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Advances in Mathematics 219 (2008) 27–62
ADVANCES IN Mathematics
www.elsevier.com/locate/aim

Orbit closures in the enhanced nilpotent cone

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Received 9 December 2007; accepted 28 April 2008

Available online 3 June 2008

Communicated by Michael J. Hopkins

Abstract

We study the orbits of $G = \mathrm{GL}(V)$ in the enhanced nilpotent cone $V \times \mathcal{N}$, where $\mathcal{N}$ is the variety of nilpotent endomorphisms of V. These orbits are parametrized by bipartitions of $n = \dim V$, and we prove that the closure ordering corresponds to a natural partial order on bipartitions. Moreover, we prove that the local intersection cohomology of the orbit closures is given by certain bipartition analogues of Kostka polynomials, defined by Shoji. Finally, we make a connection with Kato's exotic nilpotent cone in type C, proving that the closure ordering is the same, and conjecturing that the intersection cohomology is the same but with degrees doubled.

Keywords: Enhanced nilpotent cone; Intersection cohomology; Bipartitions; Kostka polynomials

1. Introduction

Many features of the representation theory of an algebraic group are known to be controlled by the geometry of its nilpotent cone. In particular, the Springer correspondence, as developed by Borho and MacPherson and Lusztig, relates the local intersection cohomology of the nilpotent orbit closures to composition multiplicities in representations of the associated Weyl group (see the survey article [23]). The correspondence in types B/C is more complicated than that in type A,

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0001-8708/$ – see front matter
doi:10.1016/j.aim.2008.04.008

in a number of respects: for instance, Weyl group representations are no longer in bijection with nilpotent orbits, and the concise algebraic description of the Weyl group action on the total cohomology of Springer fibres (see [6,10]) is lost.

Recently, S. Kato has constructed an "exotic Springer correspondence" in type C (see [17]) which appears to evade these complications. He replaces the nilpotent cone with the exotic nilpotent cone $\mathfrak{N} = W \times \mathfrak{N}_0$, where W is the symplectic vector space and $\mathfrak{N}_0$ is the variety of self-adjoint nilpotent endomorphisms of W. He had introduced this exotic nilpotent cone in [16], to generalize the Kazhdan–Lusztig–Ginzburg geometrization of affine Hecke algebras to the case of unequal parameters. Thus, questions in the representation theory of the Coxeter group of type B/C, and of the affine Hecke algebras of type B/C with unequal parameters, are related to the problem of computing local intersection cohomology of orbit closures in $\mathfrak{N}$. (Part of this problem, the computation of IC of orbit closures in $\mathfrak{N}_0$, was done in [14].)

As a step towards solving this problem, we consider an analogous but simpler variety, the *enhanced nilpotent cone* $V \times \mathcal{N}$, where $\mathcal{N}$ is the ordinary nilpotent cone, *i.e.*, the variety of nilpotent endomorphisms of the vector space V. This enhanced nilpotent cone can be viewed as a subvariety of the exotic nilpotent cone, of the kind which plays an important role in [16]; it is also important in the theory of mirabolic character sheaves being developed by Finkelberg, Ginzburg and Travkin (see [7,29,8]). The group acting on it is merely $G = \mathrm{GL}(V)$, so the geometry has the flavour of type A, whereas the combinatorics arising is of type B/C, in accordance with Kato's picture. The great advantage of the enhanced nilpotent cone over the exotic is that there is a standard way to construct resolutions of singularities of the orbit closures, which turn out to be semismall; these allow us to determine the closure ordering and the local intersection cohomology.

On the ordinary (type A) nilpotent cone, the combinatorics of the closure order is of course well known, as is Lusztig's identification of the local intersection cohomology with Kostka polynomials (see [19]). These results motivate and guide the developments of the present paper. The "mirabole" of this story is that the G-action on pairs of a vector and a nilpotent endomorphism is so similar to that on nilpotent endomorphisms *tout court*.

Here are the principal results of the paper in detail.

- Section 2. **Parametrization of orbits.** We show that G-orbits in $V \times \mathcal{N}$ are parametrized by the bipartitions $(\mu; \nu)$ of n, where $n = \dim V$ (Proposition 2.3). We also show that the orbit $\mathcal{O}_{\mu;\nu}$ has dimension $n^2 - b(\mu;\nu)$, and that its point stabilizers are connected (Proposition 2.8). The same parametrization of orbits was independently obtained in [29]. The finiteness of the number of orbits has been known since [2] (see also [9, 2.1]), and [17] proved analogous results for the exotic nilpotent cone.
- Section 3. **Orbit closures.** We construct resolutions of singularities of the orbit closures $\overline{\mathcal{O}_{\mu;\nu}}$ (Proposition 3.3), and use them to show (Theorem 3.9) that the closure ordering corresponds to a natural partial order on bipartitions (Definition 3.6), which appeared previously in [25].
- Section 4. **Fibres of the resolutions of singularities.** We show that the resolutions of singularities are semismall, and deduce that the local intersection cohomology can be determined from the cohomology of the fibres of the resolutions (Theorem 4.5). We then show that these fibres can be paved by affine spaces (Theorem 4.7), which implies the vanishing of odd-degree cohomology.

Section 5. **Intersection cohomology and Kostka polynomials.** In the main result of the paper (Theorem 5.2), we prove that for $(v, x) \in \mathcal{O}_{\rho;\sigma}$,

$$t^{b(\mu;\nu)} \sum_i \dim \mathcal{H}^{2i}_{(v,x)} \mathrm{IC}(\overline{\mathcal{O}_{\mu;\nu}}) t^{2i} = \widetilde{K}_{(\mu;\nu),(\rho;\sigma)}(t),$$

where the right-hand side is a type-B Kostka polynomial which was defined by Shoji in [24].

Section 6. **Connections with Kato's exotic nilpotent cone.** After recalling Kato's result that the orbits in the exotic nilpotent cone are parametrized by bipartitions, we prove that the closure ordering in the exotic nilpotent cone is the same as for the enhanced nilpotent cone (Theorem 6.3), and we conjecture (Conjecture 6.4) that the local IC is also the same but with degrees doubled (this is the relationship which is known to hold between $\mathfrak{N}_0$ and $\mathcal{N}$). We also explain that this conjecture may be equivalent to one made by Shoji in [25].

2. Parametrization of orbits

The following notation will be in force throughout the paper:

$$\begin{aligned}
&\mathbb{F} \text{ is an algebraically closed field,}\\
&V \text{ is an } n\text{-dimensional vector space over } \mathbb{F},\\
&G = \mathrm{GL}(V), \quad \text{and}\\
&\mathcal{N} = \{x \in \mathrm{End}(V) \mid x \text{ is nilpotent}\}.
\end{aligned}$$

Given $x \in \mathcal{N}$, we regard V as an $\mathbb{F}[x]$-module in the obvious way, where $\mathbb{F}[x]$ is the subalgebra of $\mathrm{End}(V)$ generated by x. All complexes of sheaves will be G-equivariant constructible complexes of $\overline{\mathbb{Q}}_\ell$-sheaves, where ℓ is a fixed prime not equal to the characteristic of $\mathbb{F}$.

Our conventions for partition combinatorics follow [22] in most respects. A *partition* is a nonincreasing sequence of nonnegative integers $\lambda = (\lambda_1, \lambda_2, \ldots)$ with finitely many nonzero terms. Its *size*, denoted $|\lambda|$, is the sum of its terms: $\sum_i \lambda_i$. Its *length*, denoted $\ell(\lambda)$, is the number of nonzero terms. The *transpose partition* $\lambda^{\mathbf{t}}$ is defined by $\lambda^{\mathbf{t}}_i = |\{j \mid \lambda_j \geqslant i\}|$. The set of all partitions of size n is denoted $\mathcal{P}_n$; this is a poset under the dominance order $\leqslant$, defined so that $\lambda \leqslant \mu$ is equivalent to

$$\lambda_1 + \lambda_2 + \cdots + \lambda_k \leqslant \mu_1 + \mu_2 + \cdots + \mu_k, \quad \text{for all } k \geqslant 1.$$

(Note that we never relate partitions of different size in this way, so $\lambda \leqslant \mu$ entails $|\lambda| = |\mu|$.) Addition of partitions is defined termwise: to say that $\lambda = \mu + \nu$ is to say that $\lambda_i = \mu_i + \nu_i$ for each i. Finally, given a partition λ, we define

$$n(\lambda) = \sum_i (i-1)\lambda_i = \sum_i \binom{\lambda^{\mathbf{t}}_i}{2}.$$

This function is clearly additive: $n(\mu + \nu) = n(\mu) + n(\nu)$.

It is well known that G-orbits in $\mathcal{N}$ are in bijection with $\mathcal{P}_n$, via the Jordan normal form. Explicitly, the orbit $\mathcal{O}_\lambda$ corresponding to $\lambda \in \mathcal{P}_n$ consists of all $x \in \mathcal{N}$ for which there exists a basis

$$\{v_{ij} \mid 1 \leqslant i \leqslant \ell(\lambda) \text{ and } 1 \leqslant j \leqslant \lambda_i\} \quad \text{for } V \text{ such that} \quad x v_{ij} = \begin{cases} v_{i,j-1} & \text{if } j > 1, \\ 0 & \text{if } j = 1. \end{cases}$$

We will refer to such a basis $\{v_{ij}\}$ as a *Jordan basis* for x, and to λ as the *Jordan type* of x; this terminology applies when x is a nilpotent endomorphism of any finite-dimensional vector space, not necessarily our chosen vector space V.

To classify the G-orbits in $V \times \mathcal{N}$ we introduce some analogous definitions.

Definition 2.1. A *bipartition* is an ordered pair of partitions, written $(\mu; \nu)$. The set of bipartitions $(\mu; \nu)$ with $|\mu| + |\nu| = n$ is denoted $\mathcal{Q}_n$. Following [12, §5.5.3], for any $(\mu; \nu) \in \mathcal{Q}_n$, we define

$$b(\mu; \nu) = 2n(\mu) + 2n(\nu) + |\nu|.$$

Definition 2.2. Let $(v, x) \in V \times \mathcal{N}$, and let λ be the Jordan type of x. A *normal basis* for (v, x) is a Jordan basis $\{v_{ij}\}$ for x such that

$$v = \sum_{i=1}^{\ell(\mu)} v_{i,\mu_i},$$

where μ is a partition such that $\nu_i = \lambda_i - \mu_i$ defines a partition ν. The bipartition $(\mu; \nu) \in \mathcal{Q}_n$ is the *type* of the normal basis.

The following result (which holds over non-algebraically closed fields as well) was independently proved by Travkin [29, Theorem 1].

Proposition 2.3. *The set of G-orbits in $V \times \mathcal{N}$ is in one-to-one correspondence with $\mathcal{Q}_n$. The orbit corresponding to $(\mu; \nu)$, denoted $\mathcal{O}_{\mu;\nu}$, consists of pairs (v, x) for which there exists a normal basis of type $(\mu; \nu)$.*

Proof. This proposition follows from the next two lemmas. Lemma 2.4 states that for any pair (v, x), a normal basis exists. It is obvious that for any $(\mu; \nu) \in \mathcal{Q}_n$, there exists a pair possessing a normal basis of that type, and any two such pairs are in the same G-orbit. To complete the proof, we must show that the type of the normal basis is determined uniquely by (v, x). But the partition $\mu + \nu$ is determined as the Jordan type of x, and Lemma 2.5 shows that the partition $(\nu_1 + \mu_2, \nu_2 + \mu_3, \ldots)$ of size $n - \mu_1$ is also determined. Knowing these two partitions, one can successively determine $\mu_1, \nu_1, \mu_2, \nu_2$, and so forth. □

Lemma 2.4. *For any $(v, x) \in V \times \mathcal{N}$, there exists a normal basis for (v, x) of some type $(\mu; \nu) \in \mathcal{Q}_n$.*

Proof. Let λ be the Jordan type of x, and let $\{v_{ij}\}$ be a Jordan basis for x. Write $v = \sum_{i,j} c_{ij} v_{ij}$. For $1 \leqslant i \leqslant \ell(\lambda)$, let $\mu_i \in \{0, 1, \ldots, \lambda_i\}$ be minimal such that $c_{ij} = 0$ whenever $\mu_i < j \leqslant \lambda_i$, and set $\nu_i = \lambda_i - \mu_i$. If $\mu_i \neq 0$, we change the basis of the ith Jordan block as follows. Define

$$v'_{i,\lambda_i} = \sum_{j=1}^{\mu_i} c_{ij} v_{i,j+\nu_i} \quad \text{and} \quad v'_{ij} = x^{\lambda_i - j} v'_{i,\lambda_i} \quad \text{for } 1 \leqslant j \leqslant \lambda_i - 1,$$

and then redefine v_{ij} to be v'_{ij}. This gives a new Jordan basis for x with the property that

$$v = \sum_{\substack{1 \leqslant i \leqslant \ell(\lambda) \\ \mu_i \neq 0}} v_{i,\mu_i}.$$

If $(\mu_1, \mu_2, \ldots)$ and $(\nu_1, \nu_2, \ldots)$ are partitions, we are finished. If they are not, we must adjust our basis in an appropriate way. Arguing by induction on $\ell(\lambda)$, we can assume that $\mu_2 \geqslant \mu_3 \geqslant \cdots$ and $\nu_2 \geqslant \nu_3 \geqslant \cdots$ hold, so the only possible problems are that $\mu_1 < \mu_2$ or that $\nu_1 < \nu_2$. Since $\lambda_1 \geqslant \lambda_2$, these cases are mutually exclusive.

If $\mu_1 < \mu_2$, we move the second Jordan block by redefining $v_{21}, v_{22}, \ldots, v_{2,\lambda_2}$ to be

$$v_{21} - v_{11}, v_{22} - v_{12}, \ldots, v_{2,\lambda_2} - v_{1,\lambda_2}.$$

After this change, we still have a Jordan basis for x, but the component of v in the first Jordan block is now $v_{1,\mu_1} + v_{1,\mu_2}$ (or v_{1,μ_2}, if $\mu_1 = 0$). Changing the basis of the first Jordan block as above, we can make this component v_{1,μ_2}. So we have effectively redefined μ_1 to equal μ_2 and ν_1 to equal $\lambda_1 - \mu_2$, and thus removed the problem (without making $\nu_1 < \nu_2$).

If $\nu_1 < \nu_2$, we move the first Jordan block by redefining $v_{11}, \ldots, v_{1,\lambda_1}$ to be

$$v_{11}, \ldots, v_{1,\lambda_1-\lambda_2}, v_{1,\lambda_1-\lambda_2+1} - v_{21}, v_{1,\lambda_1-\lambda_2+2} - v_{22}, \ldots,$$
$$v_{1,\mu_1} - v_{2,\lambda_2-\nu_1}, \ldots, v_{1,\lambda_1} - v_{2,\lambda_2}.$$

After this change, the component of v in the second Jordan block is $v_{2,\mu_2} + v_{2,\lambda_2-\nu_1}$ (or $v_{2,\lambda_2-\nu_1}$, if $\mu_2 = 0$). Changing the basis of the second Jordan block as above, we can make this component $v_{2,\lambda_2-\nu_1}$. So we have effectively redefined μ_2 to equal $\lambda_2 - \nu_1$ and ν_2 to equal ν_1. The inequalities $\mu_1 \geqslant \mu_2 \geqslant \cdots$ remain true, but it is possible that we now have $\nu_2 < \nu_3$; if so, we repeat this procedure with the second and third Jordan blocks, and continue until we arrive at the desired result. □

Lemma 2.5. *Suppose $\{v_{ij}\}$ is a normal basis for $(v, x) \in V \times \mathcal{N}$ of type $(\mu; \nu) \in \mathcal{Q}_n$. For $1 \leqslant i \leqslant \ell(\mu + \nu)$ and $1 \leqslant j \leqslant (\mu + \nu)_i$, define $w_{ij} = \sum_{k=1}^{i} v_{k,j-\mu_i+\mu_k}$.*

(1) *$\{w_{ij} \mid 1 \leqslant j \leqslant \mu_i - \mu_{i+1}\}$ is a basis for the $\mathbb{F}[x]$-submodule $\mathbb{F}[x]v$ of V, and therefore $\dim \mathbb{F}[x]v = \mu_1$.*
(2) *$\{w_{i,j+\mu_i-\mu_{i+1}} + \mathbb{F}[x]v \mid 1 \leqslant j \leqslant \nu_i + \mu_{i+1}\}$ is a Jordan basis for the induced endomorphism $x|_{V/\mathbb{F}[x]v}$, whose Jordan type is $(\nu_1 + \mu_2, \nu_2 + \mu_3, \ldots)$.*

Proof. If the basis elements $\{v_{ij}\}$ are drawn in a shifted array, where the ith row is shifted to the left by μ_i places, then w_{ij} is the sum of v_{ij} and all basis elements directly above it. Hence $\{w_{ij} \mid 1 \leqslant i \leqslant \ell(\mu+\nu), 1 \leqslant j \leqslant (\mu+\nu)_i\}$ is another basis of V. For example:

$$\begin{array}{l} \mu = (32^2) \\ \nu = (21^4) \end{array} \qquad \begin{array}{ccccc} v_{11} & v_{12} & v_{13} & v_{14} & v_{15} \\ & v_{21} & v_{22} & v_{23} & \\ & v_{31} & v_{32} & v_{33} & \\ & & & v_{41} & \\ & & & v_{51} & \\ \uparrow & \uparrow & \uparrow & & \\ x^2 v & xv & v & & \end{array}$$

By definition, we have $v = w_{\ell(\mu),\mu_{\ell(\mu)}}$, the sum of the 0th column of the shifted array. Moreover, x takes each basis vector v_{ij} to the one to the left of it, or to 0 if $j = 1$. Hence $x^s v = w_{\mu^{\mathbf{t}}_{s+1},\mu_{\mu^{\mathbf{t}}_{s+1}}-s}$ is the sum of the $(-s)$th column of the shifted array, for $1 \leqslant s < \mu_1$, and $x^{\mu_1} v = 0$. Part (1) follows. It is also easy to see that

$$x w_{ij} = \begin{cases} w_{i,j-1} & \text{if } j \geqslant \mu_i - \mu_{i+1} + 2, \\ w_{i,j-1} = x^{\mu_i - j + 1} v & \text{if } 2 \leqslant j \leqslant \mu_i - \mu_{i+1} + 1, \\ x^{\mu_i} v & \text{if } j = 1, \end{cases}$$

from which part (2) follows. □

We note some easy facts about this parametrization of G-orbits in $V \times \mathcal{N}$.

Lemma 2.6. *For any $\lambda \in \mathcal{P}_n$, the following hold.*

(1) *The union of the orbits $\mathcal{O}_{\mu;\nu}$ where $\mu + \nu = \lambda$ is precisely $V \times \mathcal{O}_\lambda$.*
(2) *The orbit $\mathcal{O}_{\emptyset;\lambda}$ is precisely $\{0\} \times \mathcal{O}_\lambda$.*
(3) *The orbit $\mathcal{O}_{\lambda;\emptyset}$ consists of all (v, x) where $x \in \mathcal{O}_\lambda$ and $v \in V \setminus \ker(x^{\lambda_1 - 1})$.*

Proof. Parts (1) and (2) are obvious. Part (3) follows from the observation that if $(v, x) \in \mathcal{O}_{\mu;\nu}$ with $\mu + \nu = \lambda$ and $\nu \neq \emptyset$, then $\mu_1 \leqslant \lambda_1 - 1$, so $x^{\lambda_1 - 1} v = 0$ by part (1) of Lemma 2.5. □

We also need to describe the stabilizers of our group action.

