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The distinguishing number of the hypercube

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Abstract

The distinguishing number of a graph G is the minimum number of colors for which there exists an assignment of colors to the vertices of G so that the group of color-preserving automorphisms of G consists only of the identity. It is shown, for the d-dimensional hypercubic graphs H_d, that $D(H_d)=3$ if $d \in \{2,3\}$ and $D(H_d)=2$ if $d \geqslant 4$. It is also shown that $D(H_d^2)=4$ for $d \in \{2,3\}$ and $D(H_d^2)=2$ for $d \geqslant 4$, where H_d^2 denotes the square of the d-dimensional hypercube. This solves the distinguishing number for hypercubic graphs and their squares.

Keywords: Symmetry breaking; Distinguishing number; Automorphism groups; Hypercube; Graph coloring

1. Introduction

Definition 1. An r-labeling of a graph G is a function $\Phi : V(G) \to \{1,2,\ldots,r\}$. Once G is labeled, we refer to the labeled version of G as (G,Φ).

Definition 2. A permutation π of $V(G)$ is an automorphism of the labeled graph (G,Φ) if π preserves not only the adjacencies of G, but the labels as well. In other words, automorphism π has for each vertex $v \in V(G)$, $\Phi(v)=\Phi(\pi(v))$. Let $\mathrm{Aut}(G,\Phi)$ denote the group of label-preserving automorphisms of G under Φ.

The *distinguishing number* of a graph G, denoted $D(G)$, was first defined by Albertson and Collins in 1996 [1]. It is based on the notion of a distinguishing labeling.

Definition 3. An r-labeling (G,Φ) is distinguishing, if $\mathrm{Aut}(G,\Phi)$ consists solely of the identity element.

Definition 4. The distinguishing number of a graph G, is the minimum cardinality r such that there exists a distinguishing r-labeling of G.

Throughout this paper, we will also refer to labels as colors, and look for the minimum number of colors c for which the graph has a distinguishing labeling with c colors.

It is immediate that $D(K_n)=n$ for the complete graph K_n on n vertices, and that $D(P_n)=2$ for $n \geqslant 2$, where P_n is the n-vertex path. A classical result gives that for the cycle with n vertices, C_n, $D(C_n)=3$ if $n=3,4,5$ and $D(C_n)=2$ for $n \geqslant 6$.

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Albertson and Collins [1] proved several results about distinguishing number, including if $\mathrm{Aut}(G)$ is abelian, then $D(G) \leqslant 2$, and if $\mathrm{Aut}(G) \cong D_n$, where D_n is the dihedral group, then $D(G) \leqslant 3$. Other work on distinguishing number appears in [2,4,5].

In this paper, the distinguishing number of the d-dimensional hypercubic graphs H_d, are determined. Here, H_2 denotes the square (isomorphic to C_4), H_3 denotes the cube, and in general, H_d is defined to be isomorphic to the following graph: vertices correspond to all possible d-dimensional 0-1 vectors, and there is an edge between two vertices if and only if their corresponding vectors differ in exactly one coordinate position. (By convention, throughout the paper, we will number the coordinates of the d-dimensional hypercube from 1 to d, going from left to right.) In particular, we show that $D(H_d) = 3$ if $d \in \{2,3\}$ and $D(H_d) = 2$ if $d \geqslant 4$. This completes the characterization of the distinguishing number of the class of graphs H_d.

We then look at the square of the hypercube, where H_d^2 denotes the graph with the same vertex set as H_d, where H_d^2 has an edge between u and v if and only if they are connected by an edge or a path of length two in H_d. It is shown that for $d \geqslant 4$, $D(H_d^2) = 2$, and for $d \in \{2,3\}$, $D(H_d^2) = 4$.

2. The distinguishing number of the hypercube

In this section, we prove the following theorem.

Theorem 5.

(1) $D(H_d) = 3$ *if* $d \in \{2,3\}$,
(2) $D(H_d) = 2$ *if* $d \geqslant 4$.

We first prove that $D(H_d) \leqslant 2$, where $d \geqslant 4$. Since the automorphism group of H_d is non-trivial for $d > 1$, we immediately have that $D(H_d) > 1$, so this result implies part 2 of the theorem. The result is completed by a case analysis of the square and the cube.

2.1. The case $d \geqslant 4$

Lemma 6. $D(H_d) \leqslant 2$, *when* $d \geqslant 4$.

