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Optimal trading strategy for an investor: the case of partial information

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Abstract

We shall address here the optimization problem of an investor who wants to maximize the expected utility from terminal wealth. The novelty of this paper is that the drift process and the driving Brownian motion appearing in the stochastic differential equation for the security prices are not assumed to be observable for investors in the market. Investors observe security prices and interest rates only. The drift process will be modelled by a Gaussian process, which in a special case becomes a multi-dimensional mean-reverting Ornstein–Uhlenbeck process. The main result of the paper is an explicit representation for the optimal trading strategy for a wide range of utility functions. © 1998 Elsevier Science B.V. All rights reserved.

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1. Introduction

In this paper we solve a utility maximization problem of an investor who wants to maximize the expected utility from the terminal value of his/her portfolio on the finite time interval $[0,T]$. We assume that there are N risky securities $(S_1(t),\dots,S_N(t))$ available in the market whose dynamics are given by Eq. (2.1), and there is a fixed interest rate r. This problem has been widely studied, for example by Cox and Huang (1989), Cox et al. (1985), Duffie and Zame (1989), He and Pearson (1991), Karatzas et al. (1991, 1987) or Ocone and Karatzas (1991). The special feature of this paper is that we shall *not* assume that investors can observe the drift process μ_t and the Brownian motion appearing in the stochastic differential equation for the security prices. We shall call this situation the case of *partial information* to distinguish it from the case of "full information" studied in the above papers. Clearly, it is more realistic to assume that investors have only partial information since prices and interest rates are published and available to the public, but drifts and paths of Brownian motions are mere mathematical tools for model creation, but certainly not observable. The fact

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that investors have only partial information will be modelled by requiring that trading strategies are adapted to the filtration generated by the security prices, which is smaller than the original filtration.

The problem of partial information was discussed already in Lakner (1995) where a formula was presented for the optimal level of terminal wealth, and the existence of a corresponding trading strategy has been shown. The main objective of the present paper is to work out explicit formula for the optimal trading strategy as well. The drift process μ will be a Gaussian process modelled by a system of linear stochastic differential equations where the driving Brownian motion is independent from the one appearing in the equation for the security prices, and in a special case μ becomes a multidimensional Ornstein–Uhlenbeck process with mean-reverting drift. The formula for the optimal trading strategy will involve the process m_t which is the conditional expectation of the drift μ_t given the available information. Two specific examples will be worked out, one for the logarithmic and the other for the power utility function. With the logarithmic utility function the optimal trading strategy can be written in a feedback form which can be formally "derived" from the corresponding formula in the full information case by substituting m for μ. However, it will be shown that with the power utility function the formal substitution of m for μ in the feedback form of the optimal trading strategy in the full information case does not yield the correct formula for the optimal trading strategy in the partial information case. (See also Browne and Whitt (1996) for similar example in a discrete time model.) One can find additional related information in Gennote (1986), Dothan and Feldman (1986), Detemple (1991), and in the dissertation of Honda (1998).

The computation of the optimal trading strategy basically amounts to finding the integrand in the stochastic integral representation of the optimal terminal wealth. The technique used here involves the gradient operator D, as in Ocone and Karatzas (1991), in which the optimal trading strategy under full information is computed using the same technique. We are using that paper as our basic reference for information on the gradient operator.

The optimal trading strategy has been worked out for the "Bayesean" case by Browne and Whitt (1996) for the logarithmic utility, and by Lakner (1994) for general utility functions. The word Bayesean means here that μ is an unobserved random variable with a known prior distribution.

The organization and basic content of the paper is the following. In Section 2 we describe the market model and recall the general formula for the optimal terminal wealth. This will involve a process ζ which is the conditional expectation of the Radon–Nicodym derivative of the "martingale measure" with respect to the original probability measure. In Section 3 we show that ζ satisfies a stochastic differential equation which yields an explicit representation for ζ_t. This will now involve the above-mentioned conditional expectation m_t of the drift μ_t given the available information. In Section 4 we specify the dynamics of μ_t which allows us to compute m_t using the well-known Kalman–Bucy filter.

Next the main theorem is stated, which presents our formula for the optimal trading strategy. This formula involves the previously described processes ζ and m and the deterministic conditional covariance function of μ_t. We specialize the formula for the optimal trading strategy for the logarithmic and the power utility functions. In Section 5 the proof of the main theorem will be presented. The proof itself will be broken down to several lemmas. The appendix contains the proof of a lemma and a proposition in Section 4.

2. The model

Let (Ω, F, P), $\mathscr{F}=\{\mathscr{F}_t;\ 0\leqslant t\leqslant T\}$ be a complete filtered probability space with a fixed terminaltime $T>0$. There are N risky securities on this space with the N-dimensional price process $S=\{S_t=(S_1(t),\dots,S_N(t))^*;\ t\in[0,T]\}$ (the asterisk signifies transposition). The dynamics of these processes are determined by the system of stochastic differential equations

$$\mathrm{d}S_i(t)=\mu_i(t)S_i(t)\,\mathrm{d}t+S_i(t)\sum_{j=1}^{N}\sigma_{ij}\,\mathrm{d}w_j^{(1)}(t). \tag{2.1}$$

In the above equation the drift $\mu=\{\mu_t=(\mu_1(t),\dots,\mu_N(t))^*;\ t\in[0,T\}$ is an adapted, measurable N-dimensional process such that

$$\int_0^T \|\mu_u\|^2\,\mathrm{d}u<\infty, \quad \text{a.s.}, \tag{2.2}$$

where $\|\cdot\|$ is the Euclidean norm. The process $w^{(1)}=\{w_t^{(1)}=(w_1^{(1)}(t),\dots,w_N^{(1)}(t))^*;\ t\in[0,T]\}$ is an N-dimensional Brownian motion, and $\sigma=(\sigma_{ij})_{i,j=1,N}$ is a nonsingular matrix of constants. Let r be a constant deterministic interest rate. We suppose that the initial prices $S_i(0)$, $i=1,\dots,N$ are deterministic positive constants. Let $\mathscr{F}^S=\{\mathscr{F}_t^S;\ t\leqslant T\}$ be the augmented filtration generated by the price process S. In this paper we shall assume that only $\mathscr{F}^S$-adapted processes are observable, so agents in this market do *not* observe the Brownian motion $w^{(1)}$ and the drift process μ. The constant interest rate r, the initial price vector S_0 and the volatility matrix σ are known to all agents acting in the market. We define the positive local martingale $Z=\{Z_t;\ t\leqslant T\}$ by the equation

$$\mathrm{d}Z_t=-(\mu_t-r\mathbf{1})^*(\sigma^*)^{-1}Z_t\,\mathrm{d}w_t^{(1)}, \tag{2.3}$$

$$Z_0=1, \tag{2.4}$$

where $\mathbf{1}$ is the N-dimensional vector with all entries equal to 1. Eqs. (2.3)–(2.4) have the unique solution

$$Z_t=\exp\left\{-\int_0^t(\mu_u-r\mathbf{1})^*(\sigma^*)^{-1}\,\mathrm{d}w_u^{(1)}-\frac{1}{2}\int_0^t\|\sigma^{-1}(\mu_u-r\mathbf{1})\|^2\,\mathrm{d}u.\right. \tag{2.5}$$

Assumption 2.1. We shall assume that Z is a martingale.

