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Existence of Monotone Positive Solution of Neutral Partial Difference Equation

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This paper is concerned with a class of neutral partial difference equations. The conditions for the existence of monotone eventually positive solutions are established which improve and extend some of the criteria existing in the literature. Comparison theorems are also derived. Results are obtained on the existence of a monotone eventually positive solution of dual equation. © 2000 Academic Press

Key Words: neutral partial difference equation; positive solution; existence; comparison theorems.

1. INTRODUCTION

Qualitative theory for discrete dynamics systems with one dimension, that is, for an ordinary delay difference equation which parallels the qualitative theory of ordinary delay differential equations, has been investigated by a number of authors in recent years. Some of the results obtained have been collected in the book by Gyori and Ladas [6, Chapter 7]. Nonlinear discrete dynamics systems with two dimensions, that is, delay

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partial difference equations, are equally important in terms of physical applications in population dynamics with spatial migrations, chemical reaction [7], etc.

This paper is concerned with a class of neutral partial difference equations of the form

$$\sum_{i=1}^{u}\Big[T(\Delta_1,\Delta_2)\big(\theta(\lambda)y_{m,n}+cy_{m-k_i,n-l_i}\big) +p_i(m,n)y_{m-\sigma_i,n-\tau_i}\Big]=0, \tag{1}$$

where $T(\Delta_1,\Delta_2)=\Delta_1+\Delta_2+I$, $\Delta_1 y_{m,n}=y_{m+1,n}-y_{m,n}$, and $\Delta_2 y_{m,n}=y_{m,n+1}-y_{m,n}$, and $Iy_{m,n}=y_{m,n}$, the delay k_i, l_i, σ_i, τ_i are nonnegative integers, $i=1,2,\dots,u$, $p_i(m,n)\geq 0$ on N_0^2. The notation N_i is used to denote the ray $\{i,i+1,\dots\}$ of integers. $\theta(\lambda)>0$ and λ is a parameter, and $c>0$.

Assume that $\{y_{m,n}\}$ is a solution of (1), where m and n are positive integers. Defined for $m\geq M$, $n\geq N$, if we can always get a solution $\{y_{m,n}\}>0$, then $\{y_{m,n}\}$ is a eventually positive solution of (1). Otherwise, we call $\{y_{m,n}\}$ eventually negative.

When $c=0$, $u=1$, and $\theta(\lambda)=1$, (1) can be rewritten in the form

$$y_{m+1,n}+y_{m,n+1}-y_{m,n}+\frac{1}{u}\sum_{i=1}^{u}p_i(m,n)y_{m-\sigma_i,n-\tau_i}=0. \tag{2}$$

Recently, the oscillation and asymptotic behavior of (2) have been investigated by Agarwal and Wong [1, 11, 12], Cheng, Zhang, and co-workers [2–5, 16, 17], and Liu, Yu, and co-workers [8–10, 13–15]. It should be noticed that in the study of (2), the existence of positive solutions of (2) is a very difficult problem. In this paper, by the method of monotone sequence, we obtain the criteria of existence of positive solution and the comparison theorem.

2. EXISTENCE OF POSITIVE SOLUTION

It should be noticed that (1) is the special case of the difference inequality

$$\sum_{i=1}^{u}\Big[T(\Delta_1,\Delta_2)\big(\theta(\lambda)y_{m,n}+cy_{m-k_i,n-l_i}\big) +p_i(m,n)y_{m-\sigma_i,n-\tau_i}\Big]\leq 0. \tag{3}$$

So the eventually positive solution of (1) can be deduced by the eventually positive solutions of (2).

Let

$$\alpha = \max\{\sigma_i, k_i \mid 1 \le i \le u\},$$
$$\beta = \max\{\tau_i, l_i \mid 1 \le i \le u\},$$

M, N are nonnegative integers.

