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On generalized competition index of a primitive tournament

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ABSTRACT

For positive integers k and m and a digraph D, the k-step m-competition graph $C_m^k(D)$ of D has the same set of vertices as D and an edge between vertices x and y if and only if there exist m distinct vertices $v_1, v_2, \ldots, v_m$ in D such that there exist directed walks of length k from x to v_i and from y to v_i for $1 \le i \le m$. The m-competition index of a primitive digraph D is the smallest positive integer k such that $C_m^k(D)$ is a complete graph. In this paper, we study the m-competition indices of primitive tournaments and provide an upper bound for the m-competition index of a primitive tournament.

1. Preliminaries and notations

In this paper, we follow the terminology and notation used in [1,3–6]. Let $D = (V, E)$ denote a *digraph* (directed graph) with vertex set $V = V(D)$, arc set $E = E(D)$, and order n. Loops are permitted but multiple arcs are not. A *walk* from x to y in a digraph D is a sequence of vertices $x, v_1, \ldots, v_t, y \in V(D)$ and a sequence of arcs $(x, v_1), (v_1, v_2), \ldots, (v_t, y) \in E(D)$, where the vertices and arcs are not necessarily distinct. A *closed walk* is a walk from x to y where $x = y$. A *cycle* is a closed walk from x to y with distinct vertices except for $x = y$.

The *length of a walk* W is the number of arcs in W. The notation $x \overset{k}{\to} y$ is used to indicate that there exists a walk from x to y of length k. The notation $x \to y$ indicates that there exists an arc (x, y). The *distance* from vertex x to vertex y in D is the length of the shortest walk from x to y, and it is denoted by $d_D(x, y)$. An l-cycle is a cycle of length l.

A digraph D is called *strongly connected* if for each pair of vertices x and y in $V(D)$, there exists a walk from x to y. For a strongly connected digraph D, the *index of imprimitivity* of D is the greatest common divisor of the lengths of the cycles in D, and it is denoted by $l(D)$. If D is a trivial digraph of order 1, $l(D)$ is undefined. For a strongly connected digraph D, D is *primitive* if $l(D) = 1$.

If D is a primitive digraph of order n, there exists some positive integer k such that there exists a walk of length exactly k from each vertex x to each vertex y. The smallest such k is called the *exponent* of D, and it is denoted by exp(D). For a positive integer m where $1 \le m \le n$, we define the *m-competition index* of a primitive digraph D, denoted by $k_m(D)$, as the smallest positive integer k such that for every pair of vertices x and y, there exist m distinct vertices $v_1, v_2, \ldots, v_m$ such that $x \overset{k}{\longrightarrow} v_i$ and $y \overset{k}{\longrightarrow} v_i$ for $1 \le i \le m$ in D.

Kim [7] introduced the m-competition index as a generalization of the competition index presented in [6]. Akelbek and Kirkland [1,2] introduced the scrambling index of a primitive digraph D, denoted by $k(D)$. In the case of primitive digraphs, the definitions of the scrambling index and competition index are identical. Furthermore, we have $k(D) = k_1(D)$.

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For a positive integer k and a primitive digraph D, we define the *k-step outneighborhood* of a vertex x as

$$N^+(D^k : x) = \left\{ v \in V(D) | x \xrightarrow{k} v \right\}.$$

We define the *k-step outneighborhood* of a vertex set X as $N^+(D^k : X) = \cup_{x \in X} N^+(D^k : x)$. We define the *k-step common outneighborhood* of vertices x and y as $N^+(D^k : x, y) = N^+(D^k : x) \cap N^+(D^k : y)$.

