2 \), assume

\[ \Gamma_k = \{γ ∈ C([0, 1] × B_k, E): γ(t, u) \text{ is odd in } u \text{ and } γ(t, ·)|_{∂B_k} = \text{id} \text{ for each } t ∈ [0, 1], \]

\[ γ(t, D) ⊂ D \text{ for all } t ∈ [0, 1]\]

\[ a_k(λ) = \max_{∂B_k} G_λ, \quad b_k(λ) = \inf_{N_k} G_λ, \quad c_k(λ) = \inf_{γ ∈ Γ_k} \max_{γ([0, 1], B_k) ∩ S} G_λ. \]

If (A₁) and the following (A₃) hold:

(A₃) \( a_k(λ) < b_k(λ) \) for any \( λ ∈ Λ, N_k ⊂ S, \)

\( G_λ \) is even for any \( λ ∈ Λ, \) then for almost all \( λ ∈ Λ, \) there is a sequence \( \{u_m\} \) depending on \( λ \) such that

\[ \sup_m ||u_m|| < ∞, \quad u_m ∈ S, \quad G_λ'(u_m) → 0, \quad G_λ(u_m) → c_k(λ) ∈ \left[b_k(λ), \max_{u ∈ B_k} G(u)\right]. \]

In particular, if J' is compact, then for almost all \( λ ∈ Λ, G_λ \) has a sign-changing critical point \( u_λ ∈ S \) and \( G_λ(u_λ) ∈ [b_k(λ), \max_{u ∈ B_k} G(u)]\).

The solutions of (1.1) are corresponding to the critical points of the following C¹-functional:

\[ G(u) = \frac{1}{2} ||u||² - \int_Ω F(x, u) dx = \frac{1}{2} ||u||² - J(u) \]

where \( F(x, t) = \int_0^t f(x, s) ds \). The gradient of G at u is given by

\[ G'(u) = u - J'(u). \]

Then \( \langle J'(u), v \rangle = \int_Ω f(x, u)v dx, \quad ∀ v ∈ V, \)

\[ |f(x, t)| ≤ C(1 + |t|^{s-1}), \quad ∀ x ∈ Ω, \quad ∀ t ∈ R, \]

when \( N ≥ 3, s ∈ (2, 2^n) \), when \( N ≤ 2, s ∈ (2, +∞) \), by [12, Theorem 6.3.2], \( G ∈ C¹(E, R) \) and J' is compact.
3. Main results

Suppose

\((g_1)\) \( f \in C(\mathbb{O} \times R, R) \);
\((g_2)\) there exists \( \eta > 2 \) such that \( \forall x \in \Omega, \forall t \in R, \)
\[ 0 \leq \eta F(x,t) \leq f(x,t) t. \]

Moreover \( f(x,t) = o(|t|) \) as \( t \to 0 \) uniformly in \( x \in \Omega \).

It is easy seen that \((g_1)\) and \((g_2)\) hold for nonlinearity of the form
\[ f(x,t) = 1/|x| + 1/|t|^{p-2} t \]
where \( p \in (2, 2^* ) (N \geq 3), p \in (2, +\infty ) (N \leq 2). \)

\((h_1)\) \( \mu_l t^2 - W_1(x) \leq 2F(x,t) \leq \mu_{l+1} t^2 + W_2(x), \) a.e. \( x \in \Omega, t \in R, \) where \( W_1, W_2 \in L^1(\Omega), l \geq 2. \)

This assumption implies the following double resonance case:

\[ \mu_l \leq \liminf_{|t|\to\infty} \frac{2F(x,t)}{t^2} \leq \limsup_{|t|\to\infty} \frac{2F(x,t)}{t^2} \leq \mu_{l+1}, \quad \text{a.e. } x \in \Omega, \]

as well as jumping and oscillating between \( \mu_l, \mu_{l+1}. \) Furthermore, if we assume

\((h_2)\) \( f(x,t), t \geq 0, \) for a.e. \( x \in \Omega, t \in R; f(x,t) = o(|t|) \) as \( |t| \to 0 \) uniformly for \( x \in \Omega, \)

then we have

**Theorem 3.1.** Assume \((g_1)\) and \((g_2)\) hold, then \((1.1)\) has four solutions: one naught solution, one positive solution, one negative solution and one sign-changing solution.

**Theorem 3.2.** Assume \((g_2)\) and \((h_1)\) hold, then \((1.1)\) has at least a sign-changing solution.

**Remark.** \( f \) has subcritical growth: \( |f(x,t)| \leq C(1 + |t|^{s-1}), \forall x \in \Omega, \forall t \in R, s \in (2, 2^*) (N \geq 3), s \in (2, +\infty ) (N \leq 2), \) but by \((g_2)\) \( F \) is superquadratic because \( \eta > 2. \) It is easy seen that this subcritical condition and \((g_2)\) hold for nonlinearity of the form
\[ f(x,t) = |t|^{p-2} t \]
where \( p \in (2, 2^* ) (N \geq 3), p \in (2, +\infty ) (N \leq 2). \)

**Theorem 3.3.** Assume \((h_1)\) and \((h_2)\) hold. Moreover if

\((h_3)\) \( \mu_l < L = \liminf_{|t|\to\infty} \frac{f(x,t)}{t} \leq \limsup_{|t|\to\infty} \frac{f(x,t)}{t} \leq \mu_{l+1} \ a.e. \ x \in \Omega; \)
\((h_4)\) there exists \( \alpha > 0 \) such that

\[ \lim_{|t|\to\infty} \frac{f(x,t) - 2F(x,t)}{|t|^\alpha} = \beta(x) \quad \text{a.e. } x \in \Omega, \]

where \( \int_{\Omega} \beta(x)|w(x)|^\alpha \, dx > 0 \) on the set \( \{ w \in X_{l+1} : \|w\| = 1 \}, \) then \((1.1)\) has at least one sign-changing solution.

Suppose

\[ \lim_{t\to\infty} \frac{f(x,t)}{t} = b_+(x), \quad \lim_{t\to-\infty} \frac{f(x,t)}{t} = b_-(x), \]
uniformly for $x \in \Omega$. For $k \geq 2$,

(a1) there is a constant $F_0 > \mu_k$ such that

$$4F(x, t) \leq F_0 t^2 \quad \text{for all } x \in \Omega, t \in R.$$

(a2) $\forall (x, t) \in \Omega \times R$, $2F(x, t) \geq \mu_k - 1 < 2 - W_0(x)$, where $F(x, t) = \int_0^t f(x, s) ds$, $0 < \int_\Omega W_0(x) dx < \infty$.

Choose $\mu_l$ such that

$$\mu_l \geq \frac{64\mu_k^2}{\mu_k(\mu_k - \mu_k - 1)} F_0,$$  \hspace{1cm} (3.1)

then exists positive constant $C_{l-1}$ such that $\|u\| \leq C_{l-1} \|u\| u \in V_{l-1}$.