Assume that $\{y_{m,n}\}$ is the monotone decreasing positive solution of (3). This is the same as that $y_{m,n} > 0$ for $m \ge M - \alpha$, $n \ge N - \beta$, and

$$\Delta_1 y_{m,n} = y_{m+1,n} - y_{m,n} \le 0,$$
$$\Delta_2 y_{m,n} = y_{m,n+1} - y_{m,n} \le 0. \tag{4}$$

We noticed that (3) equals the difference inequality

$$\Delta_2\Delta_1(\theta(\lambda)y_{m,n}) \ge \theta(\lambda)y_{m+1,n+1} + \sum_{i=1}^{u} 1/u\left[T(\Delta_1,\Delta_2)(cy_{m-k_i,n-l_i}) + p_i(m,n)y_{m-\sigma_i,n-\tau_i}\right]. \tag{5}$$

In fact, due to

$$T(\Delta_1,\Delta_2)y_{m,n} = y_{m+1,n} + y_{m,n+1} - y_{m,n}, \tag{6}$$
$$\Delta_2\Delta_1(y_{m,n})y_{m+1,n+1} = y_{m+1,n} - y_{m,n+1} + y_{m,n}, \tag{7}$$

from (5), we have

$$\theta(\lambda)(y_{m+1,n+1} - y_{m+1,n} - y_{m,n+1} + y_{m,n}) \ge \theta(\lambda)y_{m+1,n+1} + \sum_{i=1}^{u} 1/u\left[T(\Delta_1,\Delta_2)(cy_{m-k_i,n-l_i}) + p_i(m,n)y_{m-\sigma_i,n-\tau_i}\right].$$

It follows that

$$\theta(\lambda)(y_{m+1,n} + y_{m,n+1} - y_{m,n}) \le -\frac{1}{u}\sum_{i=1}^{u}\left[T(\Delta_1,\Delta_2)(cy_{m-k_i,n-l_i}) + p_i(m,n)y_{m-\sigma_i,n-\tau_i}\right],$$

that is,

$$u\theta(\lambda)T(\Delta_1,\Delta_2)(y_{m,n}) \leq -\sum_{i=1}^{u}\left[T(\Delta_1,\Delta_2)(cy_{m-k_i,n-l_i}) + p_i(m,n)y_{m-\sigma_i,n-\tau_i}\right],$$

which implies that

$$\sum_{i=1}^{u}\left[T(\Delta_1,\Delta_2)\big(\theta(\lambda)y_{m,n} + cy_{m-k_i,n-l_i}\big) + p_i(m,n)y_{m-\sigma_i,n-\tau_i}\right] \leq 0.$$

Hence from (5), we can obtain that (3) holds. If we deduce it in the opposite direction, from (3), it is easy to see that (5) holds.

Summing both sides of (5) from m, n to infinity, we get

$$\sum_{(i,j)=(m,n)}^{\infty}\Delta_2\Delta_1\big(\theta(\lambda)y_{i,j}\big) \geq \sum_{(i,j)=(m,n)}^{\infty}\big(\theta(\lambda)y_{i+1,j+1}\big) + \sum_{(i,j)=(m,n)}^{\infty}\sum_{r=1}^{u}\frac{1}{u}\left[T(\Delta_1,\Delta_2)(cy_{i-k_r,j-l_r}) + p_r(i,j)y_{i-\sigma_r,j-\tau_r}\right]. \tag{8}$$

On the other hand, for $m \leq \overline{m} < \infty$, $n \leq \overline{n} < \infty$, by means of (4), we have

$$\begin{aligned}\theta(\lambda)\sum_{(i,j)=(m,n)}^{(\overline{m},\overline{n})}\Delta_2\Delta_1(y_{i,j}) &= \theta(\lambda)\left(\sum_{i=m}^{\overline{m}}\Delta_1 y_{i,\overline{n}+1} - \sum_{i=m}^{\overline{m}}\Delta_1 y_{i,n}\right)\\ &\leq \left(-\theta(\lambda)\sum_{i=m}^{\overline{m}}\Delta_1 y_{1,n}\right)\\ &= \theta(\lambda)(-y_{\overline{m}+1,n} + y_{m,n}) \leq \theta(\lambda)y_{m,n}.\end{aligned} \tag{9}$$

From (8) and (9), we obtain

$$\theta(\lambda) y_{m,n} \geq \sum_{(i,j)=(m,n)}^{\infty} \theta(\lambda) y_{i+1,j+1} + \sum_{r=1}^{u} \sum_{(i,j)=(m,n)}^{\infty} \frac{1}{u} \Big[T(\Delta_1, \Delta_2)(c y_{i-k_r, j-l_r}) + p_r(i,j) y_{i-\sigma_r, j-\tau_r} \Big].$$