Proof. In H_d, for $i = \{0,\dots,d\}$ let v_i denote the vertex whose leftmost i coordinates are 0, and whose remaining coordinates are 1, and let $V = \{v_i \mid i = 0,\dots,d\}$. We also refer to the subgraph induced by V as the *spine* of the hypercube. In addition, define w to be the vertex whose leftmost and rightmost coordinates are 1, and whose remaining coordinates are 0. Then w is adjacent to v_{d-1} but non-adjacent to all the other vertices $v_i \in V$. Color the vertices $V \cup \{w\}$ with color 1, and all other vertices of H_d with color 2. First, we claim that the vertices of color 1 can be uniquely identified as follows. The subgraph induced by the vertices of color 1 is a “Y” graph with the vertices in the path V forming the stem and 1 branch of the Y, and the edge from v_{d-1} to w forming the other branch of the Y. The edge length of the branches is exactly 1, while $d \geqslant 4$ implies that the edge length of the stem is $\geqslant 3$. Thus the vertices of the stem, including v_{d-1} can be uniquely identified based on their distance from v_0, the base of the stem. It remains to distinguish v_d from w. But while these vertices are equidistant from v_0 in the induced subgraph of color 1, this is not the case if we are allowed to shortcut through vertices of color 2 in H_d; in particular $d(v_0, v_d) > d(v_0, w)$ in H_d. (As usual $d(a,b)$ here denotes the distance from a to b; the length of the shortest path.) Thus all vertices of color 1 are uniquely identified within H_d.

Any pair of vertices of color 2 can now be distinguished as follows: let a and b be two vertices colored 2, and let j denote the index of a coordinate where the 0-1 vectors of a and b differ. Consider the vertex v_j of color 1. There are two cases. The first case is that $d(a, v_j) \neq d(b, v_j)$. But since v_j is uniquely identified (as shown in the previous paragraph), then a and b are distinguished. Otherwise, $d(a, v_j) = d(b, v_j)$. Without loss of generality, assume a has a 1 in position j, while b has a 0. Then $d(a, v_{j-1}) < d(a, v_j) = d(b, v_j) < d(b, v_{j-1})$. So $d(a, v_{j-1}) < d(b, v_{j-1})$, and again we have found a uniquely identified vertex in V, for which a and b's distance to that vertex is different. □

We remark that in fact, the spine can be used to recover the coordinates of a vertex a in H_d more directly. In particular, it can be shown that $d(a, v_{i-1}) < d(a, v_i)$ if and only if a's coordinate vector has a 1 in position i, and $d(a, v_{i-1}) > d(a, v_i)$ if and only if a's coordinate vector has a 0 in position i. A generalization of this idea is used to prove our results in Section 3.

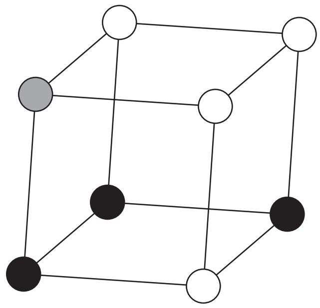

Fig. 1. A 3-distinguishing labeling of the cube.

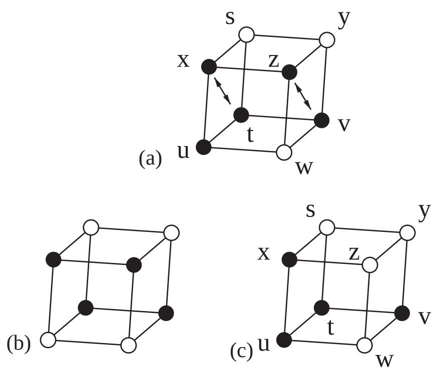

Fig. 2. (a–c) 2-colorings of the cube with remaining symmetry.

2.2. The square and the cube

We first remark that since H_2, the square, is isomorphic to C_4, it has already been established that $D(H_2)=3$. It remains only to show that there is a distinguishing labeling of H_3 using 3 colors, and no labeling using 2 colors will distinguish H_3. Fig. 1 gives a labeling of the cube with 3 colors that is easy to check is distinguishing. The proof that $D(H_3)>2$ proceeds easily by case analysis, but we include it for completeness.

Lemma 2. $D(H_3)>2$.

Proof. By way of contradiction, assume that there is a 2-distinguishing coloring of H_3 using the colors W and B, and let w denote the number of vertices colored W and b the number of vertices colored B. There are nine different values (b,w) for which $b+w=8$; without loss of generality, we consider only the five cases with $w \leqslant b$. Trivially, $w>0$; if $w=1$, then the three neighbors of the vertex colored W cannot be distinguished. If $w=2$ then the two vertices colored W cannot be distinguished from each other.

