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Attractivity of fractional functional differential equations☆

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ABSTRACT

In this paper, some attractivity results for fractional functional differential equations are obtained by using the fixed point theorem. By constructing equivalent fractional integral equations, research on the attractivity of fractional functional and neutral differential equations is skillfully converted into a discussion about the existence of fixed points for equivalent fractional integral equations. Two examples are also provided to illustrate our main results.

1. Introduction

Consider the initial value problem (IVP for short) of the following fractional functional differential equation:

$$\begin{cases} {}^{C}D^{\alpha}x(t) = f(t, x_t), & t \in (t_0, \infty),\ t_0 \geq 0,\ 0 < \alpha < 1, \\ x(t) = \phi(t), & t_0 - \tau \leq t \leq t_0, \end{cases} \tag{1}$$

where D^{α} is the Caputo fractional derivative, $f : J \times C([-\tau, 0], R) \to R$, where $J = (t_0, \infty)$, is a given function satisfying some assumptions that will be specified later, $\tau > 0$, and $\phi \in C([t_0 - \tau, t_0], R)$. If $x \in C([t_0 - \tau, \infty), R)$, then, for any $t \in [t_0, \infty)$, define x_t by $x_t(\theta) = x(t + \theta)$ for $-\tau \leq \theta \leq 0$.

In recent years, there has been a significant development in solving ordinary and partial differential equations involving fractional derivatives [1]; see, for example, the monographs [2–4]; also, see [5–8] for initial value problems of nonlinear fractional differential equations; [9–12] for boundary value problems of nonlinear fractional differential equations; [1,13,14] for fractional functional differential equations; [15–18] for fractional impulsive differential equations; [19–21] for fractional evolution equations; [22,23] for fractional difference equations; and [24–26] for fractional partial differential equations.

However, there are few works on the stability of solutions for fractional differential equations. Some local asymptotical stability, Mittag-Leffler stability, and linear matrix inequality (LMI) stability are discussed in [27–30], and the attractivity of nonlinear fractional differential equations is discussed by Deng [27] using the principle of contraction mappings; but there is no work on the attractivity of fractional functional differential equations.

In this paper, we investigate the attractivity of solutions for fractional functional differential equations by using fixed point theorems: research on the attractivity is skillfully converted into a discussion about the existence of fixed points. A similar idea can be found in integer-order differential equations [31–35].

The rest of the paper is organized as follows. In Section 2, we recall some useful preliminaries. In Section 3, we give some sufficient conditions of the attractivity of the solutions. Finally, two examples are given to illustrate our main results.

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2. Preliminaries

Let us recall the following known definitions. For more details, see [1,14].

Definition 2.1. The fractional integral of order γ with the lower limit zero for a function f is defined as

$$I^{\gamma}f(t)=\frac{1}{\Gamma(\gamma)}\int_{t_0}^{t}\frac{f(s)}{(t-s)^{1-\gamma}}\mathrm{d}s,\quad t>t_0,\ \gamma>0,$$

provided that the right-hand side is point-wise defined on $[t_0,\infty)$, where $\Gamma(\cdot)$ is the gamma function.

Definition 2.2. The Riemann–Liouville derivative of order γ with the lower limit zero for a function $f\in C^{n}([t_0,\infty),R)$ can be written as

$$D^{\gamma}f(t)=\frac{1}{\Gamma(n-\gamma)}\frac{\mathrm{d}^n}{\mathrm{d}t^n}\int_{t_0}^{t}\frac{f(s)}{(t-s)^{\gamma+1-n}}\mathrm{d}s,\quad t>t_0,\ n-1<\gamma<n.$$

Definition 2.3 (*[2]*)**.** Caputo's derivative of order γ for a function $f\in C^{n+1}([t_0,\infty),R)$ can be written as

$${}^{C}D^{\gamma}f(t)=D^{\gamma}f(t)-\sum_{k=0}^{n-1}\frac{f^{(k)}(t_0)}{\Gamma(k-\gamma+1)}t^{k-\gamma},\quad t>t_0,\ n-1<\gamma<n.$$

Remark 2.4. (1) If $f(t)\in C^{n+1}([0,\infty),R)$, then

$${}^{C}D^{\gamma}f(t)=\frac{1}{\Gamma(n-\gamma)}\int_{0}^{t}\frac{f^{(n)}(s)}{(t-s)^{\gamma+1-n}}\mathrm{d}s=I^{n-\gamma}f^{(n)}(t),\quad t>0,\ n-1<\gamma<n.$$

(2) The Caputo derivative of a constant is equal to zero, i.e., if $x(t)=c$, then ${}^{C}D^{\alpha}c=0$. However, $D^{\alpha}c=\frac{ct^{-\alpha}}{\Gamma(1-\alpha)}$.

Definition 2.5. The solution $x(t)$ of IVP (1) is said to be attractive if there exists a constant $b_0(t_0)>0$ such that $|\phi(s)|\le b_0\ (s\in[t_0-\tau,t_0])$ implies that $x(t,t_0,\phi)\to 0$ as $t\to\infty$.

Theorem 2.6 (*Schauder Fixed Point Theorem, [36]*)**.** *If U is a nonempty, closed, bounded convex subset of a Banach space X and $T:U\to U$ is completely continuous, then T has a fixed point in U.*

3. Main results

Let $\|x_t\|=\sup_{-\tau\le\theta\le 0}|x(t+\theta)|$ for $t\in J$. $L^p(J,R)$ is the L^p space with the norm $\|x\|_p=\left(\int_{t_0}^{\infty}|x(t)|^p\mathrm{d}t\right)^{1/p}$ for $1\le p<\infty$.

In this section, we always assume that $f(t,x_t)$ satisfies the following condition.

