JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS 66, 237–259 (1978)

Probabilistic Inequality Constraints in Stochastic Optimal Control Theory*

M. K. OZGOREN,[†] R. W. LONGMAN[‡]

Columbia University, New York, New York 10027

AND

C. A. COOPER[§]

Bell Laboratories, Holmdel, New Jersey 07733

Submitted by G. Leitmann

There are many optimal control problems in which it is necessary or desirable to constrain the range of values of state variables. When stochastic inputs are involved, these inequality constraint problems are particularly difficult. Sometimes the constraints must be modeled as hard constraints which can never be violated, and other times it is more natural to prescribe a probability that the constraints will not be violated. This paper treats general problems of the latter type, in which probabilistic inequality constraints are imposed on the state variables or on combinations of state and control variables. A related class of problems in which the state is required to reach a target set with a prescribed probability is handled by the same methods. It is shown that the solutions to these problems can be obtained by solving a comparatively simple bilinear deterministic control problem.

1. INTRODUCTION

It is often necessary to constrain the range of values of state variables in optimal control problems when stochastic inputs are present. When the constraints are imposed by nature, they often must be modeled as hard constraints which can never be violated. If instead, the constraints are ones which the control system designer wishes to impose, it can be more natural to pose the control problem so that the desired inequalities on the state are satisfied with some prescribed probability.

* This work was supported by the National Science Foundation under Grant GK37440. Preliminary version presented at the 1975 Conference on Information Sciences and Systems, Johns Hopkins University.

† Previously Graduate Research Assistant. Now Assistant Professor, Middle East Technical University, Ankara, Turkey.

‡ Associate Professor of Mechanical Engineering.

§ Member of Technical Staff.

0022-247X/78/0661–0237$02.00/0

This paper presents a method of handling stochastic optimal control problems with linear systems and quadratic costs when subject to data-unconditional probabilistic inequality constraints on combinations of the state variables, the control variables, or both. Optimal control problems in which it is desired to reach a target set with a prescribed probability are treated as a special case, and an example is given. It is shown that the optimal feedback control laws for these stochastic problems can be found in terms of a comparatively simple bilinear deterministic control problem. Extension of these results to data-conditional probabilistic inequality constraints is discussed.

The problem of maximizing the likelihood of reaching a specified terminal point treated in [1] can be compared to a special case of our target set problem, where the target set approaches a point by a suitable choice of probabilistic inequality constraints. In both cases, the a priori probability of reaching the terminal state is to be maximized. In [2] a specified a posteriori state mean and covariance are obtained at a terminal time. Deterministic control problems with inequality constraints have been considered by others (for example, see [3–5]).

The method presented here is immensely more tractable than solving stochastic optimal control problems subject to hard constraints, which can involve repeated solution of the Fokker–Planck equation. Not only does this method apply to probabilistic inequality constraint problems, but it can also be used for the approximate treatment of problems with hard constraints. With this viewpoint, the method can be considered as a stochastic analogue to the penalty function approach used for deterministic optimal control problems with inequality constraints [4].

2. Stochastic Control Problems with Probabilistic State Inequality Constraints or Probabilistic Target Sets

In this section two classes of stochastic optimal control problems will be defined. The first constrains the state of the system to remain in certain regions of the state space with prescribed probabilities. The second requires that the final value of the state be in a prescribed target set with given probability.

Probabilistic State Inequality Constraint Problem. *Given the linear time-varying stochastic system*

$$
\begin{aligned}
dx(t) &= A(t)\,x(t)\,dt + B(t)\,u(t)\,dt + f(t)\,dt + dv(t),\\
dy(t) &= C(t)\,x(t)\,dt + dw(t), \qquad (1)\\
E[x(0)] = \bar{x}_0\,, &\qquad E[(x(0)-\bar{x}_0)\,(x(0)-\bar{x}_0)'] = P_0\,,
\end{aligned}
$$

where $x(t) \in \mathscr{R}^n$, $u(t) \in \mathscr{R}^m$, $y(t) \in \mathscr{R}^k$; $m \leqslant n$; *and* $t \in [0, T]$. *The* $v(t) \in \mathscr{R}^n$ *and* $w(t) \in \mathscr{R}^k$ *are scaled Weiner processes with rates*

$$\begin{aligned} E[v(t)\, v'(s)] &= \tau V(\tau), \\ E[w(t)\, w'(s)] &= \tau W(\tau), \\ E[v(t)\, w'(s)] &= \tau H(\tau), \end{aligned} \tag{2}$$

for all t, $s \in [0, T]$, *and where* $\tau = \min(t, s)$. $W(t)$ *is positive definite on* $[0, T]$. *The matrices* $A(t)$, $B(t)$, *and* $C(t)$ *are continuous on* $[0, T]$. *The system* (A, B) *is totally state controllable on* $[0, T]$. *The vector* $f(t)$ *is an endogenous driving term continuous on* $[0, T]$, *and* $u(t)$ *is the control which is unconstrained.*

Let the terminal time T *be fixed, and the terminal state be free. Then find a control* u *which minimizes the functional*

$$J = E\left\{\tfrac{1}{2}\int_0^T [x'(t)\, Q(t)\, x(t) + u'(t)\, R(t)\, u(t)]\, dt + \tfrac{1}{2}x'(T) F x(T)\right\}, \tag{3}$$

where $Q(t)$ *and* $R(t)$ *are continuous symmetric matrices, and* $Q(t) \geqslant 0$, $F \geqslant 0$, $R(t) > 0$ *for all* $t \in [0, T]$. *The minimization is subject to the following probabilistic inequality constraints*

$$\begin{aligned} \Pr\{K_i'(t)\, x(t) \geqslant \alpha_i(t)\} &\geqslant \gamma_i(t), & i &\in Y_1, \\ \Pr\{K_i'(t)\, x(t) \geqslant \alpha_i(t)\} &\leqslant \gamma_i(t), & i &\in Y_2, \\ \Pr\{\alpha_i(t) \leqslant K_i'(t)\, x(t) \leqslant \beta_i(t)\} &\geqslant \gamma_i(t), & i &\in Y_3, \\ \Pr\{\alpha_i(t) \leqslant K_i'(t)\, x(t) \leqslant \beta_i(t)\} &\leqslant \gamma_i(t), & i &\in Y_4, \end{aligned} \tag{4}$$

where the $K_i(t) \in \mathscr{R}^n$. $K_i(t)$, $\alpha_i(t)$, $\beta_i(t)$, *and* $\gamma_i(t)$ *are continuously differentiable on* $[0, T]$, *and must be chosen so that the constraints are self-consistent. The* $K_i(t) \neq 0$ *and* $\gamma_i(t) \in (0, 1)$ *for all* $t \in [0, T]$. *The* Y_j *are either the null set or mutually exclusive finite sets of integers. The initial value* $x(0)$ *must be such that the inequality constraints* (4) *are satisfied at* $t = 0$.

The probabilistic inequality constraints cannot be overspecified in the sense that it becomes impossible to satisfy all constraints simultaneously. This meaning of the term "self-consistent" is made more precise in the sequel, and a test for self-consistency is given in Section 5. Note that constraints of the forms $\Pr(K_i'x \leqslant \alpha_i\} \geqslant \gamma_i^*$ and $\Pr\{K_i'x \leqslant \alpha_i\} \leqslant \gamma_i^*$ are equivalent to constraints of the forms associated with Y_1 and Y_2, respectively, with $\gamma_i^* = 1 - \gamma_i$. Note also that observability is not required.

PROBABILISTIC TARGET SET PROBLEM. *Given the time-varying stochastic system of Eqs.* (1), *find a control* u *which minimizes* (3) *and which transfers the state from* $x(0)$ *to an* $x(T)$ *satisfying the following probabilistic target set*

$$\Pr\{K_i'x(T) \geqslant \alpha_i\} \geqslant \gamma_i\,, \qquad i \in Y_1\,,$$

$$\Pr\{K_i'x(T) \geqslant \alpha_i\} \leqslant \gamma_i\,, \qquad i \in Y_2\,,$$

$$\Pr\{\alpha_i \leqslant K_i'x(T) \leqslant \beta_i\} \geqslant \gamma_i\,, \qquad i \in Y_3\,,$$

$$\Pr\{\alpha_i \leqslant K_i'x(T) \leqslant \beta_i\} \leqslant \gamma_i\,, \qquad i \in Y_4\,,$$

where the $K_i \in \mathscr{R}^n$. K_i, α_i, β_i, and γ_i are bounded and must be chosen so that the inequalities are self-consistent; and the $\gamma_i \in (0, 1)$. The sets Y_j are either the null set or mutually exclusive finite sets of integers.