(a3) $2F(x, t) \leq \mu_k - \mu_k + 1 - t^2$, for all $x \in \Omega$ and $|t| \leq r_0$, where

$$r_0 \geq C_{l-1} \left( \frac{48\mu_k}{\mu_k - \mu_k - 1} \int_\Omega W_0(x) dx \right)^{\frac{1}{2}}.$$

(a4) $H(x, t) = f(x, t) t - 2F(x, t) > 0$ for all $x \in \Omega$ and $t \neq 0$, $H(x, t)$ is convex in $t$.

**Theorem 3.4.** Assume (a1)–(a4) and (h2) hold and $\mu_k < b_\pm(x)$ for all $\forall x \in \Omega$, then (1.1) has one sign-changing solution.

**Theorem 3.5.** Assume (h2) and

(b1) $\liminf_{|t| \to \infty} \frac{f(x, t)}{t} = \infty$ uniformly for $x \in \Omega$.

(b2) $f(x, t)$ is odd in $t$.

(b3) $\frac{f(x, t)}{t}$ is nondecreasing in $t > 0$, (1.1) has infinitely many sign-changing solution.

It is easy seen that (b1)–(b3) and (h2) hold for nonlinearity of the form

$$f(x, t) = |t|^{p-2} t$$

where $p \in (2, 2^N) (N \geq 3)$, $p \in (2, +\infty) (N \leq 2)$.

4. **Proof of theorems**

For $\mu_0 > 0$, assume

$$D_0(\mu_0) = \{u \in V: \text{dist}(u, P) < \mu_0\},$$

$$-D_0(\mu_0) = \{u \in V: \text{dist}(u, -P) < \mu_0\},$$

$$P = \{u \in V: u(x) \geq 0 \text{ a.e. } x \in \Omega\}.$$

**Lemma 4.1.** Assume (g2) holds, then $G$ satisfies PS condition.

**Proof.** Assume $\{u_n\} \subset V$, $|G(u_n)| \leq C$, $G'(u_n) \to 0$. It suffices to prove that $\{u_n\}$ is bounded. By (g2)

$$\eta C + \|G'(u_n)\| \|u_n\| \geq \eta G(u_n) - \langle G'(u_n), u_n \rangle \geq \frac{\eta - 2}{2} \|u_n\|^2,$$

thus $\{u_n\}$ is bounded. \(\square\)
Lemma 4.2. Assume \((g_2)\) holds, then there exists \(\epsilon_0 > 0\) such that

(i) \(J'(\partial D_0(\epsilon_0)) \subset D_0(\epsilon_0)\), and if \(u \in D_0(\epsilon_0)\) is the solution of (1.1), then \(u \in P\);

(ii) \(J'(\partial(-D_0(\epsilon_0))) \subset -D_0(\epsilon_0)\), and if \(u \in -D_0(\epsilon_0)\) is the solution of (1.1), then \(u \in -P\).

Proof. Let \(u^\pm = \max\{\pm u, 0\}\). \(\forall u \in V\), by the definition of \(V\) and the Sobolev embedding theorem, if \(s \in (2, 2^*)\), there exists \(C_s > 0\) such that

\[
\|u^+\|_s \leq \inf_{w \in (-P)} \|u - w\|_s \leq C_s \inf_{w \in (-P)} \|u - w\|_s = C_s \text{dist}(u, -P).
\] (4.1)

By \(|f(x, t)| \leq C(1 + |t|^s-1)\) and \((g_2)\): \(\forall \epsilon > 0\), there exists \(C_\epsilon > 0\), such that

\[
f(x, t) t \leq \epsilon t^2 + C_\epsilon |t|^s, \quad \forall x \in \Omega, \forall t \in \mathbb{R}.
\] (4.2)

Assume \(v = J'(u)\). Then by (4.1) and (4.2), for \(\epsilon\) small enough,

\[
\text{dist}(v, -P) \|v^+\| \leq \|v^+\|^2
\]

\[
= \langle v, v^+ \rangle
\]

\[
\leq \int_{\Omega} f(x, u^+) v^+ dx
\]

\[
\leq \int_{\Omega} \left(\epsilon |u^+| + C_\epsilon |u^+|^s-1 \right) |v^+| dx
\]

\[
\leq \left(\frac{1}{2} \text{dist}(u, -P) + C \text{dist}(u, -P)^s-1\right) \|v^+\|.
\]

That is,

\[
\text{dist}(J'(u), -P) \leq \frac{1}{2} \text{dist}(u, -P) + C(\text{dist}(u, -P)^{s-1}).
\] (4.3)

So there exists \(\epsilon_0 > 0\) such that \(\text{dist}(J'(u), -P) \leq \frac{3}{4} \epsilon_0\) for every \(u \in \partial(-D_0(\epsilon_0))\). Thus \(J'(\partial(-D_0(\epsilon_0))) \subset -D_0(\epsilon_0)\).

If \(u \in D_0(\epsilon_0)\) is the solution of (1.1), then \(G'(u) = u - J'(u) = 0\), that is, \(J'(u) = u\). By (4.3), \(u \in -P\), (i) holds. (ii) can be proved analogously. \(\square\)

Lemma 4.3. Assume \((g_2)\) holds, then

\[
\inf_{D_0(\epsilon) \cap -D_0(\epsilon)} G(u) = d_0 > -\infty.
\]

Proof. By \((g_2)\), (4.2) and Holder inequality

\[
G(u) \geq -\int_{\Omega} F(x, u(x)) dx
\]

\[
\geq -\frac{1}{\eta} \int_{\Omega} f(x, u(x)) u(x) dx
\]

\[
\geq -\frac{C}{\eta} (\|u\|_2^2 + \|u\|_p^p).
\]

According to (4.1), \(\|u^+\|_s \leq C_s \text{dist}(u, -P) \leq C_s \epsilon_0\), \(\|u^-\|_s \leq C_s \text{dist}(u, P) \leq C_s \epsilon_0\), so

