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The Hopf algebra of odd symmetric functions

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Abstract

We consider a q -analogue of the standard bilinear form on the commutative ring of symmetric functions. The $q = -1$ case leads to a \mathbb{Z} -graded Hopf superalgebra which we call the algebra of odd symmetric functions. In the odd setting, we describe counterparts of the elementary and complete symmetric functions, power sums, Schur functions, and combinatorial interpretations of associated change of basis relations.

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Keywords: Symmetric functions; Hopf algebras; Schur functions

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1. Introduction

1.1. Symmetric groups and categorification

The ring Λ of symmetric functions plays a fundamental role in several areas of mathematics. It decategorifies the representation theory of the symmetric groups, for it can be identified with the direct sum of Grothendieck groups of group rings of the symmetric groups:

$$\Lambda \cong \bigoplus_{n \geq 0} K_0(\mathbb{C}[S_n]),$$

where any characteristic 0 field can be used instead of \mathbb{C} . Multiplication and comultiplication in Λ come from induction and restriction functors for inclusions $S_n \times S_m \subset S_{n+m}$, and Schur functions are the images of simple S_n -modules in the Grothendieck group under this isomorphism. The elementary and complete symmetric functions e_n and h_n are the images of the sign and trivial representations of S_n , respectively.

The ring $\Lambda(n)$ of symmetric functions in n variables is the quotient of Λ by the ideal generated by e_m , over all $m > n$; it is naturally isomorphic to the representation ring of polynomial representations of $GL(n)$, with Schur functions s_λ for partitions λ with at most n rows given by the symbols of the corresponding irreducible representations of $GL(n)$. Good accounts of the above in the literature can be found in [2,6,21,28].

These structures are deep and serve as a foundation as well as a model example for many further developments in representation theory. One such development starts with the nilHecke ring NH_n , the ring of endomorphisms of $\mathbb{Z}[x_1, \dots, x_n]$ generated by the divided difference operators ∂_i and the operators of multiplication by x_i . This ring is related to the geometry of flag varieties; see [14] and references there. More recently, NH_n has appeared in the categorification of quantum \mathfrak{sl}_2 [18,26]; cyclotomic quotients of this ring categorify weight spaces of irreducible representations of quantum \mathfrak{sl}_2 . The ring NH_n admits a graphical interpretation, with its elements described by diagrams on n strands. The generator x_i is presented by n vertical strands with a dot on the i -th strand, the generator ∂_i by the intersection of the i -th and $(i + 1)$ -st strands. See [11,18] for some uses of this diagrammatic representation.

The subring of NH_n generated by the divided difference operators ∂_i is known as the nilCoxeter ring; see [9] and references therein. An odd counterpart of the nilCoxeter ring, the LOT (Lipshitz–Ozsváth–Thurston) ring, recently appeared in the bordered Heegaard Floer homology [19] and should play a role in the categorification of quantum superalgebras [10]. In this odd version far away crossings (∂_i and ∂_j for $|i - j| > 1$) anticommute rather than commute, with the other defining relations remaining the same ($\partial_i^2 = 0$, $\partial_i \partial_{i+1} \partial_i = \partial_{i+1} \partial_i \partial_{i+1}$). It is natural to add dot generators to the LOT ring, making all far away generators (dots and crossings) anticommute, and suitably modifying the other defining relations for the nilHecke algebra. The resulting “odd nilHecke” algebra ONH_n on n -strands shares many similarities with the nilHecke algebra NH_n ; for instance, it acts on the space of polynomials in n anticommuting

variables via multiplication by these variables and by odd analogues of the divided difference operators. The odd nilHecke algebra and its action on skew-symmetric polynomials appear in the very recent work of Kang et al. [8], where it is used, in particular, as a building block for super-analogues of the quiver Hecke algebras. In the forthcoming paper [5], we will develop the odd counterpart of the diagrammatical calculus [12] for NH_n .

The nilHecke algebra NH_n is isomorphic to the matrix algebra of size $n! \times n!$ with coefficients in the ring of symmetric functions in n variables, and the combinatorics of symmetric functions can be rethought from the point of view of NH_n ; see [12,23] for instance. A straightforward argument shows that the odd nilHecke algebra is isomorphic to the matrix algebra of size $n! \times n!$ with coefficients in the ring $\Lambda^{-1}(n)$ with generators e_1, \dots, e_n and defining relations

$$\begin{aligned}
 e_i e_j &= e_j e_i \quad \text{if } i + j \text{ is even,} \\
 e_i e_j + (-1)^i e_j e_i &= e_{j-1} e_{i+1} + (-1)^i e_{i+1} e_{j-1} \quad \text{if } i + j \text{ is odd.}
 \end{aligned}$$

We would like to think of $\Lambda^{-1}(n)$ as the odd counterpart of the ring $\Lambda(n)$ of n -variable symmetric functions and will pursue this approach in [5], defining the odd Schur function basis of $\Lambda^{-1}(n)$ via its embedding in ONH_n , and showing that cyclotomic quotients of ONH_n are Morita equivalent to suitable quotients of $\Lambda^{-1}(n)$ which should be odd counterparts of the cohomology rings of complex Grassmannians. The ring $\Lambda^{-1}(n)$ can be thought of as an odd counterpart of the cohomology ring of $Gr(n, \infty)$, the Grassmannian of complex n -planes in \mathbb{C}^∞ . We prefer to use “odd” rather than “super” here, since $\Lambda^{-1}(n)$ is not isomorphic to the cohomology ring of any super topological space and hints at genuinely quantum geometry.

1.2. Outline of this paper

By analogy with the even case, if we send n to infinity, the resulting limit algebra Λ^{-1} should be a Hopf superalgebra, the odd analogue of the algebra Λ^1 of symmetric functions (which was called Λ above). In the present paper, we develop an approach to Λ^{-1} that bypasses odd nilHecke algebras. Fix a scalar q . We first define a q -Hopf algebra Λ' on generators h_1, h_2, \dots . A q -Hopf algebra is a Hopf algebra in the category of graded vector spaces with the braiding given by q to the power the product of the degrees (in the terminology of [1]). The q -Hopf algebra Λ' appears in Section 17.3.4 of [1] and has a natural bilinear form, which is nondegenerate over $\mathbb{Q}(q)$. The bilinear form degenerates for special values of q , and we can form the quotient of Λ' by the kernel of the bilinear form for any such special value. We denote this quotient by Λ^q or just Λ when q is understood from the context (usually $q = -1$). The case $q = 1$ results in the familiar Hopf algebra of symmetric functions. Here we study the next case in simplicity, that of $q = -1$. The resulting quotient Λ^{-1} is a Hopf superalgebra which is neither cocommutative nor commutative as a superalgebra. Sections 2.1 and 2.2 are devoted to establishing this basic language and the bialgebra structure on Λ^{-1} , as well as setting up a graphical interpretation of the bilinear form.

The Hopf algebra Λ^1 has an automorphism ω which swaps e_n and h_n . Since Λ^1 has the extremely rigid structure of a positive self-adjoint Hopf algebra [31], this automorphism is uniquely determined by the fact that it preserves the bilinear form, preserves the set of positive elements, and switches e_2 and h_2 . On the categorified level, applying ω amounts to taking the tensor product with the sign representation. The case of Λ^{-1} is more complicated, due in part to the lack of commutativity and cocommutativity and the lack of a positivity structure. In Section 2.3, we study several (anti-)automorphisms, some involutory, each of which bears some

of the properties of ω . We obtain the antipode S from these, completing the structure of a Hopf superalgebra on Λ^{-1} .

The q -Hopf algebra Λ' is isomorphic to the graded dual of the q -Hopf algebra of quantum quasi-symmetric functions introduced by Thibon and Ung [30]. In Section 2.4, we explain this isomorphism.

The classical monomial sums in Λ^1 derive much of their importance from the fact that they form a dual basis to the basis of complete symmetric functions. In Section 3.1, we introduce the dual bases to the bases of odd complete and elementary symmetric functions, which we call the odd monomial and forgotten symmetric functions. Signed analogues of the classical combinatorial relations between the complete, elementary, monomial, and forgotten bases are derived as well.

Any Hopf algebra H has an associated Lie algebra of primitives $P(H)$, consisting of those $x \in H$ such that $\Delta(x) = x \otimes 1 + 1 \otimes x$. For example, if \mathfrak{g} is a Lie algebra, then the primitives of its universal enveloping algebra are a copy of \mathfrak{g} itself. Under certain conditions on H which are satisfied in the cases $H = \Lambda^1, \Lambda^{-1}, \Lambda'$, the space of primitives can be computed as the perpendicular space to I^2 , where I is the algebra ideal of positively graded elements [31]. In Λ^1 , the primitives are the classical power sum functions; in the characteristic zero case, viewed as characters of symmetric groups, their products represent scalar multiples of the indicator functions on corresponding conjugacy classes of S_n . The space of primitives of Λ^1 is one-dimensional in each positive degree. In Section 3.2, we compute and study the space of primitives of Λ^{-1} . This space is one-dimensional in even degrees and in degree 1, and zero in other degrees. Its even degree parts generate the center of Λ^{-1} as an algebra.

The most remarkable basis of Λ^1 is the basis of Schur functions. In terms of power series they are described as generating functions for semistandard Young tableaux of a given shape or as a ratio of determinants; in terms of the nilHecke ring they are the result of applying the longest divided difference operator to a single monomial; in terms of the symmetric group they are the images of the characters of irreducible representations; in terms of $GL(n)$, they are the characters of the irreducible polynomial representations. As a result, they are an orthonormal integral basis of Λ^1 [21]. Section 3.3 constructs odd analogues of the Schur functions from the combinatorial perspective. These functions are defined by using odd Kostka numbers which count semistandard Young tableaux with signs, a slight generalization of a notion studied by Stanley [29] and others [13,16]. Using results of Reifegerste [24] and Sjöstrand [27] on the behavior of tableau sign under the RSK correspondence, we are able to prove that the odd Schur functions are all orthogonal and of norm ± 1 . In the sequel to this paper [5], we and Aaron Lauda construct the odd Schur functions inside the odd nilHecke ring in terms of odd divided difference operators.

Finally, the Appendix consists of some numerical data related to Λ' and Λ^{-1} .

The definition of Λ^{-1} is just one step beyond the work [30] of Thibon and Ung. It seems that, despite the multitude of papers on noncommutative and quasi-symmetric functions, very little has been written about the quantum case, and we did not find any mention of Λ^{-1} and its interesting structure in the literature.

2. The definition of the q -Hopf algebra Λ

2.1. The category of q -vector spaces and a bilinear form

We work over a commutative ground ring \mathbb{k} . Let Λ' be a free associative \mathbb{Z} -graded \mathbb{k} -algebra with generators h_1, h_2, \dots (it is convenient to assume $h_0 = 1$ and $h_i = 0$ for $i < 0$) of degrees

$\deg(h_n) = n$. The grading gives a vector space decomposition

$$\Lambda' = \bigoplus_{n \geq 0} \Lambda'_n.$$

We choose $q \in \mathbb{k}$ and define a multiplication in $\Lambda'^{\otimes 2}$ by

$$(x_1 \otimes x_2)(y_1 \otimes y_2) = q^{\deg(x_2) \deg(y_1)} x_1 y_1 \otimes x_2 y_2$$

on homogeneous elements. If q is invertible, consider the braided monoidal category $\mathbb{k}\text{-gmod}_q$ of \mathbb{Z} -graded \mathbb{k} -modules with the braiding functor given by

$$V \otimes W \longrightarrow W \otimes V, \quad v \otimes w \longmapsto q^{\deg(v) \deg(w)} w \otimes v$$

on homogeneous elements. The braiding structure is symmetric if $q \in \{1, -1\}$. For q not invertible, the above gives $\mathbb{k}\text{-gmod}_q$ a lax braided structure. Following [1], we refer to bialgebra and Hopf algebra objects in $\mathbb{k}\text{-gmod}_q$ as q -bialgebras and q -Hopf algebras, respectively.

The algebra Λ' can be made into a q -bialgebra by defining the comultiplication on generators to be

$$\Delta(h_n) = \sum_{m=0}^n h_m \otimes h_{n-m}.$$

Given our convention on the h 's, we can also write

$$\Delta(h_n) = \sum_{m \in \mathbb{Z}} h_m \otimes h_{n-m}.$$

The braiding structure implies, for instance, that

$$\Delta(h_n h_k) = \sum_{m,r} (h_m \otimes h_{n-m})(h_r \otimes h_{k-r}) = \sum_{m,r} q^{(n-m)r} h_m h_r \otimes h_{n-m} h_{k-r}.$$

The counit is the obvious one, with $\epsilon(x) = 0$ if $\deg(x) > 0$. The q -bialgebra Λ' is cocommutative if and only if $q = 1$.

For a sequence (or composition) $\alpha = (a_1, \dots, a_k)$, define $h_\alpha = h_{a_1} \dots h_{a_k}$. Let also $|\alpha| = a_1 + \dots + a_k$ and $S_\alpha = S_{a_1} \times \dots \times S_{a_k} \subset S_{|\alpha|}$ be the parabolic subgroup of the symmetric group $S_{|\alpha|}$ associated to the composition α .