(H_0) $f(t,x_t)$ is Lebesgue measurable with respect to t on $[t_0,\infty)$, and $f(t,\varphi)$ is continuous with respect to φ on $C([-\tau,0],R)$.

By condition (H_0), IVP (1) is equivalent to the following equation [1]:

$$x(t)=\begin{cases}\phi(t_0)+\dfrac{1}{\Gamma(\alpha)}\displaystyle\int_{t_0}^{t}(t-s)^{\alpha-1}f(s,x_s)\mathrm{d}s, & t>t_0,\\ \phi(t), & t\in[t_0-\tau,t_0];\end{cases}\tag{2}$$

that is,

$$x(t)=\begin{cases}\dfrac{1}{\Gamma(\alpha)}\displaystyle\int_{t_0}^{t}(t-s)^{\alpha-1}\left[\frac{\phi(t_0)}{\Gamma(1-\alpha)}(s-t_0)^{-\alpha}+f(s,x_s)\right]\mathrm{d}s, & t>t_0,\\ \phi(t), & t\in[t_0-\tau,t_0].\end{cases}$$

Define the operator

$$Ax(t)=\begin{cases}\phi(t_0)+\dfrac{1}{\Gamma(\alpha)}\displaystyle\int_{t_0}^{t}(t-s)^{\alpha-1}f(s,x_s)\mathrm{d}s, & t>t_0,\\ \phi(t), & t\in[t_0-\tau,t_0].\end{cases}\tag{3}$$

Obviously, $x(t)$ is a solution of IVP (1) if it is a fixed point of the operator A.

Lemma 3.1. *Assume that $f(t, x_t)$ satisfies conditions* (H_0) *and the following.*

(H_1) *There is a constant $\gamma_1 > 0$ such that*

$$\left|\phi(t_0) + \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}(t-s)^{\alpha-1}f(s, x_s)\mathrm{d}s\right| \le (t-t_0)^{-\gamma_1}$$

for $t \in J$.

(H_2) *There exists a constant $\alpha_1 \in (0, \alpha)$ such that $f \in L^{\frac{1}{\alpha_1}}(J \times C([-\tau, 0], R), R)$.*

Then IVP (1) has at least one solution $x \in C([t_0 - \tau, \infty), R)$.

Proof. Define the set

$$S_1 = \{x | x \in C([t_0 - \tau, \infty), R), |x(t)| \le (t - t_0)^{-\gamma_1} \text{ for } t \ge \tilde{t} > t_0\}.$$

It is easy to prove that S_1 is a closed, bounded, and convex subset of R, where $\tilde{t}$ is a constant.

To prove that IVP (1) has a solution, we shall prove that the operator A has a fixed point in S_1.

We first show that A maps S_1 in S_1.

For $t > t_0$, apply condition (H_1) to the operator A. We have $|Ax(t)| \le (t - t_0)^{-\gamma_1}$; then $AS_1 \subset S_1$.

Nextly, we show that A is continuous.

For any $x^m, x \in S_1, m = 1, 2, \ldots$ with $\lim_{m\to\infty} |x^m - x| = 0$, by the continuity of $f(t, x_t)$, we have $\lim_{m\to\infty} f(t, x^m_t) = f(t, x_t)$ for $t > t_0$.

Let $\varepsilon > 0$ be given, there exists a $T > t_0$ such that $t \ge T$ implies that $(t - t_0)^{-\gamma_1} < \frac{\varepsilon}{2}$.

Let $\nu = \frac{\alpha-1}{1-\alpha_1}$; then $1 + \nu > 0$, since $\alpha_1 \in (0, \alpha)$. For $t_0 < t \le T$, we have

$$\begin{aligned}
|(Ax^m)(t) - (Ax)(t)| &\le \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}(t-s)^{\alpha-1}|f(s, x^m_s) - f(s, x_s)|\mathrm{d}s \\
&\le \frac{1}{\Gamma(\alpha)}\left\{\int_{t_0}^{t}[(t-s)^{\alpha-1}]^{\frac{1}{1-\alpha_1}}\mathrm{d}s\right\}^{1-\alpha_1}\left[\int_{t_0}^{t}|f(s, x^m_s) - f(s, x_s)|^{\frac{1}{\alpha_1}}\mathrm{d}s\right]^{\alpha_1} \\
&\le \frac{1}{\Gamma(\alpha)}\left(\frac{1}{1+\nu}(t-t_0)^{1+\nu}\right)^{1-\alpha_1}\left[\int_{t_0}^{T}|f(s, x^m_s) - f(s, x_s)|^{\frac{1}{\alpha_1}}\mathrm{d}s\right]^{\alpha_1} \\
&\le \frac{1}{\Gamma(\alpha)}\left(\frac{1}{1+\nu}(T-t_0)^{1+\nu}\right)^{1-\alpha_1}(T-t_0)^{\alpha_1}\sup_{t_0\le s\le T}|f(s, x^m_s) - f(s, x_s)| \\
&\to 0 \quad \text{as } m \to \infty.
\end{aligned}$$

For $t > T$, we have

$$\begin{aligned}
|(Ax^m)(t) - (Ax)(t)| &= \left|\frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}(t-s)^{\alpha-1}f(s, x^m_s)\mathrm{d}s - \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}(t-s)^{\alpha-1}f(s, x_s)\mathrm{d}s\right| \\
&\le 2(t-t_0)^{-\gamma_1} \le \varepsilon.
\end{aligned}$$

Then, for $t > t_0$, it is clear that

$$|(Ax^m)(t) - (Ax)(t)| \to 0 \quad \text{as } m \to \infty,$$

which implies that A is continuous for $t > t_0$.

Lastly, we prove that AS_1 is equicontinuous.

Let $\varepsilon > 0$ be given. There is a $T > t_0$ such that $(t - t_0)^{-\gamma_1} < \frac{\varepsilon}{2}$ for $t > T$. Let $t_1, t_2 > t_0$ and $t_2 > t_1$.