Suppose first that $w=3$ (and $b=5$). We show that such a coloring cannot contain a B-colored copy of H_2 as follows: the four vertices that do not participate in the monochromatic H_2 must consist of precisely 3 vertices colored W and one B vertex, and the two nonadjacent W vertices cannot be distinguished. Thus, there must exist a square with 3 vertices colored B and 1 vertex colored W. Fix such a square, and call the non-adjacent vertices colored B in this square u and v, call the other B vertex t, and the W-colored vertex w. Among the other four vertices, call u's neighbor x, v's neighbor y, t's neighbor s, and w's neighbor z (see Fig. 2(a)). Then in order for u and v to be distinguishable, x and y must receive different colors; without loss of generality (else reverse the names of u,x and v,y), let x get color B and y get color W. Vertex s must get color W, else u,x,s,t form a B-colored copy of H_2. Since there are precisely 3 vertices colored W by assumption, z gets color B, completing the coloring in Fig. 2(a). But then the automorphism that swaps $z \leftrightarrow v$ and $x \leftrightarrow t$ is color-preserving; a contradiction.

The last case is $w=b=4$. If the four vertices colored w (and the four vertices colored b) form an independent set, the graph has color-preserving symmetries, so we may assume that some vertex colored B has at least one B-colored neighbor. In fact, we may assume that some vertex a colored B has at least two neighbors colored B; else the graph must be colored as in Fig. 2(b) and has remaining symmetries. If all three of a's neighbors are colored B, then the three neighbors are indistinguishable. If the two of a's neighbors colored B have their common neighbor colored B (forming a monochromatic copy of H_2), then all four vertices colored B are indistinguishable. The remaining two cases are (a) the fourth vertex colored B is not adjacent to any of the other B-colored vertices, resulting in a's two B-colored neighbors being indistinguishable, or (b) The vertices colored B induce a path of vertex-length four, isomorphic to the graph in Fig. 2(c). But the graph in Fig. 2c also has a color-preserving automorphism, namely, using the labels in the figure, $s \leftrightarrow w$, $y \leftrightarrow z$, $x \leftrightarrow v$, $u \leftrightarrow t$. □

This concludes the proof of Theorem 1. We remark that the automorphism in Fig. 2(a) is not geometrically realizable in a solid cube. Thus, restricting to the set of symmetries of the cube, the vertices of a geometric cube can be 2-distinguished.

3. The square of the hypercube

The number of coordinates where two d-dimensional 0-1 vectors differ is commonly referred to as their Hamming distance. Thus H_d is precisely the graph with vertices corresponding to all d-dimensional 0-1 vectors, and edges between vertices u and v if their corresponding vectors have Hamming distance one, and H_d^2 is defined as the graph with vertices corresponding to all d-dimensional 0-1 vectors, with edges between vertices u and v if the Hamming distance between their corresponding vectors is either one or two.

Definition 8. For vertices u and v in H_d^2, let $d(u,v)$ denote their distance in the graph H_d^2. Let $h(u,v)$ denote the Hamming distance between their corresponding d-dimensional coordinate vectors.

Lemma 9. *The distance $d(u,v)$ between u and v in H_d^2 is $\lceil h(u,v)/2 \rceil$.*

We first show the following:

Theorem 10. $D(H_d^2)=2$, *for* $d \geqslant 6$.

Proof. Similar to the proof in Section 2.1, for $i=\{0,\ldots,d\}$ let v_i denote the vertex whose leftmost i coordinates are 0, and whose remaining coordinates are 1, and let $V=\{v_i \mid i=0,\ldots,d\}$ (see Fig. 3). Let w denote the vertex whose rightmost two coordinates are 0, and whose remaining coordinates are 1. Let z denote the vertex whose coordinates alternate $10\overline{10}$ (where $\overline{10}$ denotes the alternating pattern 1 then 0 repeated as necessary to complete z's coordinates). Color the set $V \cup \{w,z\}$ with color 1 and the remaining vertices in H_d^2 with color 2. We show for every vertex in H_d^2 we can uniquely recover its unique (0,1)-coordinate label, implying that each vertex can be distinguished under this coloring.

