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On the L_p Norm for Some Approximation Operators

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1. INTRODUCTION

The Bernstein polynomials for a bounded function f on $[0, 1]$ are defined by

$$B_n(f, x) = \sum_{k=0}^{n} f(k/n)\, p_{n,k}(x),$$

where $p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}$. It is well known that if $f \in C[0, 1]$ and $\omega(\delta)$ $(\delta > 0)$ is the modulus of continuity of f on $[0, 1]$, then

$$\max_{0 \leqslant x \leqslant 1} |B_n(f, x) - f(x)| \leqslant \tfrac{5}{4}\, \omega(n^{-1/2}).$$

For a step function f of bounded variation with finitely many steps in every closed subinterval of $(0, 1)$, Hoeffding [7] showed that

$$\lim_{n \to \infty} \sqrt{n} \int_0^1 |B_n(f, x) - f(x)|\, dx = \sqrt{\frac{2}{\pi}}\, J(f), \tag{1}$$

where

$$J(f) = \int_0^1 x^{1/2}(1-x)^{1/2}\, |df(x)|.$$

However, the critical condition that f be a step function makes (1) merely a pathological result. Nevertheless, the interesting limit is a deep result.

The purpose of this note is to give an asymptotic limit of the L_p norm for a class of approximation operators. The results are obtained for the Feller operator $L_n(f, x)$ (cf. Khan [10]) which contains various well-known

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operators (see also Hahn [6] and Levikson [12]). Section 2 gives some preliminaries and the main results are established in Section 3. Several special cases are discussed in Section 4.

2. Preliminaries

The author (cf. Khan [10]) extended the well-known properties of Bernstein polynomials to the Feller operator $L_n(f, x)$. Many well-known approximation operators such as Bernstein, Szasz, Weierstrass, Baskakov, and Meyer–Konig–Zeller, etc., are all special cases of $L_n(f, x)$. Let $X_1, X_2, \ldots$ be iid (independent and identically distributed) random variables with mean x and variance $\sigma^2(x)$ $(\sigma(x) > 0)$ where x is a continuous parameter taking values in an interval $I \subseteq \mathbb{R} = (-\infty, \infty)$. In what follows $\sigma(x)$ is assumed to be continuous in $x \in I$. Let f be a bounded continuous function on $\mathbb{R}$. Set $S_n = X_1 + \cdots + X_n$ and define the Feller operator by

$$L_n(f, x) = Ef(S_n/n) = \int_{-\infty}^{\infty} f\left(\frac{t}{n}\right) dF_{n,x}(t), \tag{2}$$

where E denotes expectation and $F_{n,x}(t)$ is the distribution function of S_n depending on x. $F_{n,x}(t)$ is assumed to be continuous in x. Various properties of $L_n(f, x)$ can be found in Khan [10].

For $p > 0$ and $x \in I$ define

$$D_n^p(f, x) = \int_{-\infty}^{\infty} \left| f\left(\frac{t}{n}\right) - f(x) \right|^p dF_{n,x}(t) = E\,|f(S_n/n) - f(x)|^p. \tag{3}$$

Clearly, $D_n^p(f, x)$ is bounded, and it is continuous in x for $F_{n,x}(t)$ is assumed to be continuous in x. Note that $D_n(f, x) = D_n^1(f, x)$ is precisely the quantity dictating the properties of $L_n(f, x)$. For example, $\sup_{x \in I} D_n(f, x) \to 0$ (as $n \to \infty$) $\Rightarrow L_n(f, x) \to f(x)$ uniformly in $x \in I$. Moreover, it is known that $\max_{a \leqslant x \leqslant b} |L_n(f, x) - f(x)| \leqslant \max_{a \leqslant x \leqslant b} D_n(f, x) \leqslant K\omega(n^{-1/2})$ where $\omega(\delta)$ is the modulus of continuity of f on finite $[a, b]$. The object here is to find the asymptotic rate of the related L_p norm.

Let $G(x)$ be a distribution function on I. The L_p norm is defined by

$$\|D_n(f)\|_p = \left(\int_I D_n^p(f, x)\, dG(x) \right)^{1/p}, \qquad p > 0. \tag{4}$$

However, for the sake of notational simplicity the results will be stated for the quantity $\|D_n(f)\|_p^p$.