Definition 2.7. For $x \in \mathcal{N}$, define

$$E^x = \{y \in \mathrm{End}(V) \mid xy = yx\} \quad \text{and} \quad G^x = G \cap E^x.$$

For $(v, x) \in V \times \mathcal{N}$, define

$$E^{(v,x)} = \{y \in E^x \mid yv = 0\} \quad \text{and} \quad G^{(v,x)} = \{g \in G^x \mid gv = v\}.$$

The first four parts of the next result are well known, but we include them for ease of comparison.

Proposition 2.8. *Let $(\mu;\nu)\in\mathcal{Q}_n$, and let $\lambda=\mu+\nu$. Let $(v,x)\in\mathcal{O}_{\mu;\nu}$, and let $\{v_{ij}\}$ be a normal basis for (v,x).*

(1) *E^x has basis*

$$\left\{y_{i_1,i_2,s}\;\middle|\;1\leqslant i_1,i_2\leqslant\ell(\lambda),\ \max\{0,\lambda_{i_1}-\lambda_{i_2}\}\leqslant s\leqslant\lambda_{i_1}-1\right\},$$

where

$$y_{i_1,i_2,s}v_{ij}=\begin{cases}v_{i_2,j-s}, & \text{if } i=i_1,\ s+1\leqslant j\leqslant\lambda_i,\\ 0, & \text{otherwise}.\end{cases}$$

(2) $\dim E^x=n+2n(\lambda)$.
(3) *G^x is a connected algebraic group of dimension $n+2n(\lambda)$.*
(4) $\dim\mathcal{O}_\lambda=n^2-n-2n(\lambda)$.
(5) $E^xv=\operatorname{span}\{v_{ij}\mid 1\leqslant i\leqslant\ell(\mu),1\leqslant j\leqslant\mu_i\}$.
(6) $\dim E^{(v,x)}=b(\mu;\nu)$.
(7) *$G^{(v,x)}$ is a connected algebraic group of dimension $b(\mu;\nu)$.*
(8) $\dim\mathcal{O}_{\mu;\nu}=\dim\mathcal{O}_\lambda+|\mu|=n^2-b(\mu;\nu)$.

Proof. Part (1) is straightforward, and (2) follows easily from (1). The well-known proof of part (3) is that G^x is the principal open subvariety of E^x defined by the polynomial function det, which clearly does not vanish identically. Part (4) follows because $\dim\mathcal{O}_\lambda=\dim G-\dim G^x$. To prove part (5), we note that for i_1,i_2,s as in part (1),

$$y_{i_1,i_2,s}v=\begin{cases}v_{i_2,\mu_{i_1}-s}, & \text{if } s+1\leqslant\mu_{i_1},\\ 0, & \text{otherwise}.\end{cases}$$

In the first case, $v_{i_2,\mu_{i_1}-s}$ is in the required subspace because $\max\{0,\lambda_{i_1}-\lambda_{i_2}\}\geqslant\mu_{i_1}-\mu_{i_2}$; moreover, every basis element v_{ij} with $1\leqslant j\leqslant\mu_i$ occurs in this way for $i_1=i_2=i$ and $s=\mu_i-j$, so we have the desired equality. Consequently, $\dim E^xv=|\mu|$, which implies

$$\dim E^{(v,x)}=\dim E^x-\dim E^xv=n+2n(\lambda)-|\mu|=b(\mu;\nu),$$

as required for part (6). Part (7) follows because $G^{(v,x)}$ is the principal open subvariety of $1_V+E^{(v,x)}$ defined by det. Finally, we have $\dim\mathcal{O}_{\mu;\nu}=\dim G-\dim G^{(v,x)}=n^2-b(\mu;\nu)$, which is part (8). □

We can now state an elegant alternative characterization of $\mathcal{O}_{\mu;\nu}$, which is prominent in the treatment of Travkin [29]. It is evident *a priori* that E^xv is an x-stable subspace of V, so there are induced endomorphisms $x|_{E^xv}$ and $x|_{V/E^xv}$.

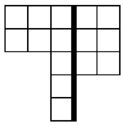
Corollary 2.9. *Let $(v,x)\in V\times\mathcal{N}$ and $(\mu;\nu)\in\mathcal{Q}_n$. Then $(v,x)\in\mathcal{O}_{\mu;\nu}$ if and only if the Jordan type of $x|_{E^xv}$ is μ and the Jordan type of $x|_{V/E^xv}$ is ν.*

Proof. Proposition 2.8(5) shows the "only if" direction; but the pair $(\mu;\nu)$ for which $(v,x)\in\mathcal{O}_{\mu;\nu}$ is uniquely determined by (v,x), so the "if" direction also holds. □

A convenient way to represent a bipartition $(\mu;\nu)$ is as the 'back-to-back union' of the Young diagrams of μ and of ν, with a solid vertical line dividing the two. The columns of this diagram form the following composition of n:

$$\mu^{\mathbf{t}}_{\mu_1}, \mu^{\mathbf{t}}_{\mu_1-1}, \dots, \mu^{\mathbf{t}}_2, \mu^{\mathbf{t}}_1, \nu^{\mathbf{t}}_1, \nu^{\mathbf{t}}_2, \dots, \nu^{\mathbf{t}}_{\nu_1-1}, \nu^{\mathbf{t}}_{\nu_1}. \tag{2.1}$$

For example, we represent

$((3^2 1^3); (2^3))$ as  .

If we are dealing with $(v,x) \in \mathcal{O}_{\mu;\nu}$ and have chosen a normal basis $\{v_{ij}\}$ for (v,x), then we can identify each basis element v_{ij} with the jth box of the ith row of the diagram, as in the proof of Lemma 2.5. We then have that v is the sum of the basis elements in the column immediately to the left of the dividing line, and Proposition 2.8(5) says that $E^x v$ is the span of all the basis elements to the left of the dividing line (*i.e.*, in the first μ_1 columns). More generally, we define a partial flag

$$0 = W_0^{(v,x)} \subset W_1^{(v,x)} \subset \dots \subset W_{\mu_1+\nu_1}^{(v,x)} = V$$

by the rule:

$$W_k^{(v,x)} = \begin{cases} x^{\mu_1-k} E^x v, & \text{if } k < \mu_1, \\ E^x v, & \text{if } k = \mu_1, \\ (x^{k-\mu_1})^{-1}(E^x v), & \text{if } k > \mu_1. \end{cases} \tag{2.2}$$

Clearly $W_k^{(v,x)}$ is the span of the basis elements in the first k columns.

3. Orbit closures

Our attention now turns to the Zariski closures of the G-orbits in $V \times \mathcal{N}$. Some easy facts (compare Lemma 2.6) are:

Lemma 3.1. *Let $(\mu;\nu) \in \mathcal{Q}_n$, and let $\lambda = \mu + \nu$.*

(1) $\overline{\mathcal{O}_{\mu;\nu}} \subseteq V \times \overline{\mathcal{O}_\lambda}$.
(2) $\overline{\mathcal{O}_{\emptyset;\lambda}} = \{0\} \times \overline{\mathcal{O}_\lambda}$.
(3) $\overline{\mathcal{O}_{\lambda;\emptyset}} = V \times \overline{\mathcal{O}_\lambda}$.

Proof. Part (1) follows from the fact that $\mathcal{O}_{\mu;\nu} \subseteq V \times \mathcal{O}_\lambda$. Part (2) is obvious. In part (3), the inclusion $\subseteq$ is obvious, and the right-hand side is an irreducible variety of the same dimension as the left-hand side. □

The closures $\overline{\mathcal{O}_{\mu;\nu}}$ are in general singular varieties, and our first aim is to define resolutions of their singularities. Motivated by a standard construction for the closures $\overline{\mathcal{O}_\lambda}$, we consider partial flags whose successive codimensions are given by the composition (2.1).

Definition 3.2. For any $(\mu;\nu) \in \mathcal{Q}_n$, define a partial flag variety

$$\begin{aligned}\mathcal{F}_{\mu;\nu} = \big\{0 = V_0 \subset V_1 \subset V_2 \subset \cdots \subset V_{\mu_1+\nu_1} = V \;\big|\; &\dim V_{\mu_1-i} = |\mu| - \mu^{\mathbf{t}}_1 - \cdots - \mu^{\mathbf{t}}_i \text{ for } 0 \leqslant i \leqslant \mu_1,\\ &\dim V_{\mu_1+i} = |\mu| + \nu^{\mathbf{t}}_1 + \cdots + \nu^{\mathbf{t}}_i \text{ for } 0 \leqslant i \leqslant \nu_1\big\},\end{aligned}$$

and two related varieties

$$\widehat{\mathcal{F}_{\mu;\nu}} = \big\{\big(x,(V_k)\big) \in \mathcal{N} \times \mathcal{F}_{\mu;\nu} \;\big|\; x(V_k) \subseteq V_{k-1} \text{ for } 1 \leqslant k \leqslant \mu_1+\nu_1\big\},$$
$$\widetilde{\mathcal{F}_{\mu;\nu}} = \big\{\big(v,x,(V_k)\big) \in V \times \mathcal{N} \times \mathcal{F}_{\mu;\nu} \;\big|\; v \in V_{\mu_1},\ \big(x,(V_k)\big) \in \widehat{\mathcal{F}_{\mu;\nu}}\big\}.$$

We have obvious actions of G on these varieties. Let

$$\psi_{\mu;\nu} : \widehat{\mathcal{F}_{\mu;\nu}} \to \mathcal{N} \quad \text{and} \quad \pi_{\mu;\nu} : \widetilde{\mathcal{F}_{\mu;\nu}} \to V \times \mathcal{N}$$

denote the projection maps, which are G-equivariant.

The statements relating to $\psi_{\mu;\nu}$ in the following result are known, but included for ease of reference; they hold for general compositions of n, not just those of the form (2.1), but this is the only case we will need.

Proposition 3.3. *For any $(\mu;\nu) \in \mathcal{Q}_n$,*

1. *the varieties $\widehat{\mathcal{F}_{\mu;\nu}}$ and $\widetilde{\mathcal{F}_{\mu;\nu}}$ are nonsingular and irreducible;*
2. *the projections $\psi_{\mu;\nu} : \widehat{\mathcal{F}_{\mu;\nu}} \to \mathcal{N}$ and $\pi_{\mu;\nu} : \widetilde{\mathcal{F}_{\mu;\nu}} \to V \times \mathcal{N}$ are proper;*
3. *the image of $\psi_{\mu;\nu} : \widehat{\mathcal{F}_{\mu;\nu}} \to \mathcal{N}$ is the closure $\overline{\mathcal{O}_{\mu+\nu}}$;*
4. *the restriction of $\psi_{\mu;\nu}$ to $\psi^{-1}_{\mu;\nu}(\mathcal{O}_{\mu+\nu})$ is an isomorphism onto $\mathcal{O}_{\mu+\nu}$;*
5. *the image of $\pi_{\mu;\nu} : \widetilde{\mathcal{F}_{\mu;\nu}} \to V \times \mathcal{N}$ is the closure $\overline{\mathcal{O}_{\mu;\nu}}$; and*
6. *the restriction of $\pi_{\mu;\nu}$ to $\pi^{-1}_{\mu;\nu}(\mathcal{O}_{\mu;\nu})$ is an isomorphism onto $\mathcal{O}_{\mu;\nu}$.*

In summary, $\psi_{\mu;\nu} : \widehat{\mathcal{F}_{\mu;\nu}} \to \overline{\mathcal{O}_{\mu+\nu}}$ is a resolution of singularities of $\overline{\mathcal{O}_{\mu+\nu}}$, and $\pi_{\mu;\nu} : \widetilde{\mathcal{F}_{\mu;\nu}} \to \overline{\mathcal{O}_{\mu;\nu}}$ is a resolution of singularities of $\overline{\mathcal{O}_{\mu;\nu}}$.

Proof. The partial flag variety $\mathcal{F}_{\mu;\nu}$ is a homogenous variety for G with parabolic stabilizers. Let $P_{\mu;\nu}$ denote one of these stabilizers, say the stabilizer of the partial flag $(V^0_k) \in \mathcal{F}_{\mu;\nu}$, and let $U_{\mu;\nu}$ be the unipotent radical of $P_{\mu;\nu}$. Then we have an isomorphism

$$P_{\mu;\nu}/U_{\mu;\nu} \cong \mathrm{GL}_{\mu^{\mathbf{t}}_1} \times \cdots \times \mathrm{GL}_{\mu^{\mathbf{t}}_{\mu_1}} \times \mathrm{GL}_{\nu^{\mathbf{t}}_1} \times \cdots \times \mathrm{GL}_{\nu^{\mathbf{t}}_{\nu_1}}. \tag{3.1}$$

As a consequence, we have

$$\begin{aligned}\dim \mathcal{F}_{\mu;\nu} = \dim G - \dim P_{\mu;\nu} = \dim U_{\mu;\nu} &= \frac{\dim G - \dim(P_{\mu;\nu}/U_{\mu;\nu})}{2}\\ &= \frac{n^2 - (\mu^{\mathbf{t}}_1)^2 - \cdots - (\mu^{\mathbf{t}}_{\mu_1})^2 - (\nu^{\mathbf{t}}_1)^2 - \cdots - (\nu^{\mathbf{t}}_{\nu_1})^2}{2}\end{aligned}$$

$$= \frac{n^2 - n}{2} - n(\mu) - n(\nu).$$

The projection $\widehat{\mathcal{F}_{\mu;\nu}} \to \mathcal{F}_{\mu;\nu}$ is well known to be a vector bundle; the fibre over (V_k^0) is exactly $\mathrm{Lie}(U_{\mu;\nu})$, and a common notation for $\widehat{\mathcal{F}_{\mu;\nu}}$ is $G \times_{P_{\mu;\nu}} \mathrm{Lie}(U_{\mu;\nu})$. In particular, we have

$$\dim \widehat{\mathcal{F}_{\mu;\nu}} = 2 \dim \mathcal{F}_{\mu;\nu} = n^2 - n - 2n(\mu) - 2n(\nu) = \dim \mathcal{O}_{\mu+\nu}. \tag{3.2}$$

Clearly the projection $\widetilde{\mathcal{F}_{\mu;\nu}} \to \widehat{\mathcal{F}_{\mu;\nu}}$ is also a vector bundle, of rank $|\mu|$ since the fibre over $(x, (V_k))$ is just V_{μ_1}. So

$$\dim \widetilde{\mathcal{F}_{\mu;\nu}} = n^2 - |\nu| - 2n(\mu) - 2n(\nu) = \dim \mathcal{O}_{\mu;\nu}, \tag{3.3}$$

where the last equality uses Proposition 2.8(8). Since the total space of a vector bundle over a nonsingular irreducible variety is nonsingular and irreducible, part (1) is proved.

Part (2) follows from the fact that $\mathcal{F}_{\mu;\nu}$ is a projective variety, and $\widehat{\mathcal{F}_{\mu;\nu}}$ and $\widetilde{\mathcal{F}_{\mu;\nu}}$ are closed subvarieties of $\mathcal{N} \times \mathcal{F}_{\mu;\nu}$ and $V \times \mathcal{N} \times \mathcal{F}_{\mu;\nu}$ respectively.

It follows that the images of $\psi_{\mu;\nu}$ and $\pi_{\mu;\nu}$ are G-stable irreducible closed subvarieties of $\mathcal{N}$ and $V \times \mathcal{N}$ respectively. Since G has finitely many orbits in $\mathcal{N}$ and in $V \times \mathcal{N}$, we can conclude that both images are the closure of a single G-orbit. Moreover, $\mathcal{O}_{\mu;\nu}$ is contained in the image of $\pi_{\mu;\nu}$, since for any $(v, x) \in \mathcal{O}_{\mu;\nu}$, $(v, x, (W_k^{(v,x)})) \in \widetilde{\mathcal{F}_{\mu;\nu}}$; this also shows that $\mathcal{O}_{\mu+\nu}$ is contained in the image of $\psi_{\mu;\nu}$. Since we have the dimension equalities (3.2) and (3.3), parts (3) and (5) follow.

Part (4) asserts that for any $x \in \mathcal{O}_{\mu+\nu}$ there is a unique pair $(x, (W_k)) \in \psi_{\mu;\nu}^{-1}(x)$, and the map $\mathcal{O}_{\mu+\nu} \to \mathcal{F}_{\mu;\nu} : x \mapsto (W_k)$ is a morphism of varieties. This statement is part of the theory of Richardson orbits in $\mathrm{Lie}(G)$, for which see [4, Theorem 5.2.3 and Corollary 5.2.4], for example. There is a dense $P_{\mu;\nu}$-orbit $\mathcal{O}$ in $\mathrm{Lie}(U_{\mu;\nu})$, and its G-saturation is known to be $\mathcal{O}_{\mu+\nu}$. If x_0 is a fixed element of $\mathcal{O}$, then its stabilizer G^{x_0} is contained in $P_{\mu;\nu}$, and the unique conjugate P' of $P_{\mu;\nu}$ satisfying $x_0 \in \mathrm{Lie}(U_{P'})$ is $P_{\mu;\nu}$ itself. Hence for an arbitrary element $gx_0 \in \mathcal{O}_{\mu+\nu}$, the unique conjugate P' of $P_{\mu;\nu}$ satisfying $gx_0 \in \mathrm{Lie}(U_{P'})$ is $gP_{\mu;\nu}g^{-1}$, and the map $gx_0 \mapsto gP_{\mu;\nu}$ is a morphism of varieties from $\mathcal{O}_{\mu+\nu} \cong G/G^{x_0}$ to $G/P_{\mu;\nu}$, as required.

For part (6) we need to show that for any $(v, x) \in \mathcal{O}_{\mu;\nu}$ there is a unique triple in the fibre $\pi_{\mu;\nu}^{-1}(v, x)$, namely $(v, x, (W_k^{(v,x)}))$, and moreover that the map $(v, x) \mapsto (W_k^{(v,x)})$ is a morphism of varieties from $\mathcal{O}_{\mu;\nu}$ to $\mathcal{F}_{\mu;\nu}$. Both claims clearly follow from part (4). □

We can now give an alternative characterization of $\overline{\mathcal{O}_{\mu;\nu}}$, which should be compared with Corollary 2.9.