We define the *local m-competition index* of vertices x and y as $k_m(D : x, y) = \min\{k : |N^+(D^t : x, y)| \geq m$ where $t \geq k\}$. We also define the *local m-competition index* of x as $k_m(D : x) = \max_{y \in V(D)}\{k_m(D : x, y)\}$. Then, we have

$$k_m(D) = \max_{x \in V(D)} k_m(D : x) = \max_{x,y \in V(D)} k_m(D : x, y).$$

From the definitions of $k_m(D)$, $k_m(D : x)$, and $k_m(D : x, y)$, we have $k_m(D : x, y) \leq k_m(D : x) \leq k_m(D)$. On the basis of the definitions of the m-competition index and the exponent of D of order n, we can write $k_m(D) \leq \exp(D)$, where m is a positive integer with $1 \leq m \leq n$. Furthermore, we have $k_n(D) = \exp(D)$ and

$$k(D) = k_1(D) \leq k_2(D) \leq \cdots \leq k_n(D) = \exp(D).$$

This is a generalization of the scrambling index and exponent. There exist many researches about exponents and their generalization; for example, [9].

An *n-tournament* T_n is a digraph with n vertices in which every pair of vertices is joined by exactly one arc. Assigning an orientation to each edge of a complete graph results in a tournament.

Proposition 1 (*Moon and Pullman [8]*)**.** *An n-tournament T_n is primitive if and only if T_n is irreducible (strongly connected) and $n > 3$.*

Theorem 2 (*Moon and Pullman [8]*)**.** *Each vertex of a strongly connected tournament T_n, when $n \geq 3$, is contained in at least one simple cycle of each length between 3 and n, inclusive.*

Some results about $k_1(T_n)$ and $k_n(T_n)$ have been presented in [6,8].

Theorem 3 (*Kim [6]*)**.** *For a primitive n-tournament T_n where $n \geq 5$, we have*

$$k_1(T_n) \leq 3$$

and the bound is sharp for all $n \geq 5$.

Proposition 4 (*Kim [6]*)**.** *Let $n \geq 5$. For a positive integer k where $2 \leq k \leq 3$, there exists a primitive n-tournament T_n with $k_1(T_n) = k$.*

Theorem 5 (*Moon and Pullman [8]*)**.** *For a primitive n-tournament T_n, we have*

$$k_n(T_n) \leq n + 2.$$

Proposition 6 (*Moon and Pullman [8]*)**.** *Let $n \geq 6$. For a positive integer k where $3 \leq k \leq n + 2$, there exists a primitive n-tournament T_n such that $k_n(T_n) = k$.*

2. Main results

Some results on the generalized exponents of tournaments have been presented in [9]. In this paper, we study $k_m(T_n)$ for a strongly connected n-tournament T_n where $n \geq 5$ and $1 \leq m \leq n$ as another generalization of an exponent.

Definition 7. Suppose $A, B \subset V(D)$ and $x \in V(D)$. We define the following notation:
$A \Rightarrow B$: $a \to b$ for any $a \in A$ and for any $b \in B$.
$x \Rightarrow B$: $x \to b$ for any $b \in B$.
$A \Rightarrow x$: $a \to x$ for any $a \in A$.

Lemma 8. *Let T_n be a primitive n-tournament and A be a nonempty subset of $V(T_n)$. Then, $|N^+(T_n : A)| \geq |A|$ and $|N^+(T_n : A) \cap A| \geq |A| - 1$.*

Proof. If $A = V(T_n)$, then $N^+(T_n : A) = V(T_n)$. We have the result. Suppose $A \neq V(T_n)$. Then, $|N^+(T_n : A) \setminus A| \geq 1$ since T_n is strongly connected, and $|A \setminus N^+(T_n : A)| \leq 1$. Therefore, $|N^+(T_n : A)| \geq |A|$ and $|N^+(T_n : A) \cap A| \geq |A| - 1$. This establishes the result. □

Lemma 9. *Let T_n be a primitive n-tournament where $n \geq 5$. Then,*

$$k_2(T_n) \leq 4.$$

Proof. Let x and y be two vertices in $V(T_n)$. Then, $N^+(T_n^3 : x, y) \neq \phi$ by Theorem 3. If $|N^+(T_n^3 : x, y)| \geq 2$, then we have $|N^+(T_n^4 : x, y)| \geq 2$ by Lemma 8. Then, $k_2(T_n : x, y) \leq 4$. Suppose that $N^+(T_n^3 : x, y) = \{u\}$. Then there exists a vertex v such that $u \to v$ since T_n is strongly connected. Denote $A = V(T_n) \setminus \{u, v\}$ and $B = N^+(T_n : v) \cap A$. In this notion, we have