For compositions α, β such that $|\alpha| = |\beta| = n$, denote by

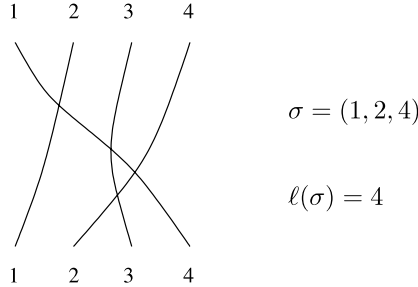
$${}_\beta S_\alpha = S_\beta \setminus S_n / S_\alpha$$

the set of double cosets of the subgroups S_α and S_β in S_n . Each double coset c has a unique minimal length representative $\sigma(c) \in S_n$. Denote by $\ell(c) = \ell(\sigma(c))$ the length of this representative.

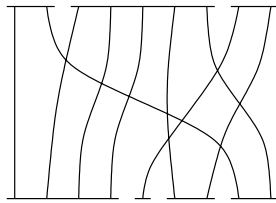
Elements of S_n admit a graphical description. A permutation $\sigma \in S_n$ can be presented by n curves in the plane connecting points $1, \dots, n$ on a horizontal line to points $\sigma(1), \dots, \sigma(n)$ on a parallel line above the former line such that

- the curves have no critical points with respect to the height function (that is, they never flatten out or turn around);
- any two curves intersect at most once (the curves starting at $i < j$ intersect if and only if $\sigma(i) > \sigma(j)$);
- there are no triple intersections.

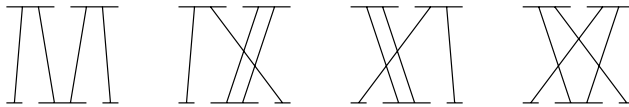
Such diagrams are considered as their combinatorial type, that is, up to rel boundary homotopy through diagrams satisfying the conditions above. If the curves are in general position, then the length $\ell(\sigma)$ is the number of intersection points of these curves, equal to the number of pairs $i < j$ such that $\sigma(i) > \sigma(j)$.



Minimal double coset representatives $\sigma(c)$, $c \in {}_{\beta}S_{\alpha}$ are singled out by the following condition. Draw intervals (or platforms) of “size” a_1, \dots, a_k from left to right at the bottom of the permutation diagram and platforms b_1, \dots, b_r at the top ($\alpha = (a_1, \dots, a_k)$, $\beta = (b_1, \dots, b_r)$), so that the first a_1 lines from the left start off at the first bottom platform, the next a_2 lines at the second platform, and so forth, and likewise for the top platforms. Then $\sigma = \sigma(c)$ for some double coset c if and only if any two lines that start or end in the same platform do not intersect. An example is depicted below, with $\alpha = (4, 1, 2, 2)$, $\beta = (2, 5, 2)$, $\ell(\sigma) = 9$.



If $\alpha = (1, 2, 1)$ and $\beta = (2, 2)$, there are four cosets, with minimal length representatives $1, (2, 3, 4), (1, 3, 2),$ and $(1, 3, 4, 2)$:



Fix $q \in \mathbb{k}$ and define a symmetric \mathbb{k} -bilinear form on Λ' taking values in \mathbb{k} by

$$(h_{\beta}, h_{\alpha}) = \begin{cases} \sum_{c \in {}_{\beta}S_{\alpha}} q^{\ell(c)} & \text{if } |\alpha| = |\beta|, \\ 0 & \text{otherwise.} \end{cases} \tag{2.1}$$

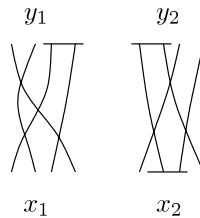
The weight spaces Λ'_n of degree n are pairwise orthogonal relative to this form.

Example 2.1. The inner product $(h_2 h_2, h_1 h_2 h_1) = 1 + 2q^2 + q^3$; see the four diagrams above. Each double coset in ${}_{\beta}S_{\alpha}$ contributes q to the power equal to the number of crossings in the diagram of the coset.

We extend this form to $\Lambda'^{\otimes 2}$ by

$$(y_1 \otimes y_2, x_1 \otimes x_2) = (y_1, x_1)(y_2, x_2). \tag{2.2}$$

One may wonder why the factor $q^{\deg(y_2)\deg(x_1)}$ does not appear in this formula given that y_2 seems to move past x_1 . Powers of q are also absent in [20, Proposition 1.2.3] in a very similar situation. The graphical interpretation of the bilinear form provides a reason: we think of the tensor product of elements as occurring horizontally, while the diagrams used in the computation of the bilinear form occur vertically. The picture below shows a diagram contributing to the inner product $(y_1 \otimes y_2, x_1 \otimes x_2)$ for suitable x_1, x_2, y_1, y_2 . This diagram is a disjoint union of two diagrams, the one on the left contributing to (y_1, x_1) , the right one contributing to (y_2, x_2) .



No strands from distinct tensor factors ever cross, justifying Eq. (2.2). From this point of view, it would be more natural to write $\binom{y}{x}$ rather than (y, x) ; we will not do so for obvious reasons. In this notation, Eq. (2.2) would become

$$\binom{y_1 \otimes y_2}{x_1 \otimes x_2} = \binom{y_1}{x_1} \binom{y_2}{x_2},$$

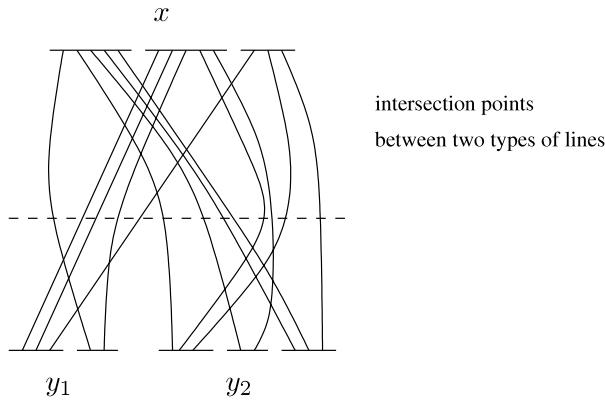
with no change in the relative position of the four variables on the two sides of the equation.

Proposition 2.2. *For all $x, y_1, y_2 \in \Lambda'$,*

$$(y_1 \otimes y_2, \Delta(x)) = (y_1 y_2, x). \tag{2.3}$$

In other words, multiplication and comultiplication are adjoint operators relative to these forms on Λ' and $\Lambda'^{\otimes 2}$.

Proof. It is enough to check the adjointness when x, y_1, y_2 are products of h_n 's. The inner product $(y_1 y_2, x)$ is computed as a sum over diagrams (of double cosets) with platforms at the bottom corresponding to the terms of x and platforms at the top corresponding to those of y_1 followed by those of y_2 . In a given diagram, lines from each platform of x will split into those going into y_1 , respectively y_2 , platforms. These two types of lines will intersect, and the intersection points will contribute powers of q , which are matched by the powers of q in $\Delta(x)$ coming from the definition of multiplication in $\Lambda'^{\otimes 2}$; see the diagram below.



Let $\mathcal{I} \subset \Lambda'$ be the radical of (\cdot, \cdot) . Then $\mathcal{I} = \bigoplus_{n \geq 0} \mathcal{I}_n$, where $\mathcal{I}_n = \mathcal{I} \cap \Lambda'_n$. Define $\Lambda = \Lambda' / \mathcal{I}$ and let Λ_n denote the subspace of elements of Λ which are homogeneous of degree n . \square

To emphasize the dependence on q one can also write Λ^q instead of Λ and Λ_n^q instead of Λ_n . We will use the shorter notation whenever possible.

Proposition 2.3. \mathcal{I} is a q -bialgebra ideal in Λ' :

$$\mathcal{I}\Lambda' = \Lambda'\mathcal{I} = \mathcal{I}, \quad \Delta(\mathcal{I}) \subset \mathcal{I} \otimes \Lambda' + \Lambda' \otimes \mathcal{I}.$$

Proof. These properties of \mathcal{I} follow at once from adjointness of multiplication and comultiplication. \square

Corollary 2.4. Λ inherits a q -bialgebra structure from that of Λ' .

If $q = 0$, the bilinear form degenerates and $\dim(\Lambda_n) = 1$ for all $n \geq 0$.

If $q = 1$, the inner product $(h_\beta, h_\alpha) = |\beta S_\alpha|$ is the number of double cosets, and it coincides with the standard inner product on the bialgebra of symmetric functions $\mathbb{k}[h_1, h_2, \dots]$ in infinitely many variables x_1, x_2, \dots , with h_n being the n -th complete symmetric function. In this case the ideal \mathcal{I} is generated by commutators $[h_n, h_m] = h_n h_m - h_m h_n$ over all n, m , and the bialgebra Λ^1 is the maximal commutative quotient of Λ' . Note that we defined Λ' as a free associative (not commutative) algebra. The bilinear form in the $q = 1$ case forces commutativity but nothing else. Nondegeneracy of the form on the maximal commutative quotient follows from the result that the elements $h_\lambda = h_{\lambda_1} \dots h_{\lambda_r}$ are linearly independent over all partitions λ of n . This is proved by introducing elementary symmetric functions e_n via the inductive relation

$$\sum_{k=0}^n (-1)^k h_k e_{n-k} = 0,$$

defining $e_\lambda = e_{\lambda_1} \dots e_{\lambda_r}$, and then checking that the matrix of the bilinear form is upper-triangular with ones on the diagonal with respect to the bases $\{h_\lambda\}_{\lambda \vdash n}$ and $\{e_{\lambda^T}\}_{\lambda \vdash n}$ for any total order on partitions refining the dominance order, where λ^T is the dual (or transpose) partition of λ .

2.2. *Odd complete and elementary symmetric functions*

From now on, unless stated otherwise, we take $q = -1$. In this case we call $\Lambda = \Lambda^{-1}$ the bialgebra of *odd symmetric functions*. Choosing $q = -1$ makes $\mathbb{k}\text{-gmod}_q$ the category of

\mathbb{Z} -graded super-vector spaces, so that an algebra in $\mathbb{k}\text{-gmod}_{-1}$ is a \mathbb{Z} -graded superalgebra, and likewise for (-1) -bialgebras and (-1) -Hopf algebras. The super-grading is the mod 2 reduction of the \mathbb{Z} -grading.

When $q = -1$, Eq. (2.1) takes the form

$$(h_\beta, h_\alpha) = \begin{cases} \sum_{c \in \beta} (-1)^{\ell(c)} & \text{if } |\alpha| = |\beta|, \\ 0 & \text{otherwise.} \end{cases} \tag{2.4}$$

If $\lambda = (\lambda_1, \dots, \lambda_r)$ is a partition, the product

$$h_\lambda = h_{\lambda_1} h_{\lambda_2} \cdots h_{\lambda_r}$$

is called an *odd complete symmetric function*. By analogy with the even ($q = 1$) case, inductively define elements $e_n \in \Lambda$ by $e_0 = 1$ and

$$\sum_{k=0}^n (-1)^{\binom{k+1}{2}} e_k h_{n-k} = 0. \tag{2.5}$$

Eq. (2.5) is equivalent to the equation

$$\sum_{k=0}^n (-1)^{\binom{k+1}{2}} h_{n-k} e_k = 0.$$

This can be checked by a straightforward calculation or by applying the involution $\psi_1 \psi_2$ of Section 2.3. The *odd elementary symmetric functions* are defined to be products

$$e_\lambda = e_{\lambda_1} e_{\lambda_2} \cdots e_{\lambda_r}$$

for partitions $\lambda = (\lambda_1, \dots, \lambda_r)$.

Define h_α and e_α for a composition $\alpha = (a_1, \dots, a_r)$ similarly, as

$$h_\alpha = h_{a_1} \cdots h_{a_r}, \quad e_\alpha = e_{a_1} \cdots e_{a_r}.$$

We will write $\binom{\alpha}{2} = \binom{a_1}{2} + \cdots + \binom{a_r}{2}$ and call $\ell(\alpha) = r$ the length of the composition α . An easy inductive argument shows that another equivalent definition of e_n is

$$e_n = (-1)^{\binom{n+1}{2}} \sum_{|\alpha|=n} (-1)^{\ell(\alpha)} h_\alpha. \tag{2.6}$$

The sum is over all 2^{n-1} compositions α of n . Observe that

$$(-1)^{\binom{\lambda}{2} + |\lambda|} (-1)^{\lambda_2^T + \lambda_4^T + \lambda_6^T + \cdots} = (-1)^{|\lambda|}$$

for any partition λ . Here, λ_i^T means the i -th row length of λ^T (equivalently, the i -th column height of λ). This will be useful later in studying odd Schur functions.

Eq. (2.5) can be solved for e_n in terms of h_1, h_2, \dots, h_n , so (2.5) makes sense as a definition of e_n . The first few e_n 's are

$$\begin{aligned} e_1 &= h_1, \\ e_2 &= h_2 - h_1^2, \\ e_3 &= h_3 - h_1^3, \end{aligned}$$

$$e_4 = -h_4 + h_2^2 - h_2h_1^2 + h_1^4,$$

$$e_5 = h_5 - 2h_4h_1 - h_3h_1^2 + h_2^2h_1 + h_1^5.$$

Since Eq. (2.5) also allows one to solve for h_n in terms of e_1, e_2, \dots, e_n , any element of Λ is a linear combination of words in the e_k . It is convenient to set $e_k = 0$ for $k < 0$.

Proposition 2.5. *We have the following.*

1. *The comultiplication on e_n is*

$$\Delta(e_n) = \sum_{k=0}^n e_k \otimes e_{n-k}. \tag{2.7}$$

2. *If α is a composition of n , then*

$$(h_\alpha, e_n) = \begin{cases} 1 & \text{if } \alpha = (1, 1, \dots, 1), \\ 0 & \text{otherwise.} \end{cases} \tag{2.8}$$

Since (\cdot, \cdot) is nondegenerate and Λ is finite dimensional in each degree, property (2.8) uniquely characterizes the elements $e_n \in \Lambda$.