In certain circumstances, the specification of a probabilistic target set is quite natural. It may be very difficult to require that every attempt reach the desired target set; and an alternative is to specify a probabilistic target set upon which some reasonable fraction of attempts are required to impinge.

3. Deterministic Control Problems

Two bilinear deterministic control problems with matrix-valued state variables are defined and are later related to the stochastic problems of Section 2. The rather interesting cost functional for these problems is necessarily nonnegative in spite of the linear terms in state Ξ (see Section 5).

Bilinear Deterministic Constraint Problem. *Given the matrix-valued differential equations*

$$\begin{aligned} \dot{\Xi}(t) &= [A(t) + B(t)\,\Psi(t)]\,\Xi(t) + \Xi(t)[A(t) + B(t)\,\Psi(t)]' + \Gamma(t), \\ \dot{z}(t) &= A(t)\,z(t) + B(t)\,\omega(t) + f(t), \\ \Xi(0) &= \Xi_0\,,\ z(0) = z_0\,, \qquad t \in [0, T], \end{aligned} \tag{5}$$

where Ξ and z are symmetric $n \times n$ and $n \times 1$ state matrices, respectively. Also, $\Gamma(t)$ is symmetric, positive semidefinite, and continuous on $[0, T]$; and Ξ_0 is symmetric and positive semidefinite.

Let the terminal time T be fixed, and the terminal state be free. Then find an $m \times n$ control matrix Ψ and an $m \times 1$ control matrix ω which minimizes the functional

$$\begin{aligned} I = \Big\{ \tfrac{1}{2} \int_0^T & [z'(t)\,Q(t)\,z(t) + \omega'(t)\,R(t)\,\omega(t) + \operatorname{tr} Q(t)\,\Xi(t) \\ & + \operatorname{tr} R(t)\,\Psi(t)\,\Xi(t)\,\Psi'(t)]\,dt + \tfrac{1}{2} z'(T) F z(T) + \tfrac{1}{2} \operatorname{tr} F\Xi(T) \Big\}\,. \end{aligned} \tag{6}$$

The minimization is subject to the following deterministic inequality constraints which must be self-consistent:

$$K_i'(t)\, z(t) - g_i(t, \Xi(t)) \geqslant 0, \qquad i \in \Sigma_1,$$

$$K_i'(t)\, z(t) - h_i(t, \Xi(t)\} \leqslant 0, \qquad i \in \Sigma_2,$$

$$K_i'(t)\, z(t) - g_i(t, \Xi(t)) \geqslant 0 \quad \text{or} \quad K_i'(t)\, z(t) - h_i(t, \Xi(t)) \leqslant 0, \qquad i \in \Sigma_3. \tag{7}$$

The initial conditions Ξ_0 and z_0 must satisfy these constraints at $t = 0$, and Σ_1, Σ_2, Σ_3 are finite integer sets. Let each constraint be defined on a prescribed subset $\mathscr{I}_i \subseteq [0, T]$. For all $i \in \Sigma_1 \cup \Sigma_2 \cup \Sigma_3$, K_i is continuously differentiable on $\mathscr{I}_i$. For these same i, g_i and h_i are continuously differentiable on $\mathscr{I}_i$ with respect to their first argument t, and twice continuously differentiable on the class of positive semidefinite matrices with respect to each element of the second argument Ξ. The matrices A, B, Q, F, R, and f satisfy the same requirements as in the Probabilistic State Inequality Constraint Problem.

BILINEAR DETERMINISTIC TARGET SET PROBLEM. *Given the system equations* (5), *find control matrices Ψ and ω which minimize* (6) *and which transfer the system from Ξ_0, z_0 to a terminal state $\Xi(T)$, $z(T)$ satisfying the deterministic target set defined by the following inequalities which must be self-consistent.*

$$K_i' z(T) - g_i^*(\Xi(T)) \geqslant 0, \qquad i \in \Sigma_1,$$

$$K_i' z(T) - h_i^*(\Xi(T)) \leqslant 0, \qquad i \in \Sigma_2,$$

$$K_i' z(T) - g_i^*(\Xi(T)) \geqslant 0, \quad \text{or} \quad K_i' z(T) - h_i^*(\Xi(T)) \leqslant 0, \qquad i \in \Sigma_3.$$

The Σ_1, Σ_2, Σ_3 are finite integer sets, and for $i \in \Sigma_1 \cup \Sigma_2 \cup \Sigma_3$ the g_i^* and h_i^* are continuously differentiable on the class of positive semidefinite matrices with respect to each element of the argument $\Xi(T)$.

4. CONVERSION OF STOCHASTIC CONTROL PROBLEMS TO DETERMINISTIC CONTROL PROBLEMS

The relationships between the stochastic problems of Section 2 and the deterministic problems of Section 3 will be established. The following lemma will be used to convert the probabilistic constraints to a deterministic form.

CONVERSION LEMMA. *Let r be a Gaussianly distributed random variable with variance σ. Let α and β be finite numbers, and let $\gamma \in (0, 1)$. Then:*

(i) *The probabilistic inequality*

$$\Pr\{r \geqslant \alpha\} \geqslant \gamma$$

is satisfied if and only if the expected value $\bar{r} = E(r)$ *satisfies*

$$\bar{r} \geqslant \alpha + 2^{1/2}\sigma \operatorname{erf}^{-1}(2\gamma - 1),$$

where erf *is the error function.*

(ii) *The inequality* $\Pr\{r \geqslant \alpha\} \leqslant \gamma$ *is satisfied if and only if* $\bar{r}$ *satisfies*

$$\bar{r} \leqslant \alpha + 2^{1/2}\sigma \operatorname{erf}^{-1}(2\gamma - 1).$$

(iii) *Define*

$$\Phi(s) = \tfrac{1}{2} \operatorname{erf}[(\beta - s)/2^{1/2}\sigma] - \tfrac{1}{2} \operatorname{erf}[(\alpha - s)/2^{1/2}\sigma],$$
$$r^* = \tfrac{1}{2}(\alpha + \beta).$$

Concerning the probabilistic inequality

$$\Pr\{\alpha \leqslant r \leqslant \beta\} = \Phi(\bar{r}) \geqslant \gamma, \tag{8}$$

three cases apply:

(a) *If* $\Phi(r^*) > \gamma$, *then* (8) *is satisfied if and only if* $\alpha^* \leqslant \bar{r} \leqslant \beta^*$, *where* $\beta^* > \alpha^*$ *are the two roots of* $\Phi(s) - \gamma = 0$.

(b) *If* $\Phi(r^*) = \gamma$, *then* (8) *is equivalent to the equality* $\bar{r} = r^*$.

(c) *If* $\Phi(r^*) < \gamma$, *then the probabilistic inequality* (8) *cannot be satisfied.*

(iv) *Concerning the probabilistic inequality*

$$\Pr\{\alpha \leqslant r \leqslant \beta\} \leqslant \gamma, \tag{9}$$

two cases apply:

(a) *If* $\Phi(r^*) > \gamma$, *then* (9) *is satisfied if and only if* $\bar{r}$ *satisfies one of the two inequalities*

$$\bar{r} \leqslant \alpha^*, \qquad \bar{r} \geqslant \beta^*.$$

(b) *If* $\Phi(r^*) \leqslant \gamma$, *then* (9) *is satisfied for all values of* $\bar{r}$.

Proof. For (i) write

$$\begin{aligned}\Pr\{r \geqslant \alpha\} &= \frac{1}{(2\pi)^{1/2}\,\sigma} \int_\alpha^\infty \exp[-(r - \bar{r})^2/2\sigma^2]\, dr \\ &= \tfrac{1}{2}\{1 - \operatorname{erf}[(\alpha - \bar{r})/2^{1/2}\sigma]\} \\ &\geqslant \gamma,\end{aligned}$$

or

$$\operatorname{erf}[(\alpha - \bar{r})/2^{1/2}\sigma] \leqslant 1 - 2\gamma.$$

Since the error function is a monotonically increasing function of its argument, and since $1 - 2\gamma$ is in the interval $(-1, 1)$, the inverse error function of both sides can be taken while preserving the direction of the inequality. This gives the desired result. The proof of (ii) is analogous.