Next we shall define a trading strategy for an agent acting in this market. Let $\pi_i(t)$ be the amount of money invested in the ith security at time t.

Definition 2.2. A trading strategy $\pi=\{\pi_t=(\pi_1(t),\dots,\pi_N(t))^*;\ 0\leqslant t\leqslant T\}$ is an N-dimensional, measurable, $\mathscr{F}^S$-adapted process such that

$$\int_0^T \|\pi_t\|^2\,\mathrm{d}t<\infty,\quad \text{a.s.} \tag{2.6}$$

We emphasize that a trading strategy is required to be $\mathscr{F}^S$-adapted, thus investors indeed observe the security prices only, not the drift μ or the Brownian motion $w^{(1)}$. Let X_t be the wealth at time t of an agent who follows the trading strategy π. The initial wealth $X_0=x_0$ is a deterministic constant. The process $X=\{X_t;\ t\in[0,T]\}$ is assumed to evolve according to the dynamics

$$\mathrm{d}X_t=\pi_t^*\mu_t\,\mathrm{d}t+\pi_t^*\sigma\,\mathrm{d}w_t^{(1)}+(X_t-\pi_t^*\mathbf{1})r\,\mathrm{d}t. \tag{2.7}$$

Ito's rule implies that the discounted wealth $\mathrm{e}^{-rt}X_t$ has the form

$$\mathrm{d}(\mathrm{e}^{-rt}X_t)=\mathrm{e}^{-rt}\pi_t^*\sigma\,\mathrm{d}\tilde{w}_t, \tag{2.8}$$

where

$$\tilde{w}_t=w_t^{(1)}+\int_0^t \sigma^{-1}(\mu_u-r\mathbf{1})\,\mathrm{d}u. \tag{2.9}$$

By Girsanov's Theorem and Assumption 2.1, the N-dimensional process $\tilde{w}=\{\tilde{w}_t=(\tilde{w}_1(t),\dots,\tilde{w}_N(t))^*;\ 0\leqslant t\leqslant T\}$ is a Brownian motion under the probability measure $\tilde{P}$ where

$$\frac{\mathrm{d}\tilde{P}}{\mathrm{d}P}=Z_T. \tag{2.10}$$

We denote by $\tilde{E}$ the expectation operator corresponding to the measure $\tilde{P}$.

Definition 2.3. A trading strategy π is called *admissible* if $X_t\geqslant 0$, a.s., $t\in[0,T]$.

Definition 2.4. A function $U:[0,\infty)\mapsto\Re\cup\{-\infty\}$ is called a *utility function* if it is continuous, strictly increasing, strictly concave on its domain, continuously differentiable on $(0,\infty)$ with derivative function $U'(\cdot)$ satisfying the relation

$$\lim_{x\to\infty} U'(x)=0. \tag{2.11}$$

Our *optimization problem* is to maximize the expected utility from terminal wealth, i.e.,

$$\max E[U(X_T)]$$

over all admissible trading strategies.

We define the N-dimensional return process $R=\{R_t=(R_1(t),\dots,R_N(t))^*;\ t\in[0,T]\}$ by

$$\mathrm{d}S_i(t)=S_i(t)\,\mathrm{d}R_i(t),\quad i=1,\dots,N, \tag{2.12}$$

so we have the following decompositions for the return process:

$$\mathrm{d}R_t=\mu_t\,\mathrm{d}t+\sigma\,\mathrm{d}w_t^{(1)} \tag{2.13}$$

and

$$\mathrm{d}R_t=r\mathbf{1}\,\mathrm{d}t+\sigma\,\mathrm{d}\tilde{w}_t. \tag{2.14}$$

Relations (2.12) and (2.14) imply that S, R, and $\tilde{w}$ each generate the same filtration. Thus $\mathscr{F}^S$ is continuous (Karatzas and Shreve, 1988, Corollary 2.7.8).

Let $\zeta=\{\zeta_t,\ t\in[0,T]\}$ be the optional projection of the P-martingale Z to $\mathscr{F}^S$, so

$$\zeta_t=E[Z_t|\mathscr{F}^S],\quad \text{a.s.},\ t\in[0,T]. \tag{2.15}$$

We note that ζ is a martingale with respect to $(P,\mathscr{F}^S)$, and for every $\mathscr{F}_t^S$-measurable random variable V, $\mathscr{F}_u$-measurable random variable Y, and $\mathscr{F}_u^S$-measurable random variable W with $0\leqslant t\leqslant u\leqslant T$

$$\tilde{E}V=E\zeta_tV, \tag{2.16}$$

$$\tilde{E}[Y|\mathscr{F}_t^S]=\frac{1}{\zeta_t}E[Z_uY|\mathscr{F}_t^S] \tag{2.17}$$

and

$$\tilde{E}[W|\mathscr{F}_t^S]=\frac{1}{\zeta_t}E[\zeta_uW|\mathscr{F}_t^S]. \tag{2.18}$$

The last identity implies that $1/\zeta$ is a $(\tilde{P},\mathscr{F}^S)$-martingale. Since $\mathscr{F}^S$ is generated by $\tilde{w}$, so $1/\zeta$, and also ζ, must be continuous.

Let the function $I:(0,\infty)\mapsto[0,\infty)$ be the pseudo inverse function of the strictly decreasing derivative of the utility function

$$I(y)=\inf\{x\geqslant 0:\ U'(x)\leqslant y\}. \tag{2.19}$$

The above defined function I actually becomes the inverse function of U' if $\lim_{x\to 0}U'(x)=\infty$. However, we did not make this assumption.

We recall the following theorem from Lakner (1995).