$A \Rightarrow u, u \rightarrow v, v \Rightarrow B$, and $A \setminus B \Rightarrow v$. By Theorem 2, we also have

$$A \Rightarrow u \xrightarrow{3} u \quad \text{and} \quad u \xrightarrow{4} u.$$

We will show that $v \xrightarrow{4} u$. Note that $|A| \geq 3$ and $B \neq \phi$ because T_n is strongly connected.

Case 1. $|B| \geq 3$.

There exists a walk of length 2 in B since T_n is a tournament. Let $a_1 \xrightarrow{2} a_2$ be such walk where $a_1, a_2 \in B$. Then, $v \rightarrow a_1 \xrightarrow{2} a_2 \rightarrow u$.

Case 2. $|B| = 2$.

Let $B = \{a, b\}$ where $a \rightarrow b$. Then, $|A \setminus B| \geq 1$ since $|A| \geq 3$. If $A \setminus B \Rightarrow \{a, b\}$, then T_n is not strongly connected because there exists no walk from a to a vertex in $A \setminus B$. Therefore, it is impossible that $A \setminus B \Rightarrow \{a, b\}$.

Subcase 2.1. $N^+(T_n : b) \cap (A \setminus B) = \phi$.

There exists a vertex $c \in N^+(T_n : a) \cap (A \setminus B)$ such that $a \rightarrow c \rightarrow b$. Then, $v \rightarrow a \rightarrow c \rightarrow b \rightarrow u$.

Subcase 2.2. $N^+(T_n : b) \cap (A \setminus B) \neq \phi$.

There exists a vertex $c \in N^+(T_n : b) \cap (A \setminus B)$ such that $a \rightarrow b \rightarrow c$. Then, $v \rightarrow a \rightarrow b \rightarrow c \rightarrow u$.

Case 3. $|B| = 1$.

Let $B = \{a\}$. Then, $|A \setminus B| \geq 2$ since $|A| \geq 3$. We also have $N^+(T_n : a) \cap (A \setminus B) \neq \phi$ since T_n is strongly connected.

Subcase 3.1. $|N^+(T_n : a) \cap (A \setminus B)| \geq 2$.

There exist two vertices $b, c \in N^+(T_n : a) \cap (A \setminus B)$ such that $a \rightarrow b \rightarrow c$. Then, $v \rightarrow a \rightarrow b \rightarrow c \rightarrow u$.

Subcase 3.2. $|N^+(T_n : a) \cap (A \setminus B)| = 1$.

Let $N^+(T_n : a) \cap (A \setminus B) = \{b\}$. It is impossible that $A \setminus \{a, b\} \Rightarrow b$ since T_n is strongly connected. Therefore, there exists a vertex $c \in A \setminus \{a, b\}$ such that $a \rightarrow b \rightarrow c$ because $A \setminus \{a, b\} \neq \phi$. Then, $v \rightarrow a \rightarrow b \rightarrow c \rightarrow u$.

In all cases, $v \xrightarrow{4} u$. Then, $u \in N^+(T_n^4 : x, y)$. Therefore, $\{u, v\} \subset N^+(T_n^4 : x, y)$ and $k_2(T_n : x, y) \leq 4$. This establishes the result. □