Corollary 3.4. *If $(v, x) \in V \times \mathcal{N}$, then $(v, x) \in \overline{\mathcal{O}_{\mu;\nu}}$ if and only if there exists a $|\mu|$-dimensional subspace W of V such that:*

(1) $v \in W$,
(2) $x(W) \subseteq W$,
(3) *the Jordan type μ' of $x|_W$ satisfies $\mu' \leqslant \mu$, and*
(4) *the Jordan type ν' of $x|_{V/W}$ satisfies $\nu' \leqslant \nu$.*

Proof. By part (5) of Proposition 3.3, $(v, x) \in \overline{\mathcal{O}_{\mu;\nu}}$ if and only if there exists a partial flag $(V_k) \in \mathcal{F}_{\mu;\nu}$ such that $(v, x, (V_k)) \in \widetilde{\mathcal{F}_{\mu;\nu}}$. Setting $W = V_{\mu_1}$, we see that this is equivalent to the existence of a $|\mu|$-dimensional subspace W of V satisfying conditions (1), (2), and the following:

(3′) there is a partial flag $0 = W_0 \subset W_1 \subset \cdots \subset W_{\mu_1} = W$ such that

$$x(W_k) \subseteq W_{k-1} \quad \text{and} \quad \dim W_k = \mu^{\mathbf{t}}_{\mu_1-k+1} + \cdots + \mu^{\mathbf{t}}_{\mu_1}, \quad \text{for } k = 1, \ldots, \mu_1;$$

(4′) there is a partial flag $0 = U_0 \subset U_1 \subset \cdots \subset U_{\nu_1} = V/W$ such that

$$x(U_k) \subseteq U_{k-1} \quad \text{and} \quad \dim U_k = \nu^{\mathbf{t}}_1 + \cdots + \nu^{\mathbf{t}}_k, \quad \text{for } k = 1, \ldots, \nu_1.$$

By the $\nu = \emptyset$ and $\mu = \emptyset$ special cases of Proposition 3.3, the condition (3′) is equivalent to $x|_W \in \overline{\mathcal{O}_\mu}$, where $\mathcal{O}_\mu$ denotes the $\mathrm{GL}(W)$-orbit of nilpotent endomorphisms of W whose Jordan type is μ, and (4') is equivalent to $x|_{V/W} \in \overline{\mathcal{O}_\nu}$, where $\mathcal{O}_\nu$ is defined similarly. Finally, the closure relation among nilpotent orbits for the general linear group is well known to be given by the dominance order on partitions. □

Beware that the existence of a $|\mu|$-dimensional subspace W of V such that (1)–(4) hold with equality in (3) and (4) does not imply that $(v, x) \in \mathcal{O}_{\mu;\nu}$. (The criterion in Corollary 2.9 refers to the specific subspace $W = E^x v$.)

Example 3.5. Two salient examples when $n = 4$ are as follows. Firstly, suppose that $(v, x) \in \mathcal{O}_{(1^2);(2)}$, and let $\{v_{11}, v_{12}, v_{13}, v_{21}\}$ be a normal basis for (v, x); we have $v = v_{11} + v_{21}$. Then $W = \mathrm{span}\{v_{11}, v_{12}, v_{21}\}$ is a three-dimensional x-stable subspace containing v, such that the Jordan type of $x|_W$ is (21) and that of $x|_{V/W}$ is (1). By Corollary 3.4, we have $(v, x) \in \overline{\mathcal{O}_{(21);(1)}}$, and hence $\mathcal{O}_{(1^2);(2)} \subset \overline{\mathcal{O}_{(21);(1)}}$. Secondly, suppose that $(v, x) \in \mathcal{O}_{(2^2);\emptyset}$, and let $\{v_{11}, v_{12}, v_{21}, v_{22}\}$ be a normal basis for (v, x); we have $v = v_{12} + v_{22}$. Then $W = \mathrm{span}\{v_{11}, v_{12} + v_{22}, v_{21}\}$ is a three-dimensional x-stable subspace containing v, such that the Jordan type of $x|_W$ is (21) and that of $x|_{V/W}$ is (1). By Corollary 3.4, we have $(v, x) \in \overline{\mathcal{O}_{(21);(1)}}$, and hence $\mathcal{O}_{(2^2);\emptyset} \subset \overline{\mathcal{O}_{(21);(1)}}$.

We now define the partial order which, we will show, corresponds to the closure ordering on G-orbits in $V \times \mathcal{N}$.

Definition 3.6. For $(\rho; \sigma), (\mu; \nu) \in \mathcal{Q}_n$, we say that $(\rho; \sigma) \leqslant (\mu; \nu)$ if and only if the following inequalities hold for all $k \geqslant 0$:

$$\rho_1 + \sigma_1 + \rho_2 + \sigma_2 + \cdots + \rho_k + \sigma_k \leqslant \mu_1 + \nu_1 + \mu_2 + \nu_2 + \cdots + \mu_k + \nu_k, \quad \text{and}$$
$$\rho_1 + \sigma_1 + \cdots + \rho_k + \sigma_k + \rho_{k+1} \leqslant \mu_1 + \nu_1 + \cdots + \mu_k + \nu_k + \mu_{k+1}.$$

This coincides with the partial order used by Shoji for his “limit symbols” with $e = 2$ (see [25]). Note that the inequalities of the first kind simply say that $\rho + \sigma \leqslant \mu + \nu$ for the dominance order. Obviously $\rho \leqslant \mu$ and $\sigma \leqslant \nu$ together imply $(\rho; \sigma) \leqslant (\mu; \nu)$, but the converse is false.

To clarify the partial order, we describe its covering relations: for $(\rho; \sigma), (\mu; \nu) \in \mathcal{Q}_n$, we say that $(\mu; \nu)$ covers $(\rho; \sigma)$ if $(\rho; \sigma) < (\mu; \nu)$ and there is no $(\tau; \upsilon) \in \mathcal{Q}_n$ such that $(\rho; \sigma) < (\tau; \upsilon) < (\mu; \nu)$.

Lemma 3.7. *For $(\rho;\sigma), (\mu;\nu) \in \mathcal{Q}_n$, $(\mu;\nu)$ covers $(\rho;\sigma)$ if and only if one of the following holds.*

(1) *$\sigma = \nu$, and for some $\ell > k \geqslant 2$ we have*

$$\rho_k = \mu_k - 1, \quad \rho_\ell = \mu_\ell + 1, \quad \rho_i = \mu_i \quad \textit{for } i \neq k, \ell,$$
$$\textit{either} \quad \ell = k+1 \quad \textit{or} \quad \mu_k - 1 = \mu_{k+1} = \cdots = \mu_{\ell-1} = \mu_\ell + 1,$$
$$\textit{and} \quad \nu_{k-1} = \nu_k = \cdots = \nu_\ell.$$

(2) *$\rho = \mu$, and for some $\ell > k \geqslant 1$ we have*

$$\sigma_k = \nu_k - 1, \quad \sigma_\ell = \nu_\ell + 1, \quad \sigma_i = \nu_i \quad \textit{for } i \neq k, \ell,$$
$$\textit{either} \quad \ell = k+1 \quad \textit{or} \quad \nu_k - 1 = \nu_{k+1} = \cdots = \nu_{\ell-1} = \nu_\ell + 1,$$
$$\textit{and} \quad \mu_k = \mu_{k+1} = \cdots = \mu_{\ell+1}.$$

(3) *For some $\ell \geqslant k \geqslant 1$ we have*

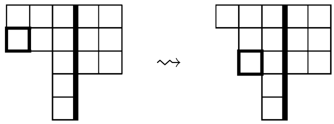
$$\rho_i = \mu_i - 1 \quad \textit{and} \quad \sigma_i = \nu_i + 1 \quad \textit{for } k \leqslant i \leqslant \ell,$$
$$\rho_i = \mu_i \quad \textit{and} \quad \sigma_i = \nu_i \quad \textit{for } i < k \textit{ and } i > \ell,$$
$$\mu_k = \mu_{k+1} = \cdots = \mu_\ell > \mu_{\ell+1},$$
$$\textit{and} \quad \nu_{k-1} > \nu_k = \nu_{k+1} = \cdots = \nu_\ell \quad (\textit{ignore } \nu_{k-1} > \nu_k \textit{ if } k = 1).$$

(4) *For some $\ell \geqslant k \geqslant 1$ we have*

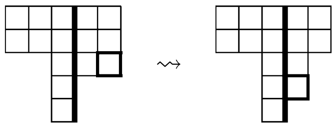
$$\sigma_i = \nu_i - 1 \quad \textit{and} \quad \rho_{i+1} = \mu_{i+1} + 1 \quad \textit{for } k \leqslant i \leqslant \ell,$$
$$\sigma_i = \nu_i \quad \textit{and} \quad \rho_{i+1} = \mu_{i+1} \quad \textit{for } i < k \textit{ and } i > \ell, \textit{ and } \rho_1 = \mu_1,$$
$$\nu_k = \nu_{k+1} = \cdots = \nu_\ell > \nu_{\ell+1},$$
$$\textit{and} \quad \mu_k > \mu_{k+1} = \cdots = \mu_{\ell+1}.$$

Proof. As with the covering relations in $\mathcal{P}_n$, the situation becomes clearer if we think diagrammatically. The relation $(\rho;\sigma) < (\mu;\nu)$ just says that the composition $(\rho_1, \sigma_1, \rho_2, \sigma_2, \ldots)$ is dominated by (and not equal to) the composition $(\mu_1, \nu_1, \mu_2, \nu_2, \ldots)$. This is equivalent to saying that the diagram of $(\rho;\sigma)$ can be obtained from that of $(\mu;\nu)$ by a sequence of moves of boxes, where at each step we move an outside corner box to an inside corner which is either in a lower row or on the right-hand end of the same row. At the start and end of such a sequence, the boxes on either side of the dividing line form the shape of a partition; $(\mu;\nu)$ covers $(\rho;\sigma)$ exactly when there is no such sequence which can be broken into two sequences with this property (in other words, for every such sequence of moves starting at $(\mu;\nu)$ and ending at $(\rho;\sigma)$, the intermediate shapes are not diagrams of bipartitions).

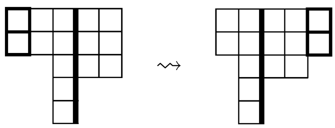
The four types of covering relations in the statement correspond to the following operations on diagrams. In type (1), a single box moves down on the μ side of the dividing line, from an outside corner to the first available inside corner, there being no inside or outside corners on the ν side between these two positions:



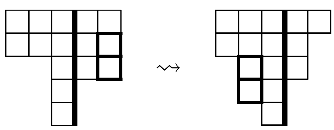
Type (2) is analogous, but with the box moving on the ν side of the dividing line:



In type (3), a column of boxes (possibly a single box) moves directly to the right, from an outside corner on the μ side to an inside corner on the ν side:



In type (4), a column of boxes (possibly a single box) moves to the left and down one row, from an outside corner on the ν side to an inside corner on the μ side:



It is easy to see that none of these operations can be broken into two steps while respecting the shape constraints.

Conversely, we must show that if $(\rho; \sigma) < (\mu; \nu)$, then we can apply one of these operations to $(\mu; \nu)$ to obtain a bipartition $(\mu'; \nu')$ which satisfies $(\mu'; \nu') \geqslant (\rho; \sigma)$. We will specify a suitable operation case by case, leaving the verification that $(\mu'; \nu') \geqslant (\rho; \sigma)$ to the reader.

We can assume that the composition $(\rho_1, \sigma_1, \rho_2, \sigma_2, \ldots)$ first differs from the composition $(\mu_1, \nu_1, \mu_2, \nu_2, \ldots)$ in one of the ρ–μ positions, since the alternative possibility can be reduced to this by inserting a single sufficiently large number N at the start of both compositions (such an insertion respects the partial order, and interchanges type (1) operations with type (2), and type (3) with type (4)). Thus we have some $i \geqslant 1$ such that $\rho_i < \mu_i$, and the first $i - 1$ parts of ρ (respectively, σ) equal those of μ (respectively, ν). Let j be the largest integer such that $\mu_j = \mu_i$, and let j' be the largest integer such that $\nu_{j'} = \nu_i$ (or set $j' = \infty$ if $\nu_i = 0$). We now have six cases.

Case I. If $i = 1$, or if $i > 1$ and $\nu_{i-1} > \nu_j$, let $k \geqslant i$ be the smallest integer such that $\nu_k = \nu_j$, and perform an operation of type (3) with this k and $\ell = j$.

Case II. If $i > 1$ and $\nu_{i-1} = \nu_j > \nu_{j+1}$ (which forces $j' = j$), perform an operation of type (4) with $k = \ell = j$.

Case III. If $i > 1$, $\nu_{i-1} = \nu_{j+1}$ (which forces $j' \geqslant j+1$), and $\mu_{j+1} \leqslant \mu_j - 2$, perform an operation of type (1) with $k = j$ and $\ell = j+1$.

Case IV. If $i > 1$, $\nu_{i-1} = \nu_{j+1}$, $\mu_{j+1} = \mu_j - 1$, $j' < \infty$, and $\mu_{j'+1} = \mu_j - 1$, perform an operation of type (4) with $k = j$ and $\ell = j'$.

Case V. If $i > 1$, $\nu_{i-1} = \nu_{j+1}$, $\mu_{j+1} = \mu_j - 1$, $j' < \infty$, and $\mu_j - 1 = \mu_{j'} > \mu_{j'+1}$, perform an operation of type (4) with $k = \ell = j'$.

Case VI. If $i > 1$, $\nu_{i-1} = \nu_{j+1}$, $\mu_{j+1} = \mu_j - 1$, and either $j' = \infty$ or $\mu_{j'} < \mu_j - 1$, let ℓ be the smallest integer such that $\mu_\ell < \mu_j - 1$. Let k be $\ell - 1$ (if $\mu_\ell < \mu_j - 2$) or j (if $\mu_\ell = \mu_j - 2$). Perform an operation of type (1) with this k and ℓ. This concludes the list of cases to be considered. □

Notice that for fixed $\lambda \in \mathcal{P}_n$, $\{(\mu; \nu) \in \mathcal{Q}_n \mid \mu + \nu = \lambda\}$ is an interval in $\mathcal{Q}_n$, in which all covering relations are of type (3).

Example 3.8. Table 1 shows the Hasse diagram of the poset $\mathcal{Q}_4$. The numbers in the left-hand column are the dimensions of the corresponding orbits, and the labels of the covering relations are the types from Lemma 3.7.

Theorem 3.9. *For $(\rho; \sigma), (\mu; \nu) \in \mathcal{Q}_n$, $\mathcal{O}_{\rho;\sigma} \subseteq \overline{\mathcal{O}_{\mu;\nu}}$ if and only if $(\rho; \sigma) \leqslant (\mu; \nu)$.*

Proof. We first prove the "only if" direction. Assume that $\mathcal{O}_{\rho;\sigma} \subseteq \overline{\mathcal{O}_{\mu;\nu}}$. Since $\overline{\mathcal{O}_{\mu;\nu}} \subseteq V \times \overline{\mathcal{O}_{\mu+\nu}}$, we have $\mathcal{O}_{\rho+\sigma} \subseteq \overline{\mathcal{O}_{\mu+\nu}}$, which as we know implies the dominance condition $\rho + \sigma \leqslant \mu + \nu$. All that remains is to prove the inequalities of the second kind, namely that for any $k \geqslant 0$,

$$\rho_1 + \sigma_1 + \cdots + \rho_k + \sigma_k + \rho_{k+1} \leqslant \mu_1 + \nu_1 + \cdots + \mu_k + \nu_k + \mu_{k+1}.$$

Our proof, like one of the standard proofs of the closure relation for ordinary nilpotent orbits, rests on the fact that if $x \in \mathcal{N}$ has Jordan type λ, then $\lambda_1 + \cdots + \lambda_k$ is the maximum possible dimension of an $\mathbb{F}[x]$-submodule $\mathbb{F}[x]\{w_1, \ldots, w_k\}$ generated by k elements $w_1, \ldots, w_k$ of V. Thanks to Lemma 2.5, this implies that for $(v, x) \in \mathcal{O}_{\mu;\nu}$, $\mu_1 + \nu_1 + \cdots + \mu_k + \nu_k + \mu_{k+1}$ is the maximum possible dimension of $\mathbb{F}[x]\{v, w_1, \ldots, w_k\}$ for $w_1, \ldots, w_k \in V$. So the desired inequality amounts to saying that for fixed N, the condition

$$\dim \mathbb{F}[x]\{v, w_1, \ldots, w_k\} \leqslant N \quad \text{for any } w_1, \ldots, w_k \in V \tag{3.4}$$

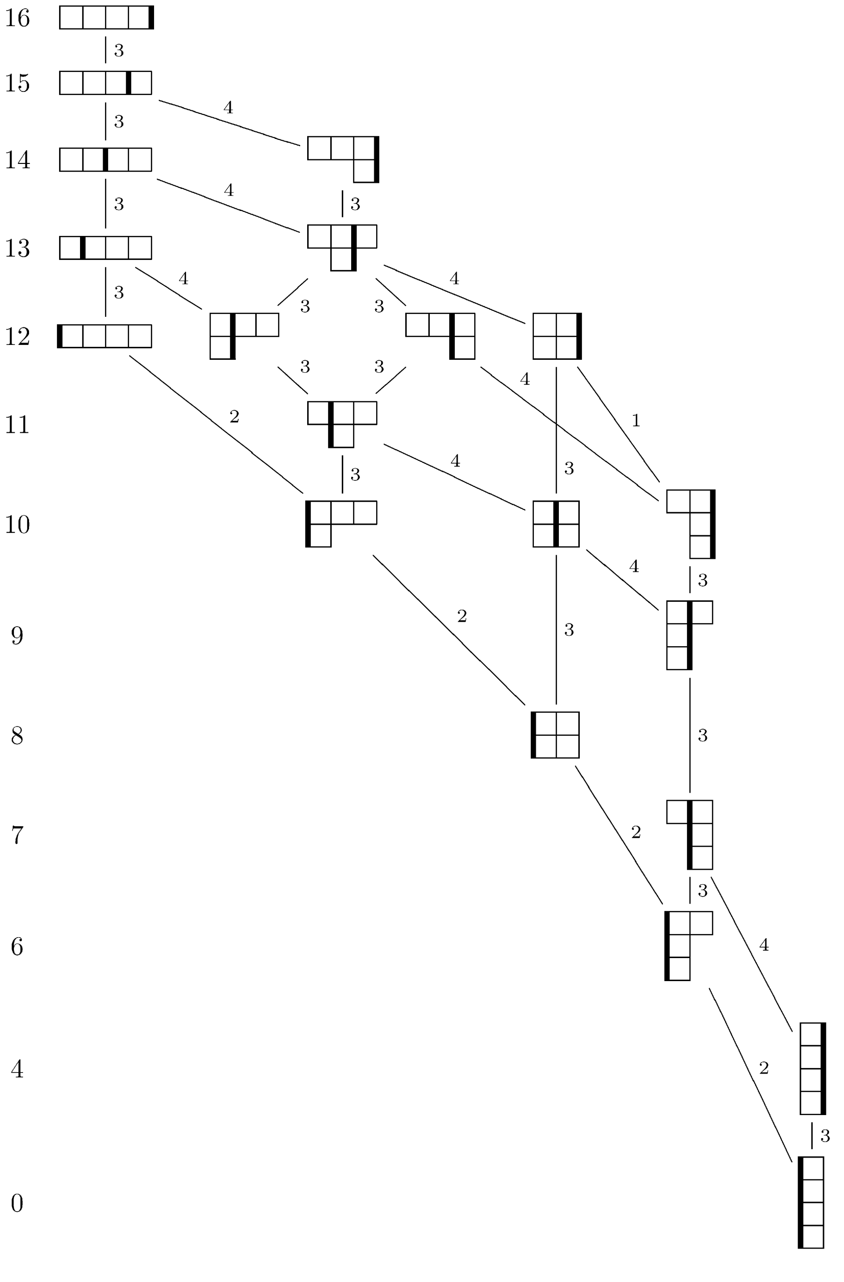
is a closed condition on (v, x) (*i.e.*, it determines a closed subvariety of $V \times \mathcal{N}$). But no matter what $v, x, w_1, \ldots, w_k$ are, $\mathbb{F}[x]\{v, w_1, \ldots, w_k\}$ is guaranteed to be spanned by the $(k+1)n$ vectors

$$v, xv, \ldots, x^{n-1}v, w_1, xw_1, \ldots, x^{n-1}w_1, \ldots, w_k, xw_k, \ldots, x^{n-1}w_k.$$

So $\dim \mathbb{F}[x]\{v, w_1, \ldots, w_k\}$ is the rank of the $n \times (k+1)n$ matrix which has these vectors as columns, and the condition (3.4) is equivalent to the vanishing of all $(N+1) \times (N+1)$ minors of this matrix. This is a collection of polynomial equations in the coordinates of v and x and the indeterminate coordinates of $w_1, \ldots, w_k$, so we are done.