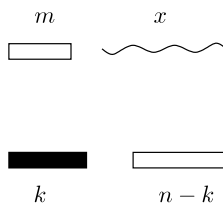
Proof. We prove both statements by simultaneous induction on n , the cases $n = 0, 1$ being clear. To prove the second statement, it suffices to prove

$$(h_mx, e_n) = \begin{cases} (x, e_{n-1}) & \text{if } m = 1, \\ 0 & \text{otherwise} \end{cases}$$

for any $x \in \Lambda$. First, a calculation: for $k < n$, the inductive hypothesis implies

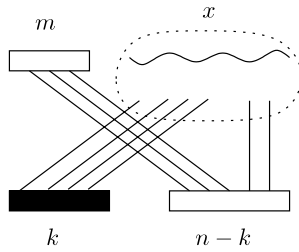
$$(h_mx, e_k h_{n-k}) = (-1)^{km} (x, e_k h_{n-k-m}) + (-1)^{(k-1)(m-1)} (x, e_{k-1} h_{n-k-m+1}). \tag{2.9}$$

To derive this equation, it is useful to first extend the bilinear form diagrammatics of Section 2.1: we represent h_n 's by white platforms of size n and e_n 's by black platforms of size n . Now we start proving (2.9) by drawing $e_k h_{n-k}$ below h_mx .

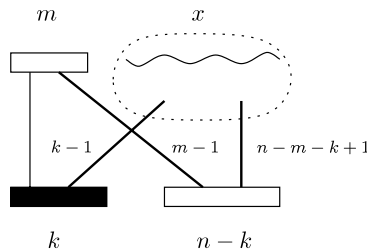


Strictly speaking, a diagram containing a black platform representing e_k stands for a linear combination of diagrams in which the black platform is replaced by groups of white platforms which come from writing e_k as a linear combination of h_α 's.

By the inductive hypothesis applied to $k < n$, at most one line can connect the bottom left black platform of width k (representing e_k) with the top left white platform of width m (representing h_m). If no lines connect these two platforms, all lines from h_m will be connected to h_{n-k} (necessarily requiring $n - k \geq m$), while all lines from e_k will go into x , creating km intersection points that contribute $(-1)^{km}$; see below. The contribution from these diagrams will total $(-1)^{km} (x, e_k h_{n-k-m})$. The dotted curve in the figure below encloses the area producing the factor $(x, e_k h_{n-k-m})$.



If one line connects the e_k with the h_m platform, the remaining $k - 1$ lines from the black platform go into x , while $m - 1$ lines from h_m enter h_{n-k} . These two types of lines intersect and contribute $(-1)^{(k-1)(m-1)}$ to the sum. In the diagram below we denote each of these bunches of “parallel” lines by a single line labeled $k - 1$, respectively $m - 1$. The dotted curve below encloses the area contributing the factor $(x, e_{k-1}h_{n-m-k+1})$.



This computation proves (2.9). Therefore

$$\begin{aligned}
 -(-1)^{\binom{n+1}{2}}(h_m x, e_n) &\stackrel{(2.5)}{=} \sum_{k=0}^{n-1} (-1)^{\binom{k+1}{2}}(h_m x, e_k h_{n-k}) \\
 &\stackrel{(2.9)}{=} \sum_{k=0}^{n-1} (-1)^{\binom{k+1}{2}} [(-1)^{km}(x, e_k h_{n-k-m}) + (-1)^{(k-1)(m-1)} \\
 &\quad \times (x, e_{k-1} h_{n-k-m+1})] \\
 &= (-1)^{\binom{n}{2}+m(n-1)}(x, e_{n-1} h_{1-m}).
 \end{aligned}$$

The third equality follows because all terms but one cancel in pairs. Since $h_i = 0$ for $i < 0$, the second statement follows. The first statement follows from the second, since adjointness of multiplication and comultiplication implies (recall that the bilinear form is symmetric)

$$(\Delta(e_n), h_\lambda \otimes h_\mu) = (e_n, h_\lambda h_\mu) = \begin{cases} 1 & \lambda = (1^k), \mu = (1^\ell), k + \ell = n, \\ 0 & \text{otherwise.} \end{cases} \quad \square$$

The proposition implies that unlike the h_n 's, the e_n 's do not all have norm 1. The elements $e_0 = 1$ and $e_1 = h_1$ both have norm 1, and

$$\begin{aligned}
 -(-1)^{\binom{n+1}{2}}(e_n, e_n) &= \sum_{k=0}^{n-1} (-1)^{\binom{k+1}{2}}(e_n, e_k h_{n-k}) \quad (2.5) \\
 &= (-1)^{\binom{n}{2}}(e_n, e_{n-1} h_1) \quad (2.8) \\
 &= (-1)^{\binom{n}{2}}(\Delta(e_n), e_{n-1} \otimes h_1) \\
 &= (-1)^{\binom{n}{2}}(e_{n-1}, e_{n-1}) \quad (2.7), (2.8).
 \end{aligned}$$

Solving the resulting recurrence relation, we find

$$(e_n, e_n) = (-1)^{\binom{n}{2}}. \tag{2.10}$$

The bilinear form can be evaluated on products of h 's and e 's with the help of diagrammatics as follows (these diagrammatics parallel the graphical calculus developed in [4] in the even case). Let $h_n^+ = h_n$ and $h_n^- = e_n$. Let $\alpha = (a_1, \dots, a_r)$, $\beta = (b_1, \dots, b_s)$ be compositions of n and let ϵ, η be tuples of signs of lengths r, s respectively. We want to compute

$$(h_\beta^\eta, h_\alpha^\epsilon) = (h_{b_1}^{\eta_1} \dots h_{b_s}^{\eta_s}, h_{a_1}^{\epsilon_1} \dots h_{a_r}^{\epsilon_r}).$$

In a rectangular region, draw platforms of widths a_1, \dots, a_r along the bottom and of widths b_1, \dots, b_s along the top. Color a platform white (respectively, black) if its corresponding sign is 1 (respectively, -1). Then connect platforms by strands subject to the following rules.

- As described in the previous subsection, a platform of width k has k strands attached to it, and strands are generic curves (no height critical points, and no triple intersections).
- The depicted permutation is a minimal double coset representative for $S_\beta \setminus S_n / S_\alpha$.
- For all black platforms P and white platforms P' , there is at most one strand connecting P to P' .
- Such diagrams are considered up to identification in S_n . That is, diagrams are considered modulo rel boundary homotopy through generic diagrams, and Reidemeister III moves.

Let $\text{diag}(\beta, \alpha, \eta, \epsilon)$ be the set of all such diagrams, up to the described equivalence. To each diagram $D \in \text{diag}(\beta, \alpha, \eta, \epsilon)$ representing a permutation $\sigma \in S_n$, assign a sign ± 1 in the following way.

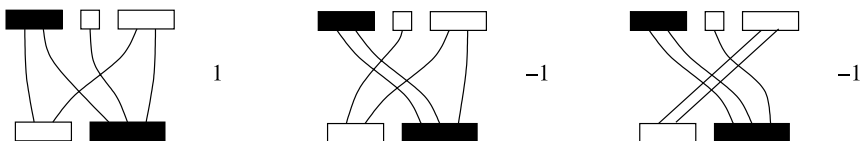
- Assign a sign $(-1)^{\ell(\sigma)}$, where $\ell(\sigma)$ is the Coxeter length of σ . That is, $\ell(\sigma)$ equals the number of crossings in the minimal double coset representative diagram; equivalently, $\ell(\sigma)$ is the number of pairs $i < j$ such that $\sigma(i) > \sigma(j)$.
- For each pair of black platforms, assign a sign $(-1)^{\binom{k}{2}} = (-1)^{\frac{1}{2}k(k-1)}$, where k is the number of strands connecting these two platforms (by Eq. (2.10)).

For each diagram $D \in \text{diag}(\beta, \alpha, \eta, \epsilon)$, let $\text{sign}(D)$ be this sign (the product of the two factors just described). The results above imply the following.

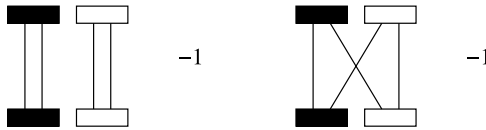
Proposition 2.6. *The product $(h_\beta^\eta, h_\alpha^\epsilon)$ is given by*

$$(h_\beta^\eta, h_\alpha^\epsilon) = \sum_{D \in \text{diag}(\beta, \alpha, \eta, \epsilon)} \text{sign}(D).$$

Example 2.7. Consider the product $(e_2 h_1 h_2, h_2 e_3) = 1 - 1 - 1 = -1$. The three contributing diagrams and their signs are shown below. Each diagram has even number of crossings, and the nontrivial signs come from having two black boxes connected by a pair of lines in the second and third diagrams.



Example 2.8. Consider the product $(e_2h_2, e_2h_2) = -2$. The two contributing diagrams and their signs are given below.



An equivalent description of the sign is as follows: for each pair of black platforms connected by k strands, introduce the longest element of S_k among those k strands. The sign of the diagram is then obtained just by counting crossings. So if the diagram has all white platforms, a minimal coset representative is still used. If it has all black platforms, a maximal coset representative diagram is used instead. If platforms of both colors are used, the double coset representative chosen is neither minimal nor maximal in general, but is chosen as above.

White and black boxes of size one represent the same element $h_1 = e_1$ of Λ .

Proposition 2.9. When $a + b$ is even,

$$h_a h_b = h_b h_a. \tag{2.11}$$

When $a + b$ is odd,

$$h_a h_b + (-1)^a h_b h_a = (-1)^a h_{a+1} h_{b-1} + h_{b-1} h_{a+1}. \tag{2.12}$$

When $b = 1$ and $a = 2k$ is even, the odd degree relation takes the form

$$h_1 h_{2k} + h_{2k} h_1 = 2h_{2k+1}.$$

Proof. We prove both relations simultaneously by induction on the total degree $a + b$. First suppose that $a + b$ is even. It suffices to prove $(h_a h_b - h_b h_a, e_k x) = 0$ for $k = 1, 2$ since the products $(h_a h_b, e_k x)$ and $(h_b h_a, e_k x)$ equal zero for any $k > 2$ by Proposition 2.5. Computing graphically,

$$\begin{aligned} (h_a h_b, e_1 x) &= (h_{a-1} h_b, x) + (-1)^a (h_a h_{b-1}, x), \\ (h_a h_b, e_2 x) &= (-1)^{a-1} (h_{a-1} h_{b-1}, x), \end{aligned} \tag{2.13}$$

and likewise for $(h_b h_a, e_k x)$. When $k = 1$ the difference $(h_a h_b - h_b h_a, e_1 x)$ vanishes by the odd degree relation in degree $a + b - 1$, and when $k = 2$ the difference vanishes by the even degree relation in degree $a + b - 2$.

For $a + b$ odd, put together terms for $h_a h_b$, $(-1)^a h_b h_a$, $(-1)^a h_{a+1} h_{b-1}$, and $h_{b-1} h_{a+1}$ as in Eq. (2.13). The result vanishes with $k = 1$ by some cancellation and the even degree relation in degree $a + b - 1$, and with $k = 2$ by the odd degree relation in degree $a + b - 2$. \square

Since any element of Λ is a linear combination of words in the e_k 's as well as a linear combination of words in the h_k 's, the same argument with these families of elements switched proves the following.

Proposition 2.10. When $a + b$ is even,

$$e_a e_b = e_b e_a. \tag{2.14}$$

When $a + b$ is odd,

$$e_a e_b + (-1)^a e_b e_a = (-1)^a e_{a+1} e_{b-1} + e_{b-1} e_{a+1}. \tag{2.15}$$

There are similar relations involving both h 's and e 's.

Proposition 2.11. *When $a + b$ is even,*

$$h_a e_b = e_b h_a. \tag{2.16}$$

When $a + b$ is odd,

$$h_a e_b + (-1)^a e_b h_a = (-1)^a h_{a+1} e_{b-1} + e_{b-1} h_{a+1}. \tag{2.17}$$

Proof. The proof is along the same lines as that of Proposition 2.9, but with the slight complication that terms with $k > 2$ do contribute. As in that proof, we prove both relations simultaneously by induction on the total degree $a + b$.

For $a + b$ even, we compute

$$\begin{aligned} (h_a e_b, e_k x) &= (h_{a-k} e_b + (-1)^{b-k+1} h_{a-k+1} e_{b-1}, x), \\ (e_b h_a, e_k x) &= ((-1)^{kb} e_b h_{a-k} + (-1)^{(k-1)(b-1)} e_{b-1} h_{a-k+1}, x). \end{aligned}$$

We want to show that the difference of the two left arguments on the right-hand side is zero. For k even, this difference vanishes by applying (2.16) twice in degree $a + b - k$. For k odd, it vanishes by applying (2.17).