Theorem 2.5. *Suppose that for every constant $x\in(0,\infty)$*

$$\tilde{E}[I(x\zeta_T)]<\infty. \tag{2.20}$$

Then the optimal level of terminal wealth is

$$\hat{X}_T=I(y\mathrm{e}^{-rT}\zeta_T), \tag{2.21}$$

where the constant y is uniquely determined by

$$\tilde{E}[\mathrm{e}^{-rT}I(y\mathrm{e}^{-rT}\zeta_T)]=x_0. \tag{2.22}$$

The optimal wealth process $\hat{X}$ and the trading strategy $\hat{\pi}$ is implicitly determined by

$$\mathrm{e}^{-rt}\hat{X}_t = \tilde{E}[\mathrm{e}^{-rT} I(y\mathrm{e}^{-rT}\zeta_T) \,|\, \mathscr{F}_t^S] = x_0 + \int_0^T \mathrm{e}^{-rt}\hat{\pi}_t^* \sigma \,\mathrm{d}\tilde{w}_t. \tag{2.23}$$

3. Explicit representation of the optimal terminal wealth level

Formula (2.21) for the optimal level of terminal wealth involves the random variable ζ_T and we shall find a way to compute it in this section. We introduce the conditional mean vector and covariance matrix of μ_t

$$m_t = E[\mu_t | \mathscr{F}_t^S] \tag{3.1}$$

and

$$\gamma(t) = E[(\mu_t - m_t)(\mu_t - m_t)^* | \mathscr{F}_t^S], \tag{3.2}$$

with the understanding that the processes m and γ are measurable versions of the appropriate conditional expectations. The result for the representation of ζ_t will be formulated in the following theorem.

Theorem 3.1. *Suppose that*

$$E[\|\mu_t\| < \infty], \quad t \in [0, T] \tag{3.3}$$

and the N-dimensional process m is continuous. Then the process $1/\zeta$ satisfies the stochastic differential equation

$$\mathrm{d}\left(\frac{1}{\zeta_t}\right) = \frac{1}{\zeta_t}(m_t - r\mathbf{1})^*(\sigma^*)^{-1}\,\mathrm{d}\tilde{w}_t, \tag{3.4}$$

and we have the representation

$$\zeta_t = \exp\left\{-\int_0^t (m_u - r\mathbf{1})^*(\sigma^*)^{-1}\,\mathrm{d}\tilde{w}_u + \frac{1}{2}\int_0^t \|\sigma^{-1}(m_u - r\mathbf{1})\|^2\,\mathrm{d}u\right\}. \tag{3.5}$$

Proof. From Eqs. (2.5) and (2.9) it follows that

$$Z_t = \exp\left\{-\int_0^t (\mu_u - r\mathbf{1})^*(\sigma^*)^{-1}\,\mathrm{d}\tilde{w}_u + \frac{1}{2}\int_0^t \|\sigma^{-1}(\mu_u - r\mathbf{1})\|^2\,\mathrm{d}u\right\}, \tag{3.6}$$

thus the process $1/Z$ satisfies the equation

$$\mathrm{d}\left(\frac{1}{Z_t}\right) = \frac{1}{Z_t}(\mu_t - r\mathbf{1})^*(\sigma^*)^{-1}\,\mathrm{d}\tilde{w}_t. \tag{3.7}$$

Liptser and Shiryayev I (1977, p.185, Theorem 5.14) guarantee that

$$\tilde{E}\left[\int_0^t (\mu_u - r\mathbf{1})^*(\sigma^*)^{-1}\frac{1}{Z_u}\,\mathrm{d}\tilde{w}_u \middle| \mathscr{F}_t^S\right] = \int_0^t \tilde{E}\left[(\mu_u - r\mathbf{1})^*(\sigma^*)^{-1}\frac{1}{Z_u}\middle| \mathscr{F}_u^S\right]\mathrm{d}\tilde{w}_u, \tag{3.8}$$

provided that the following two conditions hold:

$$\tilde{E}\left[\left|\sum_{i=1}^{N}(\mu_i(u)-r)s_{ij}\frac{1}{Z_u}\right|\right]<\infty, \quad j=1,\dots,N,\ u\in[0,T] \tag{3.9}$$

and

$$\int_0^T\left(\tilde{E}\left[\frac{1}{Z_u}\sum_{i=1}^{N}(\mu_i(u)-r)s_{ij}\middle|\mathscr{F}_u^S\right]\right)^2 \mathrm{d}u<\infty, \text{ a.s.}, \quad j=1,\dots,N, \tag{3.10}$$

where s_{ij} is the (i,j)-entry of the matrix $(\sigma^*)^{-1}$. However, Eq. (3.9) follows from Eq. (3.3) because

$$\tilde{E}\left|\sum_{i=1}^{N}(\mu_i(u)-r)s_{ij}\frac{1}{Z_u}\right| = E\left|\sum_{i=1}^{N}(\mu_i(u)-r)s_{ij}\right|<\infty, \quad j=1,\dots,N, \quad u\in[0,T]. \tag{3.11}$$

The left-hand side of Eq. (3.10) can be written as

$$\int_0^T\frac{1}{\zeta_u^2}\left(E\left[\sum_{i=1}^{N}(\mu_i(u)-r)s_{ij}\middle|,\mathscr{F}_u^S\right]\right)^2 \mathrm{d}u=\int_0^T\frac{1}{\zeta_u^2}\left(\sum_{i=1}^{N}(m_i(u)-r)s_{ij}\right)^2 \mathrm{d}u, \tag{3.12}$$

and this last expression is almost surely finite because of the continuity of m and ζ. Now Eqs. (3.7) and (3.8) imply that

$$\tilde{E}\left[\frac{1}{Z_t}\middle|\mathscr{F}_t^S\right]=1+\int_0^t\tilde{E}\left[(\mu_u-r\mathbf{1})^*(\sigma^*)^{-1}\frac{1}{Z_u}\middle|\mathscr{F}_u^S\right]\mathrm{d}\tilde{w}_u. \tag{3.13}$$

By Eq. (2.17) the left-hand side of this last identity is equal to $1/\zeta_t$, and by Eqs. (2.17) and (3.1) the right-hand side is equal to

$$1+\int_0^t\frac{1}{\zeta_u}(m_u-r\mathbf{1})^*(\sigma^*)^{-1}\,\mathrm{d}\tilde{w}_u,$$

so Eq. (3.4) follows. Identity (3.5) is an obvious consequence of Eq. (3.4), thus our proof is complete.

Now Eq. (3.5) represents a formula for ζ, but it is still not explicit enough because it involves the process m for which we still do not have a computable representation. We can say more about m only if we specify the dynamics of the drift μ, and this will be done in the next section.

4. Explicit formula for the optimal trading strategy

For the rest of this paper we shall assume that the N-dimensional drift process μ is the solution of the stochastic differential equation

$$\mathrm{d}\mu_t=\alpha(\delta-\mu_t)\,\mathrm{d}t+\beta\,\mathrm{d}w_t^{(2)}, \tag{4.1}$$

where $w^{(2)}$ is an N-dimensional Brownian motion with respect to $(\mathscr{F}, P)$, independent of $w^{(1)}$ under P, α and β are known $N \times N$ matrices of real numbers, and δ is a known N-dimensional vector of real numbers. We shall assume that β is invertible, and that μ_0 follows an N-dimensional normal distribution with mean vector m_0 and covariance matrix γ_0. The vector m_0 and the matrix γ_0 are assumed to be known to all agents in the market. We note that if α is a diagonal matrix with positive entries in the diagonal, then μ will be an N-dimensional Ornstein–Uhlenbeck process with mean-reverting drift.