First consider the subgraph S induced by the vertices of color 1. Since $d \geqslant 6$, z can be defined as the unique vertex of degree 0 in S since for all $v \in V \cup \{w\}$, $h(z,v)>2$, so $d(z,v)>1$. Vertex w is the unique vertex of degree 1 in S, since

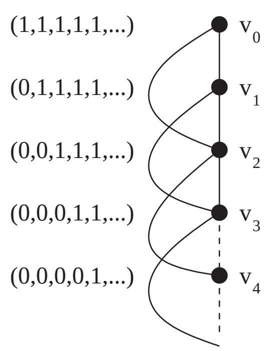

Fig. 3. The set V in H_d^2.

$d>4$ implies it is adjacent only to v_0, and every vertex in V is adjacent to at least 2 other vertices in V. So we can identify w uniquely, and hence v_0 as well. Now the two neighbors of v_0 in V, v_1 and v_2 are distinguished from all the other vertices in V as v_0's neighbors, and distinguished from each other since v_1 has one additional neighbor in V, v_3, whereas v_2 has two, v_3 and v_4. Let positive integer $k>3$, and assume by induction that all vertices in V of index $<k$ have been uniquely identified. We can identify v_k as the only color 1 vertex not yet identified that is adjacent to v_{k-2}. Thus all color 1 vertices can be uniquely identified.

Now consider a vertex x of color 2. Notice we can use the vertex labels v_i for the vertices of color 1 on the spine V, since the previous paragraph shows how we can reconstruct them from the coloring.

We use the following lemma to prove the theorem; the proof is deferred to after the proof of the theorem:

Lemma 11. *Let $x \in H_d^2$ be of color 2.*

(1) $d(x,v_{i-1}) < d(x,v_{i+1})$, *if and only if x's label has 1s in the ith and $(i+1)$st coordinate.*
(2) $d(x,v_{i-1}) > d(x,v_{i+1})$, *if and only if x's label has 0s in the ith and $(i+1)$st coordinate.*
(3) $d(x,v_{i-1}) = d(x,v_{i+1})$ *if and only if (x's label has 0 in the ith and 1 in the $(i+1)$st coordinate, or x's label has 1 in the ith and 0 in the $(i+1)$st coordinate).*

For $1 \leqslant j \leqslant d-1$, if we know whether x's label in the jth coordinate is a 0 or a 1, we can determine x's label in the $(j+1)$st coordinate using Lemma 11, by comparing $d(x,v_{j-1})$ and $d(x,v_{j+1})$. If $d(x,v_{j-1}) < d(x,v_{j+1})$, or ($d(x,v_{j-1}) = d(x,v_{j+1})$ and x has a 0 in the jth coordinate) then x has a 1 in the $(j+1)$st coordinate; if $d(x,v_{j-1}) > d(x,v_{j+1})$ or ($d(x,v_{j-1}) = d(x,v_{j+1})$ and x has a 1 in the jth coordinate) then x has a 0 in the $(j+1)$st coordinate. In the same fashion, for $2 \leqslant j \leqslant d-1$, if we know whether x's label in the jth coordinate is a 0 or a 1, we can determine x's label in the $(j-1)$st coordinate as follows: if $d(x,v_{j-1}) < d(x,v_{j+1})$, or ($d(x,v_{j-1}) = d(x,v_{j+1})$ and x has a 0 in the jth coordinate) then x has a 1 in the $(j-1)$st coordinate. If $d(x,v_{j-1}) > d(x,v_{j+1})$, or ($d(x,v_{j-1}) = d(x,v_{j+1})$ and x has a 1 in the jth coordinate) then x has a 0 in the $(j-1)$st coordinate.

We consider two cases. For the first case, consider x of color 2 for which there exists an i, $1 \leqslant i \leqslant d-1$, such that $d(x,v_{i-1}) \neq d(x,v_{i+1})$. Then, by Lemma 11 if $d(x,v_{i-1}) < d(x,v_{i+1})$, x's label has 1s in the ith and $(i+1)$st coordinate, and if $d(x,v_{i-1}) > d(x,v_{i+1})$, x's label has 0s in the ith and $(i+1)$st coordinates. If $i=d-1$, then we know the value of x's rightmost coordinate, else proceeding by induction, we can determine all coordinates l of x's label, with $i \leqslant l \leqslant d$. Similarly, if $i=1$ then we know the value of x's leftmost coordinate, else knowing x's ith coordinate label, we can determine x's $(i-1)$st coordinate. Proceeding by induction, we can determine all coordinates l of x's label with $1 \leqslant l \leqslant i$. So in this case, the label of x can be fully recovered.