To prove (iii), note that

$$\begin{aligned}
\Pr\{\alpha \leqslant r \leqslant \beta\} &= \frac{1}{(2\pi)^{1/2}\,\sigma} \int_\alpha^\beta \exp[-(r - \bar{r})^2/2\sigma^2]\, dr \\
&= \tfrac{1}{2} \operatorname{erf}[(\beta - \bar{r})/2^{1/2}\sigma] - \tfrac{1}{2} \operatorname{erf}[(\alpha - \bar{r})/2^{1/2}\sigma] \\
&= \Phi(\bar{r}).
\end{aligned}$$

Consider the continuously differentiable function $\Phi(r^* + \zeta)$ as a function of ζ. Then

$$\frac{\partial \Phi}{\partial \zeta} = -\frac{1}{(2\pi)^{1/2}\,\sigma} \left\{ \exp\left[\left(\frac{\beta - \alpha}{2} - \zeta\right)^2 \Big/ 2\sigma^2\right] - \exp\left[\left(\frac{\alpha - \beta}{2} - \zeta\right)^2 \Big/ 2\sigma^2\right] \right\},$$

which is positive for $-\infty < \zeta < 0$, zero for $\zeta = 0$, and negative for $0 < \zeta < +\infty$. Hence, $\Phi(r^* + \zeta)$ is a monotonically increasing function of ζ for $-\infty < \zeta < 0$ and a monotonically decreasing function for $0 < \zeta < +\infty$, and it takes on a global maximum at $\zeta = 0$. Since $\lim_{|\zeta|\to\infty} \Phi(r^* + \zeta) = 0$, the function $\Phi(r^* + \zeta)$ assumes each value in the interval $(0, \Phi(r^*))$ exactly once for $\zeta > 0$, and once for $\zeta < 0$. Furthermore, using $\operatorname{erf}(\xi) = -\operatorname{erf}(-\xi)$ it is seen that $\Phi(r^* + \zeta) = \Phi(r^* - \zeta)$, so that the two values of ζ are identical in absolute value. Then $\Pr\{\alpha \leqslant r \leqslant \beta\} \geqslant \gamma$ implies $\Phi(\bar{r}) - \gamma \geqslant 0$. If $\Phi(r^*) > \gamma$ there are two values of $\bar{r}$, α^* and β^*, for which $\Phi(\bar{r}) - \gamma = 0$ and all values of $\bar{r}$ in $[\alpha^*, \beta^*]$ satisfy the inequality. If $\Phi(r^*) = \gamma$ only $\bar{r} = r^*$ satisfies the inequality, and if $\Phi(r^*) < \gamma$ there is no expected value of r for which the probabilistic inequality is satisfied. Since each of the steps can be reversed, (iii) is proved. Similarly $\Pr\{\alpha \leqslant r \leqslant \beta\} \leqslant \gamma$ implies $\Phi(\bar{r}) - \gamma \leqslant 0$, and the inequality is satisfied for $\bar{r} \leqslant \alpha^*$ and for $\beta^* \leqslant \bar{r}$, provided $\Phi(r^*) > \gamma$. If $\Phi(r^*) \leqslant \gamma$, then there are no real roots of $\Phi(\bar{r}) - \gamma = 0$, and all $\bar{r}$ satisfy the inequality. ▮

The following theorem shows that for a restricted class of admissible controls, the solution to the stochastic control problem with probabilistic state inequality constraints can be obtained by solving a deterministic problem.

CONVERSION THEOREM. *Let the set of admissible controls in the Probabilistic State Inequality Constraint Problem be restricted to the class of affine functions*

$\mu(t, \hat{x}) = M(t)\hat{x} + N(t)$, *where* $\mu: [0, T] \times \mathscr{R}^n \to \mathscr{R}^m$, $M(t)$ *and* $N(t)$ *are piecewise continuous on* $[0, T]$. *Define*

$$\hat{x}(t) = E[x(t) \mid \mathscr{Y}_t],$$

where $\mathscr{Y}_t$ *is the* σ*-field generated by* $\{y(\tau), 0 \leqslant \tau \leqslant t\}$. *Then any optimal control* $u(t)$ *satisfying the Probabilities State Inequality Constraint Problem can be expressed as*

$$u(t) = \omega(t) + \Psi(t)[\hat{x}(t) - z(t)],$$

where $\omega(t)$ *and* $\Psi(t)$ *are optimal controls for a Bilinear Deterministic Constraint Problem, and* $z(t)$ *is the corresponding optimal solution.*

Proof. Let the following quantities be defined:

$$\begin{aligned}
\bar{x}(t) &= E[x(t) \mid \mathscr{Y}_0],\\
\xi(t) &= x(t) - \bar{x}(t),\\
\hat{\xi}(t) &= E[\xi(t) \mid \mathscr{Y}_t],\\
\tilde{\xi}(t) &= \xi(t) - \hat{\xi}(t),\\
\bar{u}(t) &= E[u(t) \mid \mathscr{Y}_0],\\
\tilde{u}(t) &= u(t) - \bar{u}(t),\\
\Pi(t) &= E[\hat{\xi}(t)\, \hat{\xi}'(t)],\\
P(t) &= E[\tilde{\xi}(t)\, \tilde{\xi}'(t)].
\end{aligned}$$

One can then obtain

$$\begin{aligned}
\bar{u}(t) &= M(t)\, \bar{x}(t) + N(t),\\
\tilde{u}(t) &= M(t)\, \hat{\xi}(t).
\end{aligned} \tag{10}$$

(i) *Conversion of the Cost Functional*

After the order of the expectation and the integration in (3) are interchanged, the cost functional can be written as

$$\begin{aligned}
J = \Big\{ &\tfrac{1}{2} \int_0^T [\bar{x}'Q\bar{x} + \bar{u}'R\bar{u} + \operatorname{tr} QE(\xi\xi') + \operatorname{tr} RE(\tilde{u}\tilde{u}')]\, dt\\
&+ \tfrac{1}{2}\, \bar{x}'(T)F\bar{x}(T) + \tfrac{1}{2} \operatorname{tr} FE[\xi(T)\, \xi'(T)] \Big\}.
\end{aligned} \tag{11}$$

The remaining expectations can be expressed as $E(\xi\xi') = \Pi + P$ and $E(\tilde{u}\tilde{u}') =$

$M\Pi M'$ after noting that $E(\hat{\xi}\tilde{\xi}') = 0$. Then the desired form of the cost functional is

$$J = \Big\{ \tfrac{1}{2} \int_0^T [\bar{x}'Q\bar{x} + \bar{u}'R\bar{u} + \operatorname{tr} Q(\Pi + P) + \operatorname{tr} RM\Pi M']\, dt \\ + \tfrac{1}{2}\bar{x}'(T)F\bar{x}(T) + \tfrac{1}{2} \operatorname{tr} F[\Pi(T) + P(T)] \Big\}, \tag{12}$$

which is to be minimized with respect to the control variables $\bar{u}$ and M.