For $a + b$ odd, we compute

$$\begin{aligned} (h_a e_b, h_k x) &= (h_{a-k} e_b + (-1)^{a-k+1} h_{a-k+1} e_{b-1}, x), \\ (e_b h_a, h_k x) &= ((-1)^{kb} e_b h_{a-k} + (-1)^{(k-1)(b-1)} e_{b-1} h_{a-k+1}, x), \\ (h_{a+1} e_{b-1}, h_k x) &= (h_{a-k+1} e_{b-1} + (-1)^{a-k} h_{a-k+2} e_{b-2}, x), \\ (e_{b-1} h_{a+1}, h_k x) &= ((-1)^{k(b+1)} e_{b-1} h_{a-k+1} + (-1)^{(k-1)b} e_{b-2} h_{a-k+2}, x). \end{aligned}$$

Considering the linear combination $h_a e_b + (-1)^a e_b h_a$, the argument which is paired with x is

$$h_{a-k} e_b - (-1)^{(k+1)b} e_b h_{a-k} + (-1)^{b-k} h_{a-k+1} e_{b-1} + (-1)^{k(b-1)} e_{b-1} h_{a-k+1}. \tag{2.18}$$

For the linear combination $h_{a+1} e_{b-1} + (-1)^a e_{b-1} h_{a+1}$, we get

$$\begin{aligned} &(-1)^{k(b+1)} e_{b-1} h_{a-k+1} + (-1)^{b+1} h_{a-k+1} e_{b-1} + (-1)^{(k-1)b} e_{b-2} h_{a-k+2} \\ &+ (-1)^k h_{a-k+2} e_{b-2}. \end{aligned} \tag{2.19}$$

For k even, the difference of expressions (2.18) and (2.19) vanishes by applying (2.17) twice in degree $a + b - k$. For k odd, the difference vanishes by applying (2.16) twice. \square

Corollary 2.12. Λ_n is a free \mathbb{k} -module, of which the families $\{h_\lambda\}_{\lambda \vdash n}$ and $\{e_\lambda\}_{\lambda \vdash n}$ are both bases. Hence Λ_n has the same graded rank as in the even ($q = 1$) case, namely the number of partitions of n .

Proof. Any element of Λ is a linear combination of words in the h 's. By the relations (2.11) and (2.12), only words whose subscripts are in non-increasing order are needed; that is, $\{h_\lambda\}_{\lambda \vdash n}$ is a spanning set. Now let $\Lambda_{\mathbb{Z}}$ be Λ considered over $\mathbb{k} = \mathbb{Z}$. Since it is expressed as the quotient of a free \mathbb{Z} -module by the radical of a bilinear form, $\Lambda_{\mathbb{Z}}$ is itself a free \mathbb{Z} -module. The mod 2 reduction of h_λ coincides with the mod 2 reduction of the even ($q = 1$) complete symmetric function h_λ^{even} , so the spanning set $\{h_\lambda\}_{\lambda \vdash n}$ is linearly independent in $\Lambda_{\mathbb{Z}/2}$, hence in $\Lambda_{\mathbb{Z}}$. The same argument works for the family $\{e_\lambda\}_{\lambda \vdash n}$. Now $\Lambda_{\mathbb{Z}}$ is a free \mathbb{Z} -module with the required bases, so $\Lambda = \Lambda_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{k}$ is a free \mathbb{k} -module with the required bases. \square

Corollary 2.13. *The algebra Λ can be presented by generators $h_0 = 1, h_1, h_2 \dots$ subject to defining relations (2.11) and (2.12). There is an algebra automorphism of Λ which takes h_n to e_n for all n .*

Proof. The first statement follows immediately from Proposition 2.9 and Corollary 2.12. The second follows from Proposition 2.10 and the fact that any word in the h_n 's can be expressed as a word in the e_n 's (by Eq. (2.5)). \square

The automorphism $h_n \mapsto e_n$ is later denoted ψ_1 .

An alternate argument deduces that $\{e_\lambda\}_{\lambda \vdash n}$ is a basis of $(\Lambda_{\mathbb{Z}})_n$ from the fact that $\{h_\lambda\}_{\lambda \vdash n}$ is and the following “semi-orthogonality” property. We call a minimal coset representative *lite* if any two platforms in its diagram are connected by at most one strand.

Proposition 2.14. 1. *For all partitions λ ,*

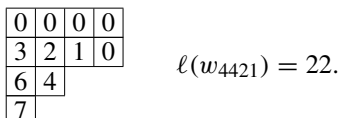
$$(h_\lambda, e_{\lambda^T}) = (-1)^{\ell(w_\lambda)}, \tag{2.20}$$

where $\ell(w_\lambda)$ is the Coxeter length of the unique lite minimal double coset representative w_λ in $S_\lambda \setminus S_n/S_{\lambda^T}$.

2. *If λ is a partition and α is a composition with $\alpha > \lambda^T$ in the lexicographic order, then*

$$(h_\lambda, e_\alpha) = (e_\lambda, h_\alpha) = 0. \tag{2.21}$$

A combinatorial description of $\ell(w_\lambda)$ is the number of strictly southwest–northeast pairs of boxes in the Young diagram corresponding to the partition λ . One way to compute this quickly is to label each box of the Young diagram corresponding to λ with the number of boxes which are strictly to the northeast. Then $\ell(w_\lambda)$ is the sum of these numbers. For example,



Before proving the proposition, we briefly recall the lexicographic order and the dominance partial order. Let $\alpha = (a_1, \dots, a_r)$ and $\beta = (b_1, \dots, b_s)$ be compositions of n . We say $\alpha < \beta$ in the *lexicographic order* (also called the *dictionary order*) if $a_i < b_i$, where i is the minimal index j at which $a_j \neq b_j$.

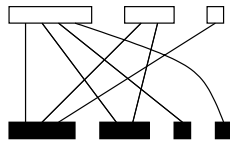
Restricting our attention to partitions $\lambda = (\lambda_1, \dots, \lambda_r), \mu = (\mu_1, \dots, \mu_s)$ of n , the lexicographic order refines the following partial order: we say $\lambda \leq \mu$ in the *dominance partial order* if

$$\lambda_1 + \lambda_2 + \dots + \lambda_i \leq \mu_1 + \mu_2 + \dots + \mu_i \quad \text{for all } i = 1, 2, \dots, n,$$

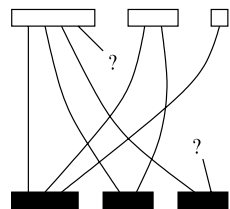
where we pad λ, μ by trailing zeros when necessary. The dominance partial order is a total order if and only if $n \leq 5$ and is graded if and only if $n \leq 6$. The lowest degree dominance-incomparable pairs are $\{(3, 1, 1, 1), (2, 2, 2)\}$ and $\{(4, 1, 1), (3, 3)\}$.

If all partitions of n are listed lexicographically, it is not true that reversing the order swaps partitions whose corresponding Young diagrams are transposes of each other (this first occurs at $n = 6$). One can, however, refine the dominance partial order in such a way that this property holds.

Proof of Proposition 2.14. The proof, except for the determination of the sign in Eq. (2.20), is exactly as in the case of classical symmetric functions. To compute an inner product (h_λ, e_α) , we must sign-count minimal double coset representative diagrams connecting a λ -arrangement of white platforms and a α -arrangement of black platforms such that no pair of a black and a white diagram are connected by more than one strand. For $\alpha = \lambda^T$, α_i equals the number of rows of λ of size at least i ; diagrammatically, the i -th black platform has exactly one strand going to each white platform of size at least i . Hence there is a unique lite diagram connecting these two platform arrangements and it counts as $(-1)^{\ell(w)}$: in this diagram, the strands of the i -th black platform go to the first α_i white platforms, and these white platforms are precisely those white platforms which accept at least i strands, and vice versa, switching black and white and switching λ and α . For example, the unique lite diagram in computing (h_{421}, e_{3211}) is



For $\alpha > \lambda^T$, let i be minimal such that $\alpha_i > \lambda_i^T$. Then filling a potential diagram as above, at the stage of connecting the i -th black platform, there are fewer than α_i white platforms which can still accept a new strand, so we are forced to send two strands from this black platform to the same white platform (the pigeonhole principle). So the diagram is zero. For example, consider the next step in filling the unfinished diagram below for $\lambda = (4, 2, 1)$, $\alpha = (3, 2, 2)$:



Connecting the strands marked “?” would result in a non-lite diagram. \square

For the remainder of this subsection, let $\mathbb{k} = \mathbb{Z}$. Recall the representation theoretic interpretation of the even ($q = 1$) analogue of Proposition 2.14: view Λ as

$$K_0(S) = \bigoplus_{n \geq 0} K_0(\mathbb{C}[S_n])$$

via the Frobenius characteristic map. Since the algebra $\mathbb{C}[S_n]$ is semisimple, its usual and split Grothendieck groups are isomorphic; the simples and the indecomposables coincide, and they are all projective. The multiplication and comultiplication in $K_0(S)$ come from induction and restriction between parabolic subgroups, and the bilinear form is determined by

$$([V], [W]) = \dim \text{Hom}_{S_n}(V, W)$$

when V, W are representations of S_n . Under this identification,

$$h_\lambda \text{ corresponds to } [I_\lambda] = [\text{Ind}_{S_\lambda}^{S_n}(L_{(n)})],$$

$$e_\lambda \text{ corresponds to } [I_\lambda^-] = [\text{Ind}_{S_\lambda}^{S_n}(L_{(1^n)})],$$

where $|\lambda| = n$, S_λ is the parabolic subgroup $S_{\lambda_1} \times \cdots \times S_{\lambda_r} \subseteq S_n$, and L_λ is the irreducible representation of S_n corresponding to the partition λ . In particular, $L_{(n)}$ is the one-dimensional trivial representation and $L_{(1^n)}$ is the one-dimensional sign representation. The $q = 1$ analogue of the first statement of the proposition (which is that $(h_\lambda, e_{\lambda^T}) = 1$) follows from the fact that the induced representations I_λ and $I_{\lambda^T}^-$ share a unique irreducible summand, which occurs with multiplicity 1; this is the irreducible representation L_λ . The $q = 1$ analogue of the second statement follows from the absence of common irreducible summands in certain induced representations; this can be used to prove that all the L_λ 's constructed in this way are distinct. We do not know an analogous representation theoretic interpretation (categorification) of the odd ($q = -1$) case. As the sign in the first statement of the [Proposition 2.14](#) makes evident, dg-algebras or similar structures will likely be necessary to categorify Λ .

Introduce the notation

$$H_{\geq \lambda} = \text{span}_{\mathbb{Z}}\{h_\mu : \mu \geq \lambda\},$$

$$E_{\geq \lambda} = \text{span}_{\mathbb{Z}}\{e_\mu : \mu \geq \lambda\}$$

and likewise with \geq replaced by one of $\{\leq, >, <\}$ (lexicographic order), as subspaces of Λ_n for $n = |\lambda|$.

The following lemma will be useful in studying odd Schur functions.

Lemma 2.15. *For any partition $\lambda \vdash n$, the bilinear form is nondegenerate when restricted to the subspaces $H_{\geq \lambda}$ and $E_{> \lambda^T}$ of Λ_n .*

Proof. By Eq. (2.21) and nondegeneracy of the bilinear form, $(H_{\geq \lambda})^\perp = E_{> \lambda^T}$. If $H_{\geq \lambda} \cap E_{> \lambda^T} = \{0\}$, then it follows that

$$\Lambda_n = H_{\geq \lambda} \oplus E_{> \lambda^T}$$

is an orthogonal decomposition, so that

$$\det(\cdot, \cdot)|_{H_{\geq \lambda}} \det(\cdot, \cdot)|_{E_{> \lambda^T}} = \det(\cdot, \cdot) = \pm 1,$$

which implies that both factors on the left hand side are ± 1 . And since $H_{\geq \lambda} \cap E_{> \lambda^T} = \{0\}$ after reducing mod 2, the intersection must have been zero over \mathbb{Z} : any nonzero element of $H_{\geq \lambda} \cap E_{> \lambda^T}$ which is zero mod 2 must be divisible by 2. But then the result of dividing this element by 2 would also be in $H_{\geq \lambda} \cap E_{> \lambda^T}$. \square

The previous lemma does not hold with \geq replaced by \leq . For instance, $(h_{11}, h_{11}) = 0$.

2.3. (Anti-)automorphisms, generating functions, and the antipode

If α is a composition, we write α^{rev} for the composition obtained by reverse-ordering α . We also use this notation for partitions λ (though, of course, λ^{rev} is rarely a partition).

We introduce three (anti-)automorphisms of Λ which will be of use to us. Their definitions and basic properties are the following.

$$\psi_1(h_n) = e_n$$

bialgebra automorphism (not an involution),

$$\psi_2(h_n) = (-1)^{\binom{n+1}{2}} h_n$$

algebra involution (not a coalgebra homomorphism),

$$\psi_3(h_n) = h_n$$

algebra anti-involution (not a coalgebra homomorphism).

All three of these maps lift to Λ' at $q = -1$. By Proposition 2.10 and Corollary 2.13, ψ_1 is a well defined algebra automorphism. Since the lifts of ψ_2 and ψ_3 to Λ' preserve the defining relations (2.11) and (2.12), ψ_2 and ψ_3 are themselves algebra automorphisms as well.

Perhaps, the only one of the ψ_i whose introduction requires comment is ψ_2 . It is useful because the h_k 's and the $\psi_2(e_k)$'s satisfy a family of relations analogous to familiar ones in the even case:

$$\sum_{k=0}^n (-1)^{k(n-k)} \psi_2(e_{n-k}) h_k = 0. \tag{2.22}$$

In order to see the difference in meaning between this equation and Eq. (2.5), define the generating functions $H(t) = \sum_t h_k t^k$ and $E(t) = \sum_t e_k t^k$, with t a variable of degree 1. Then Eq. (2.22) is equivalent to the equation

$$\psi_2(E(t))H(t) = 1 \tag{2.23}$$

holding in the ring $\Lambda[t]$. The meaning here is that t commutes (respectively, skew commutes) with h_k and e_k for k even (respectively, odd); that is, we adjoin a super-central variable t and extend ψ_2 to $\Lambda[t]$ by $\psi_2(t) = t$. We do not know of a generating function interpretation of Eq. (2.5).