Lemma 10. *Let T_n be a primitive n-tournament where $n \geq 5$. Then,*

$$k_3(T_n) \leq 5.$$

Proof. Let x and y be two vertices in $V(T_n)$. By Lemma 9, we have $|N^+(T_n^4 : x, y)| \geq 2$. If $|N^+(T_n^4 : x, y)| \geq 3$, then we have $|N^+(T_n^5 : x, y)| \geq 3$ by Lemma 8. Then, $k_3(T_n : x, y) \leq 5$. Suppose that $N^+(T_n^4 : x, y) = \{u, v\}$. If $|N^+(T_n : \{u, v\})| \geq 3$, then $k_3(T_n : x, y) \leq 5$. Suppose that $N^+(T_n : \{u, v\}) = \{v, w\}$ for a vertex w. Denote $A = V(T_n) \setminus \{u, v, w\}$ and $B = N^+(T_n : w) \cap A$. In this notion, we have $A \Rightarrow \{u, v\}, u \rightarrow v \rightarrow w, w \Rightarrow B$, and $A \setminus B \Rightarrow w$. By Theorem 2, we have

$$A \Rightarrow u \xrightarrow{4} u, u \xrightarrow{5} u, \quad \text{and} \quad w \xrightarrow{3} w \Rightarrow B \Rightarrow u.$$

Note that $B \neq \phi$ because T_n is strongly connected.

Case 1. $n = 5$.

Let $A = \{a, b\}$ where $a \rightarrow b$. Then, we have $w \rightarrow a$ since T_n is strongly connected.

Subcase 1.1. $w \rightarrow u$.

We have $v \xrightarrow{3} v \rightarrow w \rightarrow u$. Then, $u \in N^+(T_n^5 : x, y)$ and $\{u, v, w\} \subset N^+(T_n^5 : x, y)$. Therefore, $k_3(T_n : x, y) \leq 5$.

Subcase 1.2. $u \rightarrow w$.

There are two cases such as $w \rightarrow b$ or $b \rightarrow w$. In these cases, we can check $k_3(T_n) = 5$.

Case 2. $n \geq 6$.

We have $|A| \geq 3$. We will show that $v \xrightarrow{5} u$.

Subcase 2.1. $|B| \geq 3$.

There exists a walk of length 2 in B since T_n is a tournament. Let $a_1 \xrightarrow{2} a_2$ be such walk where $a_1, a_2 \in B$. Then, $v \rightarrow w \rightarrow a_1 \xrightarrow{2} a_2 \rightarrow u$.

Subcase 2.2. $|B| = 2$.

Let $B = \{a, b\}$ where $a \rightarrow b$. Then, $|A \setminus B| \geq 1$ since $|A| \geq 3$. If $A \setminus B \Rightarrow \{a, b\}$, then T_n is not strongly connected because there exists no walk from a to a vertex in $A \setminus B$. Therefore, it is impossible that $A \setminus B \Rightarrow \{a, b\}$.

Subcase 2.2.1. $N^+(T_n : b) \cap (A \setminus B) = \phi$.

There exists a vertex $c \in N^+(T_n : a) \cap (A \setminus B)$ such that $a \rightarrow c \rightarrow b$. Then, $v \rightarrow w \rightarrow a \rightarrow c \rightarrow b \rightarrow u$.

Subcase 2.2.2. $N^+(T_n : b) \cap (A \setminus B) \neq \phi$.

There exists a vertex $c \in N^+(T_n : b) \cap (A \setminus B)$ such that $a \rightarrow b \rightarrow c$. Then, $v \rightarrow w \rightarrow a \rightarrow b \rightarrow c \rightarrow u$.

Subcase 2.3. $|B| = 1$.

Let $B = \{a\}$. Then, $|A \setminus B| \geq 2$ since $|A| \geq 3$. We have $N^+(T_n : a) \cap (A \setminus B) \neq \phi$ since T_n is strongly connected.

Subcase 2.3.1. $|N^+(T_n : a) \cap (A \setminus B)| \geq 2$.

There exist two vertices $b, c \in N^+(T_n : a) \cap (A \setminus B)$ such that $a \rightarrow b \rightarrow c$. Then, $v \rightarrow w \rightarrow a \rightarrow b \rightarrow c \rightarrow u$.

Subcase 2.3.2. $|N^+(T_n : a) \cap (A \setminus B)| = 1$.