\[
\inf_{D_0(\epsilon) \cap -D_0(\epsilon)} G(u) = d_0 > -\infty.
\] \(\square\)
Proof of Theorem 3.1. The \( f(x, t) \) of Theorem 3.1 satisfies the condition of [12, Theorem 7.4.2], so as the same of (7.4.14) of [12, Theorem 7.4.2], there are two positive constants \( M_1 \) and \( M_2 \) such that \( \forall t \in R, \forall x \in \Omega \),
\[
F(x, t) \geq M_1 |t|^\eta - M_2.
\]
For any finitely dimensional subspace \( V_0 \) of \( V \), we have, \( \forall v \in V_0 \), there exists \( M_3 > 0 \) such that
\[
G(u) = \frac{1}{2} \|u\|^2 - \int_\Omega F(x, u) \, dx
\leq \frac{1}{2} \|u\|^2 - M_1 \|u\|^\eta + M_2 |\Omega|
\leq \frac{1}{2} \|u\|^2 - M_3 \|u\|^\eta + M_2 |\Omega|
\]
where \( |\Omega| \) denote the measure of \( \Omega \). Since \( \eta < 2 \), by Young inequality, there are two positive number \( M_4 \) and \( M_5 \) such that
\[
G(u) \leq -M_4 \|u\|^2 + M_5, \quad \forall v \in V_0. \tag{4.4}
\]
Since \( \varphi_2 \in V \) is sign-changing, that is, \( \varphi_2^+ \neq 0, \varphi_2^- \neq 0 \). It is clear that \( \varphi_2^+ \) and \( \varphi_2^- \) are linearly independent. Let \( V_0 = \{ t \varphi_2^+ + s \varphi_2^- : t \geq 0, s \geq 0 \} \), then \( V_0 \) is the finitely dimensional subspace of \( V \). Define a path \( h : [0, 1] \mapsto V \),
\[
h(t) = t \frac{R_0}{\| \varphi_2^+ \|} \varphi_2^+ + (1 - t) \frac{R_0}{\| \varphi_2^- \|} \varphi_2^-
\]
where \( R_0 = \max\{ \frac{d_0 - 2M_5 - 1}{-M_4}, 1 \} \), then by (4.4)
\[
G(h(t)) = G\left( t \frac{R_0}{\| \varphi_2^+ \|} \varphi_2^+ \right) + G\left( (1 - t) \frac{R_0}{\| \varphi_2^- \|} \varphi_2^- \right)
\leq -M_4 R_0 + 2M_5
\leq d_0 - 1. \tag{4.5}
\]
So
\[
\inf_{u \in D_0(\epsilon_0) \cap -D_0(\epsilon_0)} f(u) > \sup_{t \in [0, 1]} f(h(t)).
\]
Obviously, \( h(0) \in -D_0(\epsilon_0) \), \( h(1) \in D_0(\epsilon_0) \), thus \( h(0) \in -D_0(\epsilon_0) \setminus D_0(\epsilon_0) \). If not, \( h(0) \in -D_0(\epsilon_0) \cap D_0(\epsilon_0) \), by Lemma 4.3, \( G(h(0)) \geq d_0 \). This is a contradiction. Analogously, \( h(1) \in D_0(\epsilon_0) \setminus -D_0(\epsilon_0) \). Moreover, \( 0 \in -D_0(\epsilon_0) \cap D_0(\epsilon_0) \), by Lemmas 4.1, 4.2 and Proposition 2.1, (1.1) has four solutions: \( u_1 \in D_0(\epsilon_0) \cap (-D_0(\epsilon_0)) \), \( u_2 \in D_0(\epsilon_0) \setminus -D_0(\epsilon_0) \), \( u_3 \in (-D_0(\epsilon_0)) \setminus D_0(\epsilon_0) \), \( u_4 \in H \setminus (D_0(\epsilon_0) \cup -D_0(\epsilon_0)) \). That is, \( u_1 \) is naught solution, \( u_2 \) is positive solution, \( u_3 \) is negative solution and \( u_4 \) is sign-changing solution. \( \square \)

We prove Theorems 3.2 and 3.3 by Proposition 2.2. First let \( N = X_1 \oplus X_2 \oplus \cdots \oplus X_l (l \geq 2) \), \( M = \bigoplus_{i=l+1}^{\infty} X_i \), then \( V = N \oplus M \). We take \( z_0 \in X_l, \|z_0\| = 1 \), and define
\[
B = \{ u \in M : \|u\| \geq \delta \} \cup \{ u = sz_0 + v : v \in M, s \geq 0, \|u\| = \delta \}.
\]
Then each element of \( B \) is sign-changing.

Lemma 4.4. \( \text{dist}(B, -P \cup P) = d_1 > 0 \).

Proof. \( B \) and \( -P \cup P \) are two closed subsets of \( V \). Note that \( B \cap (-P \cup P) = \emptyset \) and \( V \) is normal space, the conclusion is readily to be shown. \( \square \)

Lemma 4.5. Assume \( (h_2) \) holds, then there exists \( \mu_0 \in (0, d_1) \) such that \( J'(\pm D_0(\mu_0)) \subset \pm D_0(\mu_0) \).
Proof. Let \( u^\pm = \max\{\pm u, 0\} \). \( \forall w \in -P \), we have \( w(x) \leq 0 \), so \( -w(x) \geq 0 \). Hence, \( \forall w \in -P, s \in (2, 2^*) \),

\[
\|u^+\|_s = \int_{\Omega} |u^+|^s \, dx
\]

\[
= \int_{u(x) \geq 0} |u^+|^s \, dx + \int_{u(x) < 0} |u^+|^s \, dx
\]

\[
= \int_{u(x) \geq 0} |u^+|^s \, dx
\]

\[
\leq \int_{u(x) \geq 0} |u^+ - w|^s \, dx + \int_{u(x) < 0} |-u^- - w|^s \, dx
\]

\[
= \int_{\Omega} \|u - w\|^s \, dx.
\]

Therefore \( \|u^+\|_s \leq \inf_{w \in (-P)} \|u - w\|_s \). Moreover, by the definition of \( V \) and Sobolev embedding theorem, when \( s \in (2, 2^*) \), the embedding \( V \hookrightarrow L^s(\Omega) \) is continuous. So for all \( u \in V \), if \( s \in (2, 2^*) \), there exists \( C_s > 0 \) such that

\[
\|u^+\|_s \leq \inf_{w \in (-P)} \|u - w\|_s \leq C_s \|u - P\| = C_s \text{dist}(u, -P).
\]  

(4.6)

By (h2), \( \forall \epsilon > 0 \), there exists \( C_\epsilon > 0 \), such that

\[
f(x, t) \leq \epsilon t^2 + C_\epsilon |t|^s, \quad x \in \Omega, \ t \in R.
\]  

(4.7)

Assume \( v = J'(u) \). By (4.6) and (4.7), for \( \epsilon \) small enough,

\[
\text{dist}(v, -P) \|u^+\| \leq \|v^+\|^2
\]

\[
= \langle v, v^+ \rangle
\]

\[
\leq \int_{\Omega} f(x, u^+) v^+ \, dx
\]

\[
\leq \int_{\Omega} (\epsilon |u^+| + C_\epsilon |u^+|^{s-1}) |v^+| \, dx
\]

\[
\leq \left( \frac{1}{2} \text{dist}(u, -P) + C \text{dist}(u, -P)^{s-1} \right) \|v^+\|.
\]

That is, \( \text{dist}(J'(u), -P) \leq \frac{1}{2} \text{dist}(u, -P) + C \text{dist}(u, -P)^{s-1} \), there is \( \mu_0 < d_1 \) (cf. Lemma 4.4) such that \( \text{dist}(J'(u), -P) \leq \frac{3}{2} \mu_0 \) for every \( u \in -D_0(\mu_0) \). In a similar way, \( \text{dist}(J'(u), P) \leq \frac{3}{2} \mu_0 \) for every \( u \in D_0(\mu_0) \). The conclusion follows. \( \square \)

Lemma 4.6. Under the assumptions of Theorem 3.2, \( G \) satisfies the \( w \)-PS condition.