Assuming that f has bounded continuous derivative $f'(x) \neq 0 \ \forall x \in I$ define

$$V_p(f) = \int_I \sigma^p(x)\, |f'(x)|^p \, dG(x). \tag{5}$$

Note that $G(x)$ can be replaced by improper distribution ($dG(x) = dx$) if the relevant integrals exist as Riemann or Lebesgue integrals.

Under some conditions it is proved that $n^{p/2} \|D_n(f)\|_p^p \to C_p V_p(f)$ as $n \to \infty$ where C_p is an absolute constant. Moreover, it is also shown that

$$\begin{aligned} n\, &\|L_n(f, x) - f(x)\| \\ &= n \int_a^b |L_n(f, x) - f(x)|\, dx \to \frac{1}{2} \int_a^b \sigma^2(x)\, |f''(x)|\, dx \qquad \text{as} \quad n \to \infty \end{aligned}$$

provided that f has continuous derivatives f' and f'' on $[a, b] \subseteq I$. These results are established in Section 3 and several special cases are illustrated in Section 4.

3. The Main Results

The asymptotic limit of the L_p norm is given by

Theorem 1. *Let $f(x)$ be a bounded continuous function on $\mathbb{R}$ with bounded continuous derivative $f'(x) \neq 0 \ \forall x \in \mathbb{R}$. Let $X_1, X_2, \ldots$ be iid random variables with mean x and variance $\sigma^2(x)$ where x is a continuous parameter with values in an interval $I \subseteq \mathbb{R}$. Assume that $E\,|X_1|^{r+\delta} < \infty$ ($r \geq 2$, $\delta > 0$) and $\phi_r(x) = E\,|X_1 - x|^r$ is G-integrable where G is a distribution function on I. Let $L_n(f, x)$, $D_n^p(f, x)$, $\|D_n(f)\|_p$, and $V_p(f)$ be defined by* (2), (3), (4), *and* (5), *respectively. Then*

$$n^{p/2} D_n^p(f, x) \to C_p \sigma^p(x)\, |f'(x)|^p \qquad as \quad n \to \infty,$$

and

$$\lim_{n \to \infty} n^{p/2} \|D_n(f)\|_p^p = C_p V_p(f), \qquad 0 < p \leq r, \tag{6}$$

where $C_p = 2^{p/2} \Gamma((p+1)/2)/\sqrt{\pi}$.

Corollary 1. *Let $f(x)$ be a continuous function on $\mathbb{R}$ with continuous* *derivative* $f'(x) \neq 0$ $\forall x$ *in finite closed interval* $[a, b] \subseteq I$. *Then under the remaining conditions of Theorem* 1,

$$\lim_{n \to \infty} n^{p/2} \int_a^b D_n^p(f, x)\, dx = C_p V_p(f), \qquad 0 < p \leqslant r, \tag{7}$$

where

$$V_p(f) = \int_a^b \sigma^p(x)\, |f'(x)|^p\, dx.$$

COROLLARY 2. *The conclusions* (6) *and* (7) *remain valid* $\forall p > 0$ *if* $E|X_1|^p < \infty$ *and* $\phi_p(x) = E|X_1 - x|^p$ *is G-integrable* $\forall p > 0$. *In particular,* (6) *and* (7) *hold* $\forall p > 0$ *if* X_1 *has density* $g_x(y) = \exp(yQ(x) - b(Q(x)))$ *relative to a* σ*-finite measure* μ *and* $\phi_p(x)$ *is G-integrable.*