Let $N^+(T_n : a) \cap (A \setminus B) = \{b\}$. It is impossible that $A \setminus \{a, b\} \Rightarrow b$ since T_n is strongly connected. Therefore, there exists a vertex $c \in A \setminus \{a, b\}$ such that $a \to b \to c$ because $A \setminus \{a, b\} \neq \phi$. Then, $v \to w \to a \to b \to c \to u$.

If $n \geq 6$, then we have $v \xrightarrow{5} u$. Therefore, $u \in N^+(T_n^5 : x, y)$ and $\{u, v, w\} \subset N^+(T_n^5 : x, y)$. We have $k_3(T_n : x, y) \leq 5$. This establishes the result. □

Lemma 11. *Let T_n be a primitive n-tournament where $n \geq 5$ and m be a positive integer such that $3 \leq m \leq n - 2$. If $k_m(T_n) \leq m + 2$, then we have*

$$k_{m+1}(T_n) \leq m + 3.$$

Proof. Suppose that there exists a primitive n-tournament T_n such that $k_m(T_n) \leq m + 2$ and $k_{m+1}(T_n) > m + 3$ for a positive integer m. Then, we can find two vertices x and y such that $|N^+(T_n^{m+3} : x, y)| < m + 1$ and $|N^+(T_n^{m+2} : x, y)| \geq m$. Let $N^+(T_n^{m+2} : x, y) = V'$. If $|V'| > m$ or $|N^+(T_n : V')| > m$, then we have $|N^+(T_n^{m+3} : x, y)| \geq m + 1$. Therefore, we have $|V'| = m$ and $|N^+(T_n : V')| = m$. Because T_n is strongly connected, $N^+(T_n : V') \setminus V' \neq \phi$. In addition, we have $|N^+(T_n : V') \cap V'| = m - 1$ by Lemma 8. Then, there exists a vertex $u \in V'$ such that $u \Rightarrow V' \setminus \{u\}$. Furthermore, there exists a vertex $v(\neq u)$ such that $N^+(T_n : V') \setminus V' = \{v\}$.

Denote $C = V' \setminus \{u\}$, $C' = \{x | x \to v\} \cap C$, $A = V(T_n) \setminus V' \setminus \{v\}$, and $B = N^+(T_n : v) \cap A$. Note that $B \neq \phi$ and $C \neq \phi$ because T_n is strongly connected. In this notion, we have

$$A \Rightarrow V', \qquad u \Rightarrow C, \qquad v \Rightarrow B, \qquad A \setminus B \Rightarrow v, \quad \text{and} \quad C' \Rightarrow v.$$

Let $b \in B$. For each vertex $c \in C$, we have

$$c \xrightarrow{l} v \xrightarrow{1} b,$$

where $l \leq m - 1$ because $|C| = m - 1$. Then, we have $d_{T_n}(c, b) \leq m$. In addition, we have $A \Rightarrow C' \Rightarrow v \to b$, $u \Rightarrow C' \Rightarrow v \to b$, and $v \to b$. In all cases, we have $d_{T_n}(z, b) \leq m$ for each vertex $z \in V(T_n)$ because $m \geq 3$.

By Theorem 2, there exists a q-cycle containing b where $q = m + 3 - d_{T_n}(z, b) \geq 3$. Then, we have $z \xrightarrow{m+3} b$ for each vertex $z \in V(T_n)$. Therefore, we have $\{b, v\} \cup C \subset N^+(T_n^{m+3} : x, y)$. This is contrary to $|N^+(T_n^{m+3} : x, y)| < m + 1$. This establishes the result. □

Theorem 12. *Let T_n be a primitive n-tournament where $n \geq 5$. For a positive integer m such that $1 \leq m \leq n$, we have*

$$k_m(T_n) \leq m + 2,$$

and the bound is sharp for all $n \geq 5$.