To prove the "if" direction, we may assume that $(\mu; \nu)$ covers $(\rho; \sigma)$, and invoke Lemma 3.7. Let $(v, x) \in \mathcal{O}_{\rho;\sigma}$, and let $\{v_{ij}\}$ be a normal basis for (v, x). To prove that $(v, x) \in \overline{\mathcal{O}_{\mu;\nu}}$, it suffices to find a $|\mu|$-dimensional subspace W of V satisfying conditions (1)–(4) of Corollary 3.4.

Table 1
Hasse diagram for $\mathcal{Q}_4$



Recall from Corollary 2.9 that $E^x v = \operatorname{span}\{v_{ij} \mid 1 \leqslant i \leqslant \ell(\rho),\ j \leqslant \rho_i\}$ is $|\rho|$-dimensional, contains $v = \sum v_{i,\rho_i}$, and is preserved by x; moreover, the Jordan type of $x|_{E^x v}$ is ρ and the Jordan type of $x|_{V/E^x v}$ is σ. Speaking rather loosely, we will refer to the set $\{v_{ij} \mid j \leqslant \rho_i\}$ for fixed i as the ith Jordan block of $x|_{E^x v}$, and the set $\{v_{ij} \mid \rho_i < j \leqslant \rho_i + \sigma_i\}$ for fixed i as the ith Jordan block of $x|_{V/E^x v}$.

If the covering relation is one of the first two types in Lemma 3.7, we simply take $W = E^x v$. We have $|\rho| = |\mu|$, $\rho \leqslant \mu$ and $\sigma \leqslant \nu$, so $E^x v$ meets all our requirements. In the other two types, $E^x v$ must be modified slightly; the modifications we choose are modelled on Example 3.5.

In type (3), with k and ℓ as in Lemma 3.7, we take

$$W = \operatorname{span}\bigl(\{v_{ij} \mid j \leqslant \rho_i\} \cup \{v_{k,\rho_k+1}, v_{k+1,\rho_{k+1}+1}, \ldots, v_{\ell,\rho_\ell+1}\}\bigr) \supseteq E^x v.$$

This is clearly $|\mu|$-dimensional, contains v, and is preserved by x. It is also obvious that the Jordan type of $x|_W$ is μ, since we have lengthened by 1 the kth, $(k+1)$th, $\ldots$, and ℓth Jordan blocks of $x|_{E^x v}$. Similarly, the Jordan type of $x|_{V/W}$ is ν, since we have shortened by 1 the corresponding Jordan blocks of $x|_{V/E^x v}$.

In type (4), with k and ℓ as in Lemma 3.7, we take

$$\begin{aligned} W = \operatorname{span}\bigl(&\bigl(\{v_{ij} \mid j \leqslant \rho_i\} \setminus \{v_{k,\rho_k}, v_{k+1,\rho_{k+1}}, \ldots, v_{\ell+1,\rho_{\ell+1}}\}\bigr) \\ &\cup \{v_{k,\rho_k} + v_{k+1,\rho_{k+1}} + \cdots + v_{\ell+1,\rho_{\ell+1}}\}\bigr) \subseteq E^x v. \end{aligned}$$

This is clearly $|\mu|$-dimensional, contains v, and is preserved by x. The Jordan type of $x|_V$ is μ, because we have shortened by 1 the ith Jordan block of $x|_{E^x v}$ for $k+1 \leqslant i \leqslant \ell+1$, and we have kept the kth Jordan block the same length but replaced its generator v_{k,ρ_k} by $v_{k,\rho_k} + v_{k+1,\rho_{k+1}} + \cdots + v_{\ell+1,\rho_{\ell+1}}$. Similarly, the Jordan type of $x|_{V/W}$ is ν, since we have lengthened by 1 the ith Jordan block of $x|_{V/E^x v}$ for $k \leqslant i \leqslant \ell$, and we have kept the $(\ell+1)$th Jordan block the same length but replaced its generator $v_{\ell+1,\rho_{\ell+1}+\sigma_{\ell+1}}$ by $v_{k,\rho_k+\sigma_{\ell+1}} + v_{k+1,\rho_{k+1}+\sigma_{\ell+1}} + \cdots + v_{\ell+1,\rho_{\ell+1}+\sigma_{\ell+1}}$.

So in all cases a suitable subspace W can be found, and $(v, x) \in \overline{\mathcal{O}_{\mu;\nu}}$ as required. □

4. Fibres of the resolutions of singularities

For any $(\mu; \nu) \in \mathcal{Q}_n$, we would like to describe the intersection cohomology complex $\mathrm{IC}(\overline{\mathcal{O}_{\mu;\nu}}, \overline{\mathbb{Q}}_\ell)$, in particular the dimensions of its stalks. In the case of ordinary nilpotent orbit closures, the intersection cohomology is closely related to the cohomology of the fibres of the resolutions $\psi_{\mu;\nu} : \widehat{\mathcal{F}_{\mu;\nu}} \to \overline{\mathcal{O}_{\mu+\nu}}$, which are generalized Springer fibres of type A ("generalized" in that they involve the partial flag variety $\mathcal{F}_{\mu;\nu}$ rather than the complete flag variety; in the terminology of [3], they are examples of Spaltenstein's varieties $\mathcal{P}_x^0$). Analogously, we need to study the fibres of $\pi_{\mu;\nu} : \widetilde{\mathcal{F}_{\mu;\nu}} \to \overline{\mathcal{O}_{\mu;\nu}}$. We adopt a convenient abuse of notation for these fibres: for $x \in \overline{\mathcal{O}_{\mu+\nu}}$, $\psi_{\mu;\nu}^{-1}(x)$ will refer to the variety of partial flags $(V_k) \in \mathcal{F}_{\mu;\nu}$ such that $x(V_k) \subseteq V_{k-1}$ for $1 \leqslant k \leqslant \mu_1 + \nu_1$, not to the corresponding variety of pairs $(x, (V_k))$; and similarly, we regard $\pi_{\mu;\nu}^{-1}(v, x)$ as the closed subvariety of $\psi_{\mu;\nu}^{-1}(x)$ defined by the extra condition $v \in V_{\mu_1}$.

Recall that the resolution $\psi_{\mu;\nu}$ is semismall in the sense of Goresky and MacPherson. In fact, Spaltenstein in [26] proved a more precise statement:

Theorem 4.1. *Let $(\mu; \nu) \in \mathcal{Q}_n$. For $x \in \mathcal{O}_\pi \subseteq \overline{\mathcal{O}_{\mu+\nu}}$, $\psi_{\mu;\nu}^{-1}(x)$ has $K_{\pi^{\mathrm{t}}(\mu+\nu)^{\mathrm{t}}}$ irreducible components, all of dimension*

$$n(\pi) - n(\mu+\nu) = \frac{\dim \mathcal{O}_{\mu+\nu} - \dim \mathcal{O}_\pi}{2}.$$

Here $K_{\pi^{\mathrm{t}}(\mu+\nu)^{\mathrm{t}}}$ is the Kostka number, defined as in [22, I.§6].

For $(v,x) \in V \times \mathcal{N}$, let $P^{(v,x)}$ denote the parabolic subgroup of G which is the stabilizer of the partial flag $(W_k^{(v,x)})$ defined in Section 2. Recall that $E^x v$ is one of the subspaces in this partial flag, and that $x(W_k^{(v,x)}) \subseteq W_{k-1}^{(v,x)}$, which means that x belongs to $\mathrm{Lie}(U^{(v,x)})$, where $U^{(v,x)}$ is the unipotent radical of $P^{(v,x)}$. We can regard (v,x) as an element of the vector space $E^x v \oplus \mathrm{Lie}(U^{(v,x)})$, on which $P^{(v,x)}$ acts.

Lemma 4.2. *The $P^{(v,x)}$-orbit of (v,x) is dense in $E^x v \oplus \mathrm{Lie}(U^{(v,x)})$.*

Proof. Recall from the proof of Proposition 3.3 that the Richardson orbit of $P^{(v,x)}$ is the one containing x, so the $P^{(v,x)}$-orbit of x is dense in $\mathrm{Lie}(U^{(v,x)})$. Hence it suffices to show that the $(G^x \cap P^{(v,x)})$-orbit of v is dense in $E^x v$. But $G^x \cap P^{(v,x)}$ is dense in $E^x \cap \mathrm{Lie}(P^{(v,x)})$, and $(E^x \cap \mathrm{Lie}(P^{(v,x)}))v = E^x v$ because, in the notation of Proposition 2.8, $v_{ij} = y_{i,i,\mu_i - j} v$ for all v_{ij} in the basis of $E^x v$. □

Lemma 4.3. *Suppose that $(V_k^0) \in \pi_{\mu;\nu}^{-1}(v,x)$, and let $\mathcal{O}$ be the $P^{(v,x)}$-orbit of (V_k^0) in $\mathcal{F}_{\mu;\nu}$. Let $P_{\mu;\nu}$ denote the stabilizer in G of the partial flag (V_k^0), and $U_{\mu;\nu}$ its unipotent radical. Then $\mathcal{O} \cap \pi_{\mu;\nu}^{-1}(v,x)$ is a nonsingular locally closed subvariety of $\pi_{\mu;\nu}^{-1}(v,x)$, of dimension*

$$\dim\left(\frac{P^{(v,x)}}{U^{(v,x)}}\right) - \dim\left(\frac{P^{(v,x)} \cap P_{\mu;\nu}}{U^{(v,x)} \cap U_{\mu;\nu}}\right) - \dim\left(\frac{E^x v}{E^x v \cap V_{\mu_1}^0}\right).$$

Proof. The variety $\mathcal{O} \cap \pi_{\mu;\nu}^{-1}(v,x)$ is clearly isomorphic to

$$\left\{p \in P^{(v,x)} \mid p^{-1}.(v,x) \in \left(E^x v \cap V_{\mu_1}^0\right) \oplus \left(\mathrm{Lie}\left(U^{(v,x)}\right) \cap \mathrm{Lie}(U_{\mu;\nu})\right)\right\} / \left(P^{(v,x)} \cap P_{\mu;\nu}\right),$$

so it suffices to prove that

$$\left\{p \in P^{(v,x)} \mid p^{-1}.(v,x) \in \left(E^x v \cap V_{\mu_1}^0\right) \oplus \left(\mathrm{Lie}\left(U^{(v,x)}\right) \cap \mathrm{Lie}(U_{\mu;\nu})\right)\right\}$$

is nonsingular and has dimension

$$\dim P^{(v,x)} - \dim\left(E^x v \oplus \mathrm{Lie}\left(U^{(v,x)}\right)\right) + \dim\left(\left(E^x v \cap V_{\mu_1}^0\right) \oplus \left(\mathrm{Lie}\left(U^{(v,x)}\right) \cap \mathrm{Lie}(U_{\mu;\nu})\right)\right).$$

As is observed in a general context in [5, Lemma 2.2], this is implied by the density proved in the previous lemma. □

Proposition 4.4. *Let $(v,x) \in \mathcal{O}_{\rho;\sigma} \subseteq \overline{\mathcal{O}_{\mu;\nu}}$. Let X be the closed subvariety of $\pi_{\mu;\nu}^{-1}(v,x)$ defined by the extra condition $V_{\mu_1} = E^x v$.*

(1) *X is empty unless $\rho \leqslant \mu$ and $\sigma \leqslant \nu$, in which case it has $K_{\rho^{\mathrm{t}}\mu^{\mathrm{t}}} K_{\sigma^{\mathrm{t}}\nu^{\mathrm{t}}}$ irreducible components, all of dimension $n(\rho+\sigma) - n(\mu+\nu)$.*
(2) $\dim(\pi_{\mu;\nu}^{-1}(v,x) \setminus X) < n(\rho+\sigma) - n(\mu+\nu) + \frac{|\mu|-|\rho|}{2}$.

Proof. It is clear that X is empty unless $\dim E^x v = |\mu|$, in which case $X \cong \psi_{\mu;\emptyset}^{-1}(x|_{E^x v}) \times \psi_{\emptyset;\nu}^{-1}(x|_{V/E^x v})$, where we use $E^x v$ and $V/E^x v$ in place of the vector space V in defining $\psi_{\mu;\emptyset}$ and $\psi_{\emptyset;\nu}$ respectively. Recalling that $x|_{E^x v} \in \mathcal{O}_\rho$ and $x|_{V/E^x v} \in \mathcal{O}_\sigma$, part (1) follows from Theorem 4.1.

To prove part (2), we observe that $\pi_{\mu;\nu}^{-1}(v,x) \setminus X$ is a union of finitely many locally closed pieces $\mathcal{O} \cap \pi_{\mu;\nu}^{-1}(v,x)$ as in Lemma 4.3, because $P^{(v,x)}$ has finitely many orbits in $\mathcal{F}_{\mu;\nu}$ (and fixes $E^x v$). So it suffices to show the desired inequality for one of these pieces, where in addition to the dimension formula of Lemma 4.3 we know that $V_{\mu_1}^0 \neq E^x v$. Since $\dim E^x v = |\rho|$ and $\dim V_{\mu_1}^0 = |\mu|$,

$$\dim\left(\frac{E^x v}{E^x v \cap V_{\mu_1}^0}\right) > \frac{|\rho| - |\mu|}{2}.$$

Also, by the same argument as in the proof of Lemma 4.3,

$$\dim\left(\frac{P^{(v,x)}}{U^{(v,x)}}\right) - \dim\left(\frac{P^{(v,x)} \cap P_{\mu;\nu}}{U^{(v,x)} \cap U_{\mu;\nu}}\right)$$

is the dimension of a subvariety of $\psi_{\mu;\nu}^{-1}(x)$, and so is at most $n(\rho+\sigma) - n(\mu+\nu)$ by Theorem 4.1. The result follows. □

The special case of the next result where $\mu + \nu = (n)$ was proved independently in [8, (11)].

Theorem 4.5. *Let* $(\mu;\nu) \in \mathcal{Q}_n$.

(1) *The resolution of singularities* $\pi_{\mu;\nu} : \widetilde{\mathcal{F}_{\mu;\nu}} \to \overline{\mathcal{O}_{\mu;\nu}}$ *is semismall.*
(2) *We have an isomorphism of semisimple perverse sheaves:*

$$R(\pi_{\mu;\nu})_* \overline{\mathbb{Q}}_\ell[\dim \mathcal{O}_{\mu;\nu}] \cong \bigoplus_{\substack{\rho \leqslant \mu \\ \sigma \leqslant \nu}} K_{\rho^{\mathsf{t}}\mu^{\mathsf{t}}} K_{\sigma^{\mathsf{t}}\nu^{\mathsf{t}}} \, \mathrm{IC}(\overline{\mathcal{O}_{\rho;\sigma}}, \overline{\mathbb{Q}}_\ell)[\dim \mathcal{O}_{\rho;\sigma}],$$

where mA *denotes* $A \oplus \cdots \oplus A$ *(m copies).*
(3) *For* $(v,x) \in \overline{\mathcal{O}_{\mu;\nu}}$, *we have*

$$\dim H^i\big(\pi_{\mu;\nu}^{-1}(v,x), \overline{\mathbb{Q}}_\ell\big) = \sum_{\substack{\rho \leqslant \mu \\ \sigma \leqslant \nu \\ \overline{\mathcal{O}_{\rho;\sigma}} \ni (v,x)}} K_{\rho^{\mathsf{t}}\mu^{\mathsf{t}}} K_{\sigma^{\mathsf{t}}\nu^{\mathsf{t}}} \dim \mathcal{H}_{(v,x)}^{i-2(n(\rho+\sigma)-n(\mu+\nu))} \mathrm{IC}(\overline{\mathcal{O}_{\rho;\sigma}}, \overline{\mathbb{Q}}_\ell).$$

Proof. Part (1) asserts that for $(v,x) \in \mathcal{O}_{\rho;\sigma} \subseteq \overline{\mathcal{O}_{\mu;\nu}}$, we have

$$\dim \pi_{\mu;\nu}^{-1}(v,x) \leqslant \frac{\dim \mathcal{O}_{\mu;\nu} - \dim \mathcal{O}_{\rho;\sigma}}{2}. \tag{4.1}$$

By Proposition 2.8, this upper bound is nothing but $n(\rho+\sigma) - n(\mu+\nu) + \frac{|\mu|-|\rho|}{2}$, so the inequality follows from Proposition 4.4. Semismallness of $\pi_{\mu;\nu}$ implies that $R(\pi_{\mu;\nu})_* \overline{\mathbb{Q}}_\ell[\dim \mathcal{O}_{\mu;\nu}]$ is a semisimple perverse sheaf on $\overline{\mathcal{O}_{\mu;\nu}}$. By G-equivariance and the fact that the stabilizers $G^{(v,x)}$ are connected (Proposition 2.8(7)), we have

$$R(\pi_{\mu;\nu})_*\overline{\mathbb{Q}}_\ell[\dim\mathcal{O}_{\mu;\nu}] \cong \bigoplus_{(\tau;\upsilon)\leqslant(\mu;\nu)} m^{(\mu;\nu)}_{(\tau;\upsilon)} \operatorname{IC}(\overline{\mathcal{O}_{\tau;\upsilon}}, \overline{\mathbb{Q}}_\ell)[\dim\mathcal{O}_{\tau;\upsilon}] \tag{4.2}$$

for some nonnegative integers $m^{(\mu;\nu)}_{(\tau;\upsilon)}$. Recall that if $(\rho;\sigma) < (\tau;\upsilon)$, then

$$\mathcal{H}^i \operatorname{IC}(\overline{\mathcal{O}_{\tau;\upsilon}}, \overline{\mathbb{Q}}_\ell)|_{\mathcal{O}_{\rho;\sigma}} = 0 \quad \text{for } i \geqslant \dim\mathcal{O}_{\tau;\upsilon} - \dim\mathcal{O}_{\rho;\sigma}. \tag{4.3}$$

So taking the stalk of the $(-\dim\mathcal{O}_{\rho;\sigma})$th cohomology sheaf of both sides of (4.2) at $(v,x) \in \mathcal{O}_{\rho;\sigma}$, we find

$$\dim H^{\dim\mathcal{O}_{\mu;\nu}-\dim\mathcal{O}_{\rho;\sigma}}\big(\pi^{-1}(v,x), \overline{\mathbb{Q}}_\ell\big) = m^{(\mu;\nu)}_{(\rho;\sigma)}. \tag{4.4}$$

But Proposition 4.4 implies that

$$\dim H^{\dim\mathcal{O}_{\mu;\nu}-\dim\mathcal{O}_{\rho;\sigma}}\big(\pi^{-1}(v,x), \overline{\mathbb{Q}}_\ell\big) = \begin{cases} K_{\rho^{\mathrm{t}}\mu^{\mathrm{t}}} K_{\sigma^{\mathrm{t}}\nu^{\mathrm{t}}}, & \text{if } \rho \leqslant \mu \text{ and } \sigma \leqslant \nu, \\ 0, & \text{otherwise.} \end{cases}$$

Part (2) follows, and part (3) is an immediate consequence. □

Part (2) of Theorem 4.5 implies that the perverse sheaves $R(\pi_{\mu;\nu})_*\overline{\mathbb{Q}}_\ell[\dim\mathcal{O}_{\mu;\nu}]$ form a basis for the Grothendieck group of G-equivariant perverse sheaves on $V \times \mathcal{N}$, because the transition matrix $(K_{\rho^{\mathrm{t}}\mu^{\mathrm{t}}} K_{\sigma^{\mathrm{t}}\nu^{\mathrm{t}}})_{(\mu;\nu),(\rho;\sigma)}$ is unitriangular. In particular, the simple perverse sheaves $\operatorname{IC}(\overline{\mathcal{O}_{\rho;\sigma}}, \overline{\mathbb{Q}}_\ell)[\dim\mathcal{O}_{\rho;\sigma}]$ are the unique complexes satisfying Theorem 4.5(2). Similarly, Theorem 4.5 part (3) can be used to determine the local intersection cohomologies $\dim\mathcal{H}^i_{(v,x)}\operatorname{IC}(\overline{\mathcal{O}_{\mu;\nu}}, \overline{\mathbb{Q}}_\ell)$, if the Betti numbers of the fibres $\pi^{-1}_{\mu;\nu}(v,x)$ are known.

