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STABILITY OF CODIMENSION ONE FOLIATIONS BY COMPACT LEAVES†

J. F. PLANTE

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INTRODUCTION

SUPPOSE M IS a manifold and $\mathcal{F}$ is a foliation of M which has a compact leaf F. A natural and popular question is the following: If $\mathcal{F}'$ is a foliation which is sufficiently close to $\mathcal{F}$ in some specified topology, must $\mathcal{F}'$ have a compact leaf? One version of this problem is to assume that $\mathcal{F}'$ is arbitrary, i.e. given M must every foliation of M have a compact leaf? The only known positive answers to this latter question are for codimension one foliations of compact manifolds of dimension ≤ 3 (see [5, 7, 8]). At the other extreme is the purely local version of the question, i.e. given $\mathcal{F}$ and F must every $\mathcal{F}'$ near $\mathcal{F}$ have a compact leaf *near* F? In this direction there is a result of Stowe [11] which generalizes earlier results of Hirsch ([2], Theorem 1.1) and Thurston–Langevin–Rosenberg ([3], Theorem 2). Suppose L is a compact leaf of a codimension k foliation $\mathcal{F}$ and let $\pi_1(L) \to GL(k, \mathbb{R})$ be the action determined by the linear holonomy of L. Stowe shows that if the cohomology group $H^1(\pi_1(L); \mathbb{R}^k)$ is trivial then every foliation which is $\mathscr{C}^1$-close to $\mathcal{F}$ has a compact leaf near L.

We consider here a problem of a more intermediate nature which has been proposed by Rosenberg ([4], p. 244, problem 11). Suppose p: $E \to B$ is a smooth fibration with compact fiber F and let $\mathcal{F}$ be the foliation whose leaves are the fibers of the fibration. The problem is to find conditions on (E, p, B) which will imply that foliations sufficiently close to $\mathcal{F}$ have compact leaves. When $H^1(F; \mathbb{R}) \neq 0$ the conditions we are looking for must be of a global nature since, in this situation, the leaves of $\mathcal{F}$ have trivial linear holonomy.

Let $\mathcal{S}$ be the class of finitely generated groups such that if $G \in \mathcal{S}$ then there are subgroups $1 = G_0 \subset G_1 \subset \ldots \subset G_k = G$ such that G_{i-1} is normal in G_i ($i = 1, \ldots, k$) and the quotient G_i/G_{i-1} has a finitely generated subgroup H_i which has non-exponential growth [7] and such that whenever $\bar{g} \in G_i/G_{i-1}$ there is a natural number m such that $\bar{g}^m \in H_i$. Clearly, $\mathcal{S}$ includes polycyclic groups and finitely generated groups having non-exponential growth. Also included are all groups with the property that each G_i/G_{i-1} either is finitely generated with non-exponential growth or is isomorphic to a subgroup of $\mathbb{Q}$. On the other hand, $\mathcal{S}$ does not include all finitely generated solvable groups.

If p: $E \to S^1$ is a fibration with fiber F then the space E is homeomorphic to the quotient space $F \times [0, 1]/(x, 1) \sim (f(x), 0)$ where f: $F \to F$ is a homeomorphism. Of course, f is not unique since it can be varied by isotopy.

The main result may now be stated as follows.

THEOREM. *Let* p: $E \to S^1$ *be a smooth fibration and* $\mathcal{F}$ *the codimension one foliation of* E *by fibers. Suppose the fiber* F *is compact and* $\pi_1(F) \in \mathcal{S}$. *Suppose also that* f^*: $H^1(F; \mathbb{R}) \to H^1(F; \mathbb{R})$ *has no positive real eigenvalues (automatically true if* $H^1(F; \mathbb{R}) = 0$). *Then every foliation* $\mathcal{F}'$ *which is* $\mathscr{C}^0$ *close to* $\mathcal{F}$ *has a compact leaf.*

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In Langevin–Rosenberg[4] the above result is proved for the special case $\pi_1(F) \cong Z$ ([4], Theorem 3.1). The technique used to prove this theorem will be similar to those used to prove compact leaf theorems for 3-manifolds. The last two sections contain examples which illustrate the essential nature of each hypothesis. In particular, the last section shows that the results and examples described earlier yield a definitive stability result for compact oriented 3-manifolds foliated by transversely oriented compact leaves.