We shall also assume that $\mathrm{tr}(\gamma_0)$ and $\|\beta\|$ are "small". To be more rigorous, we shall assume that

$$\mathrm{tr}(\gamma_0) + T\|\beta\|^2 < K_1, \tag{4.2}$$

where

$$K_1 = \frac{1}{360T\|\sigma^{-1}\|^2 K} \tag{4.3}$$

and

$$K = \max_{t \leqslant T} \|\mathrm{e}^{-\alpha t}\|^2. \tag{4.4}$$

Assumption (4.2) roughly means that the variances of the components of the drift μ_t are "small" compared to the variances of the components of the return process R, which is defined in Eqs. (2.12)–(2.13) (see (A.5) in the appendix for the covariance matrix of μ_t). We note that if α is a positive-semidefinite symmetric matrix then $K = N$ because in that case we can write α in the form $\alpha = T\Lambda T^*$, where T is an orthogonal matrix and Λ is diagonal with the non-negative entries $\lambda_1, \ldots, \lambda_N$ in the diagonal. Now using elementary matrix algebra we can compute

$$\begin{aligned}
\|\mathrm{e}^{-\alpha t}\|^2 &= \mathrm{tr}(\mathrm{e}^{-\alpha t}\mathrm{e}^{-\alpha t}) = \mathrm{tr}(T\mathrm{e}^{-t\Lambda}\mathrm{e}^{-t\Lambda}T^*) \\
&= \mathrm{tr}(\mathrm{e}^{-t\Lambda}\mathrm{e}^{-t\Lambda}) = \sum_{i=1}^{N} \mathrm{e}^{-2t\lambda_i} \leqslant N.
\end{aligned}$$

Lemma 4.1. *With the above specified drift process* μ, *Assumption* 2.1. *is satisfied. Furthermore*,

$$E[Z_T^5 + Z_T^{-4}] + \tilde{E}[\zeta_T^4 + \zeta_T^{-5}] < \infty. \tag{4.5}$$

We defer the proof to the appendix.

We can use the return process R of (2.12) as the "observation" process since it generates the same filtration as the price process S. If we do so then we are in the framework of the "classical" Kalman–Bucy filter. It is well known (Liptser and Shiryayev I, 1977, Theorem 10.3) that m is the unique $\mathscr{F}^S$-measurable solution of the linear system of stochastic differential equations (4.6), and γ is the unique solution of the deterministic Riccati equation (4.7)

$$\mathrm{d}m_t = [-\alpha - \gamma(t)(\sigma\sigma^*)^{-1}]m_t\,\mathrm{d}t + \gamma(t)(\sigma\sigma^*)^{-1}\,\mathrm{d}R_t + \alpha\delta\,\mathrm{d}t, \tag{4.6}$$

$$\dot{\gamma}(t) = -\gamma(t)(\sigma\sigma^*)^{-1}\gamma(t) - \alpha\gamma(t) - \gamma(t)\alpha^* + \beta\beta^* \tag{4.7}$$

with initial condition (m_0, γ_0). It follows (and also well-known) that the conditional covariance matrix $\gamma(t)$ is deterministic. In the case when $N>1$ we do not have an explicit formula for γ. However, in terms of γ we can solve for the conditional mean m in the following way. Let $\phi:[0,T]\mapsto \Re^{N\times N}$ the fundamental solution of the deterministic system

$$\dot{\phi}(t)=[-\alpha-\gamma(t)(\sigma\sigma^*)^{-1}]\phi(t), \tag{4.8}$$

i.e., ϕ is an $N\times N$-matrix valued function satisfying (4.8) with the initial condition that $\phi(0)$ is the $N\times N$ identity matrix. Then m_t is determined in terms of γ and ϕ as

$$m_t=\phi(t)\left[m_0+\int_0^t \phi^{-1}(s)\gamma(s)(\sigma\sigma^*)^{-1}\,\mathrm{d}R_s+\int_0^t \phi^{-1}(s)\,\mathrm{d}s\ \alpha\delta\right]. \tag{4.9}$$

Remark 4.2. It is known that if $N=1$ then the Riccati equation (4.7) has an explicit solution. Eq. (4.7) becomes

$$\dot{\gamma}(t)=-\frac{1}{\sigma^2}\gamma^2(t)-2\alpha\gamma(t)+\beta^2 \tag{4.10}$$

which has the solution

$$\gamma(t)=\sqrt{C}\sigma\frac{C_1\exp\{2(\sqrt{C}/\sigma)t\}+C_2}{C_1\exp\{2(\sqrt{C}/\sigma)t\}-C_2}-\alpha\sigma^2, \tag{4.11}$$

where

$$C=\alpha^2\sigma^2+\beta^2, \tag{4.12}$$

$$C_1=\sqrt{C}\sigma+\gamma_0+\alpha\sigma^2 \tag{4.13}$$

and

$$C_2=-\sqrt{C}\sigma+\gamma_0+\alpha\sigma^2. \tag{4.14}$$

Also, in the one-dimensional case

$$\phi(t)=\exp\left\{-\alpha t-\frac{1}{\sigma^2}\int_0^t \gamma(u)\,\mathrm{d}u\right\} \tag{4.15}$$

and m_t is given explicitly by

$$m_t=\phi(t)\left[m_0+\frac{1}{\sigma^2}\int_0^t \frac{\gamma(s)}{\phi(s)}\,\mathrm{d}R_s+\alpha\delta\int_0^t \frac{1}{\phi(s)}\,\mathrm{d}s\right]. \tag{4.16}$$

We return now to the discussion of the N-dimensional case. Notice that m_t is computable via Eq. (4.9) once the deterministic functions γ and ϕ are computed, which are solutions systems of ordinary first-order differential equations. For our drift process μ specified at the beginning of this section, the conditions of Theorem 3.1 are satisfied so Eqs. (3.4) and (3.5) must hold.