We can now obtain a sheaf-theoretic analogue of Corollary 3.4, using a construction similar to Lusztig's definition of multiplication in geometric Hall algebras ([21, §3]—see [8, §4] for a definition of "Hall bimodule" based on the same idea). We need to keep track of dimensions in our notation, so we temporarily write $\mathbb{F}^n$ instead of V and $\mathcal{N}_n$ instead of $\mathcal{N}$. Define varieties

$$\mathcal{G}_{m,n-m} = \big\{(v,x,W) \bigm| v \in W \subseteq \mathbb{F}^n,\ \dim W = m,\ x \in \mathcal{N}_n,\ x(W) \subseteq W\big\},$$
$$\mathcal{H}_{m,n-m} = \big\{(v,x,W,\psi_1,\psi_2) \bigm| (v,x,W) \in \mathcal{G}_{m,n-m},\ : W \xrightarrow{\sim} \mathbb{F}^m,\ \psi_2 : \mathbb{F}^n/W \xrightarrow{\sim} \mathbb{F}^{n-m}\big\}.$$

We have obvious projection maps $\pi_{m,n-m} : \mathcal{G}_{m,n-m} \to \mathbb{F}^n \times \mathcal{N}_n$ and $q : \mathcal{H}_{m,n-m} \to \mathcal{G}_{m,n-m}$, as well as a map

$$r : \mathcal{H}_{m,n-m} \to \mathcal{N}_m \times \mathcal{N}_{n-m} : (v,x,W,\psi_1,\psi_2) \mapsto \big(\psi_1(x|_W)\psi_1^{-1}, \psi_2(x|_{\mathbb{F}^n/W})\psi_2^{-1}\big).$$

Since r is a bundle projection with a nonsingular fibre of dimension n^2+m, the pull-back

$$r^*\big(\operatorname{IC}(\overline{\mathcal{O}_\mu}, \overline{\mathbb{Q}}_\ell)[\dim\mathcal{O}_\mu] \boxtimes \operatorname{IC}(\overline{\mathcal{O}_\nu}, \overline{\mathbb{Q}}_\ell)[\dim\mathcal{O}_\nu]\big)\big[n^2+m\big]$$

is a simple perverse sheaf on $\mathcal{H}_{m,n-m}$ for any $\mu \in \mathcal{P}_m$, $\nu \in \mathcal{P}_{n-m}$. Since it is equivariant for the obvious $(\mathrm{GL}(\mathbb{F}^m) \times \mathrm{GL}(\mathbb{F}^{n-m}))$-action on $\mathcal{H}_{m,n-m}$ (of which q is the quotient projection), it must be isomorphic to $q^* A_{\mu;\nu}[m^2 + (n-m)^2]$ for some simple perverse sheaf $A_{\mu;\nu}$ on $\mathcal{G}_{m,n-m}$.

Proposition 4.6. *For any $(\mu; \nu) \in \mathcal{Q}_n$, we have*

$$\mathrm{IC}(\overline{\mathcal{O}_{\mu;\nu}}, \overline{\mathbb{Q}}_\ell)[\dim \mathcal{O}_{\mu;\nu}] \cong R(\pi_{|\mu|,|\nu|})_* A_{\mu;\nu}.$$

Proof. We have a commutative diagram

$$\begin{array}{ccccc}
\widehat{\mathcal{F}_{\mu;\emptyset}} \times \widehat{\mathcal{F}_{\emptyset;\nu}} & \longleftarrow & X & \longrightarrow & \widetilde{\mathcal{F}_{\mu;\nu}} \\
{\scriptstyle \psi_{\mu;\emptyset}\times\psi_{\emptyset;\nu}}\downarrow & & \downarrow & & \downarrow{\scriptstyle \tilde{\pi}_{\mu;\nu}} \\
\mathcal{N}_m \times \mathcal{N}_{n-m} & \xleftarrow{\;r\;} & \mathcal{H}_{m,n-m} & \xrightarrow{\;q\;} & \mathcal{G}_{m,n-m}
\end{array}$$

where $\tilde{\pi}_{\mu;\nu}(v, x, (V_k)) = (v, x, V_{\mu_1})$. Here, X and the maps emanating from it are defined so as to make both squares Cartesian. Now Theorem 4.1 implies that

$$R(\psi_{\mu;\emptyset})_* \overline{\mathbb{Q}}_\ell[\dim \mathcal{O}_\mu] \cong \bigoplus_{\rho \leqslant \mu} K_{\rho^{\mathrm{t}}\mu^{\mathrm{t}}}\, \mathrm{IC}(\overline{\mathcal{O}_\rho}, \overline{\mathbb{Q}}_\ell)[\dim \mathcal{O}_\rho],$$

$$R(\psi_{\emptyset;\nu})_* \overline{\mathbb{Q}}_\ell[\dim \mathcal{O}_\nu] \cong \bigoplus_{\sigma \leqslant \nu} K_{\sigma^{\mathrm{t}}\nu^{\mathrm{t}}}\, \mathrm{IC}(\overline{\mathcal{O}_\sigma}, \overline{\mathbb{Q}}_\ell)[\dim \mathcal{O}_\sigma]. \tag{4.5}$$

(See also [14, Remark 5.7(3)].) Consequently,

$$R(\tilde{\pi}_{\mu;\nu})_* \overline{\mathbb{Q}}_\ell\big[\dim \mathcal{O}_\mu + \dim \mathcal{O}_\nu + 2|\mu||\nu| + |\mu|\big] \cong \bigoplus_{\substack{\rho \leqslant \mu \\ \sigma \leqslant \nu}} K_{\rho^{\mathrm{t}}\mu^{\mathrm{t}}} K_{\sigma^{\mathrm{t}}\nu^{\mathrm{t}}}\, A_{\rho;\sigma}. \tag{4.6}$$

Applying $R(\pi_{|\mu|,|\nu|})_*$ to both sides, we obtain

$$R(\pi_{\mu;\nu})_* \overline{\mathbb{Q}}_\ell[\dim \mathcal{O}_{\mu;\nu}] \cong \bigoplus_{\substack{\rho \leqslant \mu \\ \sigma \leqslant \nu}} K_{\rho^{\mathrm{t}}\mu^{\mathrm{t}}} K_{\sigma^{\mathrm{t}}\nu^{\mathrm{t}}}\, R(\pi_{|\rho|,|\sigma|})_* A_{\rho;\sigma}. \tag{4.7}$$

By the above-mentioned unitriangularity in Theorem 4.5(2), the result follows. □

This proposition is essentially equivalent to [8, Theorem 1].

Another known property of the generalized Springer fibre $\psi_{\mu;\nu}^{-1}(x)$ is that it has an *affine paving* (an alpha-partition into affine spaces, in the terminology of [5, 1.3]), and consequently has no odd-degree $\overline{\mathbb{Q}}_\ell$-cohomology. The original proof, explained by Spaltenstein in [27, 5.9] in the Springer fibre case, is by induction on the length of the partial flag, relying on the fact that for fixed $x \in \mathcal{N}$ and $\pi \in \mathcal{P}_m$, a variety of the form

$$\big\{ W \subset V \;\big|\; \dim W = m,\ x(V) \subseteq W,\ x|_W \in \mathcal{O}_\pi \big\}$$

can be paved by affine spaces (which in turn follows from the fact that, under the constraint $x(V) \subseteq W$, the Jordan type of $x|_W$ is determined by that of x and the dimensions $\dim W \cap \ker(x^j)$). A naive analogue of this approach for the fibres $\pi_{\mu;\nu}^{-1}(v,x)$ fails: for example, if $(v,x) \in \mathcal{O}_{(21^2);(1^3)}$, the variety

$$\{W \subset V \mid \dim W = 5,\ v \in W,\ x(V) \subseteq W,\ (v, x|_W) \in \mathcal{O}_{(21^2);(1)}\}$$

is isomorphic to $\mathbb{A}^2 \setminus \{0\}$. Hence the need to be somewhat more careful in proving:

Theorem 4.7. *For any $(v,x) \in \overline{\mathcal{O}_{\mu;\nu}}$, the fibre $\pi_{\mu;\nu}^{-1}(v,x)$ has an affine paving.*

Proof. Let P be the maximal parabolic subgroup of G which is the stabilizer of the subspace $\mathbb{F}[x]v$. We can partition $\mathcal{F}_{\mu;\nu}$ into the orbits of P, which are well known to be of the form

$$(\mathcal{F}_{\mu;\nu})_{(d_0,d_1,\dots,d_{\mu_1+\nu_1})} = \{(V_k) \in \mathcal{F}_{\mu;\nu} \mid \dim(V_k \cap \mathbb{F}[x]v) = d_k,\ 0 \leqslant k \leqslant \mu_1+\nu_1\},$$

for integers d_k which satisfy $0 = d_0 \leqslant d_1 \leqslant \cdots \leqslant d_{\mu_1+\nu_1} = \dim \mathbb{F}[x]v$ (and also some upper bounds on $d_{k+1} - d_k$ to guarantee nonemptiness of the above set, which need not concern us). It suffices to show that each $\pi_{\mu;\nu}^{-1}(v,x) \cap (\mathcal{F}_{\mu;\nu})_{(d_k)}$ has an affine paving. But for $(V_k) \in \pi_{\mu;\nu}^{-1}(v,x)$, we have the extra information that $V_{\mu_1} \supseteq \mathbb{F}[x]v$, and that $V_k \cap \mathbb{F}[x]v$ is x-stable. For $0 \leqslant d \leqslant \dim \mathbb{F}[x]v$, let U_d denote the unique d-dimensional x-stable subspace of $\mathbb{F}[x]v$; so for $(V_k) \in \pi_{\mu;\nu}^{-1}(v,x)$, the condition $\dim(V_k \cap \mathbb{F}[x]v) = d_k$ becomes $V_k \cap \mathbb{F}[x]v = U_{d_k}$. Moreover, the fact that $x(V_k) \subseteq V_{k-1}$ forces $U_{d_k-1} = x(U_{d_k}) \subseteq U_{d_{k-1}}$, so $\pi_{\mu;\nu}^{-1}(v,x) \cap (\mathcal{F}_{\mu;\nu})_{(d_k)}$ can only be nonempty when

$$\begin{aligned} &d_0 = 0,\\ &d_k = d_{k-1} \quad \text{or} \quad d_{k-1}+1 \quad \text{for } 1 \leqslant k \leqslant \mu_1,\\ &d_{\mu_1} = d_{\mu_1+1} = \cdots = d_{\mu_1+\nu_1} = \dim \mathbb{F}[x]v. \end{aligned} \tag{4.8}$$

Henceforth we fix integers d_k satisfying these conditions.

We define a variety

$$\begin{aligned} Y = \{&0 = W_0 \subseteq W_1 \subseteq \cdots \subseteq W_{\mu_1+\nu_1} = V/\mathbb{F}[x]v \mid\\ &\dim W_{\mu_1-i} = |\mu| - \mu_1^{\mathbf{t}} - \cdots - \mu_i^{\mathbf{t}} - d_{\mu_1-i} \text{ for } 0 \leqslant i \leqslant \mu_1,\\ &\dim W_{\mu_1+i} = |\mu| + \nu_1^{\mathbf{t}} + \cdots + \nu_i^{\mathbf{t}} - d_{\mu_1+i} \text{ for } 0 \leqslant i \leqslant \nu_1,\\ &x(W_k) \subseteq W_{k-1} \text{ for } 1 \leqslant k \leqslant \mu_1+\nu_1\}. \end{aligned}$$

The prescribed dimensions here are such that we have a morphism

$$\Psi : \pi_{\mu;\nu}^{-1}(v,x) \cap (\mathcal{F}_{\mu;\nu})_{(d_k)} \to Y : (V_k) \mapsto ((V_k + \mathbb{F}[x]v)/\mathbb{F}[x]v).$$

We clearly have an alpha-partition $Y = \bigcup_{(\tau_k)} Y_{(\tau_k)}$ where $(\tau_k) = (\tau_0, \tau_1, \dots, \tau_{\mu_1+\nu_1})$ runs over sequences of partitions where $|\tau_k|$ is the prescribed dimension of W_k, and

$$Y_{(\tau_k)} = \{(W_k) \in Y \mid \text{Jordan type of } x|_{W_k} \text{ is } \tau_k\}.$$

(We need not go into the conditions on (τ_k) which ensure that $Y_{(\tau_k)}$ is nonempty.) Now Y is a generalized Springer fibre based on the vector space $V/\mathbb{F}[x]v$, although it is not quite of the form $\psi_{\tilde{\mu};\nu}^{-1}(x)$, because the successive codimensions in the partial flag $W_0 \subseteq W_1 \subseteq \cdots \subseteq W_{\mu_1}$ need not be the columns of a Young diagram arranged in non-decreasing order. Spaltenstein's argument still applies, however, and shows that each $Y_{(\tau_k)}$ has an affine paving. Hence it suffices to show that the restriction of Ψ to $\Psi^{-1}(Y_{(\tau_k)})$ is a bundle projection with base $Y_{(\tau_k)}$ and fibres isomorphic to affine space of some dimension.

Now fix $(W_k) \in Y$ and let $\widetilde{W}_k$ denote the preimage of W_k under the projection $V \to V/\mathbb{F}[x]v$. The fibre $\Psi^{-1}((W_k))$ consists of all partial flags $0 = V_0 \subset V_1 \subset \cdots \subset V_{\mu_1+\nu_1} = V$ such that

$$V_k + \mathbb{F}[x]v = \widetilde{W}_k, \qquad V_k \cap \mathbb{F}[x]v = U_{d_k}, \quad x(V_k) \subseteq V_{k-1} \quad \text{for } 1 \leqslant k \leqslant \mu_1 + \nu_1. \tag{4.9}$$

Note that these conditions force $V_k = \widetilde{W}_k$ for $k \geqslant \mu_1$. Imagine that V_k is fixed and we are choosing V_{k-1}. If $d_{k-1} = d_k$, then V_{k-1} is forced to equal $V_k \cap \widetilde{W}_{k-1}$, since this has the right dimension. If $d_{k-1} = d_k - 1$, then V_{k-1} must be a codimension-1 subspace of $V_k \cap \widetilde{W}_{k-1}$ which contains $x(V_k) + U_{d_{k-1}}$, and does not contain $x(V_k) + U_{d_k}$. But for any vector spaces $A \subset B \subseteq C$ where $\dim(B/A) = 1$, the variety

$$\{D \subset C \mid \dim(C/D) = 1,\ A \subseteq D,\ B \nsubseteq D\}$$

is isomorphic to affine space of dimension $\dim(C/B)$; in our case this dimension is

$$\dim\left(\frac{V_k \cap \widetilde{W}_{k-1}}{x(V_k) + U_{d_k}}\right) = (\dim W_{k-1} + d_k) - \big(\dim x(W_k) + d_k\big) = \dim\big(W_{k-1}/x(W_k)\big),$$

which is independent of V_k. Since an affine space bundle over affine space is itself an affine space, we can conclude that

$$\Psi^{-1}((W_k)) \cong \mathbb{A}^{f(W_k)} \quad \text{where } f(W_k) = \sum_{\substack{1 \leqslant k \leqslant \mu_1 \\ d_{k-1} = d_k - 1}} \dim\big(W_{k-1}/x(W_k)\big). \tag{4.10}$$

The dimensions $\dim(W_{k-1}/x(W_k))$ are constant as (W_k) runs over one of the $Y_{(\tau_k)}$ pieces, so the fibres $\Psi^{-1}((W_k))$ fit into a bundle as required. □

Corollary 4.8. *Let* $(\rho; \sigma), (\mu; \nu) \in \mathcal{Q}_n$.

(1) *There is a polynomial* $\Pi_{\mu;\nu}^{\rho;\sigma}(t) \in \mathbb{N}[t]$, *independent of* $\mathbb{F}$, *such that for any* $(v, x) \in \mathcal{O}_{\rho;\sigma}$,

$$\sum_i \dim H^{2i}\big(\pi_{\mu;\nu}^{-1}(v,x), \overline{\mathbb{Q}}_\ell\big)t^i = \Pi_{\mu;\nu}^{\rho;\sigma}(t),$$

$$\textit{and} \quad H^i\big(\pi_{\mu;\nu}^{-1}(v,x), \overline{\mathbb{Q}}_\ell\big) = 0 \quad \textit{for } i \textit{ odd.}$$

(2) *There is a polynomial* $\mathrm{IC}_{\mu;\nu}^{\rho;\sigma}(t) \in \mathbb{N}[t]$, *independent of* $\mathbb{F}$, *such that for any* $(v, x) \in \mathcal{O}_{\rho;\sigma}$,

$$\sum_i \dim \mathcal{H}^{2i}_{(v,x)} \mathrm{IC}(\overline{\mathcal{O}_{\mu;\nu}}, \overline{\mathbb{Q}}_\ell)\, t^i = \mathrm{IC}^{\rho;\sigma}_{\mu;\nu}(t),$$

$$\textit{and} \quad \mathcal{H}^i_{(v,x)} \mathrm{IC}(\overline{\mathcal{O}_{\mu;\nu}}, \overline{\mathbb{Q}}_\ell) = 0 \quad \textit{for } i \textit{ odd.}$$

(3) *These polynomials are related by the rule:*

$$\Pi^{\tau;\upsilon}_{\mu;\nu}(t) = \sum_{\substack{\rho \leqslant \mu\ \sigma \leqslant \nu \\ (\rho;\sigma) \geqslant (\tau;\upsilon)}} K_{\rho^{\mathrm{t}}\mu^{\mathrm{t}}} K_{\sigma^{\mathrm{t}}\nu^{\mathrm{t}}}\, t^{n(\rho+\sigma)-n(\mu+\nu)} \mathrm{IC}^{\tau;\upsilon}_{\rho;\sigma}(t).$$

(4) *We have*

$$\Pi^{\rho;\sigma}_{\mu;\nu}(t) = \mathrm{IC}^{\rho;\sigma}_{\mu;\nu}(t) = 0 \quad \textit{if } (\rho;\sigma) \nleqslant (\mu;\nu),$$

$$\Pi^{\mu;\nu}_{\mu;\nu}(t) = \mathrm{IC}^{\mu;\nu}_{\mu;\nu}(t) = 1,$$

$$\Pi^{\rho;\sigma}_{\mu;\nu}(0) = \mathrm{IC}^{\rho;\sigma}_{\mu;\nu}(0) = 1 \quad \textit{if } (\rho;\sigma) < (\mu;\nu).$$