Proof. Assume \( \{u_n\} \subset V \) such that \( |G(u_n)| \leq C \) and \( G'(u_n) \to 0 \). Without loss of generality, we suppose that \( \{\|G'(u_n)\| \|u_n\|\} \) is bounded. It suffices to prove that \( \{u_n\} \) is bounded. By (g2),

\[
\eta C + \|G'(u_n)\| \|u_n\| \geq \eta G(u_n) - \{G'(u_n), u_n\} \geq \frac{\eta - 2}{2} \|u_n\|^2.
\]

Thus \( \|u_n\| \) is bounded. \( \square \)

Lemma 4.7. Under the assumptions of Theorem 3.3, \( G \) satisfies the \( w \)-PS condition.
Similarly, there exists $\Omega_l \subset \Omega$ such that $\text{meas}(\Omega \setminus \Omega_l) < \epsilon$ and that $\limsup_{|t| \to \infty} \frac{f(x,t)}{t} \leq \mu_l$ uniformly for $x \in \Omega_l$. Therefore, there exists $C_1 > 0$ such that

$$f(x,t) \leq \frac{1}{|t|} \left( C_1 + \frac{1}{|t|} \right), \quad \forall t \neq 0, \ x \in \Omega_{l+1}. \tag{4.8}$$

Similarly, there exists $\Omega_l \subset \Omega$ such that $\text{meas}(\Omega \setminus \Omega_l) < \epsilon$ and that $\limsup_{|t| \to \infty} \frac{f(x,t)}{t} \geq L > \mu_l$ uniformly for $x \in \Omega_l$. Hence, there exists $C_2 > 0$ such that

$$f(x,t) \geq \frac{1}{|t|} \left( L - C_2 \right), \quad \forall t \neq 0, \ x \in \Omega_l. \tag{4.9}$$

Let $\Omega^* = \Omega_l \cap \Omega_{l+1}$, then $\text{meas}(\Omega^*) \epsilon < 2\epsilon$. Write $u_n$ with $u_n = u_n^+ + u_n^0 + u_n^-$, where $u_n^- \in N$, $u_n^0 \in X_{l+1}$, $u_n^+ \in \bigoplus_{i=l+2}^{\infty} X_i$. If $|u_n^+ + u_n^0| \geq |u_n^-|$ on $\Omega^*$, by (4.8) we have that

$$f(x,u_n)(u_n^+ + u_n^0 - u_n^-) \leq (\mu_{l+1} + \epsilon)(u_n^+ + u_n^0)^2 - L(u_n^-)^2 + C_1|u_n^+ + u_n^0 - u_n^-|. \tag{4.10}$$

If $|u_n^+ + u_n^0| < |u_n^-|$ on $\Omega^*$, by (4.9) we have that

$$f(x,u_n)(u_n^+ + u_n^0 - u_n^-) \leq (\mu_{l+1} + \epsilon)(u_n^+ + u_n^0)^2 - L(u_n^-)^2 + C_2|u_n^+ + u_n^0 - u_n^-|. \tag{4.11}$$

Since $\|u_n^-\|^2 - L \int_\Omega f(x,u_n)^2 \, dx \leq \frac{L}{\mu_{l+2}}\|u_n^-\|^2 : = -\gamma\|u_n^-\|^2$ for every $u_n \in N$. Therefore, by (4.10) and (4.11),

$$\langle G'(u_n), u_n^+ + u_n^0 - u_n^- \rangle = \|u_n^+\|^2 + \|u_n^0\|^2 - \|u_n^-\|^2 - \int_\Omega f(x,u_n)(u_n^+ + u_n^0 - u_n^-) \, dx$$

$$\geq \|u_n^+\|^2 + \|u_n^0\|^2 - \|u_n^-\|^2 - \int_{\Omega^*} ((\mu_{l+1} + \epsilon)(u_n^+ + u_n^0)^2 - L(u_n^-)^2) \, dx$$

$$- \int_{\Omega^*} (C_1 + C_2)|u_n^+ + u_n^0 - u_n^-| \, dx - \int_{\Omega \setminus \Omega^*} |f(x,u_n)||u_n^+ + u_n^0 - u_n^-| \, dx$$

$$\geq \|u_n^+\|^2 \left( 1 - \frac{\mu_{l+1} + \epsilon}{\mu_{l+2}} \right) - \epsilon\|u_0\|^2 + L \int_\Omega (u_n^-)^2 \, dx - \|u_n^-\|^2$$

$$- L \int_\Omega (u_n^-)^2 - \int_{\Omega \setminus \Omega^*} |f(x,u_n)||u_n^+ + u_n^0 - u_n^-| \, dx$$

$$\geq \|u_n^+\|^2 \left( 1 - \frac{\mu_{l+1} + \epsilon}{\mu_{l+2}} \right) - \epsilon\|u_0\|^2 + \gamma\|u_n^-\|^2 - \int_{\Omega^*} (C_1 + C_2)|u_n^+ + u_n^0 - u_n^-| \, dx$$

$$- L \int_{\Omega^*} (u_n^-)^2 - \int_{\Omega \setminus \Omega^*} |f(x,u_n)||u_n^+ + u_n^0 - u_n^-| \, dx,$$

which implies that $\frac{u_n^+}{\|u_n\|} \to 0$, $\frac{u_n^0}{\|u_n\|} \to 0$, $\frac{u_n^-}{\|u_n\|} \to 1$, hence $\frac{u_n}{\|u_n\|} \to w$ with $w \in X_{l+1}$, $\|w\| = 1$. So