It is well-known from the theory of filtering (Liptser and Shiryayev, II 1978, formula (12.65)) that the process

$$\bar{w}_t=\tilde{w}_t-\int_0^t \sigma^{-1}(m_u-r\mathbf{1})\,\mathrm{d}u \tag{4.17}$$

is a Brownian motion with respect to $(P, \mathscr{F}^S)$. Now we are ready to state the main theorem of the paper:

Theorem 4.3. *Suppose that U is twice continuously differentiable on $(0, \infty)$ and*

$$I(x) < K_2(1 + x^{-5}), \tag{4.18}$$

$$-I'(x) < K_2(1 + x^{-2}) \tag{4.19}$$

for some $K_2 > 0$. Then the optimal trading strategy is

$$\hat{\pi}_t = H(t)\frac{1}{\zeta_t}E\left[I'(ye^{-rT}\zeta_T)\zeta_T^2\left\{-\gamma(t)(\phi^*(t))^{-1}\right.\right.$$

$$\left.\left.\times\int_t^T \phi^*(u)(\sigma^*)^{-1}\,\mathrm{d}\bar{w}_u - m_t + r\mathbf{1}\right\}\middle|\mathscr{F}_t^S\right], \tag{4.20}$$

where

$$H(t) = e^{r(t-2T)}y(\sigma\sigma^*)^{-1}, \tag{4.21}$$

ζ is given in Eq. (3.5), *m is given in Eq.* (4.9), *and the constant y is uniquely determined by Eq.* (2.22).

We defer the proof of this theorem to the next section and instead examine two special cases.

Example 4.4 Suppose that

$$U_1(x) = \log(x + \delta), \tag{4.22}$$

for some constant $\delta > 0$, which satisfies conditions (4.18) and (4.19). Condition (2.20) is satisfied by Lemma 4.1. Formulas (4.20), (4.21) and (2.22) yield that the optimal trading strategy is given by

$$\hat{\pi}_t = e^{rt}(x_0 + \delta e^{-rT})(\sigma\sigma^*)^{-1}\frac{1}{\zeta_t}(m_t - r\mathbf{1}). \tag{4.23}$$

We can write this in a "feedback form" on the current level of wealth. Formula (2.21) becomes

$$e^{-rT}\hat{X}_T = (x_0 + \delta e^{-rT})\frac{1}{\zeta_T} \tag{4.24}$$

and since both $1/\zeta_t$ and $e^{-rt}\hat{X}_t$ are $(\tilde{P}, \mathscr{F}^S)$-martingales, this implies

$$e^{-rt}\hat{X}_t = (x_0 + \delta e^{-rT})\frac{1}{\zeta_t}. \tag{4.25}$$

Substituting this last expression into Eq. (4.23) we get the feedback form

$$\hat{\pi}_t = (\sigma\sigma^*)^{-1}(m_t - r\mathbf{1})\hat{X}_t. \tag{4.26}$$

Notice that one can formally "derive" Eq. (4.26) in the following way. Consider the case of full information when the drift and the Brownian motion appearing in the equation for the security prices are observable and the utility function is given by Eq. (4.22). In this case the optimal trading strategy has the feedback form (Ocone and Karatzas, 1991, formula (4.20))

$$(\sigma\sigma^*)^{-1}(\mu_t - r\mathbf{1})X_t, \tag{4.27}$$

and we can formally "derive" Eq. (4.26) if we substitute μ_t in Eq. (4.27) by its conditional mean m_t. Next another example will be shown which, besides having interest on its own, shows that formal substitution of m for μ in the feedback form of the optimal trading strategy in the full information case does not necessarily yield the correct formula for the optimal trading strategy in the partial information case. (We refer for another counterexample to Browne and Whitt (1996), for the Bayesean case in discrete time, i.e., when the drift is an unobservable random variable selected at time zero with a known "prior" distribution.)

The optimal trading strategy (4.26) for the logarithmic utility function could be established without using Theorem 4.3. The strength of this theorem comes from the fact that it applies to a wide range of utility functions. Here is a different example which would be difficult to solve without our general result.

Example 4.5. Suppose now that the utility function is

$$U_2(x) = \frac{1}{\lambda}x^{\lambda}, \tag{4.28}$$

where $\lambda < 0$. In this case

$$I_2(x) = x^{1/(\lambda-1)} \tag{4.29}$$

and

$$-I_2'(x) = \frac{1}{1-\lambda}x^{1/(\lambda-1)} - 1 \tag{4.30}$$

and it is clear that Eqs. (4.18) and (4.19) hold. Condition (2.20) is also satisfied because

$$\begin{aligned}\tilde{E}[I_2(x\zeta_T)] &= x^{1/(\lambda-1)}\tilde{E}[\zeta_T^{(1/\lambda-1)}] = x^{1/(\lambda-1)}E[\zeta_T^{\lambda/(\lambda-1)}] \\ &\leqslant x^{1/(\lambda-1)}(E[\zeta_T])^{\lambda/(\lambda-1)} = x^{1/(\lambda-1)} < \infty.\end{aligned}$$

Now our formula for the optimal trading strategy becomes

$$\hat{\pi}_t = \frac{1}{1-\lambda}(\sigma\sigma^*)^{-1}(m_t - r\mathbf{1})\hat{X}_t + G_t, \tag{4.31}$$

where

$$\begin{aligned}G_t = {}& y^{1/(\lambda-1)}\frac{1}{1-\lambda}\exp\left\{r\left(t + \frac{T\lambda}{1-\lambda}\right)\right\}\frac{1}{\zeta_t}(\sigma\sigma^*)^{-1}\gamma(t)(\phi(t))^{-1} \\ &\times E\left[\zeta_T^{\lambda/(\lambda-1)}\int_t^T \phi^*(u)(\sigma^*)^{-1}\,\mathrm{d}\bar{w}_u\middle|\mathscr{F}_t^S\right].\end{aligned} \tag{4.32}$$

The optimal trading strategy for this utility function under full information is

$$\frac{1}{1-\lambda}(\sigma\sigma^*)^{-1}(\mu_t - r\mathbf{1})X_t$$

(Ocone and Karatzas, 1991, formula (4.22)), and our formula (4.31) for the case of partial information cannot be derived from this by substituting m for μ because of the additional non-zero term G_t in Eq. (4.31).

One may find the constraint $\lambda < 0$ in Eq. (4.28) too restrictive. The problem with power utility functions with positive λ is that they do not satisfy Eqs. (4.18) and (4.19). In the next proposition we overcome this problem by strengthening Eq. (4.2).

Proposition 4.6. *Let* $\theta \in (0,1)$ *be arbitrary, and instead of Eq.* (4.2) *assume the stronger*

$$\operatorname{tr}(\gamma_0) + T\|\beta\|^2 < K_4, \tag{4.33}$$

where

$$K_4 = \frac{1}{8K\|\sigma^{-1}\|^2 T}\min\left\{\frac{1}{45}, \frac{(1-\theta)^2}{(\theta+3)(\theta+7)}\right\} \tag{4.34}$$

and K *is given in Eq.* (4.4). *Then for the power utility function of the form* (4.28) *with* $0 < \lambda \leqslant \theta$, *formulae* (4.31) *and* (4.32) *still yield the optimal trading strategy.*

We defer the proof of this proposition to the appendix.