Proof. For any variety X with an affine paving, the long exact sequence in cohomology with compact supports shows that $H^i_c(X, \overline{\mathbb{Q}}_\ell) = 0$ for i odd, and that $\dim H^{2i}_c(X, \overline{\mathbb{Q}}_\ell)$ is the number of spaces in the paving which have dimension i. So part (1) is a consequence of Theorem 4.7, and the observation that the paving constructed in the proof does not depend on the field $\mathbb{F}$. Parts (2) and (3) follow from part (1) via Theorem 4.5(3). The only statement in part (4) which is not automatic is that $\Pi^{\rho;\sigma}_{\mu;\nu}(0) = 1$ if $(\rho;\sigma) < (\mu;\nu)$, which is equivalent to saying that the fibre $\pi^{-1}_{\mu;\nu}(v, x)$ is connected for all $(v, x) \in \mathcal{O}_{\rho;\sigma}$. This follows from part (3). □

Example 4.9. Let $n = 4$, and take $(v, x) \in \mathcal{O}_{(1^3);(1)}$. We will describe the affine paving of

$$\pi^{-1}_{(3);(1)}(v, x) = \bigl\{0 = V_0 \subset V_1 \subset V_2 \subset V_3 \subset V_4 = V \bigm| \dim V_k = k,\ v \in V_3,\ x(V_k) \subseteq V_{k-1} \text{ for } 1 \leqslant k \leqslant 4\bigr\}$$

given by the proof of Theorem 4.7. The nonempty pieces $\pi^{-1}_{(3);(1)}(v, x) \cap (\mathcal{F}_{(3);(1)})_{(d_k)}$ in this case are exactly

$$\bigl\{(V_k) \in \pi^{-1}_{(3);(1)}(v, x) \bigm| v \in V_1\bigr\},$$

$$\bigl\{(V_k) \in \pi^{-1}_{(3);(1)}(v, x) \bigm| v \in V_2,\ v \notin V_1\bigr\}, \quad \text{and}$$

$$\bigl\{(V_k) \in \pi^{-1}_{(3);(1)}(v, x) \bigm| v \in V_3,\ v \notin V_2\bigr\}.$$

The first of these pieces is isomorphic via the appropriate map Ψ to

$$\bigl\{0 = W_0 = W_1 \subset W_2 \subset W_3 \subset V/\mathbb{F}v \bigm| x(W_k) \subseteq W_{k-1}\bigr\},$$

which is a Springer fibre of type (21) (that is, a union of two projective lines intersecting at a point). For the second of these pieces, the map Ψ is an $\mathbb{A}^1$-bundle with base

$$\bigl\{0 = W_0 \subset W_1 = W_2 \subset W_3 \subset V/\mathbb{F}v \bigm| x(W_k) \subseteq W_{k-1}\bigr\},$$

which is another Springer fibre of type (21). For the third piece, the image of Ψ is

$$Y = \{0 = W_0 \subset W_1 \subset W_2 = W_3 \subset V/\mathbb{F}v \mid x(W_k) \subseteq W_{k-1}\},$$

which is another Springer fibre of type (21). According to the proof of Theorem 4.7, we should partition Y into

$$Y' = \{(W_k) \in Y \mid x(W_2) = 0\} \quad \text{and} \quad Y'' = \{(W_k) \in Y \mid x(W_2) \neq 0\}.$$

We have $Y' \cong \mathbb{P}^1$ and $Y'' \cong \mathbb{A}^1$; the restriction of Ψ to $\Psi^{-1}(Y')$ is an $\mathbb{A}^2$-bundle, while the restriction of Ψ to $\Psi^{-1}(Y'')$ is an $\mathbb{A}^1$-bundle. Thus we have

$$\Pi^{(1^3);(1)}_{(3);(1)}(t) = (2t+1) + t(2t+1) + t^2(t+1) + t^2 = t^3 + 4t^2 + 3t + 1.$$

A similar but easier calculation shows that $\Pi^{(1^3);(1)}_{(21);(1)}(t) = t^2 + 2t + 1$. Using Corollary 4.8(3), we deduce that $\mathrm{IC}^{(1^3);(1)}_{(21);(1)}(t) = 2t + 1$ and $\mathrm{IC}^{(1^3);(1)}_{(3);(1)}(t) = t + 1$.

5. Intersection cohomology and Kostka polynomials

A famous theorem of Lusztig relates the intersection cohomology of ordinary nilpotent orbit closures of type A to Kostka polynomials (and hence to the representation theory of the symmetric group). Let $\widetilde{K}_{\lambda\pi}(t) = t^{n(\pi)} K_{\lambda\pi}(t^{-1})$ denote the (modified) Kostka polynomial—see [22, III.§6–7]. In the notation defined in Corollary 4.8, Lusztig's result [19, Theorem 2] becomes:

Theorem 5.1. *For $\pi, \lambda \in \mathcal{P}_n$, $t^{n(\lambda)} \mathrm{IC}^{\emptyset;\pi}_{\emptyset;\lambda}(t) = \widetilde{K}_{\lambda\pi}(t)$.*

Note that this also implies

$$t^{n(\lambda)} \mathrm{IC}^{\rho;\sigma}_{\lambda;\emptyset}(t) = \widetilde{K}_{\lambda,\rho+\sigma}(t), \tag{5.1}$$

since $\mathrm{IC}(V \times \overline{\mathcal{O}_\lambda}, \overline{\mathbb{Q}}_\ell) \cong (\overline{\mathbb{Q}}_\ell)_V \boxtimes \mathrm{IC}(\overline{\mathcal{O}_\lambda}, \overline{\mathbb{Q}}_\ell)$.

In [24,25], Shoji has defined Kostka polynomials $\widetilde{K}_{(\mu;\nu),(\rho;\sigma)}(t) \in \mathbb{Z}[t]$ which are indexed by pairs of bipartitions rather than pairs of partitions (see especially [25, Proposition 3.3], where it is proved that these apparently rational functions are indeed polynomials). The aim of this section is to prove the following analogue of Theorem 5.1:

Theorem 5.2. *For $(\rho;\sigma), (\mu;\nu) \in \mathcal{Q}_n$, $t^{b(\mu;\nu)} \mathrm{IC}^{\rho;\sigma}_{\mu;\nu}(t^2) = \widetilde{K}_{(\mu;\nu),(\rho;\sigma)}(t)$.*

Theorem 5.2 immediately implies the following properties of Shoji's polynomials, not proved in [25]:

Corollary 5.3.

(1) *The coefficients in $\widetilde{K}_{(\mu;\nu),(\rho;\sigma)}(t)$ are all nonnegative, and only powers of t which are congruent to $b(\mu;\nu)$ modulo 2 occur.*
(2) *$\widetilde{K}_{(\emptyset;\lambda),(\emptyset;\pi)}(t) = t^{|\lambda|} \widetilde{K}_{\lambda\pi}(t^2)$, and $\widetilde{K}_{(\lambda;\emptyset),(\rho;\sigma)}(t) = \widetilde{K}_{\lambda,\rho+\sigma}(t^2)$.*

The defining property of Shoji's polynomials involves the representation theory of the Coxeter group $W_n = W(\mathrm{B}_n)$. Recall that the set of irreducible characters $\mathrm{Irr}(W_n)$ is naturally in bijection with $\mathcal{Q}_n$ (see [22, I.B.§9] or [12, 5.5]); we write $\chi^{\mu;\nu}$ for the character labelled by $(\mu;\nu)$. The *fake degree* $R(\chi)$ of a character χ of W_n (not necessarily irreducible) is defined by

$$R(\chi) = \frac{1}{2^n n!} \sum_{w \in W_n} \frac{\chi(w)\epsilon(w) \prod_{a=1}^{n}(t^{2a}-1)}{\det(t-w)}, \tag{5.2}$$

where ϵ denotes the sign character of W_n and det means the determinant of the reflection representation. This fake degree is known to be a nonnegative polynomial in the indeterminate t, because [4, Proposition 11.1.1] implies that

$$R(\chi) = \sum_{i=0}^{n^2} \langle C^i(W_n), \chi \rangle_{W_n} t^i, \tag{5.3}$$

where $C^i(W_n)$ is (the character of) the degree-i homogeneous component of the coinvariant algebra of W_n. We define a square matrix

$$\Omega = (\omega_{(\mu;\nu),(\mu';\nu')})_{(\mu;\nu),(\mu';\nu') \in \mathcal{Q}_n} \quad \text{by } \omega_{(\mu;\nu),(\mu';\nu')} = t^{n^2} R\big(\chi^{\mu;\nu} \otimes \chi^{\mu';\nu'} \otimes \epsilon\big).$$

Then Shoji has proved (see [24, Theorem 5.4] and [25, Remark 3.2]):

Theorem 5.4. *There are unique matrices $P = (p_{(\mu;\nu),(\rho;\sigma)})$ and $\Lambda = (\lambda_{(\rho;\sigma),(\tau;\upsilon)})$ over $\mathbb{Q}(t)$ satisfying the equation $P\Lambda P^{\mathbf{t}} = \Omega$ and subject to the following additional conditions:*

$$p_{(\mu;\nu),(\rho;\sigma)} = \begin{cases} 0 & \textit{if } (\rho;\sigma) \not\leq (\mu;\nu), \\ t^{b(\mu;\nu)} & \textit{if } (\rho;\sigma) = (\mu;\nu), \end{cases} \qquad \lambda_{(\rho;\sigma),(\tau;\upsilon)} = 0 \quad \textit{if } (\rho;\sigma) \neq (\tau;\upsilon).$$

The entry $p_{(\mu;\nu),(\rho;\sigma)}$ of the unique P is $\widetilde{K}_{(\mu;\nu),(\rho;\sigma)}(t)$.

The proof consists primarily of an algorithm for computing P and Λ, a special case of the "generalized Lusztig–Shoji algorithm" (see [11, Proposition 2.2] and the discussion in [1, Section 2]). Consequently, the uniqueness statement still holds if the matrices P and Λ are assumed to have entries in an extension field K of $\mathbb{Q}(t)$. So to prove Theorem 5.2, it suffices to find such a field K and elements $\lambda_{(\tau;\upsilon)} \in K$, $(\tau;\upsilon) \in \mathcal{Q}_n$, such that

$$\sum_{(\tau;\upsilon) \in \mathcal{Q}_n} \lambda_{(\tau;\upsilon)} t^{b(\mu;\nu)+b(\mu';\nu')} \mathrm{IC}^{\tau;\upsilon}_{\mu;\nu}\big(t^2\big) \mathrm{IC}^{\tau;\upsilon}_{\mu';\nu'}\big(t^2\big) = \omega_{(\mu;\nu),(\mu';\nu')}, \tag{5.4}$$

for all $(\mu;\nu), (\mu';\nu') \in \mathcal{Q}_n$. (It then follows that in fact $\lambda_{(\tau;\upsilon)} \in \mathbb{Q}(t)$.) Using Corollary 4.8(3), we see that (5.4) is equivalent to

$$\sum_{(\tau;\upsilon)\in\mathcal{Q}_n} \lambda_{(\tau;\upsilon)} \Pi^{\tau;\upsilon}_{\mu;\nu}(t^2)\Pi^{\tau;\upsilon}_{\mu';\nu'}(t^2) = t^{-b(\mu;\nu)-b(\mu';\nu')} \sum_{\substack{\rho\leqslant\mu\\ \rho'\leqslant\mu'\\ \sigma\leqslant\nu\\ \sigma'\leqslant\nu'}} K_{\rho^{\mathsf{t}}\mu^{\mathsf{t}}} K_{\sigma^{\mathsf{t}}\nu^{\mathsf{t}}} K_{(\rho')^{\mathsf{t}}(\mu')^{\mathsf{t}}} K_{(\sigma')^{\mathsf{t}}(\nu')^{\mathsf{t}}}\, \omega_{(\rho;\sigma),(\rho';\sigma')}, \tag{5.5}$$

for all $(\mu;\nu), (\mu';\nu') \in \mathcal{Q}_n$, and this is the form we will prove.

We first want to simplify the right-hand side. Interpret W_n as the group of permutations of $\{\pm 1, \pm 2, \ldots, \pm n\}$ which commute with $i \leftrightarrow -i$. For any composition $n_1, n_2, \ldots, n_k$ of n, let $W_{(n_i)}$ denote the subgroup $W_{n_1} \times W_{n_2} \times \cdots \times W_{n_k}$ of W_n, the preimage of the Young subgroup $S_{(n_i)} = S_{n_1} \times S_{n_2} \times \cdots \times S_{n_k}$ under the natural homomorphism $W_n \to S_n$. Given two compositions $(n_i)_{i=1}^k$ and $(n'_j)_{j=1}^{k'}$, the double cosets $W_{(n'_j)} \backslash W_n / W_{(n_i)}$ are clearly in bijection with the double cosets $S_{(n'_j)} \backslash S_n / S_{(n_i)}$. These in turn (see [15, §2], for example) are in bijection with $M_{(n_i),(n'_j)}$, the set of $k \times k'$ matrices (m_{ij}) satisfying:

(1) $m_{ij} \in \mathbb{N}$, for all i, j,
(2) $\sum_j m_{ij} = n_i$, for all i, and
(3) $\sum_i m_{ij} = n'_j$, for all j.

Write $m_{\leqslant i,\leqslant j}$ for $\sum_{i'\leqslant i, j'\leqslant j} m_{i'j'}$, and similarly define $m_{<i,<j}$ and $m_{<i,>j}$. The bijection $S_{(n'_j)} \backslash S_n / S_{(n_i)} \leftrightarrow M_{(n_i),(n'_j)}$ is such that the double coset containing w corresponds to the matrix (m_{ij}) which satisfies

$$m_{\leqslant i,\leqslant j} = \bigl|\bigl\{s \leqslant n_1 + \cdots + n_i \bigm| w(s) \leqslant n'_1 + \cdots + n'_j\bigr\}\bigr|. \tag{5.6}$$

Given a bipartition $(\mu;\nu) \in \mathcal{Q}_n$, we let $W_{(\mu;\nu)}$ be the subgroup of W_n determined by the composition (2.1); given a second bipartition $(\mu';\nu')$, write $M_{(\mu;\nu),(\mu';\nu')}$ for the set of matrices determined by the two compositions.

Proposition 5.5. *The right-hand side of* (5.5) *equals*

$$\sum_{(m_{ij})\in M_{(\mu;\nu),(\mu';\nu')}} \frac{t^{2(\binom{n}{2}-n(\mu+\nu)-n(\mu'+\nu')+\sum_{i,j}\binom{m_{ij}}{2})+m_{\leqslant\mu_1,\leqslant\mu'_1}} \prod_{a=1}^n (t^{2a}-1)}{\prod_{i,j}\prod_{a=1}^{m_{ij}}(t^{2a}-1)}.$$

Proof. Recall that $\chi^{\rho;\sigma} \otimes \epsilon = \chi^{\sigma^{\mathsf{t}};\rho^{\mathsf{t}}}$, so we have

$$\omega_{(\rho;\sigma),(\rho';\sigma')} = t^{n^2} R\bigl(\chi^{\sigma^{\mathsf{t}};\rho^{\mathsf{t}}} \otimes \chi^{(\sigma')^{\mathsf{t}};(\rho')^{\mathsf{t}}} \otimes \epsilon\bigr).$$

Also $\chi^{\sigma^{\mathsf{t}};\rho^{\mathsf{t}}} = \mathrm{Ind}_{W_{|\sigma|}\times W_{|\rho|}}^{W_n}(\chi^{\sigma^{\mathsf{t}}} \boxtimes \delta\chi^{\rho^{\mathsf{t}}})$, where χ^λ denotes the irreducible character of $S_{|\lambda|}$ indexed by λ and also its pull-back to $W_{|\lambda|}$, and δ is the one-dimensional character of W_n such that $\delta\epsilon$ is the pull-back of the sign character of S_n (and we continue to write δ for its restriction to any subgroup). Using the well-known fact that the Kostka numbers give the multiplicities of irreducible characters of the symmetric group in the inductions of the trivial character from Young subgroups, we find

$$\begin{aligned}\sum_{\substack{\rho\leqslant\mu\\ \sigma\leqslant\nu}} K_{\rho^{\mathbf t}\mu^{\mathbf t}}K_{\sigma^{\mathbf t}\nu^{\mathbf t}}\,\chi^{\sigma^{\mathbf t};\rho^{\mathbf t}} &= \operatorname{Ind}_{W_{|\nu|}\times W_{|\mu|}}^{W_n}\Bigl(\sum_{\sigma^{\mathbf t}\geqslant\nu^{\mathbf t}} K_{\sigma^{\mathbf t}\nu^{\mathbf t}}\chi^{\sigma^{\mathbf t}}\boxtimes\delta\sum_{\rho^{\mathbf t}\geqslant\mu^{\mathbf t}} K_{\rho^{\mathbf t}\mu^{\mathbf t}}\chi^{\rho^{\mathbf t}}\Bigr)\\ &= \operatorname{Ind}_{W_{|\nu|}\times W_{|\mu|}}^{W_n}\Bigl(\operatorname{Ind}_{W_{\nu_1^{\mathbf t}}\times\cdots\times W_{\nu_{\nu_1}^{\mathbf t}}}^{W_{|\nu|}}(1)\boxtimes\operatorname{Ind}_{W_{\mu_1^{\mathbf t}}\times\cdots\times W_{\mu_{\mu_1}^{\mathbf t}}}^{W_{|\mu|}}(\delta)\Bigr)\\ &= \operatorname{Ind}_{W_{\mu;\nu}}^{W_n}(\delta_{\mu;\nu}),\end{aligned}$$

where $\delta_{\mu;\nu}$ is the one-dimensional character of $W_{\mu;\nu}$ which is δ on all the $W_{\mu_i^{\mathbf t}}$ components and trivial on all the $W_{\nu_i^{\mathbf t}}$ components. So the right-hand side of (5.5) equals:

$$\begin{aligned}&t^{n^2-b(\mu;\nu)-b(\mu';\nu')}R\Bigl(\operatorname{Ind}_{W_{\mu;\nu}}^{W_n}(\delta_{\mu;\nu})\otimes\operatorname{Ind}_{W_{\mu';\nu'}}^{W_n}(\delta_{\mu';\nu'})\otimes\epsilon\Bigr)\\ &\quad=\frac{t^{n^2-b(\mu;\nu)-b(\mu';\nu')}\prod_{a=1}^n(t^{2a}-1)}{|W_n|}\sum_{w\in W_n}\frac{\operatorname{Ind}_{W_{\mu;\nu}}^{W_n}(\delta_{\mu;\nu})(w)\operatorname{Ind}_{W_{\mu';\nu'}}^{W_n}(\delta_{\mu';\nu'})(w)}{\det(t-w)}\\ &\quad=\frac{t^{n^2-b(\mu;\nu)-b(\mu';\nu')}\prod_{a=1}^n(t^{2a}-1)}{|W_n||W_{\mu;\nu}||W_{\mu';\nu'}|}\sum_{\substack{w,w_1,w_2\in W_n\\ w_1ww_1^{-1}\in W_{\mu;\nu}\\ w_2ww_2^{-1}\in W_{\mu';\nu'}}}\frac{\delta_{\mu;\nu}(w_1ww_1^{-1})\delta_{\mu';\nu'}(w_2ww_2^{-1})}{\det(t-w)}\\ &\quad=\frac{t^{n^2-b(\mu;\nu)-b(\mu';\nu')}\prod_{a=1}^n(t^{2a}-1)}{|W_{\mu;\nu}||W_{\mu';\nu'}|}\sum_{\substack{\tilde w\in W_n\\ y\in W_{\mu;\nu}\cap\tilde w^{-1}W_{\mu';\nu'}\tilde w}}\frac{\delta_{\mu;\nu}(y)\delta_{\mu';\nu'}(\tilde wy\tilde w^{-1})}{\det(t-y)},\end{aligned}$$

where the last step uses the change of variables $\tilde w=w_2w_1^{-1}$, $y=w_1ww_1^{-1}$ (and w_1 becomes a free variable, cancelling the $|W_n|$ from the denominator).