Another interpretation of ψ_2 is that there is a $\mathbb{Z}/2$ -grading on Λ determined by placing h_n in degree 0 (respectively, degree 1) if $n \equiv 0, 3 \pmod{4}$ (respectively, $n \equiv 1, 2 \pmod{4}$). In terms of this grading, ψ_2 is the identity on the degree 0 part and minus the identity on the degree 1 part.

By definition, $\psi_3(h_n) = h_n$. The characterization of e_n in Eq. (2.8) and the fact that ψ_3 is norm preserving imply $\psi_3(e_n) = e_n$. Of course, this does not extend to other partitions λ . For instance,

$$\psi_3(h_2 h_1) = h_1 h_2 = 2h_3 - h_2 h_1.$$

But ψ_3 does preserve e_λ, h_λ up to “higher order terms”.

Lemma 2.16. $\psi_3(h_\lambda)$ is in $H_{\geq \lambda}$, and the coefficient of h_λ in $\psi_3(h_\lambda)$ (when expanding in the complete basis) is $\text{sign}(T_\lambda)(-1)^{\binom{\lambda}{2}}$. This sign can also be computed as follows: write the row lengths of λ in reverse order. In permuting these to get λ again, accrue a -1 each time an odd number on the left is transposed with an even number on the right. Furthermore, the same holds with h, H replaced by e, E and “complete” replaced by “elementary” (no transpositions of diagrams are necessary).

Proof. Consider all compositions of a fixed degree to be ordered lexicographically. By induction, then, it suffices to show that whenever $a < b$, $h_a h_b$ is in $H_{\geq (b,a)}$. If $a + b$ is even, then $h_a h_b = h_b h_a$ and we are done. If $a + b$ is odd, apply the odd degree h -relation:

$$h_a h_b = (-1)^a h_b h_a + h_{b+1} h_{a-1} - (-1)^a h_{a-1} h_{b+1}.$$

The first and second terms on the right-hand side are now lexicographically greater than $h_a h_b$ and in non-increasing order, so it remains to express $h_{a-1} h_{b+1}$ as a linear combination of terms lexicographically higher. To do so, apply the odd degree h -relation to $h_{a-1} h_{b+1}$, and then to the

last term in that, and so forth until the left factor’s subscript reaches one. At this point apply $h_1 h_{a+b-1} = 2h_{a+b} - h_{a+b-1} h_1$, and we are done. Going through the algorithm just described, the sign of the h_λ term in $\psi_3(h_\lambda)$ is clearly as in the statement of lemma. \square

The anti-involution ψ_3 commutes with ψ_1 and ψ_2 , but ψ_1 and ψ_2 do not commute with each other. The automorphism ψ_1 has infinite order, as

$$\psi_1^m(h_2) = h_2 - mh_1^2.$$

However, $\psi_1\psi_2$ and $\psi_1\psi_2\psi_3$ do square to the identity.

Proposition 2.17. *Let $S = \psi_1\psi_2\psi_3$. Then with the (co)multiplication and (co)unit already defined and with S as antipode, Λ has the structure of an involutory \mathbb{Z} -graded Hopf superalgebra, that is, a Hopf algebra object in the category $\mathbb{k}\text{-gmod}_{-1}$ with $S^2 = 1$. The \mathbb{Z} -grading is compatible in the sense that the super-grading is just the mod 2 reduction of the \mathbb{Z} -grading. We have*

$$\begin{aligned} \psi_1\psi_2(h_\lambda) &= (-1)^{\binom{\lambda}{2}+|\lambda|} e_\lambda, & \psi_1\psi_2(e_\lambda) &= (-1)^{\binom{\lambda}{2}+|\lambda|} h_\lambda, \\ S(h_\lambda) &= (-1)^{\binom{\lambda}{2}+|\lambda|} e_{\lambda^{\text{rev}}}, & S(e_\lambda) &= (-1)^{\binom{\lambda}{2}+|\lambda|} h_{\lambda^{\text{rev}}}. \end{aligned} \tag{2.24}$$

In the setting of ordinary vector spaces, if a bialgebra admits a Hopf antipode then this antipode is unique. The same is true in more general settings including ours; see [22].

Proof. Letting η be the unit and ϵ the counit, $(\eta \circ \epsilon)(h_\lambda) = \delta_{\lambda,(0)}$. And

$$(m \circ (S \otimes 1) \circ \Delta)(h_\lambda) = \prod_{i=1}^{\ell(\lambda)} \left(\sum_{k=0}^{\lambda_i} (-1)^{\binom{k+1}{2}} e_k h_{\lambda_i-k} \right) = \prod_{i=1}^{\ell(\lambda)} \delta_{\lambda_i,0} = \delta_{\lambda,(0)}$$

as well. The same is true for $m \circ (1 \otimes S) \circ \Delta$, since $\Delta(h_n)$ is invariant under the map which swaps its tensor factors (without factors of q). So S is the Hopf antipode.

The expressions for $\psi_1\psi_2(h_\lambda)$ and $S(h_\lambda)$ in Eq. (2.24) are immediate from the definitions of ψ_1, ψ_2, ψ_3 and the above calculation. In order to compute $\psi_1\psi_2(e_n)$ and $S(e_n)$, we proceed by induction (the $n = 1$ case is clear). Applying ψ_3 to Eq. (2.5), we have

$$\sum_{k=0}^n (-1)^{\binom{k+1}{2}} h_{n-k} e_k = 0. \tag{2.25}$$

Hence

$$(-1)^{\binom{n+1}{2}} \psi_1\psi_2\psi_3(e_n) = (-1)^{\binom{n+1}{2}} \psi_1\psi_2(e_n) \tag{2.26}$$

$$\stackrel{(2.5)}{=} - \sum_{k=0}^{n-1} (-1)^{\binom{k+1}{2}} \psi_1\psi_2(e_k h_{n-k}) \tag{2.27}$$

$$= - \sum_{k=0}^{n-1} (-1)^{\binom{n-k+1}{2}} h_k e_{n-k} \tag{2.28}$$

$$\stackrel{(2.25)}{=} h_n. \tag{2.29}$$

The third equality is by the inductive hypothesis. Since $\psi_1\psi_2$ and S are (anti-)homomorphisms, this immediately generalizes to prove the expressions for $\psi_1\psi_2(e_\lambda)$ and $S(e_\lambda)$ in Eq. (2.24). \square

Corollary 2.18. We have $\psi_2\psi_1\psi_2 = \psi_1^{-1}$ and $\psi_1\psi_2\psi_1 = \psi_2$.

Proof. Immediate from the preceding proposition. \square

Let $\text{SAut}(\Lambda)$ be the $\mathbb{Z}/2$ -graded group of algebra automorphisms and anti-automorphisms of Λ . It follows from the above that the subgroup of $\text{SAut}(\Lambda)$ generated by ψ_1, ψ_2, ψ_3 is

$$\langle \psi_i : i = 1, 2, 3 \rangle \cong (\mathbb{Z}/2) \times (\mathbb{Z}/2 * \mathbb{Z}/2).$$

The first factor is generated by ψ_3 and the factors of the free product have generators $\psi_1\psi_2$ and ψ_2 .

2.4. Relation to quantum quasi-symmetric functions

The ring $Q\Lambda_q$ of quantum quasi-symmetric functions, introduced in [30], is a noncommutative deformation of Gessel’s quasi-symmetric functions [7] whose definition uses Rosso’s quantum shuffle product [25]. There is a basis of $Q\Lambda_q$ known as the basis of ribbon Schur functions. In degree n , the ribbon Schur functions are indexed by compositions α of n and are denoted R_α .

We now recall some combinatorial notions. For a permutation $\sigma \in S_n$, we say a number $k \in \{1, 2, \dots, n - 1\}$ is a descent if $\sigma(k) > \sigma(k + 1)$. In terms of the strands diagrams of Section 2.1, k is a descent of σ if and only if the k -th and $(k + 1)$ -st strands cross in a generic diagram for σ . The associated descent composition $C(\sigma) = (i_1, \dots, i_r)$ is the composition of n such that the set of descents of σ is $\{i_1, i_1 + i_2, \dots, i_1 + \dots + i_r\}$. In other words, the first i_1 strands do not cross, the next i_2 strands do not cross, and so forth, and strands i_1 and $i_1 + 1$ must cross, strands $i_1 + i_2$ and $i_1 + i_2 + 1$ must cross, and so forth. The bilinear form on $Q\Lambda_q$ is given by

$$(R_\beta, R_\alpha) = \sum_{\substack{C(\sigma)=\alpha \\ C(\sigma^{-1})=\beta}} q^{\ell(\sigma)} \tag{2.30}$$

(sum over $\sigma \in S_n$, where $n = |\alpha| = |\beta|$). This formula appears as Eq. (39) of [30] and as an unlabeled equation near the end of Section 10.15 of [1]. In the latter reference, $Q\Lambda_q$ is defined in a more abstract manner, as the graded q -Hopf algebra associated via the bosonic Fock functor to the species of linear set compositions. The dual species gives rise to a q -Hopf algebra $N\Lambda_q$ which is graded dual to $Q\Lambda_q$. For q not an algebraic integer, $Q\Lambda_q$ and $N\Lambda_q$ are isomorphic [1, Proposition 12.38].

Define elements \tilde{h}_α of Λ' by

$$\tilde{h}_\alpha = \sum_{\beta \leq \alpha} (-1)^{\ell(\alpha) - \ell(\beta)} h_\beta. \tag{2.31}$$

In the above, for two compositions α, β of n , we say $\beta \leq \alpha$ if α refines β . Note that the change of basis $h_\alpha \mapsto \tilde{h}_\alpha$ is upper-triangular and unimodular. Observe that, by Eqs. (2.6) and (2.31),

$$e_n = (-1)^{\binom{n}{2}} \tilde{h}_{(1^n)}. \tag{2.32}$$

If we denote by $\{\widehat{R}_\alpha\}$ the basis of $N\Lambda_q$ dual to the basis $\{R_\alpha\}$ of $Q\Lambda_q$, it is easy to see that

$$\begin{aligned} \Lambda' &\rightarrow N\Lambda_q \\ \tilde{h}_\alpha &\mapsto \widehat{R}_\alpha \end{aligned}$$

defines an isomorphism between Λ' and $N\Lambda_q$, since

$$(\tilde{h}_\beta, \tilde{h}_\alpha) = \sum_{\substack{C(\sigma)=\alpha \\ C(\sigma^{-1})=\beta}} q^{\ell(\sigma)}. \tag{2.33}$$

We can give a diagrammatic interpretation of Eq. (2.33). Setting up platforms at top and bottom to compute $(\tilde{h}_\beta, \tilde{h}_\alpha)$ just as one would to compute (h_β, h_α) , the following extra restriction is placed on diagrams: *strands which start or end at adjacent positions not on the same platform must cross*. The derivation of (2.33) from (2.4) is an easy exercise using inclusion–exclusion.

Example 2.19. The computation of

$$(h_{31}, h_{22}) = 1 + q^2$$

involves two admissible diagrams. Only one of these diagrams, however, contributes to

$$(\tilde{h}_{31}, \tilde{h}_{22}) = q^2.$$

3. Other bases of Λ

3.1. Dual bases: odd monomial and forgotten symmetric functions

In the even ($q = 1$) case, the dual bases to the elementary and complete symmetric functions are the forgotten and the monomial symmetric functions, respectively. The monomial functions $\{m_\lambda\}$ get their name from the fact that when Λ is viewed in terms of power series, they are sums of monomials of the same shape,

$$m_\lambda = \sum_\alpha x^\alpha. \tag{3.1}$$

Here, $\lambda = (\lambda_1, \dots, \lambda_r, 0, \dots)$ is a partition padded with infinitely many zeros at the end, the sum ranges over all *distinct* permutations α of λ , and $x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \dots$. In terms of power series, no particularly nice description of the forgotten symmetric functions is known. As a result they are often omitted from the discussion; whence their name. From the point of view of self-adjoint Hopf algebras with a bilinear form (in the sense of [31]), however, they are just as natural a consideration as the monomial functions; see [21, I.2].

We now return to the odd ($q = -1$) case. For $n \geq 0$, define the *odd monomial symmetric functions* $\{m_\lambda\}_{\lambda \vdash n}$ to be the dual basis to $\{h_\lambda\}_{\lambda \vdash n}$ and define the *odd forgotten symmetric functions* $\{f_\lambda\}_{\lambda \vdash n}$ to be the dual basis to $\{e_\lambda\}_{\lambda \vdash n}$. In other words, we define m_λ and f_λ by the conditions

$$(h_\lambda, m_\mu) = \delta_{\lambda\mu}, \quad (e_\lambda, f_\mu) = \delta_{\lambda\mu}.$$

The monomial and forgotten functions through degree 4 are given in [Appendix A.1](#).