(ii) *The Augmented State Equations*

Since the cost functional (12) contains $\bar{x}$, Π, and P, differential equations for these quantities will now be generated. Taking the expectation of the state equations (1) gives the immediate result

$$\dot{\bar{x}}(t) = A(t)\,\bar{x}(t) + B(t)\,\bar{u}(t) + f(t), \\ \bar{x}(0) = \bar{x}_0 . \tag{13}$$

To calculate Π and P it is necessary to obtain equations for $\hat{\xi}$ and $\tilde{\xi}$. Standard results can be used to show that the best estimate $\hat{x}$ satisfies

$$d\hat{x}(t) = [A(t) - G(t)\,C(t)]\,\hat{x}(t)\,dt + B(t)\,u(t)\,dt + f(t)\,dt + G(t)\,dy(t), \tag{14}$$

$$G(t) = [P(t)\,C'(t) + H(t)]\,W^{-1}(t), \tag{15}$$

$$\hat{x}(0) = \bar{x}_0 .$$

Then

$$d\hat{\xi}(t) = A(t)\,\hat{\xi}(t)\,dt + G(t)\,C(t)\,\tilde{\xi}(t)\,dt + B(t)\,\tilde{u}(t)\,dt + G(t)\,dw(t), \\ \hat{\xi}(0) = 0, \\ d\tilde{\xi}(t) = [A(t) - G(t)\,C(t)]\,\tilde{\xi}(t)\,dt + dv(t) - G(t)\,dw(t), \\ \tilde{\xi}(0) = x_0 - \bar{x}_0 . \tag{16}$$

Let $\zeta' = [\hat{\xi}' \mid \tilde{\xi}']$, and use $\tilde{u} = M\hat{\xi}$ to express ζ in integral form as

$$\zeta(t) = Z(t, 0)\,\zeta(0) + \int_0^t Z(t, s)\,dq, \tag{17}$$

where Z is the state transition matrix corresponding to the system matrix

$$\mathscr{A} = \begin{bmatrix} A + BM & GC \\ 0 & A - GC \end{bmatrix}$$

and $q' = [(Gw)' \mid (v - Gw)']$. Observe that the covariance of $\zeta(t)$ can be partitioned as

$$L(t) = E[\zeta(t)\, \zeta'(t)] = \begin{bmatrix} \Pi(t) & 0 \\ 0 & P(t) \end{bmatrix} \tag{18}$$

and can be written as

$$L(t) = Z(t, 0) L(0)\, Z'(t, 0) + \int_0^t Z(t, s)\, \Omega(s)\, Z'(t, s)\, ds. \tag{19}$$

The scaled Weiner process $q(t)$ has the rate $E[q(t_1)\, q'(t_2)] = \tau\Omega(\tau)$ for all $t_1, t_2 \in [0, T]$ where $\tau = \min(t_1, t_2)$, and

$$\Omega = \begin{bmatrix} GWG' & GH' - GWG' \\ HG' - GWG & GWG' - HG' - GH' + V \end{bmatrix}.$$

Differentiating (19) and using (18) and (15) gives the desired differential equations for Π and P.

$$\begin{aligned} \dot{\Pi}(t) &= [A(t) + B(t)\, M(t)]\, \Pi(t) + \Pi(t)[A(t) + B(t)\, M(t)]' \\ &\quad + [P(t)\, C'(t) + H(t)]\, W^{-1}(t)[P(t)\, C'(t) + H(t)]', \qquad (20) \\ \Pi(0) &= 0, \\ \dot{P}(t) &= A(t)\, P(t) + P(t)\, A'(t) - [P(t)\, C'(t) + H(t)]\, W^{-1}(t) \\ &\quad \times [P(t)\, C'(t) + H(t)]' + V(t), \qquad (21) \\ P(0) &= P_0 . \end{aligned}$$

Since $P(t)$ is independent of the control variables $\bar{u}$ and M, and can be calculated a priori, the state variables associated with the cost functional (12) are $\bar{x}$ and Π. These variables are related to the control variables by state equations (13) and (20) of the augmented system.

(iii) *Conversion of the Inequality Constraints*

Equations (1) and (16), after using (10), form a coupled set of linear equations driven by Gaussian processes. Hence $x(t)$ is Gaussianly distributed, and the conditions of the *Conversion Lemma* are satisfied at each time t. The r of the lemma is $K_i'(t)\, x(t)$, whose variance is

$$\begin{aligned} \sigma_i^2(t, \Pi(t)) &= K_i'(t)\, E[\xi(t)\, \xi'(t)]\, K_i(t) \\ &= K_i'(t)\, [P(t) + \Pi(t)]\, K_i(t), \qquad i \in \bigcup_{j=1}^{4} Y_j . \qquad (22) \end{aligned}$$

Application of parts (i) and (ii) of the lemma gives

$$\begin{aligned} \bar{g}_i(t, \sigma_i) &= \alpha_i(t) + 2^{1/2}\sigma_i(t, \Pi(t)) \operatorname{erf}^{-1}[2\gamma_i(t) - 1], \qquad i \in Y_1, \\ \bar{h}_i(t, \sigma_i) &= \alpha_i(t) + 2^{1/2}\sigma_i(t, \Pi(t)) \operatorname{erf}^{-1}[2\gamma_i(t) - 1], \qquad i \in Y_2. \end{aligned} \tag{23}$$

Define

$$\begin{aligned} \Phi_i(s, t, \sigma_i) &= \frac{1}{2}\operatorname{erf}\left[\frac{\beta_i(t) - s}{2^{1/2}\sigma_i(t, \Pi(t))}\right] - \frac{1}{2}\operatorname{erf}\left[\frac{\alpha_i(t) - s}{2^{1/2}\sigma_i(t, \Pi(t))}\right], \\ &\qquad\qquad i \in Y_3 \cup Y_4, \\ r_i^*(t) &= \tfrac{1}{2}[\alpha_i(t) + \beta_i(t)], \qquad i \in Y_3 \cup Y_4. \end{aligned} \tag{24}$$

The intervals for which the converted constraints exist are

$$\mathscr{I}_i = \{t \mid 0 \leqslant t \leqslant T\}, \qquad i \in \bigcup_{j=1}^{3} Y_j,$$

$$\mathscr{I}_i(\Pi(\cdot)) = \{t \mid 0 \leqslant t \leqslant T, \Phi_i(r_i^*(t), t, \sigma_i(t, \Pi(t))) > \gamma_i(t)\}, \qquad i \in Y_4.$$

Those inequalities indexed by Y_4 which are not always satisfied are identified by

$$Y_{41}(\Pi(\cdot)) = \{i \mid i \in Y_4, \Phi_i(r_i^*(t), t, \sigma_i(t, \Pi(t))) > \gamma_i(t) \text{ for some } t \in [0, T]\}.$$

Application of parts (iii) and (iv) of the lemma gives $\bar{g}_i(t, \sigma_i)$ and $\bar{h}_i(t, \sigma_i)$ for $i \in Y_3 \cup Y_{41}$ as the roots $s(t)$ of

$$\Phi_i(s, t, \sigma_i) = \gamma_i(t), \tag{25}$$

with $\bar{g}_i \leqslant \bar{h}_i$. If for any $i \in Y_3$, $\Phi_i(r_i^*(t), t, \sigma_i) < \gamma_i(t)$ for any $t \in [0, T]$, then the constraints (4) are inconsistent. Since by assumption the constraints are self-consistent, the $i \in Y_3$ can be expressed as two constraints, one of the form associated with Σ_1 and one associated with Σ_2. These sets then become

$$\Sigma_1 = Y_1 \cup Y_3; \qquad \Sigma_2 = Y_2 \cup Y_3; \qquad \Sigma_3 = Y_{41}.$$

It remains to make the following identification of the variables in the *Bilinear Deterministic Constraint Problem*:

$$\Xi(t) = \Pi(t); \qquad \Xi_0 = 0; \qquad z(t) = \bar{x}(t); \qquad \Psi(t) = M(t); \qquad \omega(t) = \bar{u}(t),$$

$$\begin{aligned} \Gamma(t) &= [P(t)\, C'(t) + H(t)]\, W^{-1}(t)\, [P(t)\, C'(t) + H(t)]', \\ g_i(t, \Xi(t)) &= \bar{g}_i(t, \sigma_i(t, \Xi(t))), \\ h_i(t, \Xi(t)) &= \bar{h}_i(t, \sigma_i(t, \Xi(t))), \qquad i \in Y_1 \cup Y_2 \cup Y_3 \cup Y_{41}, \end{aligned} \tag{26}$$

and the proof of the theorem is complete. ∎

Application of the above *Conversion Theorem* to a one-dimensional stochastic system reduces the problem to that of finding the optimal control for a two-dimensional deterministic system. In general, an n-dimensional stochastic optimization problem is converted to a $\frac{1}{2}n(n+3)$-dimensional deterministic problem. This is the dimensionality penalty one must pay for the comparative ease of solution of deterministic problems.

By limiting the inequality constraint conversion to time $t = T$ in part (iii) of the proof of the *Conversion Theorem*, the following corollary is established.

COROLLARY 1. *Let the set of admissible controls be as specified in the Conversion Theorem. Then any optimal control $u(t)$ satisfying the Probabilistic Target Set Problem can be expressed as*

$$u(t) = \omega(t) + \Psi(t)[\hat{x}(t) - z(t)],$$

where $\omega(t)$ and $\Psi(t)$ are optimal controls for a Bilinear Deterministic Target Set Problem, and $z(t)$ is the corresponding optimal solution.