Now if the double coset of $\tilde w$ corresponds to the matrix $(m_{ij})\in M_{(\mu;\nu),(\mu';\nu')}$, then $W_{\mu;\nu}\cap\tilde w^{-1}W_{\mu';\nu'}\tilde w$ is a reflection subgroup of W_n isomorphic to $\prod_{i,j}W_{m_{ij}}$, in such a way that its character $y\mapsto\delta_{\mu;\nu}(y)\delta_{\mu';\nu'}(\tilde wy\tilde w^{-1})$ corresponds to the character which is δ on the factors $W_{m_{ij}}$ where $i>\mu_1$ and $j\leqslant\mu'_1$ or $i\leqslant\mu_1$ and $j>\mu'_1$, and trivial on the other factors. Using the analogue of (5.3) and the fact that ϵ occurs in $C^{n^2}(W_n)$ and $\delta\epsilon$ in $C^{n^2-n}(W_n)$, we find that

$$\begin{aligned}&\frac{1}{|W_{\mu;\nu}\cap\tilde w^{-1}W_{\mu';\nu'}\tilde w|}\sum_{y\in W_{\mu;\nu}\cap\tilde w^{-1}W_{\mu';\nu'}\tilde w}\frac{\delta_{\mu;\nu}(y)\delta_{\mu';\nu'}(\tilde wy\tilde w^{-1})}{\det(t-y)}\\ &\quad=\frac{\prod_{i\leqslant\mu_1,j\leqslant\mu'_1}t^{m_{ij}^2}\prod_{i\leqslant\mu_1,j>\mu'_1}t^{m_{ij}^2-m_{ij}}\prod_{i>\mu_1,j\leqslant\mu'_1}t^{m_{ij}^2-m_{ij}}\prod_{i>\mu_1,j>\mu'_1}t^{m_{ij}^2}}{\prod_{i,j}\prod_{a=1}^{m_{ij}}(t^{2a}-1)}\end{aligned}$$

$$= \frac{t^{n-|\mu|-|\mu'|+2m_{\leqslant\mu_1,\leqslant\mu'_1}+2\sum_{i,j}\binom{m_{ij}}{2}}}{\prod_{i,j}\prod_{a=1}^{m_{ij}}(t^{2a}-1)}.$$

Substituting this in the above and using $|W_{\mu';\nu'}\tilde{w}W_{\mu;\nu}| = \frac{|W_{\mu;\nu}||W_{\mu';\nu'}|}{|W_{\mu;\nu}\cap\tilde{w}^{-1}W_{\mu';\nu'}\tilde{w}|}$, we obtain the result. □

To analyse the left-hand side of (5.5), we return to our enhanced nilpotent cone $V\times\mathcal{N}$, choosing the base field $\mathbb{F}$ to be an algebraic closure of the finite field $\mathbb{F}_q$, where q is some prime power. It is evident from the results of Section 2 that each G-orbit $\mathcal{O}_{\mu;\nu}$ is defined over $\mathbb{F}_q$.

Proposition 5.6. *For all* $(\mu;\nu),(\mu';\nu')\in\mathcal{Q}_n$, *we have*

$$\sum_{(\tau;\upsilon)\in\mathcal{Q}_n}\left|\mathcal{O}_{\tau;\upsilon}(\mathbb{F}_q)\right|\Pi^{\tau;\upsilon}_{\mu;\nu}(q)\Pi^{\tau;\upsilon}_{\mu';\nu'}(q)$$
$$= \sum_{(m_{ij})\in M_{(\mu;\nu),(\mu';\nu')}} \frac{q^{\binom{n}{2}-n(\mu+\nu)-n(\mu'+\nu')+\sum_{i,j}\binom{m_{ij}}{2}+m_{\leqslant\mu_1,\leqslant\mu'_1}}\prod_{a=1}^{n}(q^a-1)}{\prod_{i,j}\prod_{a=1}^{m_{ij}}(q^a-1)}.$$

Proof. For any $(v,x)\in V(\mathbb{F}_q)\times\mathcal{N}(\mathbb{F}_q)$, the alpha-partition of $\pi^{-1}_{\mu;\nu}(v,x)$ defined in Theorem 4.7 is defined over $\mathbb{F}_q$, and hence

$$\left|\pi^{-1}_{\mu;\nu}(v,x)(\mathbb{F}_q)\right| = \Pi^{\tau;\upsilon}_{\mu;\nu}(q), \quad \text{where } (v,x)\in\mathcal{O}_{\tau;\upsilon}(\mathbb{F}_q). \tag{5.7}$$

Hence the left-hand side of our desired equation is the number of $\mathbb{F}_q$-points of the variety

$$Z = \bigl\{\bigl(v,x,(V_i),(V'_j)\bigr)\in V\times\mathcal{N}\times\mathcal{F}_{\mu;\nu}\times\mathcal{F}_{\mu';\nu'} \bigm| \bigl(v,x,(V_i)\bigr)\in\widetilde{\mathcal{F}_{\mu;\nu}},\ \bigl(v,x,(V'_j)\bigr)\in\widetilde{\mathcal{F}_{\mu';\nu'}}\bigr\}.$$

Now the G-orbits in $\mathcal{F}_{\mu;\nu}\times\mathcal{F}_{\mu';\nu'}$ are in bijection with $M_{(\mu;\nu),(\mu';\nu')}$: the orbit $\mathcal{O}_{(m_{ij})}$ corresponding to a matrix (m_{ij}) consists of pairs $((V_i),(V'_j))$ satisfying

$$\dim\bigl(V_i\cap V'_j\bigr) = m_{\leqslant i,\leqslant j}, \quad \text{for all } i,j. \tag{5.8}$$

So we have a partition $Z=\bigcup_{(m_{ij})\in M_{(\mu;\nu),(\mu';\nu')}} Z_{(m_{ij})}$, where

$$Z_{(m_{ij})} = \bigl\{\bigl(v,x,(V_i),(V'_j)\bigr) \bigm| \bigl((V_i),(V'_j)\bigr)\in\mathcal{O}_{(m_{ij})},\ v\in V_{\mu_1}\cap V_{\mu'_1},\ x\in\mathcal{N},\ x(V_i)\subseteq V_{i-1},\ x\bigl(V'_j\bigr)\subseteq V'_{j-1}\bigr\}.$$

Hence $|Z(\mathbb{F}_q)|=\sum_{(m_{ij})\in M_{(\mu;\nu),(\mu';\nu')}}|Z_{(m_{ij})}(\mathbb{F}_q)|$, and we need only show that $|Z_{(m_{ij})}(\mathbb{F}_q)|$ is given by the (m_{ij}) term of the right-hand side of our desired equation. By standard methods, we compute

$$\left|\mathcal{O}_{(m_{ij})}(\mathbb{F}_q)\right| = \frac{\prod_{a=1}^{n}(q^a-1)}{\prod_{i,j}\prod_{a=1}^{m_{ij}}(q^a-1)}\prod_{i,j}q^{m_{ij}m_{<i,>j}}, \tag{5.9}$$

where the fraction represents the number of ways of choosing the partial flag (V_i) and the images of each $V_i \cap V'_j$ in V_i / V_{i-1}, and the other product represents the number of ways of choosing the subspaces V'_j themselves once these images are fixed. For any $((V_i), (V'_j)) \in \mathcal{O}_{m_{ij}}(\mathbb{F}_q)$, we have

$$\begin{aligned} \left|(V_{\mu_1} \cap V_{\mu'_1})(\mathbb{F}_q)\right| &= q^{m_{\leqslant \mu_1, \leqslant \mu'_1}}, \\ \left|\left\{x \in \mathcal{N}(\mathbb{F}_q) \mid x(V_i) \subseteq V_{i-1},\ x\left(V'_j\right) \subseteq V'_{j-1}\right\}\right| &= \prod_{i,j} q^{m_{ij} m_{<i,<j}}. \end{aligned} \tag{5.10}$$

Finally, to reconcile the powers of q, note that

$$\begin{aligned} \sum_{i,j} m_{ij} m_{<i,>j} + m_{ij} m_{<i,<j} &= \frac{1}{2} \sum_{\substack{i,j,i',j' \\ i' \neq i, j' \neq j}} m_{ij} m_{i'j'} \\ &= \frac{1}{2}\left(n^2 - \sum_i \left(\sum_j m_{ij}\right)^2 - \sum_j \left(\sum_i m_{ij}\right)^2 + \sum_{i,j} m_{ij}^2\right) \\ &= \binom{n}{2} - n(\mu + \nu) - n(\mu' + \nu') + \sum_{i,j} \binom{m_{ij}}{2}, \end{aligned}$$

as required. □

Now let R be the ring of all functions $g : \mathbb{Z}_{>0} \to \overline{\mathbb{Q}}_\ell$ of the form

$$g(s) = \sum_i c_i (a_i)^s \quad \text{with } c_i \in \mathbb{Z} \text{ and } a_i \in \overline{\mathbb{Q}}_\ell \text{ (a finite sum).} \tag{5.11}$$

By well-known facts, any $g \in R$ can be expressed in the above form in a unique way, and R is an integral domain. We fix a square root $q^{1/2}$ of q in $\overline{\mathbb{Q}}_\ell$, and identify $\mathbb{Z}[t]$ with a subring of R via the map which sends a polynomial $p(t)$ to the function $s \mapsto p(q^{s/2})$. Let K denote the fraction field of R, an extension field of $\mathbb{Q}(t)$. It is easy to see that $\mathbb{Q}(t) \cap R = \mathbb{Z}[t, t^{-1}]$.

For any $(\tau; \upsilon) \in \mathcal{Q}_n$, we define an element $\lambda_{(\tau;\upsilon)} \in R$ by the rule

$$\lambda_{(\tau;\upsilon)}(s) = \left|\mathcal{O}_{\rho;\sigma}(\mathbb{F}_{q^s})\right| = \sum_i (-1)^i \operatorname{tr}\left(F^s \mid H_c^i(\mathcal{O}_{\rho;\sigma}, \overline{\mathbb{Q}}_\ell)\right),$$

where F denotes the Frobenius endomorphism of $\mathcal{O}_{\rho;\sigma}$ relative to $\mathbb{F}_q$. Comparing Propositions 5.5 and 5.6 (with q replaced by a general power q^s), we see that Eq. (5.5) holds in the field K. This completes the proof of Theorem 5.2.

We obtain the following result as a by-product of this proof.

Proposition 5.7. *For any $(\tau; \upsilon) \in \mathcal{Q}_n$, there is a polynomial $\theta_{(\tau;\upsilon)}(t) \in \mathbb{Z}[t]$ such that $|\mathcal{O}_{\tau;\upsilon}(\mathbb{F}_q)| = \theta_{(\tau;\upsilon)}(q)$ for every prime power q.*

Proof. As mentioned above, the uniqueness in (5.4) shows that $\lambda_{(\tau;\upsilon)}$ is an element of $\mathbb{Q}(t)$, and hence of $\mathbb{Q}(t) \cap R = \mathbb{Z}[t, t^{-1}]$; moreover, uniqueness implies that it does not depend on the prime power q used to define it. In addition, since $\lambda_{(\tau;\upsilon)}$ is $\mathbb{Z}$-valued, it must actually lie in $\mathbb{Z}[t]$. Proposition 5.5 shows that each side of (5.5) is unchanged under $t \mapsto -t$, so uniqueness also shows that $\lambda_{(\tau;\upsilon)}$ is unchanged under $t \mapsto -t$, which means that $\lambda_{(\tau;\upsilon)} \in \mathbb{Z}[t^2]$. This gives the statement. □

In the case of an ordinary nilpotent orbit $\mathcal{O}_\lambda$, this result is well known: we have $|\mathcal{O}_\lambda(\mathbb{F}_q)| = \frac{a_{(1^n)}(q)}{a_\lambda(q)}$, where $a_\lambda(t) \in \mathbb{Z}[t]$ is defined by [22, II.(1.6)], and it is easy to see that $\frac{a_{(1^n)}(t)}{a_\lambda(t)} \in \mathbb{Z}[t]$.

Along similar lines, we can relate our intersection cohomology to the usual Kostka polynomials via certain generalizations of Hall polynomials.

Proposition 5.8. *Let* $(\tau;\upsilon), (\rho;\sigma) \in \mathcal{Q}_n$.

(1) *There is a polynomial* $g^{\tau;\upsilon}_{\rho;\sigma}(t) \in \mathbb{Z}[t]$ *such that for any prime power* q *and* $(v,x) \in \mathcal{O}_{\tau;\upsilon}(\mathbb{F}_q)$, $g^{\tau;\upsilon}_{\rho;\sigma}(q)$ *counts the* $\mathbb{F}_q$*-points of the variety*

$$\{W \subset V \mid v \in W,\ \text{Jordan type of } x|_W \text{ is } \rho,\ \text{Jordan type of } x|_{V/W} \text{ is } \sigma\}.$$

(2) *We have*

$$\mathrm{IC}^{\tau;\upsilon}_{\rho;\sigma}(t) = \sum_{\substack{\theta\leqslant\rho\\ \psi\leqslant\sigma}} t^{-n(\rho+\sigma)} g^{\tau;\upsilon}_{\theta;\psi}(t)\widetilde{K}_{\rho\theta}(t)\widetilde{K}_{\sigma\psi}(t).$$

Proof. Note that $\widetilde{K}_{\mu\mu}(t) = t^{n(\mu)}$, so the transition matrix in (2) between IC and g is a unitriangular matrix over $\mathbb{Z}[t]$. So if we define $g^{\tau;\upsilon}_{\rho;\sigma}(q)$ by the rule in (1), all we need to prove is that

$$\mathrm{IC}^{\tau;\upsilon}_{\rho;\sigma}(q) = \sum_{\substack{\theta\leqslant\rho\\ \psi\leqslant\sigma}} q^{-n(\rho+\sigma)} g^{\tau;\upsilon}_{\theta;\psi}(q)\widetilde{K}_{\rho\theta}(q)\widetilde{K}_{\sigma\psi}(q), \tag{5.12}$$

and the fact that $g^{\tau;\upsilon}_{\rho;\sigma}(q)$ is an integer polynomial in q will automatically follow. Now (5.12) is the characteristic-function analogue of Proposition 4.6, so we mimic the proof of that result. The analogues of (4.5), which are special cases of [14, Lemma 5.5], are:

$$\begin{aligned}\Pi^{\emptyset;\theta}_{\mu;\emptyset}(q) &= \sum_{\theta\leqslant\rho\leqslant\mu} q^{-n(\mu)} K_{\rho^{\mathrm{t}}\mu^{\mathrm{t}}}\widetilde{K}_{\rho\theta}(q),\\ \Pi^{\emptyset;\psi}_{\emptyset;\nu}(q) &= \sum_{\psi\leqslant\sigma\leqslant\nu} q^{-n(\nu)} K_{\sigma^{\mathrm{t}}\nu^{\mathrm{t}}}\widetilde{K}_{\sigma\psi}(q).\end{aligned} \tag{5.13}$$

The analogue of (4.7) follows from (5.7), by classifying the $\mathbb{F}_q$-points of $\pi^{-1}_{\mu;\nu}(v,x)$ according to the Jordan types of $x|_{V_{\mu_1}}$ and $x|_{V/V_{\mu_1}}$:

$$\Pi^{\tau;\upsilon}_{\mu;\nu}(q) = \sum_{(\theta;\psi)\in\mathcal{Q}_n} g^{\tau;\upsilon}_{\theta;\psi}(q)\Pi^{\emptyset;\theta}_{\mu;\emptyset}(q)\Pi^{\emptyset;\psi}_{\emptyset;\nu}(q)$$

$$= \sum_{\substack{\theta \leqslant \rho \leqslant \mu \\ \psi \leqslant \sigma \leqslant \nu}} q^{-n(\mu+\nu)} K_{\rho^{\mathbf{t}}\mu^{\mathbf{t}}} K_{\sigma^{\mathbf{t}}\nu^{\mathbf{t}}}\, g_{\theta;\psi}^{\tau;\upsilon}(q) \widetilde{K}_{\rho\theta}(q) \widetilde{K}_{\sigma\psi}(q). \tag{5.14}$$

Using the unitriangularity in Corollary 4.8(3), we deduce (5.12). □

Note that $g_{\rho;\sigma}^{\emptyset;\pi}(t)$ is the usual Hall polynomial $g_{\rho\sigma}^{\pi}(t)$, as in [22, II.4]. The relationship between our generalized Hall polynomials and those defined in [8, §4] is that $g_{\rho;\sigma}^{\tau;\upsilon}(q) = \sum_{\mu+\nu=\rho} G_{(\mu,\nu)\sigma}^{(\tau,\upsilon)}$.

6. Connections with Kato's exotic nilpotent cone

In this section, we discuss the analogy between the enhanced nilpotent cone $V \times \mathcal{N}$ studied in this paper and the exotic nilpotent cone $\mathfrak{N}$ studied by Kato in [16–18]. (We assume that $\operatorname{char} \mathbb{F} \neq 2$.) To make a concrete connection, we choose the symplectic vector space W to be $V \oplus V^*$, with the skew-symmetric form

$$\big\langle (v, f), (v', f') \big\rangle = f'(v) - f(v'). \tag{6.1}$$

Recall that $\mathfrak{N}_0$ denotes the closed subvariety of $\mathcal{N}(W)$ consisting of elements which are self-adjoint for $\langle\cdot,\cdot\rangle$, and $\mathfrak{N} = W \times \mathfrak{N}_0$. Let K denote the symplectic group $\mathrm{Sp}(W, \langle\cdot,\cdot\rangle)$; then K clearly acts on $\mathfrak{N}_0$ and $\mathfrak{N}$.

We let $G = \mathrm{GL}(V)$ act on W in the natural way; the resulting representation $G \to \mathrm{GL}(W)$ identifies G with the subgroup $\{g \in K \mid gV = V, gV^* = V^*\}$ of K. Similarly, the map $\mathrm{End}(V) \to \mathrm{End}(W) : x \mapsto (x, x^{\mathbf{t}})$ identifies $\mathcal{N}$ with

$$\big\{ x \in \mathfrak{N}_0 \,\big|\, x(V) \subseteq V, x\big(V^*\big) \subseteq V^* \big\},$$

a G-stable closed subvariety of $\mathfrak{N}_0$. So the exotic nilpotent cone $\mathfrak{N}$ is sandwiched between two enhanced nilpotent cones: $V \times \mathcal{N}$ is a G-stable closed subvariety of $\mathfrak{N}$, and $\mathfrak{N}$ is a K-stable closed subvariety of $W \times \mathcal{N}(W)$. Kato has proved that the orbits of these three varieties match up as follows. (Here, if λ is a partition, $\lambda \cup \lambda$ denotes the partition $(\lambda_1, \lambda_1, \lambda_2, \lambda_2, \ldots)$.)