That is,

$$y_{m,n} \geq \sum_{(i,j)=(m,n)}^{\infty} y_{i+1,j+1} + \frac{1}{u} \sum_{r=1}^{u} \sum_{(i,j)=(m,n)}^{\infty} [\theta(\lambda)]^{-1} \Big[T(\Delta_1, \Delta_2)(c y_{i-k_r, j-l_r}) + p_r(i,j) y_{i-\sigma_r, j-\tau_r} \Big]. \quad (10)$$

Set

$$\Omega = \{ x \mid x = \{ x_{m,n} \}, m \geq M - \alpha, n \geq N - \beta \}.$$

Define operator $T: \Omega \to \Omega$ by

$$T(x_{m,n}) = \sum_{(i,j)=(m,n)}^{\infty} x_{i+1,j+1} + \sum_{r=1}^{u} \sum_{(i,j)=(m,n)}^{\infty} \theta^{-1}(\lambda)/u \big[T(\Delta_1, \Delta_2)(c y_{i-k_r, j-l_r}) \big] + \sum_{r=1}^{u} \sum_{(i,j)=(m,n)}^{\infty} \theta^{-1}(\lambda)/u \big[p_r(i,j) y_{i-\sigma_r, j-\tau_r} \big], \quad \text{for } m \geq M, \quad n \geq N,$$

otherwise,

$$T(x_{m,n}) = y_{M,N},$$

and we define the sequence $\{x^{(t)}\}$ in Ω, that is,

$$x_{m,n}^{(0)} = \begin{cases} y_{m,n}, & (m,n) \in [M, M+1, \dots,) \times [N, N+1, \dots,), \\ y_{M,N}, & \text{otherwise,} \end{cases}$$

and

$$x^{(t+1)} = Tx^{(t)}, \qquad t = 0, 1, 2, \dots .$$

Then for $m \geq M$, $n \geq N$, by means of (10), we can get

$$0 \leq x_{m,n}^{(t+1)} \leq x_{m,n}^{(t)} \leq x_{m,n}^{(0)}.$$

It follows that $x^{(t)}$ converges to a nonnegative sequence $\omega = \{\omega_{m,n}\}$ as $t \to \infty$, and satisfies

$$\omega_{m,n} = \sum_{(i,j)=(m,n)}^{\infty} \omega_{i+1,j+1} + \sum_{r=1}^{u} \sum_{(i,j)=(m,n)}^{\infty} \theta^{-1}(\lambda)/u\Big[T(\Delta_1, \Delta_2)(c\,\omega_{i-k_r,j-l_r}) + p_r(i,j)\,\omega_{i-\sigma_r,j-\tau_r}\Big], \quad (11)$$

otherwise, we can get $\omega_{m,n} = y_{M,N}$. Thus from (11), we know that ω is an eventually nonnegative solution of (1). Next, we will prove that for an arbitrary couple of integer points (m,n), there exists a positive integer s such that ω is eventually positive when $\min\{\sigma_s, \tau_s, k_s, l_s\} > 0$ and $p_s(m,n) > 0$. Otherwise, $\omega_{m,n} > 0$ when $(m,n) \in \{M-\alpha, \dots, m^*\} \times \{N-\beta, \dots, n^*\}/\{(m^*, n^*)\}$, where $m^* \geq M$, $n^* \geq N$. But ω_{m^*,n^*}, then from (11), we have

$$0 = \sum_{(i,j)=(m^*,n^*)}^{\infty} \omega_{i+1,j+1} + \sum_{r=1}^{u} \sum_{(i,j)=(m^*,n^*)}^{\infty} \theta^{-1}(\lambda)/u\Big[T(\Delta_1, \Delta_2)(c\,\omega_{i-k_r,j-l_r}) + p_r(i,j)\,\omega_{i-\sigma_r,j-\tau_r}\Big].$$

It follows that $\omega_{i,j} = 0$, $\omega_{i-k_r,j-l_r} = 0$, and $p_r(i,j)\omega_{i-\sigma_r,j-\tau_r} = 0$ for $i \geq m^*$, $j \geq n^*$, $r = 1, 2, \dots, u$, which is a contradiction. It is easy to see that the positive solution $\omega_{i,j}$ of (11) is monotone decreasing, i.e., $\omega_{i+1,j} \leq \omega_{i,j}$, $\omega_{i,j+1} \leq \omega_{i,j}$. In fact, when we deduce from (11) to (1), and in view of the fact that (1) is a special case of (3), we can obtained the decreasing behavior of $\omega_{i,j}$. Then we can get the following results.