Now we know that (4.31) and (4.32) gives the optimal trading strategy for all $\lambda \in (-\infty, \theta]$ for any $0 < \theta < 1$ as long as condition (4.33) is satisfied. Let us note that if $\lambda \to 0$ then $I_2(x) \to I_1(x)$ where $I_1(x) = 1/x$, the inverse function of U_1', thus one may intuitively expect that

$$\lim_{\lambda \to 0} G_t = 0, \quad \text{a.s.}, \tag{4.35}$$

so the optimal trading strategy for the utility function U_2 converges to the optimal strategy for the logarithmic utility U_1. This is indeed the case. We need to show only that the conditional expectation in the definition of G_t converges to zero almost surely. However, this follows from the Dominated Convergence Theorem for conditional expectations, because if $\lambda \in [-1, 0]$ then

$$\zeta_T^{\lambda/(\lambda-1)} \leqslant 1 + \zeta_T^{1/2}$$

and if $\lambda \in [0, \frac{1}{2}]$ then

$$\zeta_T^{\lambda/(\lambda-1)} \leqslant 1 + \zeta_T^{-1}.$$

This implies that for $\lambda \in [-1, \frac{1}{2}]$

$$\zeta_T^{\lambda/(\lambda-1)} \int_t^T \phi^*(u)(\sigma^*)^{-1}\,\mathrm{d}\bar{w}_u \leqslant (1 + \zeta_T^{1/2} + \zeta_T^{-1}) \int_t^T \phi^*(u)(\sigma^*)^{-1}\,\mathrm{d}\bar{w}_u,$$

which has finite expectation under P by Holder's inequality and Proposition 4.1, and (4.35) now follows.

5. Proof of Theorem 4.3

The main technique in this proof is the use of the gradient operator D acting on the subset of the class of functionals of $\{\tilde{w}_t,\ t \leqslant T\}$ called $D_{1,1}$. For the exact definitions of the space $D_{1,1}$ and the operator D we refer to Ocone and Karatzas (1991) and Shigekawa (1980). We shall use the generalized version of Clark's formula (Karatzas et al., 1991) which guarantees that for every random variable $A \in D_{1,1}$ we have the stochastic integral representation

$$\tilde{E}[A \mid \mathscr{F}_t^S] = \tilde{E}A + \int_0^t \tilde{E}[(D_u A)^* \mid \mathscr{F}_u^S]\,\mathrm{d}\tilde{w}_u. \tag{5.1}$$

For an N-dimensional random variable $A \in (D_{1,1})^N$, we define DA as an $(N \times N)$-dimensional matrix with components $(DA)_{i,j} = D^i A_j$.

The following lemma spells out conditions under which the gradient operator D and the ordinary Lebesgue integral are exchangeable.

Lemma 5.1. *Let $\{u(s,\omega);\ s \leqslant T\}$ be a real-valued, continuous, measurable process such that $u(s) \in D_{1,1}$ for every $s \in [0,T]$,*

$$\sup_{s \leqslant T} \tilde{E}[|u(s)|^q] < \infty$$

for some $q > 1$, and

$$\sup_{s \leqslant T} \tilde{E}\left[\int_0^T |D_t^j u(s)|^4\,\mathrm{d}t\right] < \infty, \quad j = 1, \ldots, N.$$

Furthermore, we suppose that $s \mapsto D_t u(s,\omega)$ is left (or right) continuous for almost every $(t,\omega) \in [0,T] \times \Omega$. Then $\int_0^T u(s)\,\mathrm{d}s \in D_{1,1}$ and

$$D_t \int_0^T u(s)\,\mathrm{d}s = \int_0^T D_t u(s)\,\mathrm{d}s.$$

For the sake of brevity we omit the proof.

We cast the conditional mean m_u of Eq. (4.9) in the form

$$\begin{aligned} m_u = \phi(u)\Bigg[m_0 &+ \left(\int_0^u \phi^{-1}(s)\,\mathrm{d}s\right)\alpha\delta + \int_0^u \phi^{-1}(s)\gamma(s)(\sigma^*)^{-1}\,\mathrm{d}\tilde{w}_s \\ &+ r\left(\int_0^u \phi^{-1}(s)\gamma(s)\,\mathrm{d}s\right)(\sigma\sigma^*)^{-1}\mathbf{1}\Bigg], \end{aligned} \tag{5.2}$$

where $\phi(\cdot)$ is the fundamental solution of the system (4.8).

Lemma 5.2. *For every $u \in [0,T]$, $m_u \in (D_{1,1})^N$ and*

$$D_t m_u = \sigma^{-1}\gamma(t)(\phi^*(t))^{-1}\phi^*(u)1_{\{t \leqslant u\}}. \tag{5.3}$$

Proof. This follows from Eq. (5.2) and Ocone and Karatzas (1991), Proposition 2.3. □

Lemma 5.3. *The following four relations hold*:

$$\sup_{u\leqslant T} \tilde{E}[\|\sigma^{-1}(m_u - r\mathbf{1})\|^4] < \infty, \tag{5.4}$$

$$\|\sigma^{-1}(m_u - r\mathbf{1})\|^2 \in D_{1,1}, \tag{5.5}$$

$$D_t\|\sigma^{-1}(m_u - r\mathbf{1})\|^2 = 2(D_t m_u)(\sigma\sigma^*)^{-1}(m_u - r\mathbf{1}) \tag{5.6}$$

and

$$\sup_{u,t\leqslant T} \tilde{E}[|D_t^j\|\sigma^{-1}(m_u - r\mathbf{1})\|^2|^4] < \infty, \quad j = 1,\dots,N. \tag{5.7}$$

Proof. From Eq. (5.2) follows that $m_i(u)$ is Gaussian under $\tilde{P}$ and one can easily see that $u \mapsto \tilde{E}[\|\sigma^{-1}(m_u - r\mathbf{1})\|^4]$ is finite and continuous on $[0, T]$, thus Eq. (5.4) follows.