The author is grateful to D. Gabai for suggesting the instability result used in the last section which is significantly more general than an example given in the original version of this paper.

Proof of the Theorem. As before suppose that f: $F \to F$ is a homeomorphism such that E is obtained from $F \times [0, 1]$ by the identifications $(x, 1) \sim (f(x), 0)$. Denote by $b_1(M)$ the first Betti number of M. The following result is obtained by a straightforward computation (see [6], Lemma 3.3).

LEMMA 1. $b_1(E) \geq 1$. *Furthermore,* $b_1(E) > 1$ iff f^*: $H^1(F; \mathbb{R}) \to H^1(F; \mathbb{R})$ *has* 1 *as an eigenvalue.*

We will also be needing a result ([7] Theorem 6.3) concerning the relation between holonomy invariant measures for a codimension one foliation and the existence of compact leaves.

LEMMA 2. *Suppose $\mathcal{F}$ is a transversely oriented codimension one foliation of a compact manifold M. Suppose $\mathcal{F}$ has a holonomy invariant measure μ which is finite on compact arcs transverse to $\mathcal{F}$ and $b_1(M) \leq 1$. Then every leaf in the support of the measure is compact.*

Suppose a group G acts continuously on a space X and that μ is a Borel measure on X. We will say that μ is *G-quasi-invariant* if for each $g \in G$ there is a positive real number $a(g)$ such that for every measurable set $B \subset X$ we have $\mu(g(B)) = a(g)\mu(B)$. μ is *G-invariant* if $a(g) = 1$ for all $g \in G$. The following result is proved in [10]

LEMMA 3. *Suppose $G \subset \mathcal{S}$ and that G acts continuously on* $\mathbb{R}$. *Then there is a non-trivial G-quasi-invariant measure on* $\mathbb{R}$ *which is finite on compact sets.*

Lemma 3 is an immediate consequence of Theorem 4.4 of Ref. [10]. When μ is a G-quasi-invariant measure we get a homomorphism $A: G \to \mathbb{R}$ defined by $A(g) = \log a(g)$. Note that this homomorphism is trivial iff μ is G-invariant.

We return now to the situation of the Theorem, that is, E is a bundle over S^1 with fiber F and $\mathcal{F}$ is the foliation of E by fibers. We will assume that $\pi_1(F) \in \mathcal{S}$ and that there exist $\mathcal{F}'$ arbitrarily $\mathcal{C}^0$-close to $\mathcal{F}$ which don't have any compact leaves, and we will show that f^* has a positive real eigenvalue. Let X be a vector field transverse to $\mathcal{F}$. By reparametrizing the X-flow we can assume that it takes fibers to fibers (i.e. it preserves $\mathcal{F}$) and that f: $F \to F$ is the time-one map restricted to a particular fiber. We need only assume that the perturbed foliation $\mathcal{F}'$ is transverse to X. Note, in particular, that $\mathcal{F}'$ is transversely oriented. E is covered by $\hat{F} \times \mathbb{R}$ ($\hat{F}$ = universal cover of F) in such a way that the foliation $\mathcal{F}$ lifts to the trivial codimension one foliation (leaves are $\hat{F} \times \{t\}$). The X-flow also lifts to a flow ϕ_t: $\hat{F} \times \mathbb{R} \to \hat{F} \times \mathbb{R}$ such that $\phi_t(F \times \{s\}) = F \times \{s + t\}$. The lift $\hat{\mathcal{F}}'$ of $\mathcal{F}'$ to $\hat{F} \times \mathbb{R}$ is transverse to the foliation of $\hat{F} \times \mathbb{R}$ by orbits of ϕ_t.

LEMMA 4. *The leaf space of* $\hat{\mathcal{F}}'$ *is homeomorphic to* $\mathbb{R}$.