It is instructive to view the state inequality constraint problem and the target set problem in z, t space, instead of in z, Ξ, t space. The target set as viewed in z space can be considered control dependent through the choice of the feedback gain Ψ which determines the value of Ξ. Hence the optimal control law must pick not only an optimal control which causes the system to reach the target set, but it must choose the target set as well. For the probabilistic inequality constraint problem in z, t space, the α_i and β_i values produce a control dependent inequality constraint which might be termed the *fiducial boundary*. For properly chosen γ_i, increasing the feedback gain Ψ will decrease the covariance of $\tilde{x}$ and cause the *fiducial boundary* to expand, approaching the limits α_i and β_i as Ψ grows unbounded. Hence, the optimal control not only minimizes the cost functional subject to inequality constraints (in x, t space) but it simultaneously chooses the constraints themselves (as seen in z, t space). The controller decides how much cost it is willing to incur to expand the *fiducial boundary*. The example in Section 7 illustrates some of these concepts.

5. SELF-CONSISTENCY TEST FOR PROBABILISTIC CONSTRAINTS

In the statement of the *Probabilistic State Inequality Constraint Problem*, it is required that the set of constraints be self-consistent. This concept is defined as follows.

DEFINITION. The probabilistic inequality constraints (4) are said to be self-consistent on $[0, T]$ for system (1) if for all $t \in [0, T]$ there exist a state vector

$z(t)$ and a symmetric state matrix $\Xi(t)$, which are within the respective sets of reachable states and for which all constraints are simultaneously satisfied.

A test for self-consistency will be developed which can be applied a priori without solving the optimization problem. The following *Controllability Lemma* will be needed in the development.

CONTROLLABILITY LEMMA. *The covariance state equation*

$$\begin{aligned}\dot{\Xi}(t) &= [A(t) + B(t)\Psi(t)]\,\Xi(t) + \Xi(t)[A(t) + B(t)\Psi(t)]' + \Gamma(t),\\ \Xi(0) &= 0,\end{aligned}$$

with $\Gamma(t)$ positive definite, is totally state controllable on $(0, T]$ relative to the covariance control $\Psi(t)$ for $\Xi(t)$ restricted to the class of positive definite $n \times n$ symmetric matrices on $(0, T]$. That is, for every interval $[t_1, t_2]$ with $0 < t_1 < t_2 \leqslant T$, there exists a bounded $\Psi(t)$ which transfers the system from any positive definite $\Xi(t_1)$ to any prescribed positive definite $\Xi(t_2)$. Furthermore, no $\Psi(t)$ exists for which $v'\Xi(t)v = 0$, $v \neq 0$, for any $t \in (0, T]$.

Proof. The matrix Ξ can be written as the sum of two matrices Ξ_1 and Ξ_2 which satisfy

$$\begin{aligned}\dot{\Xi}_1(t) &= A(t)\,\Xi_1(t) + \Xi_1(t)\,A'(t) + \Gamma(t),\\ \dot{\Xi}_2(t) &= A(t)\,\Xi_2(t) + \Xi_2(t)\,A'(t) + B(t)\,U(t) + U'(t)\,B'(t), \qquad (27)\\ \Xi_1(0) &= \Xi_2(0) = 0,\\ U(t) &= \Psi(t)\,\Xi(t) = \Psi(t)[\Xi_1(t) + \Xi_2(t)]. \qquad (28)\end{aligned}$$

As stated earlier, the system (A, B) is assumed to be totally state controllable on $[0, T]$; i.e., it is completely state controllable on every interval $[t_1, t_2]$ with $0 \leqslant t_1 < t_2 \leqslant T$. Therefore, the Gram matrix

$$G(t_1, t_2) = \int_{t_1}^{t_2} X(t_1, t)\,B(t)\,B'(t)\,X'(t_1, t)\,dt$$

is nonsingular, where $X(t, t_1)$ is the state transition matrix for system matrix $A(t)$ [6].

Without loss of generality, assume the final state $\Xi_2(t_2) = 0$. The definition of Ξ_1 can always be modified to transform any $\Xi_2(t_2)$ to zero. Consider the control

$$U(t) = -\tfrac{1}{2}B'(t)\,X'(t_1, t)\,G^{-1}(t_1, t_2)\,\Xi_2(t_1)\,X'(t, t_1).$$

$\Xi_2(t_2)$ can be written in integral form as

$$\begin{aligned}\Xi_2(t_2) &= X(t_2, t_1)\,\Xi_2(t_1)\,X'(t_2, t_1)\\ &\quad + \int_{t_1}^{t_2} X(t_2, t)\,[B(t)\,U(t) + U'(t)\,B'(t)]\,X'(t_2, t)\,dt.\end{aligned}$$

Substituting $U(t)$, using symmetry of $\Xi_2(t_1)$ and $G(t_1, t_2)$, using properties of the state transition matrix, and using the definition of $G(t_1, t_2)$ shows that the control $U(t)$ produces $\Xi_2(t_2) = 0$. Therefore, relative to control matrix $U(t)$, system (27) is totally state controllable on $[0, T]$ for Ξ_2 in the class of symmetric $n \times n$ matrices. Then for every interval $[t_1, t_2]$ with $0 \leqslant t_1 < t_2 \leqslant T$, there exists a control $U(t)$ which transfers the state $\Xi(t_1)$ to any prescribed $\Xi(t_2)$.

Provided $\Xi^{-1}(t)$ exists for all $t \in [t_1, t_2]$, there exists a covariance control $\Psi(t)$ which generates this $U(t)$, and it is given by

$$\Psi(t) = U(t)\, \Xi^{-1}(t).$$

By definition, $\Xi(t)$ is at least positive semidefinite. It is easily shown that $\Xi(t)$ is necessarily positive definite on $(0, T]$. In integral form

$$\Xi(t) = \int_0^t \Theta(t, s)\, \Gamma(s)\, \Theta'(t, s)\, ds,$$

where $\Theta(t, s)$ is the state transition matrix associated with system matrix $A(t) + B(t)\, \Psi(t)$. Consider

$$\nu'\Xi(t)\, \nu = \int_0^t \nu'\Theta(t, s)\, \Gamma(s)\, \Theta'(t, s)\, \nu\, ds.$$

By assumption Γ is positive definite. Also, $\Theta(t, s)$ is necessarily nonsingular so that the integrand is positive for all $\nu \neq 0$. Therefore $\nu'\Xi(t)\nu > 0$ for all $\nu \neq 0$ and $t \in (0, T]$, and $\Xi(t)$ is positive definite on this interval. Hence, $\Xi^{-1}(t)$ exists. Then for every interval $[t_1, t_2]$ with $0 < t_1 < t_2 \leqslant T$, there exists a bounded covariance control $\Psi(t)$ which transfers the system from any positive definite $\Xi(t_1)$ to any prescribed positive definite $\Xi(t_2)$. Furthermore, no $\Psi(t)$ exists for which $\Xi(t)$ can have a zero or negative eigenvalue for any $t \in (0, T]$. ∎

SELF-CONSISTENCY THEOREM. *Let Y_4 be the null set, and let $\Gamma(t)$ be positive definite on* $[0, T]$. *For all $t \in [0, T]$, let the $\gamma_i(t)$ satisfy*

$$\gamma_i(t) > \tfrac{1}{2}, \qquad i \in Y_1,$$

$$\gamma_i(t) < \tfrac{1}{2}, \qquad i \in Y_2,$$

$$\gamma_i(t) > \frac{1}{2} \operatorname{erf}\left[\frac{\beta_i(t) - \alpha_i(t)}{2^{1/2}\sigma_i(t, 0)}\right] \leqslant \frac{1}{2}, \qquad i \in Y_3.$$

Define

$$Y_{31} = \{i \mid i \in Y_3,\ \Phi_i(r_i^*(t), t, \sigma_i(t, 0)) > \gamma_i(t)\ \forall t \in (0, T]\},$$

and for $t \in [0, T]$ define

$$S(t) = \{z(t) \mid z(t) \textit{ satisfies inequalities } (29) \textit{ at } t\},$$

where (29) *is*

$$\begin{aligned} K_i'(t)\, z(t) &> g_i(t, 0), \qquad i \in Y_1 \cup Y_{31}, \\ K_i'(t)\, z(t) &< h_i(t, 0), \qquad i \in Y_2 \cup Y_{31}. \end{aligned} \tag{29}$$

Then for system (1) *the probabilistic inequality constraints* (4) *are self-consistent if and only if* $Y_{31} = Y_3$ *and* $S(t)$ *is nonempty for all* $t \in (0, T]$.