THEOREM 1. *Assume that k_i, l_i, σ_i, and τ_i are nonnegative integers, $p_i(m,n)$ is a nonnegative real sequence, $\theta(\lambda) > 0$, and, for an arbitrary couple of integer points (m,n), there exists a positive integer s such that*

$$\min\{\sigma_s, \tau_s, k_s, l_s\} > 0 \qquad \text{and} \qquad p_s(m,n) > 0.$$

If (3) *has monotone decreasing positive solutions, so does* (1).

Next, we will seek the monotone decreasing eventually positive solution of (3). At first, we should find the monotone decreasing eventually positive solution of

$$\sum_{i=1}^{u} \left[T(\Delta_1, \Delta_2)\left(\theta(\lambda) y_{m,n} + c y_{m-k_i, n-l_i}\right) + p_i y_{m-\sigma_i, n-\tau_i} \right] \le 0, \quad (12)$$

where p_i are nonnegative real numbers, $i = 1, 2, \dots, u$.

Set

$$y = \{y_{m,n}\}$$

and

$$y_{m,n} = \left(\frac{1-\lambda}{2} \right)^{m+n}, \qquad m \ge -\alpha, \quad n \ge -\beta. \qquad (13)$$

By means of (13), from (12) we have

$$\begin{aligned}
&T(\Delta_1, \Delta_2)\left(\theta(\lambda) y_{m,n} + c y_{m-k_i, n-l_i}\right) \\
&\quad = \theta(\lambda)\left(y_{m+1,n} + y_{m,n+1} - y_{m,n}\right) \\
&\qquad + c\left(y_{m-k_i+1, n-l_i} + y_{m-k_i, n-l_i+1} - y_{m-k_i, n-l_i}\right) \\
&\quad = \theta(\lambda)(-\lambda)\left(\frac{1-\lambda}{2}\right)^{m+n} - c\lambda\left(\frac{1-\lambda}{2}\right)^{m+n-k_i-l_i} \\
&\quad = (-\lambda)\left(\theta(\lambda) + c\left(\frac{1-\lambda}{2}\right)^{-k_i-l_i}\right)\left(\frac{1-\lambda}{2}\right)^{m+n} \\
&\quad = (-\lambda)\left(\theta(\lambda) + c\left(\frac{1-\lambda}{2}\right)^{-k_i-l_i}\right) \\
&\qquad \times \left(\frac{1-\lambda}{2}\right)^{m+n-\sigma_i-\tau_i}\left(\frac{1-\lambda}{2}\right)^{\sigma_i+\tau_i} \\
&\quad = (-\lambda)\left(\theta(\lambda) + c\left(\frac{1-\lambda}{2}\right)^{-k_i-l_i}\right)\left(\frac{1-\lambda}{2}\right)^{\sigma_i+\tau_i} y_{m-\sigma_i, n-\tau_i}.
\end{aligned}$$

If

$$\lambda\left(\theta(\lambda) + c\left(\frac{1-\lambda}{2}\right)^{-k_i-l_i}\right)\left(\frac{1-\lambda}{2}\right)^{\sigma_i+\tau_i} \geq p_i, \qquad 1 \leq i \leq u, \quad (14)$$

then $y = \{y_{m,n}\}$ is the eventually positive solution of (12).

Let

$$f(\lambda) = \lambda\left(\theta(\lambda) + c\left(\frac{1-\lambda}{2}\right)^{-k_i-l_i}\right)\left(\frac{1-\lambda}{2}\right)^{\sigma_i+\tau_i}, \qquad (15)$$

and

$$\theta(\lambda) = c\left(\frac{1-\lambda}{2}\right)^{-k_i-l_i}. \qquad (16)$$

In view of

$$\lambda = (-2)\cdot\frac{1-\lambda}{2} + 1, \qquad (17)$$

then

$$f(\lambda) = \left((-2)\cdot\frac{1-\lambda}{2} + 1\right) \times\left(\left(\frac{1-\lambda}{2}\right)^{-k_i-l_i} + c\left(\frac{1-\lambda}{2}\right)^{-k_i-l_i}\right)\left(\frac{1-\lambda}{2}\right)^{\sigma_i+\tau_i}.$$