Proof. Fix a ϕ_t-orbit L in $\hat{F}\times\mathbb{R}$. It suffices to show that every leaf of $\hat{\mathcal{F}}'$ meets L exactly once. Let $\hat{F}'$ be a leaf of $\hat{\mathcal{F}}'$. It is easily checked that the map $\hat{F}'\to\hat{F}\times\{0\}$ given by projection along ϕ_t-orbits is a covering map. Since $\hat{F}$ is simply connected, it follows that this map is a homeomorphism. In particular, $\hat{F}'$ meets L in a single point. This proves Lemma 4.

Since $\pi_1(F)\in\mathcal{S}$ it follows that $\pi_1(E)\in\mathcal{S}$. Thinking of $\pi_1(E)$ as a group of covering transformations of $\hat{F}\times\mathbb{R}$ which preserve $\hat{\mathcal{F}}'$ it follows that $\pi_1(E)$ acts on $\mathbb{R}$ and, therefore, has a non-trivial quasi-invariant measure by Lemma 3. If f^* has 1 as an eigenvalue we are done, so by Lemma 1 we may assume that $b_1(E)=1$. Since $\pi_1(E)$ is isomorphic to the semi-direct product of $\pi_1(F)$ and Z, where the generator of Z acts as $f_\#$: $\pi_1(F)\to\pi_1(F)$, the commutator subgroup of $\pi_1(E)$ has finite index in $\pi_1(F)$ (thought of as a subgroup of $\pi_1(E)$). Let A: $\pi_1(E)\to\mathbb{R}$ be the homomorphism described earlier arising from the $\pi_1(E)$-quasi-invariant measure. If A were trivial then the measure would be invariant and Lemma 2 would imply the existence of compact leaves for $\mathcal{F}'$ so $A\neq 0$. On the other hand, since the commutator subgroup of $\pi_1(E)$ has finite index in $\pi_1(F)$, the restriction of A to $\pi_1(F)$ is trivial. This means that the foliation $\bar{\mathcal{F}}'$ which is induced from $\mathcal{F}'$ on the infinite cyclic covering space $(F\times\mathbb{R})$ determined by $\pi_1(F)\subset\pi_1(E)$ has a non-trivial holonomy invariant measure. Let μ denote the holonomy invariant measure for $\bar{\mathcal{F}}'$ and let $\Phi_\mu\in H^1(F\times\mathbb{R};\ \mathbb{R})$ be its corresponding cohomology class [7] (remember, all codimension one foliations in this situation are transversely oriented). The covering transformations of $F\times\mathbb{R}$ over E are generated by $\bar{f}$: $F\times\mathbb{R}\to F\times\mathbb{R}$ which is defined by $\bar{f}(x,t)=(f(x),\ t-1)$. Since $A\neq 0$ it follows that there is a positive real number $c\neq 1$ such that $\bar{f}^*\Phi_\mu=c\Phi_\mu$. Since the projection π: $F\times\mathbb{R}\to F$ is a homotopy equivalence and $\pi\circ\bar{f}=f\circ\pi$ we will have shown that f^* has a positive real eigenvalue once we know that $\Phi_\mu\neq 0$. As in the proof of Lemma 4 it can be shown each leaf of $\bar{\mathcal{F}}'$ covers F by projection along orbits of the transverse flow. If F' is a non-compact leaf in the support of μ then there is a loop of the form $\alpha*\beta$ where α is a non-trivial segment of an orbit of the transverse flow and β lies in F'. Clearly, $\Phi_\mu(\alpha*\beta)\neq 0$ and the proof of the Theorem is complete.

EXAMPLES AND OBSERVATIONS

We now describe examples which show the essential importance of the hypotheses of the Theorem proved in the last section. Call a foliation by compact leaves *stable* if every sufficiently $\mathscr{C}^0$-close foliation has a compact leaf.