Define the coefficients $M_{\lambda\mu}, M'_{\lambda\mu}, M''_{\lambda\mu}$ (indexed over ordered pairs of partitions λ, μ of some n) by

$$M_{\lambda\mu} = (e_\lambda, h_\mu), \quad M'_{\lambda\mu} = (h_\lambda, h_\mu), \quad M''_{\lambda\mu} = (e_\lambda, e_\mu). \tag{3.2}$$

The following change of basis relations are immediate consequences of (3.2):

$$\begin{aligned}
 h_\lambda &= \sum_{\mu \vdash n} M_{\lambda,\mu} f_\mu, & h_\lambda &= \sum_{\mu \vdash n} M'_{\lambda,\mu} m_\mu, \\
 e_\lambda &= \sum_{\mu \vdash n} M_{\lambda,\mu} m_\mu, & e_\lambda &= \sum_{\mu \vdash n} M''_{\lambda,\mu} f_\mu.
 \end{aligned}
 \tag{3.3}$$

Along with the results of Section 2.2, we see that these change of basis matrices have the following properties.

- $M_{\lambda,\mu}$ is equal to 0 when $\mu > \lambda^T$ in the lexicographic order and equal to ± 1 when $\mu = \lambda^T$. So the change of basis matrix is upper-left-triangular with ± 1 's on the diagonal.
- The matrix for $M'_{\lambda,\mu}$ is symmetric and has determinant equal to ± 1 .
- The matrix for $M''_{\lambda,\mu}$ is symmetric and has determinant equal to ± 1 .

Since $(e_\lambda, e_\mu) = (h_\lambda, h_\mu)$ when $q = 1$, the $q = 1$ analogues of $M'_{\lambda,\mu}$ and $M''_{\lambda,\mu}$ are equal. Their combinatorial interpretations in Proposition 3.1 below are the same when the signs are omitted. In the odd ($q = -1$) case, they differ because $(e_k, e_k) = (-1)^{\binom{k}{2}}$, as this sign comes up whenever k strands connect the same two black platforms.

The determinant of the matrix M is not hard to compute. M is upper-left-triangular by Proposition 2.14, and the anti-diagonal entry $(h_\lambda, e_{\lambda^T})$ equals $(-1)^{\ell(w_\lambda)}$. But this entry and the anti-diagonal entry $(h_{\lambda^T}, e_\lambda)$ are equal, so the determinant of M in degree n is a sign computed only from self-transpose diagrams:

$$\det(M_n) = \prod_{\lambda=\lambda^T} (-1)^{\ell(w_\lambda)}.
 \tag{3.4}$$

The determinants $\det(M'_n)$ and $\det(M''_n)$ both equal $\det(M_n)$ times the determinant of the change of basis between the e - and h -bases. Note that self-transpose Young diagrams with n boxes are in a natural bijection with partitions of n into distinct odd positive integers. Under this bijection, the sign $(-1)^{\ell(w_\lambda)}$ has a factor of -1 for each summand which is congruent to 3 modulo 4.

The proof of the following proposition, which we omit, is essentially the same as in the even ($q = 1$) case, but with the extra bookkeeping of signs. For the even case, see Proposition 37.5 of [2]. For a matrix A , define the composition $\text{row}(A)$ (respectively, $\text{col}(A)$) to consist of the row (respectively, column) sums of A . If A is an \mathbb{N} -matrix (that is, its entries are all natural numbers), there are two sorts of signs we can attach to A when counting matrices. Our natural numbers include zero: $\mathbb{N} = \{0, 1, 2, \dots\}$.

- To count *SW-NE pairs* means to accrue a sign of $(-1)^{ab}$ for every pair of entries in which an a is strictly below and strictly to the left of a b .
- To count *cables* means to accrue a sign of $(-1)^{\binom{a}{2}}$ for every entry a . Since 0- and 1-cables accrue 1's, this is not interesting for $\{0, 1\}$ -matrices.

Proposition 3.1. *The numbers defined in Eq. (3.2) have the following combinatorial interpretations.*

1. $M_{\lambda,\mu}$ equals the signed count of $\{0, 1\}$ -matrices A with $\text{row}(A) = \lambda$ and $\text{col}(A) = \mu$. The sign counts SW-NE pairs.
2. $M'_{\lambda,\mu}$ equals the signed count of \mathbb{N} -matrices A with $\text{row}(A) = \lambda$ and $\text{col}(A) = \mu$. The sign counts SW-NE pairs.

3. $M''_{\lambda,\mu}$ equals the signed count of \mathbb{N} -matrices A with $\text{row}(A) = \lambda$ and $\text{col}(A) = \mu$. The sign counts SW-NE pairs and cables.

Example 3.2. We compute the $(3, 2), (2, 2, 1)$ entry of the matrices M, M', M'' . Below are the five \mathbb{N} -matrices with row sum $(3, 2)$ and column sum $(2, 2, 1)$, and their contributions to M, M', M'' .

Matrix	Contribution to M	Contribution to M'	Contribution to M''
$\begin{pmatrix} 2 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$	0	$(-1)^0$	$(-1)^0(-1)^{\binom{3}{2}}$
$\begin{pmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \end{pmatrix}$	0	$(-1)^2$	$(-1)^2(-1)^{\binom{3}{2}+\binom{3}{2}}$
$\begin{pmatrix} 1 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix}$	0	$(-1)^2$	$(-1)^2(-1)^{\binom{3}{2}}$
$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}$	$(-1)^3$	$(-1)^3$	$(-1)^3(-1)^0$
$\begin{pmatrix} 0 & 2 & 1 \\ 2 & 0 & 0 \end{pmatrix}$	0	$(-1)^6$	$(-1)^6(-1)^{\binom{3}{2}+\binom{3}{2}}$

Therefore

$$M_{(3,2),(2,2,1)} = -1, \quad M'_{(3,2),(2,2,1)} = 3, \quad M''_{(3,2),(2,2,1)} = -1.$$

We end this section by pointing out that the above results are enough to compute the matrix of the bilinear form in any of the bases described so far. For instance, since M' is the matrix of the bilinear form in the h -basis, M is the matrix which takes the f -basis to the h -basis, and $M = M^T$, the matrix $M^{-1}M'M^{-1}$ is the matrix of the bilinear form in the f -basis.

3.2. Primitive elements, odd power symmetric functions, and the center of Λ

For this section, assume that \mathbb{k} is a field of characteristic zero.

Recall that an element x of a $(q-)$ Hopf algebra is called *primitive* if

$$\Delta(x) = 1 \otimes x + x \otimes 1.$$

In the even ($q = 1$) case, the primitive elements of Λ are spanned by the power sum functions

$$p_n = \sum_j x_j^n.$$

In the odd setting, however, there are only “half” as many.

Proposition 3.3. *The subspace of primitive elements P in Λ is spanned by the elements m_1 and m_{2k} for $k \geq 1$.*

Proof. Let $I \subset \Lambda$ be the ideal generated by all elements of positive degree. By the general theory of self-adjoint connected graded Hopf algebras with a bilinear form, $P = (I^2)^\perp$

(Lemma 1.7 of [31]). It is clear from the h -relations (Proposition 2.9) or the e -relations (Proposition 2.10) that in each degree n ,

$$I^2 \cap \Lambda_n = \begin{cases} 0 & n = 1, \\ \text{span}_{\mathbb{K}}\{h_\lambda : \lambda \neq (n)\} & n \text{ is even,} \\ \Lambda_n & n \text{ is odd and } \geq 3. \end{cases}$$

The result follows. \square

Note that $f_n = \pm m_n$, so the f_{2k} are primitive as well (the sign is the same as the coefficient of h_n in the expansion of e_n in the h -basis). We define, therefore, the n -th odd power symmetric function to be

$$p_n = m_n.$$

The first few p_n are:

$$\begin{aligned} p_1 &= h_1, \\ p_2 &= h_{11}, \\ p_3 &= h_{111} + h_{21} - h_3, \\ p_4 &= -h_{1111} - 2h_{22} + 4h_4, \\ p_5 &= h_{11111} + h_{2111} + 3h_{221} - h_{311} - 3h_{32} - 9h_{41} + 9h_5, \\ p_6 &= h_{111111} + 3h_{2211} - 3h_{33} - 6h_{411} + 6h_{51}. \end{aligned}$$

Proposition 3.4. *The element p_k belongs to the center of Λ if and only if k is even.*

Proof. We will show that $(p_k h_m, e_\lambda) = (h_m p_k, e_\lambda)$ for every $m \geq 0$ and every $\lambda \vdash (k + m)$, if and only if k is even. Let ℓ be the length of λ . The coproduct of e_λ is

$$\Delta(e_\lambda) = \prod_{i=1}^{\ell} \sum_{j=0}^{\lambda_i} e_j \otimes e_{\lambda_i-j} = \sum_{\alpha} (e_{a_1} \otimes e_{\lambda_1-a_1}) \cdots (e_{a_\ell} \otimes e_{\lambda_\ell-a_\ell}),$$

where the last sum is over all α such that $|\alpha| = k + m$ and $0 \leq a_j \leq \lambda_j$ for each j . When paired against $p_k h_m$ or $h_m p_k$, only partitions $\lambda = (k + 1, 1^{m-1})$ and $\lambda = (k, 1^m)$ yield nonzero results. It is straightforward to check that

$$\begin{aligned} (p_k h_m, e_{k+1} e_1^{m-1}) &= 1, & (h_m p_k, e_{k+1} e_1^{m-1}) &= (-1)^{k(m-1)}, \\ (p_k h_m, e_k e_1^m) &= 1, & (h_m p_k, e_k e_1^m) &= (-1)^{km}, \end{aligned}$$

using the adjointness of multiplication and comultiplication. The result follows. \square

In fact, the center is precisely the polynomial algebra generated by the p_{2k} 's. This will follow from the results of [5], but it would be nice to have a proof wholly within the framework of the Hopf algebra approach.

3.3. Odd Schur functions

We begin by reviewing some terminology from the combinatorics of Young diagrams. Let λ be a Young diagram. A *Young tableau* T of shape λ is an assignment of a positive integer to each box of λ . We say T is *standard* if its entries are strictly increasing in all rows and in all columns

and we say T is *semistandard* if its entries are non-decreasing in all rows and strictly increasing in all columns.

The *content* of a semistandard Young tableau T of shape λ , denoted $\text{cont}(T)$, is the composition $\alpha = (a_1, \dots, a_r)$ of $|\lambda|$ defined by

$$a_i = \text{the number of entries of } T \text{ equal to } i.$$

In the even ($q = 1$) case, the Schur functions $\{s_\lambda\}_{\lambda \vdash n}$ form an orthonormal basis of Λ_n . In terms of power series, they are generating functions for semistandard Young tableaux. If we denote the set of semistandard Young tableaux of shape $\lambda \vdash n$ by $\text{SSYT}(\lambda)$, one definition of the Schur function s_λ is

$$s_\lambda = \sum_{T \in \text{SSYT}(\lambda)} x^{\text{cont}(T)}.$$

Then if one defines the *Kostka number* associated to partitions λ, μ to be

$$K_{\lambda\mu} = \text{the number of semistandard Young tableaux of shape } \lambda \text{ and content } \mu,$$

it follows from the above and (3.1) that

$$s_\lambda = \sum_{\mu \vdash n} K_{\lambda\mu} m_\mu. \tag{3.5}$$

Having expressed Schur functions in terms of the dual basis to the complete functions, we have a definition which we can attempt to mimic in the odd ($q = -1$) case. An essential feature in the even case is that the Schur functions $\{s_\lambda\}_{\lambda \vdash n}$ form an orthonormal basis of Λ_n ; in the odd case, Schur functions will be orthogonal but their norms may be either 1 or -1 .

In the odd case, we define the *odd Schur functions* by a change of basis relation closely related to (3.5),

$$h_\mu = \sum_{\lambda \vdash n} K_{\lambda\mu} s_\lambda. \tag{3.6}$$

To define the coefficients $K_{\lambda\mu}$, the *odd Kostka numbers*, we first define the sign associated to a Young tableau T . For a Young tableau T , let $w_r(T)$ be its *row word*, that is, the string of numbers obtained by reading the entries of T from left to right and bottom to top. Then define $\text{sign}(T)$ to be the sign of the minimal length permutation which sorts $w_r(T)$ into non-decreasing order. The sign of a standard Young tableau was introduced by Stanley in [29]; our notion is an obvious generalization.

For a Young diagram λ , let T_λ be the unique semistandard Young tableau with shape and content both equal to λ . In other words, every first-row entry of T_λ is a 1, every second-row entry is a 2, and so forth. With these notations established, we can define

$$K_{\lambda\mu} = \text{sign}(T_\lambda) \sum_T \text{sign}(T), \tag{3.7}$$

where the sum is over all semistandard Young tableaux T of shape λ and content μ . Note that $K_{(n)\mu} = 1$ for all $\mu \vdash n$, $K_{(1^n)\mu} = \delta_{\mu, (1^n)}$, and $K_{\lambda\lambda} = 1$ for all λ . Odd Kostka numbers of the form $K_{\lambda(1^n)}$ sign-count standard Young tableaux of a given shape; up to an overall sign, this count is what Stanley calls the *sign imbalance* of the partition λ [29]. Sign imbalance is a topic of current study; see [13,16]. Tables of odd Kostka numbers are included in [Appendix A.1](#).