By calculating $f'(\lambda) = 0$, it easy to see that $f(\lambda)$ reaches its extreme value at

$$\lambda_0 = \frac{1}{\sigma_i + \tau_i - (k_i + l_i) + 1}.$$

In view of

$$f''(\lambda_0) = \frac{\sigma_i + \tau_i - k_i - l_i + 1}{\sigma_i + \tau_i - k_i - l_i}\cdot(-1) < 0,$$

we have

$$\max_{0 \le \lambda < 1} f(\lambda) = f(\lambda_0)$$
$$= \frac{c+1}{2^{(\sigma_i + \tau_i - k_i - l_i)}(\sigma_i + \tau_i - k_i - l_i)} \cdot \left(\frac{\sigma_i + \tau_i - k_i - l_i}{\sigma_i + \tau_i - k_i - l_i + 1} \right)^{\sigma_i + \tau_i - k_i - l_i}.$$

Let

$$\sigma_i + \tau_i - (k_i + l_i) \ge 1, \qquad 1 \le i \le u. \tag{18}$$

Since

$$p_i \le \frac{c+1}{2^{(\sigma_i + \tau_i - k_i - l_i)}(\sigma_i + \tau_i - k_i - l_i)} \cdot \left(\frac{\sigma_i + \tau_i - k_i - l_i}{\sigma_i + \tau_i - k_i - l_i + 1} \right)^{\sigma_i + \tau_i - k_i - l_i + 1}$$
$$= \frac{c+1}{2^{(\sigma_i + \tau_i - k_i - l_i)}} \cdot \frac{(\sigma_i + \tau_i - k_i - l_i)^{\sigma_i + \tau_i - k_i - l_i}}{(\sigma_i + \tau_i - k_i - l_i + 1)^{\sigma_i + \tau_i - k_i - l_i + 1}},$$

we set $\lambda \in [0,1)$ such that (14) holds; it is easy to see that $y_{m,n} = ((1 - \lambda)/2)^{m+n}$ is a monotone decreasing positive solution of (3). Then we have the following result.

THEOREM 2. *Assume that* k_i, l_i, σ_i, *and* τ_i *are nonnegative integers,* $p_i(m,n) \ge 0$, $i = 1, 2, \ldots, u$, *and* (16), (18), (19), *and the following inequalities hold*:

$$p_i(m,n) \le p_i \le \frac{c+1}{2^{(\sigma_i + \tau_i - k_i - l_i)}} \cdot \frac{(\sigma_i + \tau_i - k_i - l_i)^{\sigma_i + \tau_i - k_i - l_i}}{(\sigma_i + \tau_i - k_i - l_i + 1)^{\sigma_i + \tau_i - k_i - l_i + 1}}.$$

Then (3) *has a monotone decreasing eventually positive solution.*

In fact, under the conditions of Theorem 2, the monotone decreasing eventually positive solution of (12) is also the monotone decreasing eventually positive solution of (3).

EXAMPLE 1. Consider the equation and inequality

$$T(\Delta_1, \Delta_2)\left(y_{m,n} + \frac{1}{e} \cdot y_{m-1,n}\right) + \frac{1}{e^3} \cdot y_{m-1,n-1} = 0, \tag{19}$$

$$T(\Delta_1, \Delta_2)\left(y_{m,n} + \frac{1}{e} \cdot y_{m-1,n}\right) + \frac{1}{e^3} \cdot y_{m-1,n-1} \leq 0, \tag{20}$$

where $m \geq 2$, $n \geq 2$. Since

$$p = \frac{1}{e^3} < \frac{e+1}{8e}$$
$$= \frac{c+1}{2^{(\sigma+\tau-k-l)}} \cdot \frac{(\sigma+\tau-k-l)^{\sigma+\tau-k-l}}{(\sigma+\tau-k-l+1)^{\sigma+\tau-k-l+1}},$$

and by using Theorem 2, we know that (21) has an eventually positive solution. In fact,

$$\{y_{m,n}\} = \left\{\frac{1}{e^m}\right\} \tag{21}$$

is such a solution. Hence, (22) is also a monotone decreasing eventually positive solution of (20).

3. COMPARISON THEOREM

Consider the equation

$$\sum_{r=1}^{u} \left[T(\Delta_1, \Delta_2)\left(\theta(\lambda) y_{m,n} + c y_{m-k_r, n-l_r}\right) \right.$$
$$\left. + q_r(m,n) y_{m-\sigma_r, n-\tau_r}\right] = 0, \tag{22}$$

where

$$q_r(m,n) \geq p_r(m,n). \tag{23}$$

THEOREM 3. *Suppose that* (24) *holds. If* (23) *has a monotone decreasing eventually positive solution, so does* (1).