If $t_1, t_2 \in (t_0, T]$, similar to the proof of Theorem 1 in [37], by condition (H_2) we have

$$\begin{aligned}
|(Ax)(t_2) - (Ax)(t_1)| &\le \left|\frac{1}{\Gamma(\alpha)}\int_{t_0}^{t_2}(t_2-s)^{\alpha-1}f(s, x_s)\mathrm{d}s - \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t_1}(t_1-s)^{\alpha-1}f(s, x_s)\mathrm{d}s\right| \\
&\le \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t_1}[(t_1-s)^{\alpha-1} - (t_2-s)^{\alpha-1}]|f(s, x_s)|\mathrm{d}s + \frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}|f(s, x_s)|\mathrm{d}s \\
&\le \frac{1}{\Gamma(\alpha)}\left\{\int_{t_0}^{t_1}\left[(t_1-s)^{\alpha-1} - (t_2-s)^{\alpha-1}\right]^{\frac{1}{1-\alpha_1}}\mathrm{d}s\right\}^{1-\alpha_1}\left[\int_{t_0}^{t_1}|f(s, x_s)|^{\frac{1}{\alpha_1}}\mathrm{d}s\right]^{\alpha_1} \\
&\quad + \frac{1}{\Gamma(\alpha)}\left[\int_{t_1}^{t_2}(t_2-s)^{\frac{\alpha-1}{1-\alpha_1}}\mathrm{d}s\right]^{1-\alpha_1}\left[\int_{t_1}^{t_2}|f(s, x_s)|^{\frac{1}{\alpha_1}}\mathrm{d}s\right]^{\alpha_1}
\end{aligned}$$

$$
\begin{aligned}
&\le \frac{1}{\Gamma(\alpha)}\left\{\int_{t_0}^{t_1}\left[(t_1-s)^{\frac{\alpha-1}{1-\alpha_1}}-(t_2-s)^{\frac{\alpha-1}{1-\alpha_1}}\right]ds\right\}^{1-\alpha_1}\left[\int_{t_0}^{t_1}|f(s,x_s)|^{\frac{1}{\alpha_1}}ds\right]^{\alpha_1}\\
&\quad+\frac{1}{\Gamma(\alpha)}\left[\int_{t_1}^{t_2}(t_2-s)^{\frac{\alpha-1}{1-\alpha_1}}ds\right]^{1-\alpha_1}\left[\int_{t_1}^{t_2}|f(s,x_s)|^{\frac{1}{\alpha_1}}ds\right]^{\alpha_1}\\
&\le \frac{1}{\Gamma(\alpha)}\left(\frac{1}{1+\nu}\right)^{1-\alpha_1}\left[(t_1-t_0)^{\frac{\alpha-1}{1-\alpha_1}+1}+(t_2-t_1)^{\frac{\alpha-1}{1-\alpha_1}+1}\right.\\
&\quad\left.-(t_2-t_0)^{\frac{\alpha-1}{1-\alpha_1}+1}\right]^{1-\alpha_1}\left[\int_{t_0}^{T}|f(s,x_s)|^{\frac{1}{\alpha_1}}ds\right]^{\alpha_1}\\
&\quad+\frac{1}{\Gamma(\alpha)}\left(\frac{1}{1+\nu}\right)^{1-\alpha_1}\left[(t_2-t_1)^{\frac{\alpha-1}{1-\alpha_1}+1}\right]^{1-\alpha_1}\left[\int_{t_0}^{T}|f(s,x_s)|^{\frac{1}{\alpha_1}}ds\right]^{\alpha_1}\\
&\le \frac{1}{\Gamma(\alpha)}\left(\frac{1}{1+\nu}\right)^{1-\alpha_1}\left[\int_{t_0}^{T}|f(s,x_s)|^{\frac{1}{\alpha_1}}ds\right]^{\alpha_1}(t_2-t_1)^{\alpha-\alpha_1}\\
&\to 0 \quad \text{as } t_2\to t_1. \qquad (4)
\end{aligned}
$$

If $t_1, t_2 > T$, then we have

$$
\begin{aligned}
|(Ax)(t_2)-(Ax)(t_1)| &\le \left|\frac{1}{\Gamma(\alpha)}\int_{t_0}^{t_2}(t_2-s)^{\alpha-1}f(s,x_s)ds-\frac{1}{\Gamma(\alpha)}\int_{t_0}^{t_1}(t_1-s)^{\alpha-1}f(s,x_s)ds\right|\\
&\le (t_2-t_0)^{-\gamma_1}+(t_1-t_0)^{-\gamma_1}\le\varepsilon. \qquad (5)
\end{aligned}
$$

If $t_0 < t_1 < T < t_2$, note that $t_2 \to t_1$ implies that $t_2 \to T$ and $T \to t_1$, and according to the above discussion we have

$$|(Ax)(t_2)-(Ax)(t_1)| \le |(Ax)(t_2)-(Ax)(T)|+|(Ax)(T)-(Ax)(t_1)| \to 0 \quad \text{as } t_2 \to t_1. \qquad (6)$$

Combining (4)–(6), we obtain that $|(Ax)(t_2)-(Ax)(t_1)| \to 0$ as $t_2 \to t_1$ for $t_1, t_2 > t_0$. Then AS_1 is equicontinuous.