$$0 = \lim_{n \to \infty} \frac{G(u_n) - \frac{1}{2} \langle G'(u_n), u_n \rangle}{\|u_n\|^q} = \lim_{n \to \infty} \int_\Omega (f(x,u_n)u_n - 2F(x,u_n)) \, dx \leq \int_\Omega \beta(x)|w(x)|^q \, dx > 0.$$
Proof of Theorems 3.2 and 3.3. Assume
\[ D_0^{(1)} = D_0(\mu_0), \quad D_0^{(2)} = -D_0(\mu_0), \]
\[ W = D_0^{(1)} \cup D_0^{(2)}, \quad S = V \setminus W. \]
By Lemma 4.4, \( B \subset S \), that is, the condition (H3) of Proposition 2.2 holds. Lemma 4.5 says that condition (H1) of proposition is also satisfied. Since \( 0 \in D_0^{(1)} \cap D_0^{(2)} \), then (H2) holds automatically. By Lemmas 4.6 and 4.7, \( G \) satisfies w-PS condition. Moreover, note that \( \|v\|^2 \leq \mu_l \|v\|^2 \) for all \( v \in N \) and \( \mu_{l+1} \|w\|^2 \leq \|w\|^2 \) for all \( w \in M \). Combining (h1), we have that
\[ G(v) \leq \frac{1}{2} \|v\|^2 - \frac{\mu_l}{2} \|v\|^2 + \int_{\Omega} W_1(x) \, dx \leq \frac{\int_{\Omega} W_1(x) \, dx}{2}, \quad \forall v \in N, \]
and
\[ G(w) \geq \frac{1}{2} \|w\|^2 - \frac{\mu_{l+1}}{2} \|w\|^2 - \frac{\int_{\Omega} W_2(x) \, dx}{2} \geq -\frac{\int_{\Omega} W_2(x) \, dx}{2}, \quad \forall w \in M. \]
Therefore, we have
\[ \sup_{N} G = a_0 < \infty, \quad \inf_{M} b_0 > -\infty. \]
Since
\[ |f(x,t)| \leq c(1 + |t|^{s-1}), \quad \forall x \in \Omega, \quad \forall t \in \mathbb{R}, \]
\( G \) maps bounded sets to bounded sets. By Proposition 2.2, \( G \) has a critical point in \( S \). Therefore, (1.1) has a sign-changing solution.

Lemma 4.8. Assume (a2) holds, then \( G_{\lambda}(u) \leq \frac{1}{2} \int_{\Omega} W_0(x) \, dx \) for all \( u \in V_{k-1}, \lambda \in \Lambda \).

Proof. \( \forall u \in V_{k-1}, \forall \lambda \in \Lambda \), by (a2), we have that
\[ G_{\lambda}(u) \leq \frac{1}{2} \|u\|^2 - \int_{\Omega} F(x,u) \, dx \]
\[ \leq \frac{1}{2} \mu_{k-1} \int_{\Omega} u^2 \, dx - \int_{\Omega} F(x,u) \, dx \]
\[ \leq \frac{1}{2} \mu_{k-1} \int_{\Omega} u^2 \, dx - \frac{1}{2} \mu_{k-1} \int_{\Omega} u^2 \, dx + \frac{1}{2} \int_{\Omega} W_0(x) \, dx \]
\[ \leq \frac{1}{2} \int_{\Omega} W_0(x) \, dx. \]

Lemma 4.9. Assume (h2) holds and \( \forall x \in \Omega, \mu_k < b_{\pm}(x) \). Then \( G_{\lambda}(u) \to -\infty \) for \( u \in V_k \) as \( \|u\| \to \infty \) uniformly in \( \lambda \in \Lambda \).

Proof. Write \( G(u) \) as
\[ G(u) = \frac{1}{2} \|u\|^2 - \int_{\Omega} \left( \frac{1}{2} b_+(x)(u^+)^2 + \frac{1}{2} b_-(x)(u^-)^2 + P(x,u) \right) \, dx, \quad \forall u \in V \]
where \( P(x,u) = \int_{0}^{u} p(x,t) \, dt; \quad p(x,t) = f(x,t) - (b_+(x)t^+ - b_-(x)t^-), \quad t^\pm = \max\{\pm t, 0\} \). Note that \( \mu_k < b_{\pm}(x) \)
and the variational characterization of eigenvalues \( \{\mu_k\} \): \( \forall u \in V_k, \lambda \in \Lambda \), we have
\[ G_\lambda(u) \leq G(u) \]
\[ = \frac{1}{2} \|u\|^2 - \int_\Omega P(x, u) \, dx - \frac{1}{2} \left( \int_{b_-(x) \geq b_+(x)} b_+(x)(u^+)^2 + b_-(x)(u^-)^2 \right) \, dx \]
\[ = \frac{1}{2} \|u\|^2 - \int_\Omega P(x, u) \, dx - \frac{1}{2} \int_{b_-(x) \geq b_+(x)} b_+(x)u^+ \, dx - \frac{1}{2} \int_{b_-(x) < b_+(x)} (b_+(x) - b_-(x))(u^-)^2 \, dx \]
\[ \leq \frac{1}{2} \|u\|^2 - \int_\Omega P(x, u) \, dx - \frac{1}{2} \int_{b_-(x) \geq b_+(x)} b_+(x)u^+ \, dx - \frac{1}{2} \int_{b_-(x) < b_+(x)} b_-(x)u^2 \, dx \]
\[ \leq \frac{1}{2} \|u\|^2 - \int_\Omega P(x, u) \, dx \]

where \( \delta = \frac{\min\{b_+(x), b_-(x)\} - \mu_k}{2\mu_k} \). By (h2), \( \lim_{t \to \infty} p(x, t) = 0 \), therefore,
\[ \lim_{u \in V_k, \|u\| \to \infty} \frac{G_\lambda(u)}{\|u\|^2} \leq -\delta. \]

The conclusion follows. \( \square \)

**Lemma 4.10.** Assume (a1) and (a3) hold, then there exists \( \rho_0 > 0 \) independent of \( \lambda \) such that \( G_\lambda(u) \geq \frac{1}{2} \int_\Omega W_0(x) \, dx \) for all \( u \in \bigoplus_{i=1}^\infty X_i \) with \( \|u\| = \rho_0 \) and all \( \lambda \in \Lambda. \)

**Proof.** By (a1), if \( x \in \Omega, |t| \geq r_0, \)
\[ F(x, t) \leq \frac{1}{2} F_0 t^2 - \frac{1}{4} F_0 r_0^2, \] \( \forall u \in \bigoplus_{i=1}^\infty X_i, \) write \( u = v + w, \) where \( v \in \bigoplus_{i=1}^{l-1} X_i, w \in \bigoplus_{l=1}^\infty X_i. \) Let
\[ \beta_0 = \frac{\mu_k + \mu_{k-1}}{2}, \]
\[ \xi_1 = \frac{2F_0 + \mu_l}{8} w^2 + \frac{\mu_k + \beta_0}{8} v^2 - F(x, v + w). \]

If \( |v + w| \leq r_0, \) then by (a3) and the choice of \( \mu_l, \) we see that
\[ \xi_1 \geq \frac{2F_0 + \mu_l}{8} w^2 + \frac{\mu_k + \beta_0}{8} v^2 - \frac{1}{4} \beta_0 (v + w)^2 \]
\[ \geq \frac{2F_0 + \mu_l - 2\beta_0}{8} w^2 + \frac{\mu_k + \beta_0 - 2\beta_0}{8} v^2 - \frac{1}{2} \beta_0 |vw| \]
\[ \geq \left( \frac{(2F_0 + \mu_l - 2\beta_0)(\mu_k - \beta_0)}{4} - \frac{1}{2} \beta_0 \right) |vw| \geq 0. \] \( \quad \) \( \text{(4.14)} \)