Theorem 6.1. *The K-orbits in $\mathfrak{N}$ are in bijection with $\mathcal{Q}_n$, in such a way that the orbit $\mathbb{O}_{\mu;\nu}$ corresponding to $(\mu;\nu)$ contains the G-orbit $\mathcal{O}_{\mu;\nu}$, and is contained in the* $\mathrm{GL}(W)$*-orbit $\mathcal{O}_{\mu\cup\mu;\nu\cup\nu}$.*

Proof. Kato's results [16, Theorem 1.9] and [17, Theorem B] are not stated in quite these terms, so let us indicate how they imply the above statement, using Proposition 2.3 to simplify the argument. The key claim is that $\mathfrak{N} = K.(V \times \mathcal{N})$, which is equivalent to saying that for any $(v, x) \in \mathfrak{N}$, there are x-stable maximal isotropic subspaces $W_1, W_2 \subset W$ such that $v \in W_1$ and $W_1 \oplus W_2 = W$. Kato proves this in [16, Appendix A] by showing that the K-orbit of (v, x) contains an explicit "normal form," which manifestly has this property (see also [28, Proposition 3.6]). Hence every K-orbit in $\mathfrak{N}$ contains a G-orbit in $V \times \mathcal{N}$, and is contained in a unique $\mathrm{GL}(W)$-orbit in $W \times \mathcal{N}(W)$. Given the parametrization of G-orbits in $V \times \mathcal{N}$ by $\mathcal{Q}_n$ and the parametrization of $\mathrm{GL}(W)$-orbits in $W \times \mathcal{N}(W)$ by $\mathcal{Q}_{2n}$, the result will follow immediately once we show that the orbit $\mathcal{O}_{\mu;\nu}$, regarded as a subvariety of $W \times \mathcal{N}(W)$, is contained in $\mathcal{O}_{\mu\cup\mu;\nu\cup\nu}$.

Take $(v, x) \in \mathcal{O}_{\mu;\nu}$, and let $\{v_{ij}\}$ be a normal basis of V for (v, x). If $\{v_{ij}^*\}$ denotes the dual basis of V^*, then

$$x^{\mathbf{t}} v_{ij}^* = \begin{cases} v_{i,j+1}^* & \text{if } j < \mu_i + \nu_i, \\ 0 & \text{if } j = \mu_i + \nu_i. \end{cases} \tag{6.2}$$

Hence we have a Jordan basis of type $(\mu + \nu) \cup (\mu + \nu)$ for x regarded as an endomorphism of W, where each Jordan block $v_{i1}, v_{i2}, \ldots, v_{i,\mu_i+\nu_i}$ is followed by the corresponding dual basis elements in reverse order. Applying to this basis the normalization procedure in Lemma 2.4, we obtain a normal basis for $(v, x) \in W \times \mathcal{N}(W)$ of type $(\mu \cup \mu; \nu \cup \nu)$, and the proof is complete. □

Note that in proving [17, Theorem B], Kato constructs a bijection between $\mathcal{Q}_n$ and a set of "marked partitions," and uses the latter to parametrize his normal forms; his bijection is such that the normal form attached to $(\mu; \nu) \in \mathcal{Q}_n$ is in our orbit $\mathbb{O}_{\nu;\mu}$, so his bipartitions need to be switched when comparing with this paper.

In order to prove that the closure ordering on the K-orbits in $\mathfrak{N}$ is given by the same partial order as for the enhanced nilpotent cone, we need a new interpretation of the quantity $\mu_1 + \nu_1 + \cdots + \mu_k + \nu_k + \mu_{k+1}$. For any subspace $U \subset W$, $U^\perp$ denotes the perpendicular subspace under $\langle \cdot, \cdot \rangle$.

Lemma 6.2. *For any $k \geqslant 0$ and $(v, x) \in \mathbb{O}_{\mu;\nu}$, $2(\mu_1 + \nu_1 + \cdots + \mu_k + \nu_k + \mu_{k+1})$ is the maximum possible dimension of $U/(U \cap U^\perp)$ where U is an $\mathbb{F}[x]$-submodule of W of the form $\mathbb{F}[x]\{v, w_1, w_2, \ldots, w_{2k+1}\}$ for some $w_1, \ldots, w_{2k+1} \in W$.*

Proof. By K-equivariance, we can assume that $(v, x) \in \mathcal{O}_{\mu;\nu}$. As in the previous proof, let $\{v_{ij}\}$ be a normal basis of V for (v, x), and let $\{v_{ij}^*\}$ be the dual basis of V^*. We can easily see that the stated dimension is attained: set

$$\begin{aligned} U_0 &= \mathbb{F}[x]\big\{v, v_{1,\mu_1+\nu_1}, \ldots, v_{k,\mu_k+\nu_k}, v_{11}^*, \ldots, v_{k1}^*, v_{k+1,1}^*\big\} \\ &= \operatorname{span}\{v_{ij} \mid 1 \leqslant i \leqslant k,\ 1 \leqslant j \leqslant \mu_i + \nu_i\} \oplus \mathbb{F}[x] \sum_{i=k+1}^{\ell(\mu)} v_{i,\mu_i} \\ &\quad \oplus \operatorname{span}\big\{v_{ij}^* \;\big|\; 1 \leqslant i \leqslant k+1,\ 1 \leqslant j \leqslant \mu_i + \nu_i\big\}. \end{aligned}$$

(If $k \geqslant \ell(\mu)$, ignore the middle summand; and if $k \geqslant \ell(\mu + \nu)$, interpret v_{ij} and v_{ij}^* as zero for $i > \ell(\mu + \nu)$.) We have $U_0 \cap U_0^\perp = \operatorname{span}\{v_{k+1,j}^* \mid j > \mu_{k+1}\}$, so $\dim U_0/(U_0 \cap U_0^\perp) = 2(\mu_1 + \nu_1 + \cdots + \mu_k + \nu_k + \mu_{k+1})$ as required.

We now show that for any $U = \mathbb{F}[x]\{v, w_1, w_2, \ldots, w_{2k+1}\}$, the dimension of $U/(U \cap U^\perp)$ has the claimed upper bound. Notice first that $\mathbb{F}[x]v$ is an isotropic subspace of W, being contained in the maximal isotropic subspace V. The form $\langle \cdot, \cdot \rangle$ on W induces a nondegenerate skew-symmetric form on the subquotient $\widetilde{W} = (\mathbb{F}[x]v)^\perp/\mathbb{F}[x]v$, and x induces a self-adjoint nilpotent endomorphism of $\widetilde{W}$. Since $V/\mathbb{F}[x]v$ is a maximal isotropic subspace of $\widetilde{W}$ with an x-stable complementary isotropic subspace, Lemma 2.5 implies that the Jordan type of x on $\widetilde{W}$ is

$$(\nu_1 + \mu_2, \nu_1 + \mu_2, \nu_2 + \mu_3, \nu_2 + \mu_3, \ldots).$$

Let $\widetilde{U} = (U \cap (\mathbb{F}[x]v)^{\perp})/\mathbb{F}[x]v$, which is an x-stable subspace of $\widetilde{W}$. Since $U/\mathbb{F}[x]v$ is generated as an $\mathbb{F}[x]$-module by the images of $w_1, \ldots, w_{2k+1}$, x has at most $2k+1$ Jordan blocks on $U/\mathbb{F}[x]v$, and hence also has at most $2k+1$ Jordan blocks on $\widetilde{U}$ and on $\widehat{U} = \widetilde{U}/(\widetilde{U} \cap \widetilde{U}^{\perp})$. But the induced skew-symmetric form on $\widehat{U}$ is again nondegenerate, so the Jordan type of x on $\widehat{U}$ must be of the form $\pi \cup \pi$, where $\ell(\pi) \leqslant k$. Moreover, since the $\mathbb{F}[x]$-module $\widehat{U}$ is a subquotient of $\widetilde{W}$, the Young diagram of $\pi \cup \pi$ must be contained in that of $(\nu_1 + \mu_2, \nu_1 + \mu_2, \ldots)$. So

$$\dim \widehat{U} = 2|\pi| \leqslant 2(\nu_1 + \mu_2 + \cdots + \nu_k + \mu_{k+1}). \tag{6.3}$$

To relate this to $\dim U/(U \cap U^{\perp})$, notice that

$$\widetilde{U} \cap \widetilde{U}^{\perp} = \frac{U \cap (\mathbb{F}[x]v)^{\perp} \cap (U^{\perp} + \mathbb{F}[x]v)}{\mathbb{F}[x]v} = \frac{(U \cap U^{\perp}) + \mathbb{F}[x]v}{\mathbb{F}[x]v},$$

where we have used the inclusions $\mathbb{F}[x]v \subseteq U \cap (\mathbb{F}[x]v)^{\perp}$ and $U^{\perp} \subseteq (\mathbb{F}[x]v)^{\perp}$. So

$$\dim \widehat{U} = \dim \frac{U \cap (\mathbb{F}[x]v)^{\perp}}{(U \cap U^{\perp}) + \mathbb{F}[x]v} \geqslant \dim \frac{U}{U \cap U^{\perp}} - 2\mu_1, \tag{6.4}$$

since $\dim \mathbb{F}[x]v = \operatorname{codim}(\mathbb{F}[x]v)^{\perp} = \mu_1$. Combining (6.3) and (6.4), we get the desired upper bound. □

Theorem 6.3. *For $(\rho; \sigma), (\mu; \nu) \in \mathcal{Q}_n$, $\mathbb{O}_{\rho;\sigma} \subseteq \overline{\mathbb{O}_{\mu;\nu}}$ if and only if $(\rho; \sigma) \leqslant (\mu; \nu)$.*

Proof. It is clear that $K.\overline{\mathcal{O}_{\mu;\nu}} \subseteq \overline{\mathbb{O}_{\mu;\nu}}$, so the "if" direction is a consequence of the "if" direction in Theorem 3.9. For the "only if" direction, it is clear that $\mathrm{GL}(W).\overline{\mathbb{O}_{\mu;\nu}} \subseteq \overline{\mathcal{O}_{\mu\cup\mu;\nu\cup\nu}}$, so we have

$$\mathbb{O}_{\rho;\sigma} \subseteq \overline{\mathbb{O}_{\mu;\nu}} \quad \Rightarrow \quad (\rho \cup \rho; \sigma \cup \sigma) \leqslant (\mu \cup \mu; \nu \cup \nu).$$

The latter condition does not imply $(\rho; \sigma) \leqslant (\mu; \nu)$, but at least it does imply $\rho + \sigma \leqslant \mu + \nu$, which leaves only the inequalities

$$\rho_1 + \sigma_1 + \cdots + \rho_k + \sigma_k + \rho_{k+1} \leqslant \mu_1 + \nu_1 + \cdots + \mu_k + \nu_k + \mu_{k+1},$$

for all $k \geqslant 0$. By Lemma 6.2, we need to prove that for fixed N, the condition

$$\dim U/\big(U \cap U^{\perp}\big) \leqslant N \quad \text{where } U = \mathbb{F}[x]\{v, w_1, \ldots, w_{2k+1}\}, \\ \text{for any } w_1, \ldots, w_{2k+1} \in W \tag{6.5}$$

is a closed condition on (v, x) (*i.e.*, it determines a closed subvariety of $\mathfrak{N}$). But $\mathbb{F}[x]\{v, w_1, \ldots, w_{2k+1}\}$ is guaranteed to be spanned by the $(2k+2)n$ vectors

$$v, xv, x^2v, \ldots, x^{n-1}v, w_1, xw_1, \ldots, x^{n-1}w_1, \ldots, w_{2k+1}, xw_{2k+1}, \ldots, x^{n-1}w_{2k+1},$$

and the dimension involved in (6.5) is the rank of the $(2k+2)n \times (2k+2)n$ matrix formed by using these vectors as the left and right inputs of $\langle \cdot, \cdot \rangle$. So as in the proof of Theorem 3.9, the condition (6.5) amounts to a collection of polynomial equations in the coordinates of v, x, and $w_1, \dots, w_{2k+1}$, and we are done. □

The closures $\overline{\mathbb{O}_{\emptyset;\lambda}}$ are known to have the same intersection cohomology as the ordinary nilpotent orbit closures $\overline{\mathcal{O}_{\emptyset;\lambda}}$, but with all degrees doubled; a proof with a gap was given in [13], and the gap was filled in [14]. On the evidence of direct calculations for $n \leqslant 3$, we conjecture that the same holds for all $(\mu;\nu)$. In view of Theorem 5.2, this is equivalent to the following.

Conjecture 6.4.

(1) *For* $(\mu;\nu) \in \mathcal{Q}_n$, $\mathcal{H}^i \mathrm{IC}(\overline{\mathbb{O}_{\mu;\nu}}, \overline{\mathbb{Q}}_\ell) = 0$ *for* $4 \nmid i$.
(2) *For* $(\rho;\sigma), (\mu;\nu) \in \mathcal{Q}_n$ *and* $(v,x) \in \mathbb{O}_{\rho;\sigma}$,

$$t^{b(\mu;\nu)} \sum_i \dim \mathcal{H}^{4i}_{(v,x)} \mathrm{IC}(\overline{\mathbb{O}_{\mu;\nu}}, \overline{\mathbb{Q}}_\ell)\, t^{2i} = \widetilde{K}_{(\mu;\nu),(\rho;\sigma)}(t).$$

We now sketch a possible argument to show that this conjecture is equivalent to a recent conjecture of Shoji, stated below.

Step 1. It follows from the properties of the usual Springer correspondence in type A that for $\lambda \in \mathcal{P}_n$, $x \in \mathcal{N}$, and $i \geqslant 0$,

$$\dim \mathcal{H}^i_x \mathrm{IC}(\overline{\mathcal{O}_\lambda}, \overline{\mathbb{Q}}_\ell) = \left\langle H^{i+2n(\lambda)}(\mathcal{B}_x, \overline{\mathbb{Q}}_\ell), \chi^\lambda \right\rangle_{S_n}, \tag{6.6}$$

where $\mathcal{B}_x$ denotes the Springer fibre ($\psi^{-1}_{\emptyset;(n)}(x)$ in the notation used before), on whose cohomology S_n acts via the Springer representation. Since $\mathcal{B}_x$ has an affine paving, both sides vanish if i is odd. Analogously, one may hope to deduce from Kato's exotic Springer correspondence in type C that for $(\mu;\nu) \in \mathcal{Q}_n$, $(v,x) \in \mathfrak{N}$, and $i \geqslant 0$,

$$\dim \mathcal{H}^i_{(v,x)} \mathrm{IC}(\overline{\mathbb{O}_{\mu;\nu}}, \overline{\mathbb{Q}}_\ell) = \left\langle H^{i+2b(\mu;\nu)}(\mathcal{C}_{(v,x)}, \overline{\mathbb{Q}}_\ell), \chi^{\mu;\nu} \right\rangle_{W_n}, \tag{6.7}$$

where $\mathcal{C}_{(v,x)}$ is Kato's analogue of the Springer fibre. It is also expected that $\mathcal{C}_{(v,x)}$ has an affine paving, so that both sides would vanish if i is odd.

Step 2. The Springer representations in type A are isomorphic to representations defined purely algebraically. Explicitly, consider the graded S_n-module $R^\pi_\bullet = \overline{\mathbb{Q}}_\ell[x_1, \dots, x_n]/I^\pi$, where I^π is the ideal of all polynomials $p(x_1, \dots, x_n)$ such that $p(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n})$ annihilates the Specht module V^π, realized in the usual way as a subspace of the homogeneous component $\overline{\mathbb{Q}}_\ell[x_1, \dots, x_n]_{n(\pi)}$. It follows from [6] that for $x \in \mathcal{O}_\pi$ and for each i, there is an isomorphism of S_n-modules

$$H^{2i}(\mathcal{B}_x, \overline{\mathbb{Q}}_\ell) \cong R^\pi_i. \tag{6.8}$$

Analogously, one may expect that for $(v,x) \in \mathbb{O}_{\rho;\sigma}$ and for each i, there is an isomorphism of W_n-modules

$$H^{2i}(\mathcal{C}_{(v,x)}, \overline{\mathbb{Q}}_\ell) \cong R^{\rho;\sigma}_i, \tag{6.9}$$

where $R_\bullet^{\rho;\sigma}$ is associated in the same way to the Specht module $V^{\rho;\sigma}$, realized via Macdonald–Lusztig–Spaltenstein induction (see [12]) as a subspace of the homogeneous component $\overline{\mathbb{Q}}_\ell[x_1,\dots,x_n]_{b(\rho;\sigma)}$.

Assuming Steps 1 and 2 can be carried out, we see (using Corollary 5.3(1)) that Conjecture 6.4 is equivalent to the following statement:

Conjecture 6.5. *(See Shoji [25, 3.13].) For $(\rho;\sigma), (\mu;\nu) \in \mathcal{Q}_n$,*

$$\sum_i \langle R_i^{\rho;\sigma}, \chi^{\mu;\nu}\rangle_{W_n} t^i = \widetilde{K}_{(\mu;\nu),(\rho;\sigma)}(t).$$

It is not clear which of these putatively equivalent conjectures would be easier to prove. Garsia and Procesi (see [10, (I.8)]) have given a purely algebraic/combinatorial proof that

$$\sum_i \langle R_i^{\pi}, \chi^{\lambda}\rangle_{S_n} t^i = \widetilde{K}_{\lambda\pi}(t). \tag{6.10}$$

Possibly their arguments can be adapted to prove Conjecture 6.5. Alternatively, one might tackle Conjecture 6.4 by imitating Lusztig's study of intersection cohomology in [20]. In [20], Lusztig defines a certain inner product on intersection cohomology complexes, and then computes this inner product with the aid of orthogonality relations for character sheaves. An analogous inner product for $\mathfrak{N}$ is defined by

$$\begin{aligned}&\big\langle \mathrm{IC}(\overline{\mathbb{O}_{\mu;\nu}}, \overline{\mathbb{Q}}_\ell), \mathrm{IC}(\overline{\mathbb{O}_{\mu';\nu'}}, \overline{\mathbb{Q}}_\ell)\big\rangle_q \\ &\quad = \sum_{\substack{i,j\in\mathbb{Z}\\ z\in\mathfrak{N}(\mathbb{F}_q)}} (-1)^{i+j}\,\mathrm{tr}\big(F\,|\,\mathcal{H}_z^i \mathrm{IC}(\overline{\mathbb{O}_{\mu;\nu}}, \overline{\mathbb{Q}}_\ell)\big)\,\mathrm{tr}\big(F\,|\,\mathcal{H}_z^i \mathrm{IC}(\overline{\mathbb{O}_{\mu';\nu'}}, \overline{\mathbb{Q}}_\ell)\big),\end{aligned}$$

and the desired formula is

$$\big\langle \mathrm{IC}(\overline{\mathbb{O}_{\mu;\nu}}, \overline{\mathbb{Q}}_\ell), \mathrm{IC}(\overline{\mathbb{O}_{\mu';\nu'}}, \overline{\mathbb{Q}}_\ell)\big\rangle_q = q^{-(b(\mu;\nu)+b(\mu';\nu'))}\omega_{(\mu;\nu),(\mu';\nu')}(q). \tag{6.11}$$

By the same uniqueness argument as in the proof of Theorem 5.4, Eq. (6.11) would imply Conjecture 6.4, and one could also deduce as a by-product that $|\mathbb{O}_{\mu;\nu}(\mathbb{F}_q)| = \theta_{\mu;\nu}(q^2) = |\mathcal{O}_{\mu;\nu}(\mathbb{F}_{q^2})|$.

Acknowledgments

We are very grateful to S. Kato, and to M. Finkelberg, V. Ginzburg, and R. Travkin, for generously keeping us informed of their work as ours progressed. We are also indebted to T. Shoji for pointing out the connection with his paper [25].

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