We begin with an example of instability in which $\pi_1(F)\in\mathcal{S}$ but f^* has positive real eigenvalues. F will be the n-torus $T^n=\mathbb{R}^n/Z^n$ and f: $T^n\to T^n$ will be a codimension one hyperbolic toral automorphism [1]. This means that there is a hyperbolic linear isomorphism A: $\mathbb{R}^n\to\mathbb{R}^n$ such that $A(Z^n)=Z^n$ and f is the induced diffeomorphism of T^n. There is a splitting $\mathbb{R}^n=E^u\oplus E^s$ into linear subspaces each of which is invariant under A. We assume further that the space E^u is one-dimensional and if $v\in E^u$ then $Av=cv$ where $c>1$. The codimension one foliation of $\mathbb{R}^n$ by hyperplanes parallel to E^s induces a codimension one foliation $\mathcal{F}_T$ of T^n (with planar leaves). Let $\mathcal{F}_E$ be the codimension 2 foliation of $E=T^n\times[0,1]/(x,1)\sim(f(x),0)$ whose leaves are of the form (leaf of $\mathcal{F}_T)\times\{t\}$.

Let X_T be a vector field on T^n which is tangent to the line bundle determined by E^u and whose flow takes leaves of $\mathcal{F}_T$ to leaves of $\mathcal{F}_T$ (X_T is, of course, transverse to $\mathcal{F}_T$). Since $Df(X_T)=cX_T$ we may construct an action of the affine group of the line on E which preserves $\mathcal{F}_E$ and whose orbits are transverse to the leaves of $\mathcal{F}_E$ (see [9], Proposition 1.2). This means that there are vector fields X and Y such that the restriction of X to any torus fiber is a constant multiple of X_T, Y is transverse to torus fibers, and $[X, Y] = X$. Furthermore, the flow generated by any constant linear combination of X and Y preserves $\mathcal{F}_E$. Let X^ϵ denote the vector field $X + \epsilon Y$. The flow of X^ϵ, X_t^ϵ preserves $\mathcal{F}_E$ and we let $\mathcal{F}^\epsilon$ be the codimension one foliation of E whose leaves are X_t^ϵ-orbits of a leaf of $\mathcal{F}_E$. It is clear that $\mathcal{F}^\epsilon \to \mathcal{F}$ as $\epsilon \to 0$ (actually in an $\mathscr{C}^r$ topology) and the leaves of $\mathcal{F}^\epsilon$ are non-compact since the leaves of $\mathcal{F}_E$ are non-compact.

We now make some comments on the non-transversely oriented case. If $\mathcal{F}$ is a transversely oriented codimension one foliation with all leaves compact then $\mathcal{F}$ is actually a foliation by fibers of a fibration. If $\mathcal{F}$ is not transversely oriented but has all leaves compact then a 2-fold cover of $\mathcal{F}$ is a fibration. If the 2-fold transversely oriented cover of $\mathcal{F}$ is stable (any perturbation has a compact leaf) then so is $\mathcal{F}$. On the other hand, there are situations in which $\mathcal{F}$ is stable even though the 2-fold transversely orientable cover is not. If L is a leaf of $\mathcal{F}$ whose normal bundle is non-trivial and $\pi_1(L)$ is abelian then L is locally $\mathscr{C}^0$ stable by a result of Hirsch ([2], Theorem 1.2). This means, for example, that a foliation of the Klein bottle by circles which is not transversely oriented is always stable. The same phenomenon can also occur without local stability. In [8] (Corollary 3.3 and proof of Theorem 5.1) there are examples of compact 3-manifolds which admit foliations by tori except for 2 Klein bottle leaves and such that every codimension one foliation of the given manifold has a compact leaf. It is clear that local stability is not in force in these examples because there are other manifolds of the same type (obtained by attaching together two [0, 1]-bundles over the Klein bottle) for which compact leaves can be perturbed away ([8], p. 225, Remark ii).

STABILITY IN DIMENSION 3

In this section, we consider the special case in which E is a compact orientable 3-manifold. In this case F is a compact orientable surface. If F is the sphere or torus, the main theorem and examples of earlier sections give necessary and sufficient conditions for $\mathcal{F}$ to be stable. The following result, due to D. Gabai, says that all other cases are unstable.