Formulae (5.5) and (5.6) follow directly from Lemma A1 of Ocone and Karatzas (1991) once we show that the condition of that lemma is satisfied. The condition of that lemma translates to

$$\tilde{E}\|\sigma^{-1}(m_u - r\mathbf{1})\|^2 < \infty \tag{5.8}$$

and

$$\tilde{E}\left(\int_0^T \left\| 2\sum_{i,j,k=1}^{N} s_{j,i}s_{k,i}(m_j(u) - r)D_t m_k(u)\right\|^2 \mathrm{d}t\right)^{1/2} < \infty, \tag{5.9}$$

where $s_{j,i}$ is the entry in the jth row and ith column of the matrix $(\sigma^{-1})^*$. □

Inequality (5.8) follows from Eq. (5.4). By Jensen's inequality it suffices to show Eq. (5.9) without the square root. By Eq. (5.3) and the elementary inequality $\|\sum_{i=1}^n v_i\|^2 \leqslant n\sum_{i=1}^n \|v_i\|^2$ for $v_i \in \Re^N, n = 1,\dots,n$ we have

$$\tilde{E}\left[\int_0^T \left\| 2\sum_{i,j,k=1}^{N} s_{j,i}s_{k,i}(m_j(u) - r)D_t m_k(u)\right\|^2 \mathrm{d}t\right]$$

$$\leqslant 4N^3 \sum_{i,j,k=1}^{N} \left\{ s_{j,i}^2 s_{k,i}^2 \tilde{E}(m_j(u) - r)^2 \int_0^u \|(\sigma^{-1}\gamma(t)(\phi^*(t))^{-1}\phi^*(u))^{(k)}\|^2 \,\mathrm{d}t \right\} < \infty,$$

where the matrix followed by superscript (k) in the last expression represents the kth column vector of the matrix.

Next we are going to show Eq. (5.7). By Eqs. (5.6) and (5.3), for all $t \leqslant u \leqslant T$,

$$\begin{aligned}
\tilde{E}[|D_t^j\|\sigma^{-1}(m_u - r\mathbf{1})\|^2|^4] &\leqslant \tilde{E}[\|D_t\|\sigma^{-1}(m_u - r\mathbf{1})\|^2\|^4] \\
&= 16\tilde{E}\|\sigma^{-1}\gamma(t)(\phi^*(t))^{-1}\phi^*(u)(\sigma\sigma^*)^{-1}(m_u - r\mathbf{1})\|^4 \\
&\leqslant 16\|\sigma^{-1}\gamma(t)(\phi^*(t))^{-1}\phi^*(u)(\sigma^*)^{-1}\|^4 \tilde{E}\|\sigma^{-1}(m_u - r\mathbf{1})\|^4
\end{aligned}$$

and Eq. (5.7) now follows from Eq. (5.4).

We note that $m_u \in (D_{1,1})^N$ implies that

$$\sigma^{-1}(m_u - r\mathbf{1}) \in D_{1,1} \tag{5.10}$$

and

$$D_t\sigma^{-1}(m_u - r\mathbf{1}) = (D_t m_u)(\sigma^*)^{-1}. \tag{5.11}$$

We introduce the notations

$$V_1 = -\int_0^T (m_u - r\mathbf{1})^*(\sigma^*)^{-1}\,\mathrm{d}\tilde{w}_u \quad \text{and} \quad V_2 = \frac{1}{2}\int_0^T \|\sigma^{-1}(m_u - r\mathbf{1})\|^2\,\mathrm{d}u. \tag{5.12}$$

Lemma 5.4. *Both V_1 and V_2 are members of $D_{1,1}$, and*

$$D_t V_2 = \int_t^T (D_t m_u)(\sigma\sigma^*)^{-1}(m_u - r\mathbf{1})\,\mathrm{d}u, \tag{5.13}$$

$$D_t V_1 = -\int_t^T (D_t m_u)(\sigma^*)^{-1}\,\mathrm{d}\tilde{w}_u - \sigma^{-1}(m_t - r\mathbf{1}). \tag{5.14}$$

Proof. Identity (5.13) follows from Lemmas 5.1 and 5.3. Identity (5.14) follows from Proposition 2.3 of Ocone and Karatzas (1991) once we verify that the process $\{\sigma^{-1}(m_t - r\mathbf{1});\ t \leqslant T\}$ is a member of the class $\mathbf{L}^a_{1,1}$ which is defined in that paper (pp. 190, 191). Condition (i) of that definition translates to Eq. (5.10). Condition (ii) follows from the right continuity of $u \mapsto D_t(m_u - r\mathbf{1})^*(\sigma^*)^{-1}$ (see Eqs. (5.3) and (5.11)). Condition (iii) follows from Eqs. (5.4), (5.11), and (5.3). □

Lemma 5.5. *We have the following relations*:

$$\tilde{E}\int_0^T \|D_t V_1\|^4\,\mathrm{d}t < \infty \quad \textit{and} \quad \tilde{E}\int_0^T \|D_t V_2\|^4\,\mathrm{d}t < \infty. \tag{5.15}$$

Proof. The left-hand side of the first inequality can be written as

$$\tilde{E}\int_0^T \|D_t V_1\|^4 \mathrm{d}t = \tilde{E}\int_0^T \left\| \int_t^T (D_t m_u)(\sigma^*)^{-1}\,\mathrm{d}\tilde{w}_u + \sigma^{-1}(m_t - r\mathbf{1}) \right\|^4 \mathrm{d}t\ ,$$

and by Eq. (5.4) it is sufficient to show that

$$\tilde{E}\int_0^T \left\| \int_t^T (D_t m_u)(\sigma^*)^{-1}\,\mathrm{d}\tilde{w}_u \right\|^4 \mathrm{d}t < \infty. \tag{5.16}$$

However, this follows from Eq. (5.3) and some elementary calculations involving the fourth moment of the normal distribution. The second inequality in Eq. (5.15) is an easy consequence of Eqs. (5.13), (5.3), (5.4) and some straightforward calculations. □

Lemma 5.6. *The random variable ζ_T is a member of $D_{1,1}$ and*

$$D_t\zeta_T = \zeta_T[D_t V_1 + D_t V_2]. \tag{5.17}$$

Proof. We shall apply again Lemma A1 of Ocone and Karatzas (1991). By Eq. (3.5) we can write ζ_T in the form

$$\zeta_T = \exp\{V_1 + V_2\}.$$

We already know from Lemma 5.4 that V_1 and V_2 are in $D_{1,1}$, so we have to show only the condition of Lemma A1 of Ocone and Karatzas (1991), which in our case becomes

$$\tilde{E}\zeta_T < \infty \quad \text{and} \quad \tilde{E}\left(\int_0^T \|\zeta_T D_t V_1 + \zeta_T D_t V_2\|^2 \,\mathrm{d}t\right)^{1/2} < \infty. \tag{5.18}$$

The first inequality follows from Lemma 4.1, and the second is a consequence of Holder's and Jensen's inequalities, Lemma 4.1, and Eq. (5.15).