The second case is that x of color 2 has for all i, $1 \leqslant i \leqslant d-1$, $d(x,v_{i-1}) = d(x,v_{i+1})$. But this can happen (by the "only if" part of Lemma 11) only if x's coordinate label is $0101\overline{01}$ or $1010\overline{10}$. But $1010\overline{10}$ is precisely z of color 1, so we can distinguish these two vertices from each other. □

This completes the proof of Theorem 10, once we prove Lemma 11.

Proof of lemma 11. v_{i-1} and v_{i+1} are adjacent in H_d^2, so $|d(x,v_{i-1}) - d(x,v_{i+1}| \leqslant 1$. Furthermore, by definition of v_{i-1} and v_{i+1}, their labels only differ from each other in the ith and $(i+1)$st coordinate, where v_{i-1} has two 1s and v_{i+1} has two 0s. So if $h(x,v_{i-1})$ and $h(x,v_{i+1})$ differ, this difference is solely due to the contribution of these coordinates. Suppose first that $d(x,v_{i-1}) \neq d(x,v_{i+1})$. Then by Lemma 9, $|h(x,v_{i-1}) - h(x,v_{i+1})| \geqslant 2$. To achieve a difference of two in the Hamming distance, the corresponding coordinates of x must be identical to those of v_{i-1} and opposite those of v_{i+1}, whence $d(x,v_{i-1}) < d(x,v_{i+1})$, or identical to those of v_{i+1} and opposite those of v_{i-1}, whence $d(x,v_{i-1}) > d(x,v_{i+1})$.

If $d(x,v_{i-1}) = d(x,v_{i+1})$, x's i and $(i+1)$st coordinates cannot both be 1s (or both be 0s), because these coordinates would contribute 2 to the value of $h(x,v_{i-1})$ (or 2 to $h(x,v_{i+1})$, if they are both 0s) and 0 to $h(x,v_{i+1})$ (or 0 to $h(x,v_{i-1}$, if they are both 0s). But by Lemma 9 $d(x,v_{i-1}) = d(x,v_{i+1})$ implies $|h(x,v_{i-1}) - h(x,v_{i+1})| \leqslant 1$. Thus x's ith and $(i+1)$st coordinate cannot be identical.

Conversely, if x has consecutive 1s in the ith and $(i+1)$st coordinate, then these coordinates contribute 0 to $h(x,v_{i-1})$ and 2 to $h(x,v_{i+1})$, so Lemma 9 implies that $d(x,v_{i-1}) < d(x,v_{i+1})$ in H_d^2. Similarly, if x has consecutive 0s in the ith and $(i+1)$st coordinate, then these coordinates contribute 2 to the $h(x,v_{i-1})$ and 0 to $h(x,v_{i+1})$, which by Lemma 9 implies that $d(x,v_{i-1}) > d(x,v_{i+1})$ in H_d^2. If x has 01 or 10 in positions i and $i+1$, then the "if" direction of parts (a) and (b) imply that it cannot be the case that $d(x,v_{i-1}) < d(x,v_{i+1})$ or that $d(x,v_{i-1}) > d(x,v_{i+1})$. Thus it must be that $d(x,v_{i-1}) = d(x,v_{i+1})$.

This completes the proof of the lemma. □

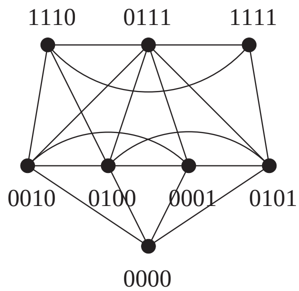

Fig. 4. The induced subgraph S of vertices colored 1.

Theorem 12. $D(H_d^2)=2$, *for* $d=5$.

Proof. Define $V=\{v_i \mid i=0,\dots,5\}$, where, v_i denotes the vertex whose leftmost i coordinates are 0, and whose remaining coordinates are 1, as above. Let $y=10100$, and color $V\cup\{y\}$ with color 1, and all other vertices with color 2. Consider S the induced subgraph of color 1 vertices. Then y is the unique vertex of degree 1 in S, adjacent only to v_5, so we can identify y uniquely, and hence v_5 as well. Now the two neighbors of v_5 in V, v_4 and v_3 are distinguished from all the other vertices in V as v_5's neighbors, and distinguished from each other since v_4 has one additional neighbor in V, v_2, whereas v_3 has two, v_2 and v_1. Now v_2 is distinguished from v_1 and v_0 by being adjacent to v_4, and v_1 is distinguished from v_0 by being adjacent to v_3. Thus all vertices of color 1 are uniquely identified. □

Based on the unique identities of the vertices in the spine V, the same argument as in the proof of Theorem 10 goes through verbatim to identify the vertices of color 2 with the exception of distinguishing 01010 from 10101. But these vertices can be distinguished from each other by the fact that 10101 is adjacent to y and 01010 is not.