Example 3.5. To compute $K_{(2,2,1),(1^5)}$:

tableau T	$\begin{array}{ c c } \hline 1 & 2 \\ \hline 3 & 4 \\ \hline 5 & \\ \hline \end{array}$	$\begin{array}{ c c } \hline 1 & 2 \\ \hline 3 & 5 \\ \hline 4 & \\ \hline \end{array}$	$\begin{array}{ c c } \hline 1 & 3 \\ \hline 2 & 4 \\ \hline 5 & \\ \hline \end{array}$	$\begin{array}{ c c } \hline 1 & 3 \\ \hline 2 & 5 \\ \hline 4 & \\ \hline \end{array}$	$\begin{array}{ c c } \hline 1 & 4 \\ \hline 2 & 5 \\ \hline 3 & \\ \hline \end{array}$
sign(T)	+	-	-	+	-

Since $\text{sign}(T_{(2,2,1)}) = 1$, we have $K_{(2,2,1),(1^5)} = -1$.

Example 3.6. To compute $K_{(3,1,1),(2,1,1,1)}$:

tableau T	$\begin{array}{ c c c } \hline 1 & 1 & 2 \\ \hline 3 & & \\ \hline 4 & & \\ \hline \end{array}$	$\begin{array}{ c c c } \hline 1 & 1 & 3 \\ \hline 2 & & \\ \hline 4 & & \\ \hline \end{array}$	$\begin{array}{ c c c } \hline 1 & 1 & 4 \\ \hline 2 & & \\ \hline 3 & & \\ \hline \end{array}$
sign(T)	-	+	-

Since $\text{sign}(T_{(3,1,1)}) = -1$, we have $K_{(3,1,1),(2,1,1,1)} = 1$.

Appendix A.1 lists the odd Schur functions through degree 5 in the basis of complete functions. The following theorem of Reifegerste and Sjöstrand refines the usual RSK correspondence (see [6]) by keeping track of signs.

Theorem 3.7 (Odd RSK Correspondence I [24,27]). *The RSK map is a bijection*

$$RSK : \left\{ \begin{array}{l} \mathbb{N}\text{-matrices } A \text{ with} \\ \text{row}(A) = \mu \text{ and } \text{col}(A) = \rho \end{array} \right\} \rightarrow \left\{ \begin{array}{l} \text{pairs } (P, Q) \text{ of semistandard Young tableaux of the} \\ \text{same shape, with } \text{cont}(P) = \mu \text{ and } \text{cont}(Q) = \rho \end{array} \right\}, \tag{3.8}$$

under which the sign of A as in the computation of $M'_{\mu\rho}$ equals $(-1)^{\binom{\lambda^T}{2}} \text{sign}(P)\text{sign}(Q)$, where $\lambda = \text{shape}(P) = \text{shape}(Q)$. In particular,

$$\begin{aligned} M'_{\mu\rho} &= \sum_{\lambda \vdash n} (-1)^{\binom{\lambda^T}{2}} K_{\lambda\mu} K_{\lambda\rho} \\ &= \sum_{\lambda \vdash n} (-1)^{\lambda_2 + \lambda_4 + \lambda_6 + \dots} K_{\lambda\mu} K_{\lambda\rho}. \end{aligned} \tag{3.9}$$

Corollary 3.8. *The Schur function s_λ can be expressed in terms of the monomial functions as*

$$(-1)^{\binom{\lambda^T}{2}} s_\lambda = \sum_{\mu \vdash n} K_{\lambda\mu} m_\mu. \tag{3.10}$$

Proof. Define matrices A, B, C , all square and indexed by all partitions of n , by

$$\begin{aligned} A_{\lambda\mu} &= M'_{\lambda\mu}, \\ B_{\lambda\mu} &= K_{\lambda\mu}, \\ C_{\lambda\mu} &= (-1)^{\binom{\lambda^T}{2}} K_{\lambda\mu}. \end{aligned}$$

The ordering on the index set can be taken to be any total ordering which refines the dominance partial order. In these terms, Eq. (3.6) says that B^T takes the Schur basis to the complete basis and Eq. (3.3) says that A takes the monomial basis to the complete basis. It follows that $(B^T)^{-1}A$ takes the monomial basis to the Schur basis. Now Eq. (3.9) says that $A = B^T C$, proving the corollary. \square

Corollary 3.9. *The Schur functions are signed-orthonormal:*

$$(s_\lambda, s_\mu) = (-1)^{\binom{\lambda^T}{2}} \delta_{\lambda,\mu}. \tag{3.11}$$

Proof. In $\Lambda \otimes \Lambda$, Eqs. (3.10) and (3.6) imply

$$\sum_{\lambda \vdash n} (-1)^{\binom{\lambda^T}{2}} s_\lambda \otimes s_\lambda = \sum_{\lambda, \mu \vdash n} K_{\lambda\mu} m_\mu \otimes s_\lambda = \sum_{\mu \vdash n} m_\mu \otimes h_\mu.$$

Since $\{m_\lambda\}_{\lambda \vdash n}$ and $\{h_\lambda\}_{\lambda \vdash n}$ are dual bases, it follows that $\{s_\lambda\}_{\lambda \vdash n}$ and $\{(-1)^{\binom{\lambda^T}{2}} s_\lambda\}_{\lambda \vdash n}$ are dual bases. \square

In order to express the Schur functions in the elementary and forgotten bases, note that the two following properties uniquely characterize the Schur functions.

1. $(s_\lambda, h_\mu) = 0$ if $\mu > \lambda$ (lexicographic order).
2. For certain integers a_μ (depending on λ),

$$s_\lambda = h_\lambda + \sum_{\mu > \lambda} a_\mu h_\mu. \tag{3.12}$$

That these uniquely determine the Schur functions follows from Lemma 2.15. The first property follows immediately from Eq. (3.10) and the second follows from Eq. (3.6). We think of these conditions as an inductive definition of s_λ , starting from $s_{(n)} = h_n$.

Proposition 3.10. *Define the elements s'_λ of Λ inductively as follows: $s'_{(1^n)} = e_n$, and the following two properties hold.*

1. $(s'_\lambda, e_\mu) = 0$ if $\mu > \lambda^T$ (lexicographic order).
2. For certain integers b_μ (depending on λ),

$$s'_\lambda = e_{\lambda^T} + \sum_{\mu > \lambda^T} b_\mu e_\mu.$$

Then $s'_\lambda = (-1)^{\ell(w_\lambda) + \text{binom}\lambda^T} 2 s_\lambda$.

Proof. By Lemma 2.15 and the property 2 preceding the statement of the Proposition, the space $(H_{\geq \lambda} \cap E_{\geq \lambda^T}) \otimes \mathbb{Q}$ is one-dimensional and spanned by s_λ . But it is also spanned by s'_λ , by property 2 in the statement of the proposition. In order to determine the constant by which they differ, we compute

$$(s_\lambda, s'_\lambda) = (h_\lambda, e_{\lambda^T}) + \sum_{\mu > \lambda} a_\mu (h_\mu, e_{\lambda^T}) + \sum_{\mu > \lambda^T} b_\mu (h_\lambda, e_\mu) + \sum_{\substack{\rho > \lambda \\ \mu > \lambda^T}} a_\rho b_\mu (h_\rho, e_\mu)$$

$$\stackrel{(2.21)}{=} (-1)^{\ell(w_\lambda)}.$$

Hence, by the signed orthonormality of Schur functions, $s'_\lambda = (-1)^{\ell(w_\lambda) + \binom{\lambda^T}{2}} s_\lambda$. \square

Lemma 3.11. *The anti-involution ψ_3 and the involution $\psi_1 \psi_2$ act on Schur functions as follows:*

$$\psi_3(s_\lambda) = \text{sign}(T_\lambda) (-1)^{\binom{\lambda}{2}} s_\lambda, \quad \psi_1 \psi_2(s_\lambda) = (-1)^{\ell(w_\lambda) + |\lambda|} s_{\lambda^T}. \tag{3.13}$$

In particular $S(s_\lambda) = \text{sign}(\lambda) (-1)^{\ell(w_\lambda) + \binom{\lambda}{2} + |\lambda|} s_{\lambda^T}$.

Proof. Since ψ_3 is norm preserving, the expression for $\psi_3(s_\lambda)$ follows from Lemma 2.16 and the two properties stated before Proposition 3.10. To prove the expression for $\psi_1 \psi_2(s_\lambda)$, we express s_λ in terms of both complete and elementary functions and then compare the results (a_μ and b_μ are integers depending on λ and on μ ; their particular values are immaterial):

$$\begin{aligned} \psi_1 \psi_2(s_\lambda) &= \psi_1 \psi_2 \left(h_\lambda + \sum_{\mu > \lambda} a_\mu h_\mu \right) \\ &= (-1)^{\binom{\lambda}{2} + |\lambda|} e_\lambda + \sum_{\mu > \lambda} (-1)^{\binom{\mu}{2} + |\mu|} a_\mu h_\mu \\ \psi_1 \psi_2(s_\lambda) &= \psi_1 \psi_2 \left((-1)^{\ell(w_\lambda) + \binom{\lambda^T}{2}} e_{\lambda^T} + \sum_{\mu > \lambda^T} b_\mu e_\mu \right) \\ &= (-1)^{\ell(w_\lambda) + |\lambda|} h_{\lambda^T} + \sum_{\mu > \lambda^T} (-1)^{\binom{\mu}{2} + |\mu|} b_\mu h_\mu. \end{aligned}$$

Since $H_{\geq \lambda} \cap E_{\geq \lambda^T}$ is generated by s_{λ^T} as in the proof of Proposition 3.10, it follows that both the above expressions for $\psi_1 \psi_2(s_\lambda)$ are equal to plus or minus s_{λ^T} . Considering the leading coefficient of either one, we see that the sign between $\psi_1 \psi_2(s_\lambda)$ and s_{λ^T} must be $(-1)^{\ell(w_\lambda) + |\lambda|}$. \square

Corollary 3.12. *The Schur function basis is related to the monomial and the complete bases as follows:*

$$\begin{aligned} (-1)^{\binom{\mu}{2} + |\mu|} e_\mu &= \sum_{\lambda \vdash n} (-1)^{\ell(w_\lambda) + |\lambda|} K_{\lambda^T \mu} s_\lambda, \\ (-1)^{\ell(w_\lambda) + \binom{\lambda^T}{2} + |\lambda|} s_\lambda &= \sum_{\mu \vdash n} (-1)^{\binom{\mu}{2} + |\mu|} K_{\lambda^T \mu} f_\mu. \end{aligned} \tag{3.14}$$

Proof. Apply $\psi_1\psi_2$ to Eqs. (3.6) and (3.10). \square

Corollary 3.13 (“*Odd RSK Correspondence II*”). *The following formula holds:*

$$\begin{aligned} (-1)^{\binom{n}{2}+|\mu|+\binom{\rho}{2}+|\rho|} M''_{\mu\rho} &= \sum_{\lambda \vdash n} (-1)^{\binom{\lambda^T}{2}} K_{\lambda^T \mu} K_{\lambda^T \rho} \\ &= \sum_{\lambda \vdash n} (-1)^{\lambda_2+\lambda_4+\lambda_6+\dots} K_{\lambda^T \mu} K_{\lambda^T \rho}. \end{aligned} \tag{3.15}$$

Proof. Argue as in the proof of Corollary 3.8. \square

Why the scare quotes around the name of the corollary? Unlike the formula for $M'_{\mu\rho}$ (Odd RSK Correspondence I, Theorem 3.7), it does not appear that the above formula can be refined to a matching of signs between particular matrices and their RSK-corresponding pairs of semistandard Young tableaux. Such a refined correspondence is possible after permuting the matrices counted in a particular $M''_{\mu\rho}$, but we do not know of a general rule governing these permutations.

We conclude this section by comparing odd Schur functions to a generalization due to Lascoux et al. [17] of the spin-weight domino symmetric functions of Carré and Leclerc [3]. We will review the definition of these functions very briefly; see [17] for details and examples. Let λ be a Young diagram. A *domino tableau* D of shape λ is a tiling of λ by 2×1 dominos and an assignment of a positive integer to each domino which is weakly increasing in rows and strictly increasing in columns (in order for λ to admit a domino tiling, it is necessary but not sufficient that $|\lambda|$ be even). In the polynomial ring $\mathbb{Z}[x_1, x_2, \dots, x_n]$, let x^D be the monomial $\prod_i x_i^{a_i}$, where a_i is the number of dominos of D labeled i . The *spin* of D , denoted $s(D)$, is defined to be one-half the number of vertically oriented dominos in D . If λ has at least one domino tiling, let $s^*(\lambda)$ be the highest possible spin of a domino tiling of λ and define the *cospin* of a domino tableau D of shape λ to be $\tilde{s}(D) = s^*(\lambda) - s(D)$. The spin $s(D)$ is a non-negative half-integer and the cospin $\tilde{s}(D)$ is always an integer.

For an indeterminate q , let

$$G_\lambda = \sum_D q^{\tilde{s}(D)} x^D, \tag{3.16}$$

the sum being over all domino tableaux D of shape λ . This is called the *spin-weight domino symmetric function* corresponding to λ ; it is an (ordinary, not odd) symmetric function. It was pointed out to us by Thomas Lam that there is likely a connection between spin-weight domino functions and odd Schur functions, but we do not know a precise relationship. Two pieces of evidence for such a connection are that both are closely related to the combinatorics of tableau signs and that the norm of an odd Schur function can be computed as

$$\langle s_\lambda, s_\lambda \rangle = (-1)^{\frac{1}{2}s^*(2\lambda)}.$$

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Appendix. Data

A.1. Bases of Λ

In this subsection, we express the monomial, forgotten, and Schur functions in terms of the complete functions for low degrees.