Meanwhile, AS_1 is relatively compact because $AS_1 \subset S_1$ is uniformly bounded. Thus A is completely continuous on S_1. By Theorem 2.6, the Schauder fixed point theorem, we deduce that A has a fixed point in S_1 which is a solution of IVP (1). □

Theorem 3.2. *Assume that conditions* (H_0)–(H_2) *hold; then the zero solution of* (1) *is attractive.*

Proof. By Lemma 3.1, the solution of (1) exists and is in S_1. All functions $x(t)$ in S_1 tend to 0 as $t \to \infty$. Then the solution of (1) tends to zero as $t \to \infty$. □

Lemma 3.3. *Assume that the function* $f(t, x_t)$ *satisfies conditions* (H_0) *and the following.*

(H_3) $\left|\frac{\phi(t_0)}{\Gamma(1-\alpha)}(t-t_0)^{-\alpha}+f(t,x_t)\right| \le l(t)\|x_t\|$ *for any* $t \in J$ *and* $x \in C([t_0-\tau,\infty),R)$, *where* $l \in C(J,R^+)$ *satisfies that*

(i) *there is a constant* $\gamma_2 > 0$ *such that* $\frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}(t-s)^{\alpha-1}l(s)(s-t_0)^{-\gamma_2}ds \le (t-t_0)^{-\gamma_2}$ *for* $t \in J$;

(ii) *there is a constant* $\alpha_2 \in (0,\alpha)$ *such that* $l \in L^{\frac{1}{\alpha_2}}(J,R^+)$.

Then IVP (1) *has at least one solution* $x \in C([t_0-\tau,\infty),R)$.

Proof. Define the set

$$S_2=\{x|x\in C([t_0-\tau,\infty),R),\ \|x_t\|\le(t-t_0)^{-\gamma_2}\ \text{for } t\ge\tilde{t}>t_0\}.$$

It is easy to show that S_2 is a closed, bounded, and convex subset of R.

To prove that (1) has a solution, we shall prove that the operator A has a fixed point in S_2.

We first show that A maps S_2 in S_2.

For $t > t_0$, applying condition (H_3), we have

$$
\begin{aligned}
|(Ax)(t)| &= \left|\phi(t_0)+\frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}(t-s)^{\alpha-1}f(s,x_s)ds\right|\\
&= \left|I^{\alpha}\left(\frac{\phi(t_0)}{\Gamma(1-\alpha)}(t-t_0)^{-\alpha}\right)+\frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}(t-s)^{\alpha-1}f(s,x_s)ds\right|
\end{aligned}
$$

$$
\begin{aligned}
&= \left|\frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}(t-s)^{\alpha-1}\left[\frac{\phi(t_0)}{\Gamma(1-\alpha)}(s-t_0)^{-\alpha}+f(s,x_s)\right]\mathrm{d}s\right| \\
&\le \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}(t-s)^{\alpha-1}\left|\frac{\phi(t_0)}{\Gamma(1-\alpha)}(s-t_0)^{-\alpha}+f(s,x_s)\right|\mathrm{d}s \\
&= \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}(t-s)^{\alpha-1}l(s)\|x_s\|\mathrm{d}s \\
&\le \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}(t-s)^{\alpha-1}l(s)(s-t_0)^{-\gamma_2}\mathrm{d}s \\
&\le (t-t_0)^{-\gamma_2}, \qquad (7)
\end{aligned}
$$

which implies that $AS_2 \subset S_2$.

Next, we show that AS_2 is equicontinuous.

Let $\varepsilon > 0$ be given. Since $\lim_{t\to\infty}(t-t_0)^{-\gamma_2} = 0$, there is a $T' > t_0$ such that $(t-t_0)^{-\gamma_2} < \frac{\varepsilon}{2}$ for $t > T'$. At the same time, $\max\{\|x_t\| : t_0 \le t \le T'\}$ exists by $x \in C([t_0-\tau,\infty), R)$. Let $\Phi = \max\{\|x_t\| : t_0 \le t \le T'\}$, and let $t_1, t_2 > t_0$ and $t_2 > t_1$.

If $t_1, t_2 \in (t_0, T']$, we have

$$
\begin{aligned}
|(Ax)(t_2)-(Ax)(t_1)| &= \left|\frac{1}{\Gamma(\alpha)}\int_{t_0}^{t_1}(t_1-s)^{\alpha-1}f(s,x_s)\mathrm{d}s - \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t_2}(t_2-s)^{\alpha-1}f(s,x_s)\mathrm{d}s\right| \\
&\le \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t_1}[(t_1-s)^{\alpha-1}-(t_2-s)^{\alpha-1}]|f(s,x_s)|\mathrm{d}s + \frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}|f(s,x_s)|\mathrm{d}s \\
&\le \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t_1}((t_1-s)^{\alpha-1}-(t_2-s)^{\alpha-1})\left[\frac{|\phi(t_0)|}{\Gamma(1-\alpha)}(s-t_0)^{-\alpha}+l(s)\|x_s\|\right]\mathrm{d}s \\
&\quad + \frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}\left[\frac{|\phi(t_0)|}{\Gamma(1-\alpha)}(s-t_0)^{-\alpha}+l(s)\|x_s\|\right]\mathrm{d}s \\
&= \frac{2|\phi(t_0)|}{\Gamma(\alpha)\Gamma(1-\alpha)}\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}(s-t_0)^{-\alpha}\mathrm{d}s \\
&\quad + \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t_1}[(t_1-s)^{\alpha-1}-(t_2-s)^{\alpha-1}]l(s)\|x_s\|\mathrm{d}s + \frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}l(s)\|x_s\|\mathrm{d}s \\
&\le \frac{2|\phi(t_0)|}{\Gamma(\alpha)\Gamma(1-\alpha)}\int_{\frac{t_1-t_0}{t_2-t_0}}^{1}(1-s)^{\alpha-1}s^{-\alpha}\mathrm{d}s \\
&\quad + \frac{\Phi}{\Gamma(\alpha)}\int_{t_0}^{t_1}[(t_1-s)^{\alpha-1}-(t_2-s)^{\alpha-1}]l(s)\mathrm{d}s + \frac{\Phi}{\Gamma(\alpha)}\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}l(s)\mathrm{d}s \\
&\le \frac{2|\phi(t_0)|}{\Gamma(\alpha)\Gamma(1-\alpha)}\int_{\frac{t_1-t_0}{t_2-t_0}}^{1}(1-s)^{\alpha-1}s^{-\alpha}\mathrm{d}s \\
&\quad + \frac{\Phi}{\Gamma(\alpha)}\left\{\int_{t_0}^{t_1}\left[(t_1-s)^{\alpha-1}-(t_2-s)^{\alpha-1}\right]^{\frac{1}{1-\alpha_2}}\mathrm{d}s\right\}^{1-\alpha_2} \\
&\quad \times\left[\int_{t_0}^{t_1}(l(s))^{\frac{1}{\alpha_2}}\mathrm{d}s\right]^{\alpha_2} + \frac{\Phi}{\Gamma(\alpha)}\left[\int_{t_1}^{t_2}(t_2-s)^{\frac{\alpha-1}{1-\alpha_2}}\mathrm{d}s\right]^{1-\alpha_2}\left[\int_{t_1}^{t_2}(l(s))^{\frac{1}{\alpha_2}}\mathrm{d}s\right]^{\alpha_2} \\
&\le \frac{2|\phi(t_0)|}{\Gamma(\alpha)\Gamma(1-\alpha)}\int_{\frac{t_1-t_0}{t_2-t_0}}^{1}(1-s)^{\alpha-1}s^{-\alpha}\mathrm{d}s \\
&\quad + \frac{\Phi}{\Gamma(\alpha)}\left(\frac{1}{1+\nu'}\right)^{1-\alpha_2}\left[\int_{t_0}^{T'}(l(s))^{\frac{1}{\alpha_2}}\mathrm{d}s\right]^{\alpha_2}(t_2-t_1)^{\alpha-\alpha_2}, \qquad (8)
\end{aligned}
$$