If \( |v + w| > r_0, \) then by (4.12), we conclude that
\[ \xi_1 \geq \left( \frac{\mu_l + 2F_0 - 4F_0}{8} w^2 + \frac{\mu_k + \beta_0 - 4F_0}{8} v^2 \right) - F_0 vw + \frac{F_0 r_0}{4} = \xi_2 + \xi_3 \]
\( \text{(4.15)} \)

where
\( \xi_2 = \frac{\mu - 2F_0}{16}w^2 + \frac{\mu_k - \beta_0}{8}v^2 - \frac{\beta_0 w}{2} \),
\( \xi_3 = \frac{\mu - 2F_0}{16}w^2 - \frac{2F_0 - \beta_0}{4}v^2 - \frac{(2F_0 - \beta_0)w}{2} + \frac{F_0 r_0^2}{4}. \) (4.16) (4.17)

Next, we estimate \( \xi_2 \) and \( \xi_3 \). If \( \frac{\mu - \beta_0}{8}\beta_0 \geq 0 \), then
\( \xi_2 \geq \frac{\mu - 2F_0}{16}w^2 + \left( \frac{\mu_k - \beta_0}{8} |v| - \frac{\beta_0 |w|}{2} \right) |v| \geq 0. \) (4.18)

If \( \frac{\mu - \beta_0}{8}|v| - \frac{\beta_0 |w|}{2} < 0 \), by the choice of \( \mu_l \), we deduce that
\( \xi_2 \geq \left( \frac{\mu_l - 2F_0}{16} - \frac{2\beta_0}{\mu_k - \beta_0} \right)w^2 + \frac{\mu_k - \beta_0}{8}v^2 \geq 0. \) (4.19)

On the other hand,
\( \xi_3 \geq \frac{\mu_l + 2F_0 - 4F_0}{16}w^2 - \left( F_0 - \frac{\beta_0}{2} \right)(|v| + |w|)|v| + \frac{F_0 r_0^2}{4} = \xi_4. \) (4.20)

Thus
\( \xi_4 \geq \frac{\mu_l - 10F_0 + 4\beta_0}{16}w^2 - \frac{3(2F_0 - \beta_0)}{4}v^2 + \frac{F_0 r_0^2}{4} \)
\( \geq - \frac{3(2F_0 - \beta_0)}{4}v^2 + \frac{F_0 r_0^2}{4}. \) (4.21)

Choose \( \rho_0 = \frac{1}{C_{l-1}} \left( \frac{F_0 r_0^2}{4(2F_0 - \beta_0)} \right)^{\frac{1}{2}} \). If \( \|u\| = \rho_0 \), then \( \|v\| \leq C_{l-1} \|v\| \leq C_{l-1} \|u\| \leq C_{l-1} \rho_0. \) Hence, \( \xi_4 \geq 0. \) Therefore, by (4.13)–(4.21), \( \xi_1 \geq 0. \) Finally
\[
G_{\lambda}(u) = G_{\lambda}(v + w)
\geq \frac{1}{4} \left( \|v\|^2 + \|w\|^2 \right) - \int_\Omega F(x, v + w) \, dx
\geq \frac{1}{8} \|v\|^2 + \frac{1}{8} \|w\|^2 + \frac{1}{8} \mu_k \|v\|^2 + \frac{1}{8} \mu_l \|w\|^2 - \int_\Omega F(x, v + w) \, dx
\geq \frac{1}{8} \left( 1 - \frac{\beta_0}{\mu_k} \right) \|v\|^2 + \frac{1}{8} \left( 1 - \frac{2F_0}{\mu_l} \right) \|w\|^2 + \int_\Omega \xi_1 \, dx
\geq \frac{1}{8} \min \left\{ \left( 1 - \frac{\beta_0}{\mu_k} \right), \left( 1 - \frac{2F_0}{\mu_l} \right) \right\} \|u\|^2
\geq \frac{1}{8} \left( 1 - \frac{\beta_0}{\mu_k} \right) \rho_0^2
\geq \frac{1}{2} \int_\Omega W_0(x) \, dx. \quad \square
\]

By Lemma 4.9, there exists \( R \geq \rho_0 \) such that \( G_{\lambda}(u) \leq 0 \) for all \( u \in V_k, \|u\| \geq R. \) Choose \( y_0 \in X_k, \|y_0\| = 1. \) Let
\( B = \bigcup_{i=k}^\infty X_i \cap \partial B_{\rho_0}(0), \) \( A = \{ u = v + sy_0: v \in V_{k-1}, s \geq 0, \|u\| = R \} \cup (V_{k-1} \cap B_R(0)). \) By the definite of \( A, B \) and link, \( A \) links \( B \) and each element of \( B \) is sign-changing. Similar to Lemma 4.4, \( \text{dist}(B, -P \cup P) = d_2 > 0. \) In the same as that of the proof of Lemma 4.5, we have that
Lemma 4.11. Under the assumptions of Theorem 3.4, then there exists \( \mu_0 \in (0, d_2) \), \( \mu_0 < \frac{1}{2} \) such that

\[
\text{dist}(J'(u), \pm P) \leq \frac{1}{5} \text{dist}(u, \pm P)
\]

for \( u \in V \) and \( \text{dist}(u, \pm P) < \mu_0 \).

Proof of Theorem 3.4. By Lemmas 4.8–4.10, for \( \lambda \in \Lambda \),

\[
a_0(\lambda) = \sup_A G_\lambda \leq \frac{1}{2} \int_\Omega W_0(x) \, dx = b_0 \leq \inf_B G_\lambda.
\]

Let

\[
D = (-D_0(\mu_0)) \cup D_0(\mu_0), \quad S = V \setminus D,
\]

then \( B \subset S \). That is, condition (A2) of Proposition 2.3 holds. By Lemma 4.11, condition (A1) of Proposition 2.3 also satisfied. Since

\[
|f(x, t)| \leq c(1 + |t|^{s-1}), \quad \forall x \in \Omega, \quad \forall t \in \mathbb{R},
\]

\( G \) maps bounded sets to bounded sets. Therefore by Proposition 2.3 and [9, Theorem 2.1], for almost all \( \lambda \in \Lambda \), \( G_\lambda \) has a sign-changing critical point \( u_\lambda \in S \) such that

\[
G'_\lambda(u_\lambda) = 0, \quad G_\lambda(u_\lambda) \in \left[ b_0, \sup_{(t, u) \in [0, 1] \times A} G((1-t)u) \right].
\]

Then we prove \( \{u_\lambda\}_{\lambda \in \Lambda} \) is bounded as follows.