Through degree 4, the odd monomial and forgotten functions are

$$\begin{array}{ll}
 m_1 = h_1 & m_{1111} = h_{1111} - h_{211} + h_{22} - h_4 \\
 m_{11} = -h_{11} + h_2 & m_{211} = -h_{1111} + h_{31} \\
 m_2 = h_{11} & m_{22} = h_{1111} + h_{22} - 2h_4 \\
 m_{111} = -h_{111} + h_3 & m_{31} = h_{211} - h_{31} \\
 m_{21} = -h_{21} + h_3 & m_4 = -h_{1111} - 2h_{22} + 4h_4 \\
 m_3 = h_{111} + h_{21} - h_3 & \\
 f_1 = h_1 & f_{1111} = h_4 \\
 f_{11} = h_2 & f_{211} = h_{31} \\
 f_2 = h_{11} & f_{22} = -h_{22} + 2h_4 \\
 f_{111} = h_3 & f_{31} = h_{211} - h_{31} \\
 f_{21} = -h_{21} + h_3 & f_4 = h_{1111} + 2e_{22} - 4h_4 \\
 f_3 = h_{111} + h_{21} - h_3. &
 \end{array}$$

Below are odd Kostka numbers through degree 5. Partitions are listed lexicographically, with shape parametrizing the rows and content parametrizing the columns.

deg. 1	(1)				
(1)	1				
deg. 2	(11)	(2)			
(11)	1	0			
(2)	1	1			
deg. 3	(111)	(21)	(3)		
(111)	1	0	0		
(21)	0	1	0		
(3)	1	1	1		
deg. 4	(1111)	(211)	(22)	(31)	(4)
(1111)	1	0	0	0	0
(211)	1	1	0	0	0
(22)	0	1	1	0	0
(31)	1	0	-1	1	0
(4)	1	1	1	1	1

deg. 5	(11111)	(2111)	(221)	(311)	(32)	(41)	(5)
(11111)	1	0	0	0	0	0	0
(2111)	0	1	0	0	0	0	0
(221)	-1	0	1	0	0	0	0
(311)	2	1	-1	1	0	0	0
(32)	1	1	0	1	1	0	0
(41)	0	1	2	0	-1	1	0
(5)	1	1	1	1	1	1	1

Using the above Kostka numbers, we can compute odd Schur functions. Through degree 5 they are, in the h -basis,

$$\begin{aligned}
 s_1 &= h_1 & s_{11111} &= h_{11111} + h_{221} - h_{311} - 2h_{41} + h_5 \\
 s_{11} &= h_{11} - h_2 & s_{2111} &= h_{2111} - h_{311} - h_{41} + h_5 \\
 s_2 &= h_2 & s_{221} &= h_{221} + h_{311} - h_{32} - 3h_{41} + 2h_5 \\
 s_{111} &= h_{111} - h_3 & s_{311} &= h_{311} - h_{32} - h_{41} + h_5 \\
 s_{21} &= h_{21} - h_3 & s_{32} &= h_{32} + h_{41} - 2h_5 \\
 s_3 &= h_3 & s_{41} &= h_{41} - h_5 \\
 s_{1111} &= h_{1111} - h_{211} + h_{22} - h_4 & s_5 &= h_5 \\
 s_{211} &= h_{211} - h_{22} - h_{31} + h_4 \\
 s_{22} &= h_{22} + h_{31} - 2h_4 \\
 s_{31} &= h_{31} - h_4 \\
 s_4 &= h_4.
 \end{aligned}$$

A.2. The bilinear form

In this subsection, we present some low degree computations regarding the bilinear forms on Λ' and Λ .

Let $[n] = 1 + q + q^2 + \dots + q^{n-1}$ be the (unbalanced) q -number and let $[n]! = [n][n-1] \dots [2][1]$ be the corresponding q -factorial. The bilinear form for unspecialized q is (the form is symmetric, so “*” stands for the matching entry above the diagonal)

deg. 1	h_1			
h_1	1			
deg. 2	h_{11}	h_2		
h_{11}	$[2]$	1		
h_2	*	1		
deg. 3	h_{111}	h_{12}	h_{21}	h_3
h_{111}	$[3]!$	$[3]$	$[3]$	1
h_{12}	*	$[2]$	$1 + q^2$	1
h_{21}	*	*	$[2]$	1
h_3	*	*	*	1

deg. 4	h_{1111}	h_{112}	h_{121}	h_{211}	h_{22}	h_{13}	h_{31}	h_4
h_{1111}	$[4]!$	$[4][3]$	$[4][3]$	$[4][3]$	$[5] + q^2$	$[4]$	$[4]$	1
h_{112}	*	$[5] + q[2]$	$[5] + q^2[2]$	$[6] + q^2$	$[3] + q^4$	$[3]$	$[1] + q^2[2]$	1
h_{121}	*	*	$[4] + q + q^3 + q^5$	$[5] + q^2[2]$	$1 + 2q^2 + q^3$	$[2] + q^3$	$[2] + q^3$	1
h_{211}	*	*	*	$[5] + q[2]$	$[3] + q^4$	$1 + q^2[2]$	$[3]$	1
h_{22}	*	*	*	*	$[2] + q^4$	$1 + q^2$	$1 + q^2$	1
h_{13}	*	*	*	*	*	$[2]$	$1 + q^3$	1
h_{31}	*	*	*	*	*	*	$[2]$	1
h_4	*	*	*	*	*	*	*	1

With $q = -1$, the bilinear form on the quotient Λ is (through degree 6)

deg. 1	h_1
h_1	1

deg. 2	h_{11}	h_2
h_{11}	0	1
h_2	1	1

deg. 3	h_{111}	h_{21}	h_3
h_{111}	0	1	1
h_{21}	1	0	1
h_3	1	1	1

deg. 4	h_{1111}	h_{211}	h_{22}	h_{31}	h_4
h_{1111}	0	0	2	0	1
h_{211}	0	1	2	1	1
h_{22}	2	2	1	2	1
h_{31}	0	1	2	0	1
h_4	1	1	1	1	1

deg. 5	h_{11111}	h_{2111}	h_{221}	h_{311}	h_{32}	h_{41}	h_5
h_{11111}	0	0	2	0	2	1	1
h_{2111}	0	1	0	1	3	0	1
h_{221}	2	0	-3	2	3	-1	1
h_{311}	0	1	2	1	2	1	1
h_{32}	2	3	3	2	1	2	1
h_{41}	1	0	-1	1	2	0	1
h_5	1	1	1	1	1	1	1

deg. 6	h_{111111}	h_{21111}	h_{2211}	h_{222}	h_{3111}	h_{321}	h_{33}	h_{411}	h_{42}	h_{51}	h_6
h_{111111}	0	0	0	6	0	0	0	0	3	0	1
h_{21111}	0	0	2	6	0	2	2	1	3	1	1
h_{2211}	0	2	4	3	2	4	4	2	2	2	1
h_{222}	6	6	3	-3	6	5	5	3	0	3	1
h_{3111}	0	0	2	6	0	-1	0	1	3	0	1
h_{321}	0	2	4	5	-1	-4	-2	2	3	-1	1
h_{33}	0	2	4	5	0	-2	0	2	3	0	1
h_{411}	0	1	2	3	1	2	2	1	2	1	1
h_{42}	3	3	2	0	3	3	3	2	1	2	1
h_{51}	0	1	2	3	0	-1	0	1	2	0	1
h_6	1	1	1	1	1	1	1	1	1	1	1

The following are the minimal polynomials for the values of q at which the q -bilinear form is degenerate, through degree 7. (The same values are plotted in Fig. 1 of [30].) Since a nontrivial

relation in degree k causes nontrivial relations in all higher degrees, we only list the new minimal polynomials in each degree. Note that all are monic and with constant coefficient 1, so their roots are units in the ring of algebraic integers. In fact, most (but not all) are roots of unity. All of the polynomials are palindromic, so for longer ones we use ellipses (...) to denote palindromic continuation. For instance, $q^5 + 7q^4 - q^3 + \dots$ would mean $q^5 + 7q^4 - q^3 - q^2 + 7q + 1$ and $q^4 - q^3 + 2q^2 + \dots$ would mean $q^4 - q^3 + 2q^2 - q + 1$. In parentheses, we give the multiplicities in various degrees in the following format: ((5, 1), (6, 4), (7, 10)) means multiplicity 1 in degree 5, multiplicity 4 in degree 6, and multiplicity 10 in degree 7.

degree 2:

- q ((2, 1), (3, 5), (4, 17), (5, 49), (6, 129), (7, 321))

degree 3:

- $q - 1$ ((3, 1), (4, 4), (5, 14), (6, 38), (7, 102))
- $q + 1$ ((3, 1), (4, 4), (5, 12), (6, 34), (7, 88))

degree 4:

- $q^6 + 2q^4 - q^3 + 2q^2 + 1$ ((4, 1), (5, 2), (6, 5), (7, 12)) (not a root of unity)

degree 5:

- $q^2 + q + 1$ ((5, 2), (6, 6), (7, 18)) (3rd root of unity)
- $q^2 - q + 1$ ((5, 1), (6, 4), (7, 11)) (6th root of unity)
- $q^{18} + q^{17} + 3q^{16} + 4q^{15} + 6q^{14} + 7q^{13} + 8q^{12} + 10q^{11} + 11q^{10} + 10q^9 + \dots$ ((5, 1), (6, 2), (7, 5)) (not a root of unity)

degree 6:

- $q^2 + 1$ ((6, 2), (7, 8)) (4th root of unity)
- $q^{10} - q^9 + q^8 - q^7 + q^6 - q^5 + \dots$ ((6, 1), (7, 2)) (22nd root of unity)
- $q^{50} + q^{49} + 2q^{48} + 2q^{47} + 5q^{46} + 4q^{45} + 8q^{44} + 6q^{43} + 11q^{42} + 9q^{41} + 16q^{40} + 16q^{39} + 21q^{38} + 14q^{37} + 23q^{36} + 24q^{35} + 30q^{34} + 23q^{33} + 30q^{32} + 28q^{31} + 38q^{30} + 30q^{29} + 34q^{28} + 30q^{27} + 39q^{26} + 34q^{25} + \dots$ ((6, 1), (7, 2)) (not a root of unity)

degree 7:

- $q^4 + q^3 + q^2 + q + 1$ ((7, 2)) (5th root of unity)
- $q^6 + q^5 + q^4 + q^3 + q^2 + q + 1$ ((7, 1)) (7th root of unity)
- $q^4 + 1$ ((7, 1)) (8th root of unity)
- $q^4 - q^3 + q^2 - q + 1$ ((7, 1)) (10th root of unity)
- $q^4 - q^2 + 1$ ((7, 1)) (12th root of unity)
- $q^{16} + q^{15} + q^{14} + q^{13} + q^{12} + q^{11} + q^{10} + q^9 + q^8 + \dots$ ((7, 1)) (17th root of unity)
- $q^{12} - q^{10} + q^8 - q^6 + q^4 - q^2 + 1$ ((7, 1)) (28th root of unity)
- $q^{102} - q^{101} + 4q^{100} - 2q^{99} + 9q^{98} - 2q^{97} + 18q^{96} - q^{95} + 34q^{94} + 2q^{93} + 58q^{92} + 13q^{91} + 88q^{90} + 36q^{89} + 134q^{88} + 64q^{87} + 204q^{86} + 99q^{85} + 298q^{84} + 155q^{83} + 405q^{82} + 238q^{81} + 537q^{80} + 330q^{79} + 705q^{78} + 442q^{77} + 887q^{76} + 584q^{75} + 1089q^{74} + 731q^{73} + 1323q^{72} + 881q^{71} + 1572q^{70} + 1050q^{69} + 1808q^{68} + 1233q^{67} + 2045q^{66} + 1401q^{65} + 2284q^{64} + 1565q^{63} + 2494q^{62} + 1716q^{61} + 2692q^{60} + 1829q^{59} + 2874q^{58} + 1926q^{57} + 2995q^{56} + 2018q^{55} + 3067q^{54} + 2070q^{53} + 3118q^{52} + 2080q^{51} + \dots$ ((7, 1)) (not a root of unity)

In light of the above data, a few remarks on the determinant of the bilinear form are in order. It is immediate from the definition of the bilinear form that this determinant is monic in q . It is not hard to see that its degree is given by

$$\begin{aligned} D &= \sum_{\alpha} \left(\frac{1}{2}n(n-1) - \sum_{i=1}^{\ell(\alpha)} \frac{1}{2}\alpha_i(\alpha_i-1) \right) \\ &= 2^{n-2}n(n-1) - \frac{1}{2} \sum_{\alpha} \sum_{i=1}^{\ell(\alpha)} \alpha_i(\alpha_i-1). \end{aligned} \quad (\text{A.1})$$

The outer summation is over all compositions α of n , and $\ell(\alpha)$ is the length of the composition α . Let

$$A_n = \sum_{\alpha} \sum_{i=1}^{\ell(\alpha)} \alpha_i(\alpha_i-1)$$

be the double summation of Eq. (A.1). Re-indexing so as to first sum over the first entry of each composition, we see

$$A_n = n(n-1) + \sum_{k=1}^{n-1} \left[2^{n-k-1}k(k-1) + A_{n-k} \right].$$

This recursion can be solved as

$$A_n = 2 + 2^n(n-2),$$

from which we conclude that the degree of the determinant of the bilinear form, as a polynomial in q , equals

$$D = 2^{n-2} \left(n^2 - 3n + 4 \right) - 1. \quad (\text{A.2})$$

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