where $\nu' = \frac{\alpha-1}{1-\alpha_2}$. Since $\int_0^1(1-s)^{\alpha-1}s^{-\alpha}\mathrm{d}s$, $\int_{t_0}^{T'}(l(s))^{\frac{1}{\alpha_2}}\mathrm{d}s$, and Φ exist, from (8) we obtain

$$|(Ax)(t_2)-(Ax)(t_1)| \to 0 \quad \text{as } t_2 \to t_1. \qquad (9)$$

From condition (H_3) it follows that

$$-\phi(t_0) - \int_{t_0}^{t} (t-s)^{\alpha-1} l(s)\|x_s\| \mathrm{d}s \le \int_{t_0}^{t} (t-s)^{\alpha-1} f(s, x_s) \mathrm{d}s \le -\phi(t_0) + \int_{t_0}^{t} (t-s)^{\alpha-1} l(s)\|x_s\| \mathrm{d}s.$$

If $t_1, t_2 > T'$, we have

$$\begin{aligned}
(Ax)(t_2) - (Ax)(t_1) &= \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t_1} (t_1 - s)^{\alpha-1} f(s, x_s) \mathrm{d}s - \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t_2} (t_2 - s)^{\alpha-1} f(s, x_s) \mathrm{d}s \\
&\le \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t_2} (t_2 - s)^{\alpha-1} l(s)\|x_s\| \mathrm{d}s + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t_1} (t_1 - s)^{\alpha-1} l(s)\|x_s\| \mathrm{d}s \\
&\le \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t_2} (t_2 - s)^{\alpha-1} l(s)(s-t_0)^{-\gamma_2} \mathrm{d}s + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t_1} (t_1 - s)^{\alpha-1} l(s)(s-t_0)^{-\gamma_2} \mathrm{d}s \\
&\le (t_2 - t_0)^{-\gamma_2} + (t_1 - t_0)^{-\gamma_2} \\
&\le \varepsilon.
\end{aligned} \tag{10}$$

Similar to (10), we have

$$(Ax)(t_2) - (Ax)(t_1) \ge -\varepsilon. \tag{11}$$

Combining (10) and (11) leads to

$$|(Ax)(t_2) - (Ax)(t_1)| \le \varepsilon. \tag{12}$$

If $t_0 < t_1 < T' < t_2$,

$$|(Ax)(t_2) - (Ax)(t_1)| = |(Ax)(t_2) - (Ax)(T)| + |(Ax)(T) - (Ax)(t_1)| \to 0 \quad \text{as } t_2 \to t_1. \tag{13}$$

Combining (9), (12) and (13), we obtain that $|(Ax)(t_2) - (Ax)(t_1)| \to 0$ as $t_2 \to t_1$ for $t_1, t_2 > t_0$.
Then AS_2 is equicontinuous.
The remaining part of the proof is similar to that of Lemma 3.1, so we omit it. □

Similar to Theorem 3.2, from Lemma 3.3 we have the following.

Theorem 3.4. *Assume that conditions* (H_0) *and* (H_3) *hold. Then the solution of* (1) *is attractive.*

Theorem 3.5. *Assume that condition* (H_0) *holds, and also assume the following.*

(H_4) *There exists a constant* $\beta_1 \in (\alpha, 1)$ *such that*

$$\left| \frac{\phi(t_0)}{\Gamma(1-\alpha)} (t-t_0)^{-\alpha} + f(t, x_t) \right| \le \frac{\Gamma(1+\alpha-\beta_1)}{\Gamma(1-\beta_1)} (t-t_0)^{-\beta_1}$$

holds for all $t \in J$.

Then IVP (1) *has at least one solution* $x \in C([t_0 - \tau, \infty), R)$ *which is attractive.*

Proof. Define the set

$$S_3 = \{x | x \in C([t_0 - \tau, \infty), R), |x(t)| \le (t - t_0)^{-\gamma_3} \text{ for } t \ge \tilde{t} > t_0\},$$

where $\gamma_3 = \alpha - \beta_1$. S_3 is a closed, bounded, and convex subset of R.