Proof. By Theorems 3 and 5, Lemmas 9–11, we have

$$k_m(T_n) \leq m + 2$$

where $1 \leq m \leq n$. Consider an n-tournament $F = (V, E)$ for $n \geq 5$, as follows.

$$V = \{v_1, v_2, \ldots, v_n\},$$
$$E = \{(v_i, v_{i+1}) | 1 \leq i \leq n - 1\} \cup \{(v_n, v_1)\} \cup \{(v_i, v_j) | 2 \leq j + 1 < i \leq n\}.$$

Because there exists an n-cycle, F is primitive. We have $N^+(F^2 : v_1, v_2) = \phi$ and for a positive integer p such that $3 \leq p \leq n + 2$, we have $N^+(F^p : v_1, v_2) = \{v_1, v_2, \ldots, v_{p-2}\}$. Therefore, we have $k_m(F) \geq m + 2$ for $1 \leq m \leq n$. This establishes the result. □

Example 13. The condition $n \geq 5$ in Theorem 12 is essential. Consider the strongly connected 4-tournament T_4 whose adjacency matrix is given as

$$\begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}.$$

Then, we have $k_1(T_4) = 4$, $k_2(T_4) = 6$, $k_3(T_4) = 8$ and, $k_4(T_4) = 9$. We have $k_m(T_4) > m + 2$ for $m = 1, 2, 3, 4$.

Theorem 14. *Let $n \geq 6$ and $1 \leq m \leq n$. For a positive integer k where $3 \leq k \leq m + 2$, there exists a primitive n-tournament T_n such that $k_m(T_n) = k$.*

Proof. If $m = 1$ or $m = n$, this holds by Propositions 4 and 6, respectively. Suppose $2 \leq m \leq n - 1$ and $3 \leq k \leq m + 2$. Let $p = n - 3 - m + k$.

Case 1. $p \geq 4$.

Consider an n-tournament F for $n \geq 6$ as follows.

$$V(F) = \{v_1, v_2, \ldots, v_n\},$$
$$E(F) = \{(v_{i+1}, v_i)|1 \leq i \leq n-1\} \cup \{(v_1, v_n)\} \cup \{(v_i, v_j)|p \leq j < i \leq n\} \cup \{(v_i, v_j)|1 \leq i \leq p-1, i+2 \leq j \leq n\}.$$

We have $N^+(F^2 : v_p, v_{p+1}) = \phi$. For $1 \leq x < y \leq n$ and $1 \leq i \leq p+1$, we have

$$N^+(F^{i+2} : v_p, v_{p+1}) = \{v_{p+2-i}, v_{p+3-i}, \ldots, v_n\},$$
$$|N^+(F^{i+2} : v_x, v_y)| \geq n-p+i-1.$$

Therefore, we have $k_m(F) = m+p+3-n = k$ because $n-m \leq p \leq n-1$.

Case 2. $p \leq 3$.

We have

$$\{(k, m)|p \leq 3\} = \{(3, n-3), (3, n-2), (3, n-1), (4, n-2), (4, n-1), (5, n-1)\}.$$

We will find tournaments with this condition. Consider n-tournaments F_1, and F_2 for $n \geq 6$, and F_3 for $n \geq 7$ as follows.

$$V(F_1) = V(F_2) = V(F_3) = \{v_1, v_2, \ldots, v_n\},$$

and

$$\begin{aligned} E(F_1) &= \{(v_{i+1}, v_i)|1 \leq i \leq n-1\} \cup \{(v_1, v_n)\} \cup \{(v_i, v_j)|3 \leq j < i \leq n\} \cup \{(v_i, v_j)|1 \leq i \leq 2, i+2 \leq j \leq n\} \\ &\quad \cup \{(v_4, v_n)\} \setminus \{(v_n, v_4)\}, \\ E(F_2) &= \{(v_{i+1}, v_i)|1 \leq i \leq n-1\} \cup \{(v_1, v_n)\} \cup \{(v_i, v_j)|2 \leq j < i \leq n\} \cup \{(v_i, v_j)|i = 1, 3 \leq j \leq n\} \\ &\quad \cup \{(v_3, v_n)\} \setminus \{(v_n, v_3)\}, \end{aligned}$$
$$E(F_3) = \{(v_i, v_{i+1})|1 \leq i \leq n-1\} \cup \{(v_n, v_1)\} \cup \{(v_i, v_j)|2 \leq j+1 < i \leq n\} \cup \{(v_2, v_n)\} \setminus \{(v_n, v_2)\}.$$

Subcase 2.1. $k_m(F_1)$.