Proof. The converted forms of the probabilistic constraints are relationships between $z(t)$ and $\Xi(t)$. From the *Controllability Lemma* the positive definite $\Xi(t)$ is totally state controllable on $(0, T]$. In particular a covariance control $\Psi(t)$ exists which keeps $\Xi(t)$ arbitrarily close (in the sense of the Euclidean norm) to the zero matrix for all $t \in [0, T]$, although there exists no control function Ψ for which $\Xi(t)$ has a zero eigenvalue for some $t \in (0, T]$. It will be shown that, under the conditions of the theorem, the probabilistic inequality constraints are least restrictive on the set of values that $z(t)$ can assume when $\Xi(t)$ approaches zero.

Equation (22) with $\Pi = \Xi$ gives σ_i^2 as the sum of two quadratic forms. Since the $K_i(t) \neq 0$, the second quadratic form is positive for all $t \in (0, T]$, and therefore

$$\sigma_i(t, \Xi(t)) > \sigma_i(t, 0), \qquad t \in (0, T]. \tag{30}$$

Furthermore, by proper choice of $\Psi(t)$, $\sigma_i(t, \Xi(t))$ can be made to assume any value satisfying this inequality.

Constraints associated with $i \in Y_1$ transform to $K_i'(t)\, z(t) \geqslant \bar{g}_i(t, \sigma_i)$, where the $\bar{g}_i$ are given by (23). By assumption $\gamma_i(t) > \frac{1}{2}$ for $i \in Y_1$, so that $(\partial \bar{g}_i/\partial \sigma_i) > 0$. Combining this and (30) shows that $g_i(t, \Xi(t)) > g_i(t, 0)$ for $\Xi \neq 0$. Since $\Xi(t)$ can be maintained arbitrarily close to the zero matrix, but $\Xi \neq 0$ for $t \in (0, T]$, the least restrictive form of the constraints is $K_i'(t)\, z(t) > g_i(t, 0)$ for $t \in (0, T]$. Since $\Xi(0) = 0$, equality is allowed at $t = 0$. For constraints associated with $i \in Y_2$, analogous arguments give the least restrictive form of these constraints as $K_i'(t)\, z(t) < h_i(t, 0)$ for $t \in (0, T]$ with equality allowed at $t = 0$.

Consider constraints associated with $i \in Y_3$, and define

$$\gamma_i^*(t, \sigma_i) = \frac{1}{2} \operatorname{erf} \left[\frac{\beta_i(t) - \alpha_i(t)}{2^{1/2} \sigma_i(t, \Xi(t))} \right].$$

Then $(\partial \gamma_i^*/\partial \sigma_i) \leqslant 0$ for $\alpha_i \geqslant 0$. This together with (30) establishes that the stated assumption $\gamma_i(t) > \gamma_i^*(t, \sigma_i(t, 0))$ implies $\gamma_i(t) > \gamma_i^*(t, \sigma_i(t, \Xi))$ for all Ξ. Note that

$$\Phi_i(\alpha_i, t, \sigma_i(t, \Xi)) = \Phi_i(\beta_i, t, \sigma_i(t, \Xi)) = \gamma_i^*(t, \sigma_i(t, \Xi)),$$

and hence α_i and β_i are roots of $\Phi_i(s, t, \sigma_i(t, \Xi)) = \gamma_i^*(t, \sigma_i(t, \Xi))$. It was shown in the proof of the *Conversion Lemma* that $\Phi_i(r_i^* + \zeta, t, \sigma_i)$ is a monotonically increasing function of ζ for $-\infty < \zeta < 0$, and a monotonically decreasing function of ζ for $0 < \zeta < +\infty$. Therefore the roots $\bar{g}_i$ and $\bar{h}_i$ of

$$\Phi_i(s, t, \sigma_i(t, \Xi)) = \gamma_i(t) \tag{31}$$

must satisfy

$$\alpha_i < \bar{g}_i \leqslant r_i^* \leqslant \bar{h}_i < \beta_i ,$$

provided these roots exist. If the roots fail to exist for any $t \in [0, T]$, then part (iii, c) of the *Conversion Lemma* applies, and the constraint cannot be satisfied. By direct calculation $(\partial\Phi_i/\partial\sigma_i) < 0$ for $\alpha_i < s < \beta_i$. Therefore the roots exist for all $t \in (0, T]$ for some reachable $\Xi(t)$, provided $i \in Y_{31}$. If $i \in Y_{31}$, by continuity arguments it is necessary that at $t = 0$, $\Phi_i(r_i^*(0), 0, \sigma_i(0, 0)) \geqslant \gamma_i(0)$. Since $\Xi(0) = 0$, equality is allowed at $t = 0$, and the roots $\bar{g}_i$ and $\bar{h}_i$ will therefore exist for all $t \in [0, T]$. If there is an i such that $i \in Y_3$ and $i \notin Y_{31}$, then there is some t for which the associated probabilistic constraint cannot be satisfied. The condition $Y_{31} = Y_3$ is therefore a necessary condition for self-consistency.

Write $s = r_i^*(t) + \zeta$ in (31) and differentiate with respect to σ_i for $i \in Y_{31}$ to obtain

$$\frac{\partial\zeta}{\partial\sigma_i} = -\frac{\partial\Phi_i/\partial\sigma_i}{\partial\Phi_i/\partial\zeta} .$$

The numerator has been shown above to be negative; and the denominator is positive for $-\infty < \zeta < 0$ and negative for $0 < \zeta < +\infty$. Therefore

$$\frac{\partial\bar{g}_i}{\partial\sigma_i} > 0, \qquad \frac{\partial\bar{h}_i}{\partial\sigma_i} < 0$$

and using (30) shows that $g_i(t, \Xi(t)) > g_i(t, 0)$ and that $h_i(t, \Xi(t)) < h_i(t, 0)$ for $t \in (0, T]$. Since $\Xi(t)$ can be maintained arbitrarily close to the zero matrix, but $\Xi \neq 0$ for $t \in (0, T]$, the least restrictive form of the constraints for $i \in Y_{31}$ is

$$g_i(t, 0) < K_i'(t)\, z(t) < h_i(t, 0)$$

for $t \in (0, T]$. If this inequality is satisfied, then by continuity $g_i(0, 0) \leqslant K_i'(0)\, z(0) \leqslant h_i(0, 0)$. Equality is allowed at $t = 0$ since $\Xi(0) = 0$.

It has now been shown that if $Y_{31} = Y_3$, the constraints associated with $i \in Y_3$ can individually be satisfied. If simultaneously $S(t)$ is nonempty for all $t \in (0, T]$, then there exist a covariance control $\Psi(t)$ and associated covariance $\Xi(t)$ whose norm is sufficiently small that all probabilistic constraints (4) (with Y_4 the null set) are simultaneously satisfied by some $z(t)$. Therefore the constraints are self-consistent. Furthermore, if $Y_{31} \neq Y_3$, there exists no $z(t)$ which satisfies the probabilistic constraints; and similarly if $S(t)$ is the empty set for some

$t \in (0, T]$, there exists no $z(t)$ which can satisfy the constraints (4) (with Y_4 the null set) no matter what the value of $\Xi(t)$. Hence the proof is complete. ∎

The driving term $\Gamma(t)$ is necessarily at least positive semidefinite. The requirement in the theorem that $\Gamma(t)$ must be positive definite might be relaxed, but the determination of the set of reachable states for $\Xi(t)$ becomes difficult; it may be possible to maintain some or all eigenvalues of Ξ at zero. This introduces the possibility of equality in (29), and whether or not equality is allowed depends on the detailed structure of $\Gamma(t)$, $A(t)$, and the $K_i(t)$.