To prove that IVP (1) has a solution which is attractive, we only prove that A maps S_3 in S_3; the remaining part of the proof is similar to that of Lemma 3.3 and Theorem 3.4.

For $t > t_0$, applying condition (H_4), we have

$$\begin{aligned}
|(Ax)(t)| &= \left| \phi(t_0) + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} f(s, x_s) \mathrm{d}s \right| \\
&\le \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} \left| \frac{\phi(t_0)}{\Gamma(1-\alpha)} (s-t_0)^{-\alpha} + f(s, x_s) \right| \mathrm{d}s \\
&\le \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} \frac{\Gamma(1+\alpha-\beta_1)}{\Gamma(1-\beta_1)} (s-t_0)^{-\beta_1} \mathrm{d}s \\
&= (t-t_0)^{\alpha-\beta_1} \\
&= (t-t_0)^{-\gamma_3},
\end{aligned} \tag{14}$$

which implies that $AS_3 \subset S_3$. □

Theorem 3.6. *Assume that condition* (H_0) *holds, and assume the following.*

(H_5) *There exist constants* $l_1 > 0$ *and* $\beta_2 > \frac{1}{1-\alpha}$ *such that*

$$\left|\frac{\phi(t_0)}{\Gamma(1-\alpha)}(t-t_0)^{-\alpha}+f(t,x_t)\right| \le l_1\|x_t\|^{\beta_2}.$$

Then IVP (1) *has at least one solution* $x \in C([t_0-\tau,\infty),R)$ *which is attractive provided that*

$$\frac{l_1\Gamma(1-\beta_2\gamma_4)}{\Gamma(1+\alpha-\beta_2\gamma_4)} \le 1,$$

where

$$\frac{\alpha}{\beta_2-1} < \gamma_4 < \frac{1}{\beta_2}.$$

Proof. From $\beta_2 > \frac{1}{1-\alpha}$, we have $\frac{\alpha}{\beta_2-1} < \frac{1}{\beta_2}$, which implies that γ_4 exists. From $\gamma_4 < \frac{1}{\beta_2}$, we have that $\Gamma(1-\beta_2\gamma_4) > 0$ and $\Gamma(1+\alpha-\beta_2\gamma_4) > 0$.

Define the set

$$S_4 = \{x | x \in C([t_0-\tau,\infty),R), \|x_t\| \le (t-t_0)^{-\gamma_4} \text{ for } t \ge \tilde{t} > t_0\}.$$

Similar to the proof of Theorem 3.5, we may deduce that A has a fixed point in S_4 which is a solution of IVP (1), and this solution is attractive. □

The attractive results of solutions of IVP (1) can be extended to the corresponding fractional neutral differential equations

$$\begin{cases} {}^C D^\alpha(x(t)-g(t,x_t)) = f(t,x_t), & t > t_0,\ 0<\alpha<1, \\ x(t) = \phi(t), & t_0-\tau \le t \le t_0. \end{cases} \tag{15}$$

Under condition (H_0), IVP (15) is equivalent to the following equation:

$$x(t) = \begin{cases} \phi(t_0) - g(t_0,x_{t_0}) + g(t,x_t) + \dfrac{1}{\Gamma(\alpha)}\displaystyle\int_{t_0}^t (t-s)^{\alpha-1} f(s,x_s)\mathrm{d}s, & t > t_0, \\ \phi(t), & t_0-\tau \le t \le t_0. \end{cases} \tag{16}$$

Define the operator B as follows:

$$Bx(t) = \begin{cases} \phi(t_0) - g(t_0,x_{t_0}) + g(t,x_t) + \dfrac{1}{\Gamma(\alpha)}\displaystyle\int_{t_0}^t (t-s)^{\alpha-1} f(s,x_s)\mathrm{d}s, & t > t_0, \\ \phi(t), & t_0-\tau \le t \le t_0. \end{cases} \tag{17}$$

For the sake of simplicity, in the following, we directly give the theorems of attractivity, and the proofs of these theorems are similar to the corresponding ones discussed above.

Theorem 3.7. *Assume that condition* (H_0) *holds, and assume the following.*

(H_1') *There is a constant* $\gamma_1 > 0$ *such that*

$$\left|\phi(t_0) - g(t_0,x_{t_0}) + g(t,x_t) + \frac{1}{\Gamma(\alpha)}\int_{t_0}^t (t-s)^{\alpha-1} f(s,x_s)\mathrm{d}s\right| \le (t-t_0)^{-\gamma_1}$$

for $t \in J$.

(H_2') g *is completely continuous and there exists a constant* $\alpha_1 \in (0,\alpha)$ *such that* $f \in L^{\frac{1}{\alpha_1}}(J \times C([-\tau,0],R),R)$.

Then IVP (15) *has at least one solution* $x \in S_1$ *which is attractive.*

Theorem 3.8. *Assume that condition* (H_0) *and the following conditions are satisfied.*

(H_3')

$$\left|\frac{k_1}{\Gamma(1-\alpha)}(\phi(t_0)-g(t_0,x_{t_0}))(t-t_0)^{-\alpha}+f(t,x_t)\right| \le l(t)\|x_t\|$$

for any $t \in J$ *and* $x \in C([t_0-\tau,\infty),R)$, *where* $k_1 \in R$ *and* $l \in C(J,R^+)$ *satisfies that*

(i) *there exist constants* $\gamma_2, l_2 > 0$ *such that*

$$\frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}(t-s)^{\alpha-1}l(s)(s-t_0)^{-\gamma_2}\mathrm{d}s \le l_2(t-t_0)^{-\gamma_2} \quad \text{for } t \in J;$$

(ii) *there is a constant* $\alpha_2 \in (0, \alpha)$ *such that* $l \in L^{\frac{1}{\alpha_2}}(J, R^+)$.