We have

$$N^+(F_1^3 : v_2, v_3) = \{v_3, v_4, \ldots, v_n\},$$
$$N^+(F_1^4 : v_2, v_3) = \{v_2, v_3, \ldots, v_n\}.$$

For $1 \leq i < j \leq n$, we have $|N^+(F_1^4 : v_i, v_j)| \geq n-1$. Therefore, we have $k_{n-1}(F_1) = 4$.

Subcase 2.2. $k_m(F_2)$.

We have

$$N^+(F_2^2 : v_2, v_4) = \{v_n\},$$
$$N^+(F_2^3 : v_2, v_4) = \{v_2, v_3, \ldots, v_n\}.$$

For $1 \leq i < j \leq n$, we have $|N^+(F_2^3 : v_i, v_j)| \geq n-1$. Therefore, we have $k_{n-3}(F_2) = k_{n-2}(F_2) = k_{n-1}(F_2) = 3$.

Subcase 2.3. $k_m(F_3)$.

We have

$$N^+(F_3^3 : v_1, v_3) \subset \{v_1, v_3, v_6\},$$
$$N^+(F_3^4 : v_1, v_3) = \{v_1, v_2, \ldots, v_{n-2}\},$$

and $k_n(F_3) = 5$. For $1 \leq i < j \leq n$, we have $|N^+(F_3^4 : v_i, v_j)| \geq n-2$. Therefore, we have $k_{n-2}(F_3) = 4$ and $k_{n-1}(F_3) = 5$.

Subcase 2.4. $k_m(F_4)$.

Let F_4 be 6-tournament whose adjacency matrix is given as

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix}.$$

We have $k_4(F_4) = 4$ and $k_5(F_4) = 5$.

This establishes the result. □

Example 15. Let F_1, F_2, F_3, and F_4 be 5-tournaments whose adjacency matrices are respectively given as

$$\begin{bmatrix}0&1&0&0&0\\0&0&1&0&0\\1&0&0&1&0\\1&1&0&0&1\\1&1&1&0&0\end{bmatrix}, \begin{bmatrix}0&1&0&0&0\\0&0&1&1&0\\1&0&0&1&0\\1&0&0&0&1\\1&1&1&0&0\end{bmatrix},$$
$$\begin{bmatrix}0&1&0&0&0\\0&0&1&1&1\\1&0&0&1&0\\1&0&0&0&1\\1&0&1&0&0\end{bmatrix}, \begin{bmatrix}0&1&0&1&0\\0&0&1&0&1\\1&0&0&1&0\\0&1&0&0&1\\1&0&1&0&0\end{bmatrix}.$$

Then, we have

$$k_2(F_3) = k_3(F_3) = k_4(F_3) = 3,$$
$$k_2(F_1) = k_3(F_2) = k_4(F_4) = k_5(F_4) = 4,$$
$$k_3(F_1) = k_4(F_2) = 5,$$
$$k_4(F_1) = k_5(F_2) = 6,$$
$$k_5(F_1) = 7.$$

By Proposition 4, there exists a 5-tournament whose 1-competition index is 3. Let m be a positive integer such that $2 \leq m \leq 4$. For a positive integer k such that $3 \leq k \leq m+2$, there exists a 5-tournament T_5 such that $k_m(T_5) = k$. Also, there exists a 5-tournament whose 5-competition index is 4, 6 or 7. However, we cannot find a 5-tournament whose 5-competition index is 3 or 5.

3. Closing remark

Akelbek and Kirkland [1] provided the concept of the scrambling index of a primitive digraph. Kim [7] introduced a generalized competition index $k_m(D)$ as another generalization of the exponent $\exp(D)$ and scrambling index $k(D)$ for a primitive digraph D. In this paper, we study $k_m(T_n)$ as an extension of the results presented in [6,8].

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