Proof. Letting $S_n = \sum_{i=1}^n X_i$ and $\zeta_n = \sum_{i=1}^n (X_i - x)$ we have

$$f(S_n/n) = f(x) + \frac{\zeta_n}{n} f'(\eta), \tag{8}$$

where $\eta = x + \theta h$ $(0 < \theta < 1)$, $h = \zeta_n/n$. Since $E|X_1|^2 < \infty$, $\eta = \eta_n \to^{\text{a.s.}} x$, and $f'(x)$ is continuous with $f'(x) \neq 0$ $\forall x$, using Cramer's theorem it follows from the central limit theorem that

$$Z_n = \frac{\sqrt{n}(f(S_n/n) - f(x))}{\sigma(x)\, |f'(x)|} \xrightarrow{d} Z \qquad \text{as} \quad n \to \infty\ \forall x \in I, \tag{9}$$

where $\xrightarrow{d}$ denotes convergence in distribution and Z has standard normal distribution. From (8) we have

$$R_n = \sqrt{n}(f(S_n/n) - f(x)) = \frac{\zeta_n}{\sqrt{n}} f'(\eta).$$

Since f' is bounded, we have

$$E|R_n|^{r+\delta} \leqslant M n^{-(r+\delta)/2} E|\zeta_n|^{r+\delta}. \tag{10}$$

Now since $X_1, X_2, \ldots$ are iid with mean x and $E|X_1|^{r+\delta} < \infty$ $(\Leftrightarrow E|X_1 - x|^{r+\delta} < \infty)$, it follows from a result of Jogdeo and Dharmadhikari [8] that

$$E|\zeta_n|^{r+\delta} \leqslant C n^{(1/2)(r+\delta)-1} \sum_{i=1}^n E|X_i - x|^{r+\delta} \leqslant C n^{(1/2)(r+\delta)} E|X_1 - x|^{r+\delta}, \tag{11}$$

where C is a constant depending only on $r+\delta$. Hence it follows from (10) that

$$E\,|R_n|^{r+\delta} \leqslant MCE\,|X_1 - x|^{r+\delta}. \tag{12}$$

Thus $E\,|R_n|^{r+\delta}$ is bounded and $\{|R_n|^r,\ n \geqslant 1\}$ is uniformly integrable sequence of random variables. Consequently, $\{|Z_n|^r,\ n \geqslant 1\}$ (Z_n defined by (9)) is also uniformly integrable. Hence it follows from moment convergence theorem (cf. Loéve [13, p. 186]) that

$$\lim_{n\to\infty} E\,|Z_n|^p = C_p = E\,|Z|^p = \frac{2^{p/2}}{\sqrt{\pi}}\,\Gamma\left(\frac{p+1}{2}\right), \qquad 0 < p \leqslant r.$$

Since

$$E\,|R_n|^p = n^{p/2}\,D_n^p(f,x) = \sigma^p(x)\,|f'(x)|^p\,E\,|Z_n|^p,$$

it follows that

$$\lim_{n\to\infty} n^{p/2}\,D_n^p(f,x) = C_p \sigma^p(x)\,|f'(x)|^p. \tag{13}$$

Now we claim that $\sigma^p(x)\,|f'(x)|^p$ is G-integrable. Since $|f'|^p$ is bounded, it is enough to show that $\sigma^p(x)$ is G-integrable. For $r \geqslant 2$ Jensen's inequality gives

$$\phi_r(x) = E(|X_1 - x|^2)^{r/2} \geqslant (E\,|X_1 - x|^2)^{r/2} = \sigma^r(x) \geqslant \sigma^p(x), \qquad r \geqslant p.$$

Since $\phi_r(x)$ $(r \geqslant 2)$ is assumed to be G-integrable, hence $\sigma^p(x)$ is G-integrable. Moreover, $E\,|R_n|^p \leqslant MCE\,|X_1 - x|^p = MC\phi_p(x)$ by (12), and since $\phi_r(x)$ $(r \geqslant p)$ is G-integrable, it follows from (13) and the dominated convergence theorem that

$$\lim_{n\to\infty} n^{p/2}\,\|D_n(f)\|_p^p = C_p V_p(f) = C_p \int_I \sigma^p(x)\,|f'(x)|^p\,dG(x), \qquad 0 < p \leqslant r.$$

This proves Theorem 1. Corollary 1 is obvious. Corollary 2 follows from Theorem 1 and the fact that the distribution with the density $g_x(y) = \exp(yQ(x) - b(Q(x)))$ admits moments of all orders (cf. Lehmann [11]).

We will now prove another asymptotic result under some additional conditions. Let $X_1, X_2, \ldots$ be iid random variables with mean x and variance $\sigma^2(x)$. Assume that the mgf (moment generating function) $\psi(\theta) = E\exp(\theta X_1)$ is finite, and let $h_n = \sum_{i=1}^n (X_i - x)/n$. It is known (cf.