(H_6) *g is completely continuous and there exist constants* $l_3 > 0$ *and* $k_2 \in R$ *such that*

$$\left|k_2(\phi(t_0) - g(t_0, x_{t_0})) + g(t, x_t)\right| \le l_3\|x_t\|.$$

Then IVP (15) *has at least one solution* $x \in S_2$ *which is attractive provided that*

$$k_1 + k_2 = 1, \quad \text{and} \quad l_2 + l_3 \le 1.$$

Theorem 3.9. *Assume that condition* (H_0) *and the following conditions are satisfied.*

(H_4') *There exist constants* $k_3 \in R$, $\beta_1 \in (\alpha, 1)$ *and* $l_4 > 0$ *such that*

$$\left|\frac{k_3}{\Gamma(1-\alpha)}(\phi(t_0) - g(t_0, x_{t_0}))(t-t_0)^{-\alpha} + f(t, x_t)\right| \le \frac{l_4\Gamma(1+\alpha-\beta_1)}{\Gamma(1-\beta_1)}(t-t_0)^{-\beta_1}$$

hold for all $t \in J$.

(H_7) *g is completely continuous and there exist constants* $l_5 > 0$ *and* $k_4 \in R$ *such that*

$$\left|k_4(\phi(t_0) - g(t_0, x_{t_0})) + g(t, x_t)\right| \le l_5(t-t_0)^{\alpha-\beta_1}.$$

Then IVP (15) *has at least one solution* $x \in S_3$ *which is attractive provided that*

$$k_3 + k_4 = 1, \quad \text{and} \quad l_4 + l_5 \le 1.$$

Theorem 3.10. *Assume that condition* (H_0) *and the following conditions are satisfied.*

(H_5') *There exist constants* $k_5 \in R$, $l_6 > 0$ *and* $\beta_2 > \frac{1}{1-\alpha}$ *such that*

$$\left|\frac{k_5}{\Gamma(1-\alpha)}(\phi(t_0) - g(t_0, x_{t_0}))(t-t_0)^{-\alpha} + f(t, x_t)\right| \le l_6\|x_t\|^{\beta_2}.$$

(H_8) *g is completely continuous and there exist constants* $l_7 > 0$ *and* $k_6 \in R$ *such that*

$$\left|k_6(\phi(t_0) - g(t_0, x_{t_0})) + g(t, x_t)\right| \le l_7\|x_t\|.$$

Then IVP (15) *has at least one solution* $x \in S_4$ *which is attractive provided that*

$$k_5 + k_6 = 1, \qquad l_7 + \frac{l_6\Gamma(1-\beta_2\gamma_4)}{\Gamma(1+\alpha-\beta_2\gamma_4)} \le 1,$$

where

$$\frac{\alpha}{\beta_2 - 1} < \gamma_4 < \frac{1}{\beta_2}.$$

4. Examples

Example 4.1. Consider the fractional functional differential equation

$$\begin{cases} {}^{C}D^{\frac{1}{2}}x(t) = \frac{1}{5}(t+1)^{-\frac{3}{4}}\sin^4\left(x\left(t-\frac{\pi}{2}\right)\right), & t > 0, \\ x(t) = \sin t, & t \in \left[-\frac{\pi}{2}, 0\right], \end{cases} \tag{18}$$

where $f(t, x_t) = \frac{1}{5}(t+1)^{-\frac{3}{4}}\sin^4\left(x\left(t-\frac{\pi}{2}\right)\right)$ and $x(0) = 0$. It is obvious that condition (H_0) holds.

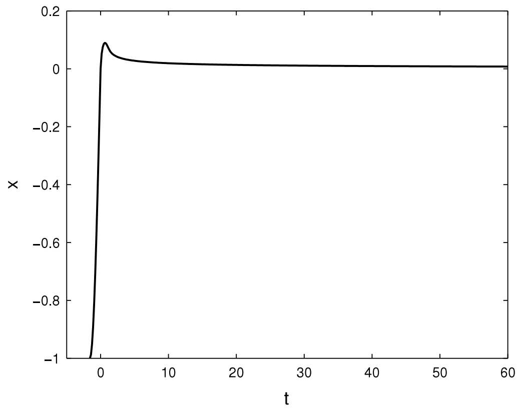

Fig. 1. Numerical solution of (18).

Let $m(t) = \frac{1}{5}(t+1)^{-\frac{3}{4}}$; then $|f(t, x_t)| \le m(t)$. It follows that

$$\begin{aligned}\frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}(t-s)^{\alpha-1}m(s)\mathrm{d}s &= \frac{1}{\Gamma\left(\frac{1}{2}\right)}\int_{0}^{t}(t-s)^{-\frac{1}{2}}\frac{1}{5}(s+1)^{-\frac{3}{4}}\mathrm{d}s\\ &\le \frac{1}{5\Gamma\left(\frac{1}{2}\right)}\int_{0}^{t}(t-s)^{-\frac{1}{2}}s^{-\frac{3}{4}}\mathrm{d}s\\ &= \frac{\Gamma\left(\frac{1}{4}\right)}{5\Gamma\left(\frac{3}{4}\right)}t^{-\frac{1}{4}}. \end{aligned} \tag{19}$$

Here, $1 = \Gamma(1) < \Gamma\left(\frac{3}{4}\right) < \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$ yields that

$$\frac{\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{3}{4}\right)} = \frac{\sqrt{2}\pi}{\left[\Gamma\left(\frac{3}{4}\right)\right]^2} < \sqrt{2}\pi < 5. \tag{20}$$

Combining inequalities (19) and (20), it follows that

$$\frac{1}{\Gamma\left(\frac{1}{2}\right)}\int_{0}^{t}(t-s)^{-\frac{1}{2}}\frac{1}{5}(s+1)^{-\frac{3}{4}}\mathrm{d}s < t^{-\frac{1}{4}}.$$

Then

$$\left|\frac{1}{\Gamma\left(\frac{1}{2}\right)}\int_{0}^{t}(t-s)^{-\frac{1}{2}}f(s, x_s)\mathrm{d}s\right| < t^{-\frac{1}{4}},$$

which, together with $x(0) = 0$, implies that condition (H_1) holds.