Assume \( \{u_\lambda\}_{\lambda \in \Lambda} \) is unbounded, then there exists \( \lambda_n \in \Lambda \) such that \( \|u_{\lambda_n}\| \to \infty \) for \( n \to \infty \). We consider \( w_{\lambda_n} = \frac{u_{\lambda_n}}{\|u_{\lambda_n}\|} \). Then, up to a subsequence, we get that

\[
w_{\lambda_n} \rightharpoonup w \quad \text{in} \quad V,
\]

\[
w_{\lambda_n} \to w \quad \text{in} \quad L^t(\Omega) \quad \text{for} \quad 2 \leq t < 2^*,
\]

\[
w_{\lambda_n}(x) \to w(x) \quad \text{a.e.} \quad x \in \Omega.
\]

If \( w \neq 0 \) in \( V \), since \( G'_{\lambda_n}(u_{\lambda_n}) = 0 \), we have that

\[
\frac{1}{2} \int_\Omega H(x, u_{\lambda_n}) = \int_\Omega \left( \frac{1}{2} f(x, u_{\lambda_n}) u_{\lambda_n} - F(x, u_{\lambda_n}) \right)
\]

\[
= G_{\lambda_n}(u_{\lambda_n}) - \frac{1}{2} \langle G'_{\lambda_n}(u_{\lambda_n}), u_{\lambda_n} \rangle
\]

\[
= G_{\lambda_n}(u_{\lambda_n})
\]

\[
\leq \sup_{(t, u) \in [0, 1] \times A} G((1-t)u).
\]

However, by (a4), \( H(x, t) \to \infty \) as \( |t| \to \infty \) for each \( x \in \Omega \). Therefore

\[
\int_\Omega H(x, u_{\lambda_n}) \, dx \geq \int_{\{w(x) \neq 0\}} H(x, u_{\lambda_n}) \, dx \to \infty
\]

as \( n \to \infty \). This is a contradiction.

If \( w = 0 \) in \( V \), we define

\[
G_{\lambda_n}(t_n u_{\lambda_n}) = \max_{t \in [0, 1]} G_{\lambda_n}(t u_{\lambda_n}).
\]

For any \( c > 0 \) and \( \bar{w}_{\lambda_n} = \sqrt{4c} w_{\lambda_n} \), we have, for \( n \) large enough, that
\[ G_{\lambda_n}(t_n u_{\lambda_n}) \geq G_{\lambda_n}(\overline{u}_{\lambda_n}) \geq 2c \lambda_n - \int_{\Omega} F(x, \overline{u}_{\lambda_n}) \geq \frac{c}{2}, \]

which implies \( \lim_{n \to \infty} G_{\lambda_n}(t_n u_{\lambda_n}) = \infty \). Evidently, \( t_n \in (0, 1) \), hence, we have \( \langle G_{\lambda_n}'(t_n u_{\lambda_n}), t_n u_{\lambda_n} \rangle = 0 \). It follows that

\[ \int_{\Omega} \left( \frac{1}{2} f(x, t_n u_{\lambda_n}) t_n u_{\lambda_n} - F(x, t_n u_{\lambda_n}) \right) dx \to \infty. \]

By the convexity of \( H(x, t) \) in \( t \), we have that

\[ \int_{\Omega} \left( \frac{1}{2} f(x, u_{\lambda_n}) u_{\lambda_n} - F(x, u_{\lambda_n}) \right) dx \geq \int_{\Omega} \left( \frac{1}{2} f(x, t_n u_{\lambda_n}) t_n u_{\lambda_n} - F(x, t_n u_{\lambda_n}) \right) dx \to \infty. \]

We get a contradiction since

\[ \int_{\Omega} \left( \frac{1}{2} f(x, u_{\lambda_n}) u_{\lambda_n} - F(x, u_{\lambda_n}) \right) dx = G_{\lambda_n}(u_{\lambda_n}) \in [b_0, \sup_{(t, u) \in [0,1] \times A} G((1-t)u)]. \]

Therefore, \( \{u_\lambda\}_{\lambda \in A} \) is bounded.

Let \( \lambda_m \to 1 \) (\( m \to \infty \)), since \( \{u_{\lambda_m}\} \) is bounded, then, up to a subsequence, we get \( u_{\lambda_m} \to u \), we will prove \( u \) is sign-changing.

Let \( u_{\lambda_m} = \max\{\pm u_{\lambda_m}, 0\} \), then

\[ \lambda_m \|u_{\lambda_m}^\pm\|^2 \leq \int_{\Omega} f(x, u_{\lambda_m}^\pm) u_{\lambda_m}^\pm dx. \]

By (h2), there exists \( C_3 > 0 \) such

\[ f(x, u)u \leq \frac{M_1}{4} |u|^2 + C_3 |u|^s, \quad x \in \Omega, u \in R. \]

Note that if \( \forall u \in V, \|u\|^2 \geq \sqrt{\mu} \|u\|_2 \). It follows that

\[ \frac{1}{2} \|u_{\lambda_m}^\pm\|^2 \leq \frac{1}{4} \|u_{\lambda_m}^\pm\|^2 + C_3 \|u_{\lambda_m}^\pm\|^s. \]

Hence, \( \|u_{\lambda_m}^\pm\| \geq C_4 > 0 \), where \( C_4 \) is a constant independent of \( \lambda_m \). So \( u \) is sign-changing. That is, \( u \) is the sign-changing solution of (1.1). \( \square \)

We are going to prove Theorem 3.5 by applying Proposition 2.4. For \( k \geq 2 \), assume

\[ E_k = \bigoplus_{j=1}^k X_j, \quad Z_k = \bigoplus_{j=k}^\infty X_j, \quad B_k = \{ u \in E_k: \|u\| \leq \rho_k \}, \quad N_k = \{ u \in Z_k: \|u\| = r_k \} \]

where \( \rho_k > r_k > 0 \). We need the following lemma.

**Lemma 4.12.** Under the assumptions of Theorem 3.5, there exists \( \rho_k > r_k > 0 \) independent of \( \lambda \) such that

\[ \max_{\partial B_k} G_{\lambda} \leq a_k \leq b_k \leq \inf_{N_k} G_{\lambda} \]

for \( \lambda \in \Lambda \). Here \( a_k \) and \( b_k \) are independent of \( \lambda \). Moreover, \( b_k \to \infty \) as \( k \to \infty \).