Chernoff [4], Khan [9, p. 506]) that for $\delta > 0$ there exists a number $\rho < 1$ such that

$$P(|h_n| \geqslant \delta) \leqslant 2\rho^n, \qquad 0 < \rho < 1. \tag{14}$$

We can now prove

THEOREM 2. *Let $X_1, X_2, \ldots$ be iid random variables with mean $x \in I \subseteq \mathbb{R}$ and variance $\sigma^2(x)$ with finite mgf $\psi(\theta) = E\exp(\theta X_1)$. Let f be a continuous function on $\mathbb{R}$ with continuous derivatives f' and f'' on finite closed interval $[a, b] \subseteq I$, and let $L_n(f, x)$ be defined by* (2). *Then*

$$\lim_{n \to \infty} n \int_a^b |L_n(f, x) - f(x)|\, dx = \frac{1}{2} \int_a^b \sigma^2(x)\, |f''(x)|\, dx. \tag{15}$$

Proof. Let $S_n = \sum_{i=1}^n X_i$ and $\zeta_n = \sum_{i=1}^n (X_i - x)$. Clearly, $E\zeta_n = 0$ and $E\zeta_n^2 = n\sigma^2(x)$. Using Taylor expansion we have

$$f(S_n/n) = f(x) + \frac{\zeta_n}{n} f'(x) + \frac{\zeta_n^2}{2n^2} f''(x) + r_n,$$

where $r_n = (\zeta_n^2/2n^2)(f''(x + \theta h_n) - f''(x))$, $0 < \theta < 1$, $h_n = \zeta_n/n$. Hence taking expectations we have

$$L_n(f, x) = f(x) + \frac{\sigma^2(x)}{2n} f''(x) + Er_n. \tag{16}$$

Note that $v(h_n) = (f''(x + \theta h_n) - f''(x)) \to^{\text{a.s.}} 0$ as $h_n \to^{\text{a.s.}} 0$ (as $n \to \infty$). Hence given $\varepsilon > 0$ we can choose $\delta > 0$ such that $|v(h_n)| < \varepsilon$ whenever $|h_n| < \delta$. This is possible by a proper choice of n for $h_n = \zeta_n/n \to^{\text{a.s.}} 0$ as $n \to \infty$. Hence for $x \in [a, b]$ we have

$$|Er_n| \leqslant E\,|r_n| = \frac{E\zeta_n^2\, |v(h_n)|}{2n^2} I\{|h_n| < \delta\} + \frac{E\zeta_n^2\, |v(h_n)|}{2n^2} I\{|h_n| \geqslant \delta\},$$

where I is the usual indicator function. Thus

$$|Er_n| \leqslant \frac{\varepsilon E\zeta_n^2}{2n^2} + \frac{M}{2n^2} E\zeta_n^2 I\{|h_n| \geqslant \delta\}. \tag{17}$$

It follows from Cauchy–Schwarz inequality that

$$E\zeta_n^2 I\{|h_n| \geqslant \delta\} \leqslant \sqrt{E\,|\zeta_n|^4 P(|h_n| \geqslant \delta)}. \tag{18}$$

Since $E|\zeta_n|^4 \leqslant Cn^2 E|X_1 - x|^4$ by (11), hence it follows from (14), (17), and (18) that

$$|Er_n| \leqslant \frac{\varepsilon\sigma^2(x)}{2n} + \frac{K}{n}\sqrt{E|X_1 - x|^4}\,\rho^{n/2}.$$

Thus

$$n|Er_n| \leqslant \varepsilon_n(x) \downarrow 0 \qquad \text{as} \quad n \to \infty. \tag{19}$$

Hence (15) follows from (16) and (19) and Theorem 2 is proved.