THEOREM (Gabai). *Suppose p: $E \to S^1$ is a smooth fibration where E is a compact oriented 3-manifold. If the fiber F has genus > 1 then the foliation of E by fibers is unstable.*

Letting $\mathcal{F}$ denote the foliation of E by fibers, the Theorem will be proved by constructing a foliation $\mathcal{F}'$ which is $\mathscr{C}^0$ close to $\mathcal{F}$ but which has no compact leaves, Denote by $\langle , \rangle$ the algebraic intersection number of 1-cycles on F and by f: $F \to F$ the attaching map which determines E.

LEMMA. *Let α, β, γ be simple closed curves on F such that $[\alpha]$, $[\beta]$, $[\gamma]$ are contained in a basis for $H_1(F; \mathbb{Z})$, $\langle \alpha, \beta \rangle = 1$, and $\langle \alpha, \gamma \rangle = \langle \beta, \gamma \rangle = 0$. Then there exists a simple closed curve δ in F such that $[\delta] = 0$ but $\langle \gamma, \delta \rangle = \langle f(\gamma), \delta \rangle = 0$.*

Proof. Set $p = \langle \alpha, f(\gamma) \rangle$, $q = \langle \beta, f(\gamma) \rangle$ and note that $\langle p\beta - q\alpha, f(\gamma) \rangle = 0$. If p or q is zero, take δ to be α or β, respectively. Otherwise, choose a simple closed curve δ such that $p\beta - q\alpha$ is homologous to a multiple of δ. This proves the lemma.

Denote by F_0 a particular fiber of E. We first describe a foliation $\bar{\mathcal{F}}$ of E which is close to $\mathcal{F}$ and has F_0 as the only compact leaf. $E - F_0$ is homeomorphic to $\mathbb{R} \times F$. Let N be a tubular neighborhood of γ in F such that $\bar{N}$ is a closed tubular neighborhood with coordinates (s, t), $s \in \gamma$, $0 \le t \le 1$. Start with the foliation of $\mathbb{R} \times (F - N)$ having leaves

$\{r\} \times (F - N)$ and make the identifications $(r, s, 0) \sim (r+1, s, 1)$ to obtain a foliation of $\mathbb{R} \times F$ having no compact leaves. Replacing F_0 yields a folation $\bar{\mathcal{F}}$ of E having F_0 as the only compact leaf. Let η be an arbitrary loop in the compact leaf F_0. The holonomy along η is trivial iff $\langle \eta, \gamma \rangle = \langle \eta, f(\gamma) \rangle = 0$. In particular, the holonomy along δ (from the Lemma) is trivial. Let Q be a small neighborhood of δ of the form $\mathbb{R} \times S^1 \times (0, 1)$ where δ corresponds to $\{0\} \times S^1 \times \{\frac{1}{2}\}$ and such that the restriction of $\bar{\mathcal{F}}$ to Q has leaves of the form $\{r\} \times S^1 \times (0, 1)$. Modify $\bar{\mathcal{F}}$ by first deleting the neighborhood $\mathbb{R} \times S^1 \times (\frac{1}{3}, \frac{2}{3})$ and then making the identifications $(r, s, \frac{1}{3}) \sim (r \pm 1, s, \frac{2}{3})$. By choosing the + or − properly (so that this second modification does not counteract the first one), we obtain a foliation $\mathcal{F}'$ which has no compact leaves and the theorem is proved.

Corollary. *Let E be a compact orientable 3-manifold which fibers over S^1 and let $\mathcal{F}$ be the codimension one foliation by fibers. The foliation $\mathcal{F}$ is stable in the following cases.*

(i) *The fiber is S^2.*

(ii) *The fiber is T^2 and trace $f^* < 2$. (f^*: $H^1(T^2; \mathbb{R}) \to H^1(T^2; \mathbb{R})$ where f: $T^2 \to T^2$ is an attaching map which determines E.)*

In all other cases $\mathcal{F}$ is unstable.

Proof. The stability in cases (i) and (ii) follows from the main theorem stated in the introduction. (Actually, results of [8] imply that *every* transversely oriented foliation of such E has a compact leaf—not just the ones close to $\mathcal{F}$.) The other cases with torus fibers are unstable by the examples of the previous section. Finally, if the fiber has genus > 1, $\mathcal{F}$ is unstable by the theorem of this section.

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University of North Carolina
Chapel Hill
NC 27514
U.S.A.