**Proof.** By (h2), there exists \( C_5 > 0 \) such that \( |F(x, u)| \leq \frac{M_1}{4} |u|^2 + C_5 |u|^s \) for \( x \in \Omega, u \in R \). Recall Gagliardo-Nirenberg inequality

\[ \|u\|^s \leq c_0 \|\nabla u\|_2^a \|u\|_2^{1-a}, \quad \forall u \in H^1(R^N), \]

\[ B_k \]
where \( \alpha = N\left(\frac{1}{2} - \frac{1}{s}\right) \), \( c_0 \) is a constant depending on \( s, N \). Note \( \mu_k \| u \|_2^2 \leq \| u \|^2 \) for all \( u \in \bigoplus_{i=k}^\infty X_i \), if \( u \in \bigoplus_{i=k}^\infty X_i \),

\[
\| u \| = r_k = \frac{\| u \|_2}{(16c_0^2C_5)^{s/2}},
\]

we have the following estimates:

\[
G_\lambda(u) \geq \frac{1}{4} \frac{\| u \|^2}{\mu_k} - \frac{\| u \|_2^2}{8} + \frac{\int \| u \|_{2s}^s \| \nabla u \|_2^{s(1-\alpha)} d\Omega}{C_5} \\
\geq \frac{1}{8} \| u \|^2 - C_5^s \mu_k \| u \|_{2s}^{1-\alpha} \\
\geq \frac{1}{8} \| u \|^2 - C_5^s \mu_k \| u \|_{2s}^{1-\alpha} \\
\geq \frac{1}{16} r_k^2 = b_k.
\]

Since \( \dim V_k < \infty \), then by \((b_1)\) and \((h_2)\),

\[
\frac{G_\lambda(u)}{\| u \|^2} \leq \frac{1}{2} - \frac{\int F(x, u)}{\| u \|^2} \to -\infty
\]

as \( u \in V_k, \| u \| \to \infty \) uniformly for \( \lambda \in \Lambda \). Then there exists \( \rho_k > r_k > 0 \) independent of \( \lambda \) such that

\[
\max_{\partial B_k} G_\lambda \leq a_k \leq 0.
\]

**Proof of Theorem 3.5.** Since each element of \( N_k \) \( (k \geq 2) \) is sign-changing, there exists \( \gamma_k > 0 \) such that \( \text{dist}(N_k, -P \cup P) = \gamma_k \). Under the assumptions of Theorem 3.5, Lemma 4.11 is also true. That is, \((A_1)\) holds. Let

\[
D = (-D_0(\mu_0)) \cup D_0(\mu_0), \quad S = V \setminus D.
\]

Then \( N_k \subset S \). Moreover, by Lemma 4.12, \((A_2)\) is also satisfied and \( J' \) is compact. Thus, by Proposition 2.4, \( G_\lambda \) has a sign-changing critical point \( u_\lambda \in S \) and \( G_\lambda(u_\lambda) \in [b_k, \max_{u \in B_k} G(u)] \), an interval independent of \( \lambda \). We will prove \( \{u_\lambda\}_{\lambda \in \Lambda} \) is bounded.

Assume \( \{u_\lambda\}_{\lambda \in \Lambda} \) is unbounded, then there exists \( \lambda_n \in \Lambda \) such that \( \| u_{\lambda_n} \| \to \infty \) for \( n \to \infty \). We consider \( w_{\lambda_n} = \frac{u_{\lambda_n}}{\| u_{\lambda_n} \|} \). Then, up to a subsequence, we get that

\[
w_{\lambda_n} \to w \quad \text{in} \quad V, \\
w_{\lambda_n} \to w \quad \text{in} \quad L^t(\Omega) \quad \text{for} \quad 2 \leq t < 2^*,
\]

\[
w_{\lambda_n}(x) \to w(x) \quad \text{a.e.} \quad \text{x} \in \Omega.
\]

If \( w \neq 0 \) in \( V \), since \( G_{\lambda_n}'(u_{\lambda_n}) = 0 \), we have that

\[
\int_{\Omega} \frac{f(x, u_{\lambda_n}) u_{\lambda_n}}{\| u_{\lambda_n} \|^2} dx \leq 1.
\]

On the other hand, by \((h_2)\), \((b_1)\) and Fatou’s lemma,

\[
\int_{\Omega} \frac{f(x, u_{\lambda_n}) u_{\lambda_n}}{\| u_{\lambda_n} \|^2} dx = \int_{w(x) \neq 0} \left| w_{\lambda_n}(x) \right|^2 \frac{f(x, u_{\lambda_n}) u_{\lambda_n}}{\| u_{\lambda_n} \|^2} dx \to \infty.
\]

This is a contradiction.

If \( w = 0 \) in \( V \), define

\[
G_{\lambda_n}(t_{\lambda_n} u_{\lambda_n}) = \max_{t \in [0,1]} G_{\lambda_n}(tu_{\lambda_n}).
\]

For any \( c > 0 \) and \( \bar{w}_{\lambda_n} = \sqrt{4c} w_{\lambda_n} \), we have, for \( n \) large enough, that
\[ G_{\lambda_n}(t_n u_{\lambda_n}) \geq G_{\lambda_n}(w_{\lambda_n}) \geq 2c_{\lambda_n} - \int_{\Omega} F(x, w_{\lambda_n}) \geq \frac{c}{2}, \]

which implies that \( \lim_{n \to \infty} G_{\lambda_n}(t_n u_{\lambda_n}) = \infty. \) Evidently, \( t_n \in (0, 1), \) hence, we have \( \langle G'_{\lambda_n}(t_n u_{\lambda_n}), t_n u_{\lambda_n} \rangle = 0. \) It follows that

\[ \int_{\Omega} \left( \frac{1}{2} f(x, t_n u_{\lambda_n}) t_n u_{\lambda_n} - F(x, t_n u_{\lambda_n}) \right) dx \to \infty. \]

If condition \((b_3)\) holds, \( h(t) = \frac{1}{2} t^2 f(x, s) s - F(x, ts) \) is increasing in \( t \in [0, 1], \) hence \( \frac{1}{2} f(x, s) s - F(x, s) \) is increasing in \( s > 0. \) Combining the oddness of \( f, \) we have that

\[ \int_{\Omega} \left( \frac{1}{2} f(x, u_{\lambda_n}) u_{\lambda_n} - F(x, u_{\lambda_n}) \right) dx \geq \int_{\Omega} \left( \frac{1}{2} f(x, t_n u_{\lambda_n}) t_n u_{\lambda_n} - F(x, t_n u_{\lambda_n}) \right) dx \to \infty. \]

Therefore, we get a contradiction since

\[ \int_{\Omega} \left( \frac{1}{2} f(x, u_{\lambda_n}) u_{\lambda_n} - F(x, u_{\lambda_n}) \right) dx = G_{\lambda_n}(u_{\lambda_n}) \in \left[ b_0, \sup_{(t, u) \in [0, 1] \times A} G((1-t)u) \right]. \]

Thus \( \{u_{\lambda_n}\}_{\lambda_n \in A} \) is bounded.

Let \( \lambda_m \to 1 \quad (m \to \infty), \) since \( \{u_{\lambda_m}\} \) is bounded, then, up to a subsequence, we get \( u_{\lambda_m} \to u. \) In the same as that of the proof of Theorem 3.4, \( u \) is sign-changing. Hence, \( u \) is the sign-changing solution of \((1.1). \) Since \( b_k \to \infty \quad (k \to \infty), \) we obtain infinitely many sign-changing solutions of \((1.1). \)

**Remark.** As far as we know, the sign-changing solutions of \((1.1)\) have not studied. In this paper, we study the existence and multiple of sign-changing solutions for problem \((1.1). \) The results include the existence of four sign-changing solutions or infinitely many sign-changing solutions for \((1.1)\) which are different from the references [1–7]. All these results are new.

**References**