Theorem 2 remains valid if $X_1, X_2, \ldots$ are iid with density $g_x(y) = \exp(yQ(x) - b(Q(x)))$ relative to a σ-finite measure μ. This is due to the fact that $g_x(y)$ admits finite mgf (cf. Lehmann [11]). We remark in passing that if f and $f' \in C[a, b]$, then it is easy to see that

$$\sqrt{n}\int_a^b |L_n(f, x) - f(x)|\, dx = o(1).$$

4. Special Cases

First of all (6) and (7) specialized to $p = 1$ give

$$\lim_{n\to\infty} \sqrt{n}\,\|D_n(f)\|_1 = \sqrt{\frac{2}{\pi}}\,V(f) = \sqrt{\frac{2}{\pi}}\int_I \sigma(x)\,|f'(x)|\, dG(x),$$

and

$$\lim_{n\to\infty} \sqrt{n}\int_a^b D_n(f, x)\, dx = \sqrt{\frac{2}{\pi}}\int_a^b \sigma(x)\,|f'(x)|\, dx,$$

which are analogous to (1). We will now identify the limits in (6) and (7) for various special operators. The main emphasis is on the limits.

(i) *Bernstein Operator.* Let $f \in C[0, 1]$ with continuous derivative $f'(x) \neq 0 \quad \forall x \in [0, 1]$, and let $X_1, X_2, \ldots$ be iid with $P(X_1 = 1) = 1 - P(X_1 = 0) = x$, $0 \leqslant x \leqslant 1$. Then (2) defines Bernstein polynomials $B_n(f, x)$ and

$$D_n^p(f, x) = \sum_{k=0}^{n} |f(k/n) - f(x)|^p\, p_{n,k}(x), \qquad \|D_n(f)\|_p^p = \int_0^1 D_n^p(f, x)\, dx,$$

where $p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}$. Since $EX_1 = x$ and $\sigma^2(x) = x(1-x)$, (6) gives

$$\lim_{n \to \infty} n^{p/2} \|D_n(f)\|_p^p = C_p \int_0^1 x^{p/2}(1-x)^{p/2} |f'(x)|^p \, dx \qquad \forall p \geqslant 0,$$

and

$$\lim_{n \to \infty} \sqrt{n}\, \|D_n(f)\|_1 = \sqrt{\frac{2}{\pi}} \int_0^1 x^{1/2}(1-x)^{1/2} |f'(x)| \, dx.$$

(ii) *Szasz Operator.* Let $X_1, X_2, \ldots$ be iid with $P(X_1 = k) = e^{-x}x^k/k!$, $k = 0, 1, 2, \ldots$, $x \geqslant 0$. Then (2) defines Szasz operator $S_n(f, x)$ and

$$D_n^p(f, x) = e^{-nx} \sum_{k=0}^{\infty} |f(k/n) - f(x)|^p \frac{(nx)^k}{k!}.$$

Since $EX_1 = \sigma^2(x) = x$, it follows from (6) that

$$\lim_{n \to \infty} n^{p/2} \|D_n(f)\|_p^p = C_p \int_0^{\infty} x^{p/2} |f'(x)|^p \, dG(x) \qquad \forall p > 0,$$

and

$$\lim_{n \to \infty} \sqrt{n} \int_a^b D_n(f, x) \, dx = \sqrt{\frac{2}{\pi}} \int_a^b x^{1/2} |f'(x)| \, dx, \qquad 0 \leqslant a < b < \infty.$$

(iii) *Weierstrass Operator.* Let $X_1, X_2, \ldots$ be iid with density $g_x(y) = (2\pi)^{-1/2} \exp(-\frac{1}{2}(y-x)^2)$, $-\infty < y,\ x < \infty$. Then (2) defines Weierstrass operator

$$W_n(f, x) = \sqrt{\frac{n}{2\pi}} \int_{-\infty}^{\infty} f(x+u) \exp\left(-\frac{nu^2}{2}\right) du,$$

and

$$D_n^p(f, x) = \sqrt{\frac{n}{2\pi}} \int_{-\infty}^{\infty} |f(x+u) - f(x)|^p \exp\left(-\frac{nu^2}{2}\right) du.$$

Since $EX_1 = x$ and $\sigma^2(x) = 1$, it follows from (6) that

$$\lim_{n \to \infty} n^{p/2} \|D_n(f)\|_p^p = C_p \int_{-\infty}^{\infty} |f'(x)|^p \, dG(x),$$

and

$$\lim_{n\to\infty} n^{p/2} \int_a^b D_n^p(f, x)\, dx = C_p \int_a^b |f'(x)|^p\, dx \qquad \forall p > 0,$$

where $[a, b]$ is a finite closed interval. In particular,

$$\lim_{n\to\infty} \sqrt{n} \int_a^b D_n(f, x)\, dx = \sqrt{\frac{2}{\pi}}\, V(f) = \sqrt{\frac{2}{\pi}} \int_a^b |f'(x)|\, dx,$$

where $V(f)$ is the total variation of f on $[a, b]$.