The limitations on the values of the γ_i are necessary in order that the smallest variance σ_i corresponds to the *fiducial boundary* which is least restrictive on $z(t)$. If the γ_i for an $i \in Y_1$ is less than $\frac{1}{2}$, then the *fiducial boundary* is outside the $K_i'x \geqslant \alpha_i$ region, and an increase in the variance σ_i will cause the *fiducial boundary* to recede from α_i making the constraint less restrictive. This gives rise to the peculiar situation where increasing the uncertainty in $x(t)$ makes it easier to satisfy the constraint. Since one would normally require a high probability for constraints with $i \in Y_1 \cup Y_3$, and a low probability for constraints with $i \in Y_2$, the given limits on the γ_i are natural.

Constraints associated with Y_4 were not considered in the *Self-Consistency Theorem*. If γ_i is sufficiently small for $i \in Y_4$ then decreasing σ_i will make the constraints less restrictive. However, another possibility exists, that by making σ_i sufficiently large case (iv, b) of the *Conversion Lemma* applies and the constraint then disappears. Hence, a sufficient increase in the uncertainty in x can eliminate these constraints while making other constraints more restrictive. Both possibilities must be considered to determine whether a probabilistic constraint set including constraints in Y_4 is self-consistent.

6. Stochastic Control Problems with Probabilistically Constrained Controls and States

In stochastic control problems it is sometimes natural to impose probabilistic constraints on the control $u(t)$ or on combinations of $u(t)$ and $x(t)$. The following general problem of this type can be treated by the techniques developed in previous sections.

Probabilistic State and Control Inequality Constraint Problem. *In the Probabilistic State Inequality Constraint Problem let constraints* (4) *be replaced by*

$$\begin{aligned}
\Pr\{L_i'(t)\,u(t) + K_i'(t)\,x(t) \geqslant \alpha_i(t)\} &\geqslant \gamma_i(t), \qquad i \in Y_1,\\
\Pr\{L_i'(t)\,u(t) + K_i'(t)\,x(t) \geqslant \alpha_i(t)\} &\leqslant \gamma_i(t), \qquad i \in Y_2,\\
\Pr\{\alpha_i(t) \leqslant L_i'(t)\,u(t) + K_i'(t)\,x(t) \leqslant \beta_i(t)\} &\geqslant \gamma_i(t), \qquad i \in Y_3,\\
\Pr\{\alpha_i(t) \leqslant L_i'(t)\,u(t) + K_i'(t)\,x(t) \leqslant \beta_i(t)\} &\leqslant \gamma_i(t), \qquad i \in Y_4,
\end{aligned} \tag{32}$$

where L_i and K_i are not simultaneously zero. Then find a control $u(t)$ which minimizes (3) *subject to constraints* (32).

A second corollary to the *Conversion Theorem* expresses the optimal control for this problem in terms of the control functions of a deterministic problem.

COROLLARY 2. *Let the set of admissible controls be as specified in the Conversion Theorem. Let the inequality constraints* (7) *of the Bilinear Deterministic Constraint Problem be replaced by*

$$L_i'(t)\,\omega(t) + K_i'(t)\,z(t) \geqslant \bar{g}_i(t, \sigma_i), \qquad i \in \Sigma_1,$$

$$L_i'(t)\,\omega(t) + K_i'(t)\,z(t) \leqslant \bar{h}_i(t, \sigma_i), \qquad i \in \Sigma_2,$$

$$L_i'(t)\,\omega(t) + K_i'(t)\,z(t) \geqslant \bar{g}_i(t, \sigma_i) \qquad or$$

$$L_i'(t)\,\omega(t) + K_i'(t)\,z(t) \leqslant \bar{h}_i(t, \sigma_i), \qquad i \in \Sigma_3,$$

where

$$\begin{aligned}&\sigma_i^2(t, \Xi(t), \Psi(t))\\ &\quad = [L_i'(t)\,\Psi(t) + K_i'(t)]\,\Xi(t)\,[L_i'(t)\,\Psi(t) + K_i'(t)]' + K_i'(t)\,P(t)\,K_i(t).\end{aligned}$$

Then any optimal control $u(t)$ satisfying the Probabilistic State and Control Inequality Constraint Problem can be expressed as

$$u(t) = \omega(t) + \Psi(t)[\hat{x}(t) - z(t)],$$

where $\omega(t)$ and $\Psi(t)$ are optimal controls for a Bilinear Deterministic Constraint Problem, and $z(t)$ is the corresponding optimal solution.

Proof. Parts (i) and (ii) of the *Conversion Theorem* apply without change. Since both $u(t)$ and $x(t)$ are Gaussian stochastic processes the *Conversion Lemma* applies to the constraints (32). After proceeding as in (12), the required variance

$$\sigma_i^2(t, \Xi(t), \Psi(t)) = E\{[L_i'(u - \bar{u}) + K_i'(x - \bar{x})]\,[L_i'(u - \bar{u}) + K_i'(x - \bar{x})]'\}$$

is as stated above. With the substitution of this σ_i, the definitions of $\bar{g}_i$, $\bar{h}_i$, and Φ_i remain as in Eqs. (23)–(25), and the proof is complete. ▌

Note that the σ_i are not only dependent on the covariance control $\Psi(t)$ through the covariance $\Xi(t)$ but now have explicit dependence on $\Psi(t)$ as well. As a result, the *Self-Consistency Theorem* does not apply to this problem.

An important special case of (32) is the probabilistic form of the constraint $|u_j| \leqslant 1$ which can be generated from the third type of constraint or as a combination of the first and second constraint types.

7. Example

Consider the following scalar probabilistic target set problem,

$$dx(t) = u(t)\,dt + dv(t), \qquad x(0) = x_0,$$
$$y(t) = x(t),$$
$$E[v(t)\,v(s)] = V\min(t, s),$$
$$p(t) = E[x(t) - z(t)]^2, \qquad p(0) = V/2f, \qquad z(t) = E[x(t)].$$

The set of admissible controls $\mathscr{U}$ is restricted to affine functions of the form $\mu(t, x) = -\psi x + N(t)$ with ψ a constant parameter, and $N(t)$ piecewise continuous. The optimal cost functional is

$$J^* = \operatorname*{Min}_{u\in\mathscr{U}} E\left\{\tfrac{1}{2}\int_0^T [qx^2(t) + u^2(t)]\,dt + \tfrac{1}{2}fx^2(T)\right\}.$$

Let $f = q^{1/2}$, and let the target set be

$$\Pr\{x(T) \geqslant \alpha\} \geqslant \gamma.$$

Note that this problem does not lie precisely within the purview of the *Probabilistic Target Set Problem* since the observations are perfect and ψ is a constant parameter. Nevertheless, with slight modifications, the same methods apply to problems of this form. Use of the *Conversion Lemma* produces the deterministic target set

$$z(t) \geqslant \alpha + \rho[p(T)]^{1/2}, \qquad \rho = 2^{1/2}\operatorname{erf}^{-1}(2\gamma - 1).$$

A procedure analogous to that in the proof of the *Conversion Theorem* results in the converted cost functional

$$J^* = \operatorname*{Min}_{\omega(t),\psi}\left\{\tfrac{1}{2}\int_0^T [qz^2(t) + \omega^2(t) + qp(t) + \psi^2 p(t)]\,dt + \tfrac{1}{2}fz^2(T) + \tfrac{1}{2}fp(T)\right\},$$

where $\omega(t) = E[u(t)]$. Direct calculations show the system equations analogous to (5) are

$$\dot{z}(t) = \omega(t), \qquad z(0) = z_0,$$
$$\dot{p}(t) = -2\psi p(t) + V, \qquad p(0) = V/2f.$$

Figure 1 gives the solution to this deterministic problem when $q = f = \alpha = \rho = T = 1$ and $V = 2$. In region I the trajectories are free trajectories in the sense that they are unaffected by the target set constraint. Free trajectories can be calculated directly without consideration of the above converted cost func-