Proof. Since (23) has a monotone decreasing eventually positive solution, while

$$\sum_{r=1}^{u} \left[T(\Delta_1, \Delta_2)\big(\theta(\lambda) y_{m,n} + c y_{m-k_r, n-l_r}\big) + p_r(m,n) y_{m-\sigma_r, n-\tau_r}\right]$$

$$\leq \sum_{r=1}^{u} \Big[T(\Delta_1, \Delta_2)\big(\theta(\lambda) y_{m,n} + c y_{m-k_r, n-l_r}\big)$$

$$+ q_r(m,n) y_{m-\sigma_r, n-\tau_r}\Big] \leq 0,$$

it follows that

$$\sum_{r=1}^{u} \Big[T(\Delta_1, \Delta_2)\big(\theta(\lambda) y_{m,n} + c y_{m-k_r, n-l_r}\big)$$

$$+ p_r(m,n) y_{m-\sigma_r, n-\tau_r}\Big] \leq 0$$

has a monotone decreasing eventually positive solution. Using Theorem 2, so does (1).

THEOREM 4. *Assume that* $f_i(x) > x$ *for* $x > 0$, $i = 1, 2, \dots, u$, *and*

$$\lim_{(m,n)\to\infty} p_i(m,n) \geq p_i, \qquad 1 \leq i \leq u. \tag{24}$$

If

$$\sum_{r=1}^{u} \Big[T(\Delta_1, \Delta_2)\big(\theta(\lambda) y_{m,n} + c y_{m-k_r, n-l_r}\big)$$

$$+ p_r(m,n) f_r(y_{m-\sigma_r, n-\tau_r})\Big] = 0 \tag{25}$$

has a monotone decreasing eventually positive solution $\{y_{m,n}\}$, *so does* (1).

Proof. Since (26) has a monotone decreasing eventually positive solution,

$$0 = \sum_{r=1}^{u} \Big[T(\Delta_1, \Delta_2)\big(\theta(\lambda) y_{m,n} + c y_{m-\sigma_r, n-\tau_r}\big)$$

$$+ p_r(m,n) f_r(y_{m-\sigma_r, n-\tau_r})\Big]$$

$$= \sum_{r=1}^{u} T(\Delta_1, \Delta_2)\big(\theta(\lambda) y_{m,n} + c y_{m-\sigma_r, n-\tau_r}\big)$$

$$+ \sum_{r=1}^{u} p_r(m,n) \frac{f_r(y_{m-\sigma_r, n-\tau_r})}{y_{m-\sigma_r, n-\tau_r}} \cdot y_{m-\sigma_r, n-\tau_r},$$

due to

$$\lim_{(m,n)\to\infty} \inf p_i(m,n)\frac{f_i(y_{m-\sigma_i,n-\tau_i})}{y_{m-\sigma_i,n-\tau_i}} \geq \lim_{(m,n)\to\infty} \inf p_i(m,n) \geq p_i, \qquad i = 1,2,\dots,u.$$

Hence, we have

$$\sum_{r=1}^{u}\Big[T(\Delta_1,\Delta_2)\big(\theta(\lambda)y_{m,n} + cy_{m-k_r,n-l_r}\big) + p_r(m,n)y_{m-\sigma_r,n-\tau_r}\Big] \leq \sum_{r=1}^{u} T(\Delta_1,\Delta_2)\big(\theta(\lambda)y_{m,n} + cy_{m-k_r,n-l_r}\big) + \sum_{r=1}^{u} p_r(m,n)\frac{f_r(y_{m-\sigma_r,n-\tau_r})}{y_{m-\sigma_r,n-\tau_r}}\cdot y_{m-\sigma_r,n-\tau_t} = 0.$$

It follows that (3) has a monotone decreasing eventually positive solution. By means of Theorem 1, so does (1).

Similarly, we can also have the following results.

THEOREM 5. *Assume that* $\lim_{m,n\to\infty} p_i(m,n) \geq p_i > 0$, $i = 1,2,\dots,u$, *and*

$$\lim_{(m,n)\to\infty} \inf \frac{f_i(x)}{x} \geq 1, \qquad i = 1,2,\dots,u.$$

If (26) *has a monotone decreasing eventually positive solution* $\{y_{m,n}\}$ *which satisfies* $\lim_{m,n\to\infty} y_{m,n} = 0$, *then* (1) *also has a monotone decreasing eventually positive solution.*

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