Let $\alpha_1 = \frac{1}{4} \in \left(0, \frac{1}{2}\right)$. We have

$$\int_{t_0}^{\infty}|f(t, x_t)|^{\frac{1}{\alpha_1}}\mathrm{d}t \le \int_{t_0}^{\infty}(m(t))^{\frac{1}{\alpha_1}}\mathrm{d}t = \int_{0}^{\infty}\left[\frac{1}{5}(t+1)^{-\frac{3}{4}}\right]^4\mathrm{d}t = \frac{1}{1250},$$

which implies that condition (H_2) holds. Thus the solution of (18) is existent and attractive by Lemma 3.1 and Theorem 3.2. The approximate solution is obtained by the Adams-type predictor–corrector method [38], which is displayed in Fig. 1 for the step size $h = 0.1$. Fig. 1 numerically illustrates the attractive result of (18).

Example 4.2. Consider the fractional functional differential equation

$$\begin{cases} {}^{C}D^{\frac{1}{2}}x(t) = \dfrac{1}{2}(t+1)^{-\frac{7}{4}}x(t-1), & t > 0,\\ x(t) = t, & t \in [-1, 0], \end{cases} \tag{21}$$

where $f(t, x_t) = \frac{1}{2}(t+1)^{-\frac{7}{4}}x(t-1)$ and $x(0) = 0$. It is easy to show that condition (H_0) holds.

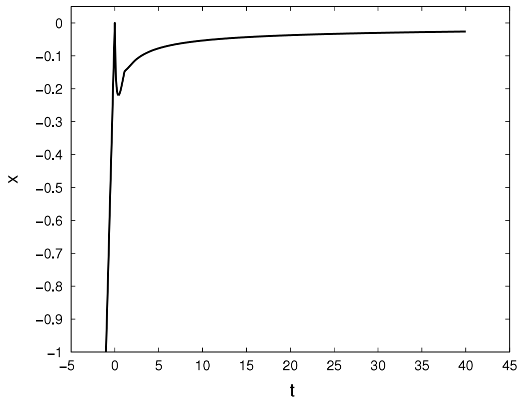

Fig. 2. Numerical solution of (21).

Let $l(t) = \frac{1}{2}(t+1)^{-\frac{7}{4}}$. Since $x(0) = 0$, we have

$$|f(t, x_t)| = l(t)|x(t-1)| \le l(t)\|x_t\|.$$

For $t \ge 1$, we have

$$\begin{aligned}\frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}(t-s)^{\alpha-1}l(s)s^{-1}\mathrm{d}s &= \frac{1}{\Gamma\left(\frac{1}{2}\right)}\int_{0}^{t}(t-s)^{-\frac{1}{2}}\frac{1}{2}(s+1)^{-\frac{7}{4}}s^{-1}\mathrm{d}s\\ &\le \frac{1}{\Gamma\left(\frac{1}{2}\right)}\int_{0}^{t}(t-s)^{-\frac{1}{2}}\frac{1}{2}s^{-\frac{7}{4}-1}\mathrm{d}s\\ &= \frac{\Gamma\left(-\frac{7}{4}\right)}{2\Gamma\left(-\frac{5}{4}\right)}t^{-\frac{9}{4}}.\end{aligned} \tag{22}$$

Since $\Gamma\left(\frac{1}{4}\right) = \left(-\frac{3}{4}\right)\left(-\frac{7}{4}\right)\Gamma\left(-\frac{7}{4}\right)$ and $\Gamma\left(\frac{3}{4}\right) = \left(-\frac{1}{4}\right)\left(-\frac{5}{4}\right)\Gamma\left(-\frac{5}{4}\right)$, together with (20) we have

$$\frac{\Gamma\left(-\frac{7}{4}\right)}{2\Gamma\left(-\frac{5}{4}\right)} = \frac{5\Gamma\left(\frac{1}{4}\right)}{42\Gamma\left(\frac{3}{4}\right)} \le \frac{25}{42} < 1. \tag{23}$$

Combining inequalities (22) and (23), it follows that

$$\frac{1}{\Gamma\left(\frac{1}{2}\right)}\int_{0}^{t}(t-s)^{-\frac{1}{2}}\frac{1}{2}(s+1)^{-\frac{7}{4}}s^{-1}\mathrm{d}s < t^{-\frac{9}{4}} < t^{-1} \quad \text{for } t \ge 1,$$

which implies that condition (i) of (H_3) holds.

Let $\alpha_2 = \frac{1}{4} \in \left(0, \frac{1}{2}\right)$. We have

$$\int_{t_0}^{\infty}(l(t))^{\frac{1}{\alpha_2}}\mathrm{d}t = \int_{0}^{\infty}\left[\frac{1}{2}(t+1)^{-\frac{7}{4}}\right]^{4}\mathrm{d}t = \frac{1}{96}, \tag{24}$$

which implies that condition (ii) of (H_3) holds. Thus the solution of (21) is existent and attractive by Lemma 3.3 and Theorem 3.4. The numerical solution of (21) is displayed in Fig. 2 for the step size $h = 0.1$. Fig. 2 numerically illustrates the attractive result of (21).

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