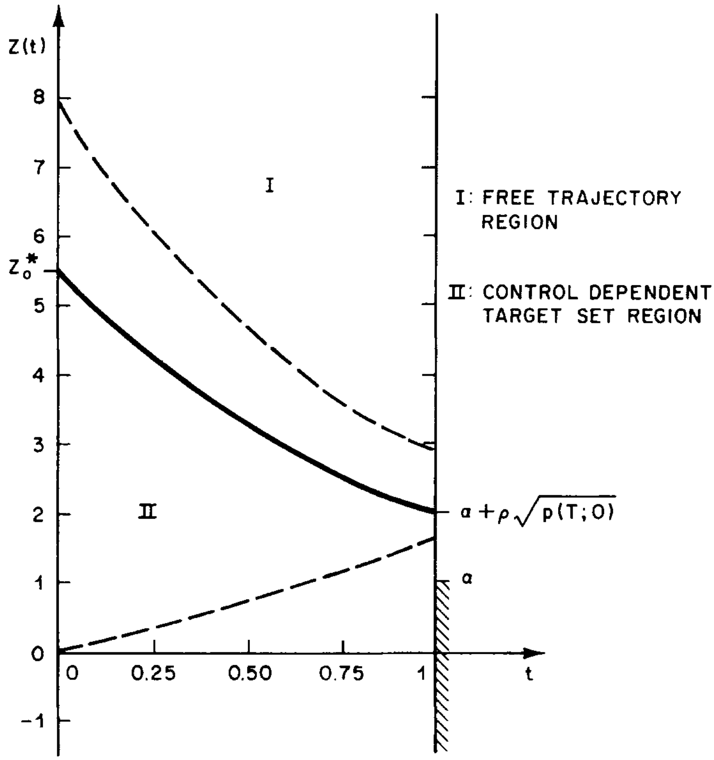

FIG. 1. Example of a probabilistic target set problem.

tional. Hence, the control is $u(t) = -fx(t)$, where f is the constant solution of the Riccati equation

$$\dot{\phi} = \phi^2 - q, \qquad \phi(T) = f.$$

Thus, $\omega(t) = -fz(t)$, $\psi = f$, and the optimal trajectories are given by $z(t) = z_0 \exp(-ft)$. The boundary of the free trajectory region is

$$z(t) = [\alpha + \rho(V/2f)^{1/2}]\, e^{f(T-t)} = z_0^* e^{-ft},$$

which is that trajectory reaching $z(T) = \alpha + \rho[p(T)]^{1/2}$ where $p(T) = p(t) = V/2f$ is the constant solution for $p(t)$ when $\psi = f$. It is indicated by a bold line in Fig. 1.

For $z_0 < z_0^*$ the covariance control ψ is adjusted to alter the size of the target set as seen in z space. Throughout region II strict equality will apply in the deterministic target set inequality, although within region I strict inequality holds. The dashed curve starting from $z_0 = 0$ in the figure is an example of an optimal trajectory in region II and can be determined as follows. Let $\psi = f + \Delta\psi$, and write $p(t; \Delta\psi)$ for $p(t)$ in order to show the dependence on the control parameter $\Delta\psi$. Then

$$p(t; \Delta\psi) = \frac{V}{2f}\left[\frac{f + \Delta\psi e^{-2(f+\Delta\psi)t}}{f + \Delta\psi}\right].$$

The solutions for $z(t)$ and $\omega(t)$ from the state and costate equations for this problem are

$$\begin{aligned} z(t) &= k(\Delta\psi)\sinh ft, \\ \omega(t) &= fk(\Delta\psi)\cosh ft, \\ k(\Delta\psi) &= \{\alpha + \rho[p(T; \Delta\psi)]^{1/2}\}/\sinh fT, \end{aligned}$$

where the boundary conditions are $z(0) = 0$, $z(T) = \alpha + \rho[p(T; \Delta\psi)]^{1/2}$. Direct substitution of p, z, and ω into the converted cost functional gives

$$J^* = \operatorname*{Min}_{\Delta\psi}\left\{\frac{1}{2}c_1k^2(\Delta\psi) + \frac{c_2[f^2 + (f + \Delta\psi)^2] - \Delta\psi^2 p(T; \Delta\psi)}{4(f + \Delta\psi)}\right\},$$

where

$$\begin{aligned} c_1 &= \frac{q + f^2}{4f}\sinh 2fT + f\sinh^2 fT + \frac{1}{2}(f^2 - q)\,T, \\ c_2 &= V(2fT + 1)/2f. \end{aligned}$$

The minimizing value of $\Delta\psi$ is approximately 1.5, and the optimal trajectory $z(t)$ is thus determined.

The increase in the covariance control from $\psi = f = 1$ in the free region decreases the covarince $p(t; \Delta\psi)$ at $t = T$ from unity to 0.4. As a result, the target set boundary decreases from $\alpha + \rho[p(T; 0)]^{1/2} = 2$ in the free region, to $\alpha + \rho[p(T; \Delta\psi)]^{1/2} = 1.63$ for this trajectory in the control dependent target set region.

8. Extensions

Generalization of the data-unconditional probabilistic inequality constraints considered so far can include data conditioning of the constraints and probabilistic polygonal constraints. Extension of our results under these generalizations will now be discussed.

Consider data-conditional probabilistic inequality constraints such as

$$\Pr\{K_i'(t)\,x(t) \geqslant \alpha_i(t) \mid \mathscr{Y}_t\} \geqslant \gamma_i(t).$$

This constraint will necessarily destroy the Gaussian property. To show this, assume that $x(t)$ is Gaussianly distributed. Then $\hat{x}(t)$, which is a linear combination of the data, is also Gaussian. Application of the *Conversion Lemma* gives the inequality

$$K_i'(t)\,\hat{x}(t) \geqslant \alpha_i(t) + 2^{1/2}\sigma_i \operatorname{erf}^{-1}[2\gamma_i(t) - 1].$$

But if this inequality is satisfied, $\hat{x}(t)$ cannot be Gaussian, which is a contradiction.

It would appear that to handle such constraints, one would first have to determine the distribution functions for the state variables, and determine how these are related to the applied control. However, it is possible to generate controls for problems with data conditional constraints by repeated application of the results of this paper. Pick a finite number of times t_k and let $u(t_k)$ be the initial value of the optimal control obtained using the *Conversion Theorem* with constraints such as

$$\Pr\{K_i'(\tau)\,x(\tau) \geqslant \alpha_i(\tau) \mid \mathscr{Y}_{t_k}\} \geqslant \gamma_i(\tau)$$

for τ ranging from t_k to T. The control sequence $u(t_k)$ obtained in this manner might be described as an open loop optimal feedback control as defined in [7].

It is possible to pose probabilistic polygonal constraint problems and probabilistic polygonal target set problems analogous to the problems of Section 2. If $S^*(t)$ is a closed polygonal set, then the probabilistic state inequality constraint can be written

$$\Pr\{x(t) \in S^*(t)\} \geqslant \gamma(t).$$

For a polygonal constraint in two dimensions this constraint can be written in terms of the tabulated $T(h, a)$ function which gives the volume of an uncorrelated bivariate normal distribution with zero means and unit variances over the area $0 \leqslant x_2 \leqslant ax_1$, $x_1 \geqslant h$. Then the probabilistic constraint can be written in the form

$$1 - \sum_i \sum_j T(h_i(\bar{x}), a_j(\bar{x})) \geqslant \gamma(t),$$

where the h_i and a_j are expressible as nonlinear functions of the coordinates of adjacent vertices of the polygon transformed to uncorrelated, zero mean, and unit variance coordinates [8, 9].

References

1. S. Horing, On the design of optimal stochastic terminal control systems, *IEEE Trans. Automatic Control* **AC-14** (1969), 86–88.
2. J. V. Breakwell and F. Tung, Minimum effort control of several terminal components, *SIAM J. Control Ser. A* 2 (1964), 295–316.

3. W. W. Hager and S. K. Mitter, Lagrange duality theory for convex control problems, *SIAM J. Control and Optimization* **14** (1976), 843–856.
4. A. E. Bryson and Y. Ho, "Applied Optimal Control," pp. 101–103, 240–242, Ginn, Waltham, Mass., 1969.
5. B. R. Barmish, Target Set and Target Tube Reachability Criteria for Approximated Discrete Systems, *in* "Proc. IEEE Conference on Decision and Control, 1976," pp. 693–697.
6. K. Ogata, "State Space Analysis of Control Systems," p. 409, Prentice-Hall, Englewood Cliffs, N.J., 1967.
7. S. E. Dreyfus, Some types of optimal control of stochastic systems, *SIAM J. Control Ser. A* **2** (1962), 120–134.
8. D. B. Owen, "Handbook of Statistical Tables," pp. 184–201, Addison-Wesley, Reading, Mass., 1962.
9. J. H. Caldwell, The bivariate normal integral, *Biometrika* **38** (1951), 475–481.

Printed in Belgium