(iv) *Gamma Operator.* Let $X_1, X_2, \ldots$ be iid with density $g_x(y) = x^{-1} e^{-y/x}$, $y \geqslant 0$, $x > 0$. Then (2) defines Gamma operator

$$G_n(f, x) = \frac{x^{-n}}{(n-1)!} \int_0^\infty f(y/n)\, y^{n-1} e^{-y/x}\, dy.$$

Since $\sigma^2(x) = x^2$, hence (6) gives

$$\lim_{n\to\infty} n^{p/2} \|D_n(f)\|_p^p = C_p \int_a^b x^p\, |f'(x)|^p\, dG(x), \qquad 0 \leqslant a < b < \infty.$$

In particular,

$$\lim_{n\to\infty} n^{p/2} \int_a^b D_n^p(f, x)\, dx = C_p \int_a^b x^p\, |f'(x)|^p\, dx, \qquad 0 \leqslant a < b < \infty,\ p > 0.$$

(v) *Baskakov Operator.* Let $X_1, X_2, \ldots$ be iid with $P(X_1 = k) = pq^k$, $k = 0, 1, 2, \ldots$ $(0 \leqslant p \leqslant 1,\ p + q = 1)$. If $p = (1 + x)^{-1}$ $(x \geqslant 0)$, then (2) defines Baskakov operator

$$B_n^*(f, x) = (1 + x)^{-n} \sum_{k=0}^{\infty} f(k/n) \binom{n+k-1}{k} \left(\frac{x}{1+x}\right)^k,$$

and (6) and (7) hold with $\sigma^2(x) = x(1 + x)$.

(vi) *Feller Operator.* This is an example where Theorem 1 holds for only restrictive p. Let $X_1, X_2, \ldots$ be iid random variables with common density

$$g_x(y) = \frac{2}{\pi(1 + (y - x)^2)^2}, \qquad -\infty < y, x < \infty.$$

In what follows $F_{n,x}(y)$ denotes the distribution function of $S_n = X_1 + \cdots + X_n$ and $g_{n,x}(y)$ is the resulting density function. Letting

$$a_k = \sum_{j=0}^{[(k+1)/2]} \binom{k+1}{2j} (-1)^j n^{k+1-2j} (y-nx)^{2j},$$

some routine calculations show that

$$g_{n,x}(y) = \frac{dF_{n,x}(y)}{dy} = \frac{n!}{\pi} \sum_{k=0}^{n} \frac{a_k}{(n-k)!\,(n^2+(y-nx)^2)^{k+1}}.$$

Then (2) defines the Feller operator $L_n(f, x)$. The random variable X_1 has all the moments of order $p \leqslant 3-\delta$ $(\delta > 0)$ but none of order $p \geqslant 3$. Hence it follows from Theorem 5 of Brown [2, p. 661] that the moment convergence used in the proof of Theorem 1 holds for $p \leqslant 3-\delta$ but fails for $p \geqslant 3$. Since $EX_1 = x$ and $\sigma^2(x) = 1$, we obtain from (6) that

$$\lim_{n \to \infty} n^{p/2} \|D_n(f)\|_p^p = C_p \int_{-\infty}^{\infty} |f'(x)|^p \, dG(x), \qquad p \leqslant 3-\delta,$$

a result with the same limit as in Example (iii) except for restrictive p.

Several other special cases can be obtained from (6) and (7). Finally, (15) can be specialized to various operators by identifying $\sigma(x)$ in the asymptotic limit. For example, in the case of Bernstein polynomials we have

$$\lim_{n \to \infty} n \int_0^1 |B_n(f, x) - f(x)| \, dx = \frac{1}{2} \int_0^1 x^{1/2}(1-x)^{1/2} \, |f''(x)| \, dx.$$

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