Theorem 13. $D(H_d^2)=2$ *for* $d=4$.

Proof. This case requires a different approach. In H_4^2, 10 vertices are neighbors of the vertex 0000 and 5 vertices are non-neighbors. These five non-neighbors are the vertices with at least 3 of their coordinates set to 1; clearly they are all mutually adjacent. We shall call the set of neighbors of 0000 the *close* set, and the set of non-neighbors of 0000 the *far* set.

Now consider the coloring that assigns color 1 to vertices 0000, 0001, 0010, 0100, 0101, 0111, 1110, and 1111, and color 2 to the remaining vertices. If S is the induced subgraph of color 1 vertices (see Fig. 4), then 1111 is the unique vertex of degree 3 in S. 0000 is the degree 4 vertex in S not adjacent to 1111; the unique identification of 0000 partitions the remaining vertices of H_4^2 into close and far sets, where the vertices in the close set are distinguished from the vertices in the far set based on their adjacency to 0000. Then 1110 is the far vertex of degree 4 and 0111 is the far vertex of degree 6.

We have shown that the labels of all color 1 vertices in the far set can be uniquely recovered; we can also distinguish between the two color 2 vertices in the far set as follows. Note that 0100 of color 1 is also uniquely identifiable based on S, as it is the only degree 6 vertex in S in the close set. Now we distinguish the color 2 vertices in the far set by the fact that 1101 is adjacent to 0100 and 1011 is not.

It remains to identify the vertices in the close set. Each of the 10 vertices in the close set is adjacent to a different subset of size three of the far set, so that each of the $\binom{5}{3}$ subsets of the far set are represented by the adjacencies of some vertex in the close set. This implies that the unique identification of the vertices in the far set suffices to uniquely identify the vertices in the close set. □

Since H_2^2 is isomorphic to K_4, its distinguishing number is 4. It is also easy to determine $D(H_3^2)$, using the fact that a graph and its complement have the same distinguishing number, and $\overline{H_3^2}$ is a perfect matching with four edges, requiring 4 colors.

Putting together all the results of this section completes the study of the distinguishing number of the square of the hypercube.

Corollary 14.

(1) $D(H_d^2) = 4$ *if* $d \in \{2, 3\}$,
(2) $D(H_d^2) = 2$ *if* $d \geqslant 4$.

4. Discussion and open problems

We have not investigated powers of the hypercube greater than the square, and their distinguishing number is an open problem. We define these graphs as follows:

Definition 15. H_d^p is defined as the graph with the same vertex set as H_d, and an edge between u and v if their corresponding vertices are of distance $\leqslant p$ in H_d.

Note that if $p \geqslant d$, then H_d^p is simply K_{2^d}, which requires 2^d colors. If $p = d-1$ then since $D(H_d^p) = D(\overline{H_d^p})$, we have that $D(H_d^p)=$ the minimum integer x, such that x colors will distinguish a perfect matching of 2^{d-1} edges, i.e. $\binom{x}{2} \geqslant 2^{d-1}$.

Problem 16. *Determine* $D(H_d^p)$ *for any* (d, p) *with* $d-1 > p > 2$.

Along these lines, the following conjecture seems reasonable; but does not seem trivial to prove.

Conjecture 17. For fixed p, when d is sufficiently large, $D(H_d^p) = 2$.

A weaker conjecture that may be easier to prove would be the following.

Conjecture 18. When d is sufficiently large as a function of p, then $D(H_d^p) \leqslant cp$ for some constant c.

Finally, Cheng and Cowen [3] defined the *local distinguishing numbers* $LD^i(G)$ for a graph G with n vertices as follows. Consider n subgraphs consisting of each vertex together with its neighbors out to distance i in G. A coloring of G is i-locally-distinguishing if there is no isomorphism between any pair of these n subgraphs that is color-preserving. (When $i = \text{diameter}(G)$, this reduces to the ordinary concept of distinguishing number). For fixed $1 \leqslant i \leqslant \text{diam}(G)$, define $LD^i(G)$ to be the minimum positive integer r for which G has an i-locally distinguishing labeling with label set $\{1, \ldots, r\}$.

Problem 19. *Determine* $LD^i(H_d)$ *for* $i = 1, \ldots, diam(H_d) - 1 = d - 1$.

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