Now we are ready to prove Theorem 4.3. We are going to show that for every $x \in (0, \infty)$

$$I(x\zeta_T) \in D_{1,1} \quad \text{and} \quad D_t I(x\zeta_T) = xI'(x\zeta_T) D_t \zeta_T. \tag{5.19}$$

Both relations follow from Ocone and Karatzas, Lemma A1, provided that the conditions

$$\tilde{E}\int_0^T \|I'(x\zeta_T) D_t \zeta_T\|^2 \,\mathrm{d}t < \infty \quad \text{and} \quad \tilde{E}I(x\zeta_T) < \infty \tag{5.20}$$

are satisfied. The second inequality is a consequence of assumption (4.18) and Lemma 4.1. The left-hand side of the first inequality in Eq. (5.20) becomes by Eqs. (5.17), (5.14), (5.13), and (4.19)

$$\begin{aligned}
&\tilde{E}\left[|I'(x\zeta_T)|^2 \zeta_T^2 \int_0^T \|D_t V_1 + D_t V_2\|^2 \,\mathrm{d}t\right] \\
&\leqslant K_2^2 \tilde{E}\left[(1 + x^{-2}\zeta_T^{-2})^2 \zeta_T^2 \int_0^T \|D_t V_1 + D_t V_2\|^2 \,\mathrm{d}t\right] \\
&\leqslant 2K_2^2 \tilde{E}\left[\zeta_T^2 \int_0^T \|D_t V_1 + D_t V_2\|^2 \,\mathrm{d}t\right] + 2K_2^2 x^{-4} \tilde{E}\left[\zeta_T^{-2} \int_0^T \|D_t V_1 + D_t V_2\|^2 \,\mathrm{d}t\right].
\end{aligned}$$

Both terms in the last expression are finite by Eq. (5.15), Holder's and Jensen's inequalities, and Lemma 4.1. We can now derive formula (4.20) for the optimal trading strategy by straightforward algebra, putting together Eqs. (2.23), (5.1), (5.19), (5.17), (5.3), and (4.17), which completes the proof of the theorem.

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Appendix

We are going to prove Lemma 4.1 through two other lemmas.

Lemma A.1. *Suppose that for some* $\gamma \in \Re$,

$$\int_0^T E[\exp\{2T\|\sigma^{-1}\|^2\gamma(2\gamma-1)\|\mu_u\|^2\}]\,\mathrm{d}u < \infty. \tag{A.1}$$

Then

$$EZ_T^\gamma < \infty. \tag{A.2}$$

Proof. We note that Z is a positive local martingale, thus by Fatou's Lemma it is a supermartingale. Therefore, we can assume without loss of generality that $\gamma < 0$ or $\gamma > 1$, because otherwise

$$EZ_T^\gamma \leqslant (EZ_T)^\gamma \leqslant 1$$

follows from the supermartingale property for Z. By Holder's inequality and Eq. (2.5)

$$\begin{aligned} EZ_T^\gamma &= E\left[\exp\left\{-\gamma\int_0^T(\mu_u - r\mathbf{1})(\sigma^*)^{-1}\,\mathrm{d}w_u^{(1)} - \gamma^2\int_0^T\|\sigma^{-1}(\mu_u - r\mathbf{1})\|^2\,\mathrm{d}u\right\}\right. \\ &\qquad \left.\times\exp\left\{\left(\gamma^2 - \frac{\gamma}{2}\right)\int_0^T\|\sigma^{-1}(\mu_u - r\mathbf{1})\|^2\,\mathrm{d}u\right\}\right] \\ &\leqslant \left(E\exp\left\{-2\gamma\int_0^T(\mu_u - r\mathbf{1})(\sigma^*)^{-1}\,\mathrm{d}w_u^{(1)} - 2\gamma^2\int_0^T\|\sigma^{-1}(\mu_u - r\mathbf{1})\|^2\,\mathrm{d}u\right\}\right)^{1/2} \\ &\qquad \times\left(E\exp\left\{(2\gamma^2 - \gamma)\int_0^T\|\sigma^{-1}(\mu_u - r\mathbf{1})\|^2\,\mathrm{d}u\right\}\right)^{1/2}. \end{aligned}$$

The first factor in the last expression is finite because the process

$$(t,\omega) \mapsto \exp\left\{-2\gamma\int_0^t(\mu_u - r\mathbf{1})(\sigma^*)^{-1}\,\mathrm{d}w_u^{(1)} - 2\gamma^2\int_0^t\|\sigma^{-1}(\mu_u - r\mathbf{1})\|^2\,\mathrm{d}u\right\}$$

is again a positive local martingale thus a supermartingale. The square of the second factor is bounded by

$$E\exp\left\{(2\gamma^2 - \gamma)\left(\int_0^T 2\|\sigma^{-1}\|^2\|\mu_u\|^2\,\mathrm{d}u + 2\|\sigma^{-1}\|^2 r^2 NT\right)\right\},$$

and by Jensen's inequality this is bounded by constant multiplier times the left-hand side of Eq. (A.1), which completes the proof of the lemma. □

Lemma A.2. *Suppose that* λ *is a positive real number* such that

$$\lambda < \frac{1}{4K(\mathrm{tr}(\gamma_0) + T\|\beta\|^2)}, \tag{A.3}$$

where the constant K *is defined in Eq.* (4.4). *Then*

$$\int_0^T E\exp\{\lambda\|\mu_u\|^2\}\,\mathrm{d}u < \infty. \tag{A.4}$$

Proof. Let $V(t)$ be the covariance matrix of μ_t which has the form

$$V(t) = \mathrm{e}^{-\alpha t}\left[\gamma_0 + \int_0^t \mathrm{e}^{\alpha s}\beta\beta^*(\mathrm{e}^{\alpha s})^*\,\mathrm{d}s\right](\mathrm{e}^{-\alpha t})^* \tag{A.5}$$

(Arnold, 1973, formula (8.2.7)). With this notation the random vector

$$V^{-1/2}(t)(\mu_t - E\mu_t)$$

follows N-dimensional standard normal distribution. Now using the identity $\|A\|^2 = \mathrm{tr}(AA^*)$ for any $N \times N$-matrix A, we compute

$$\begin{aligned} E\exp\{\lambda\|\mu_u\|^2\} &\leqslant E\exp\{2\lambda\|\mu_u - E\mu_u\|^2 + 2\lambda E\|\mu_u\|^2\} \\ &\leqslant \exp\{2\lambda\|E\mu_u\|^2\}E\exp\{2\lambda\|V^{1/2}(u)\|^2\|V^{-1/2}(u)(\mu_u - E\mu_u)\|^2\} \\ &\leqslant K_3\varphi(2\lambda\,\mathrm{tr}(V(u))), \end{aligned} \tag{A.6}$$

where

$$K_3 = \max_{u \leqslant T}\exp\{2\lambda\|E\mu_u\|^2\} \tag{A.7}$$

and $\varphi(\cdot)$ is the moment-generating function of the χ^2 distribution with parameter N. We note that $K_3 < \infty$ because

$$E\mu_u = \mathrm{e}^{-\alpha u}\left[m_0 + \int_0^u \mathrm{e}^{\alpha s}\alpha\delta\,\mathrm{d}s\right] \tag{A.8}$$

is a continuous function of $u \in [0,T]$. One can see by looking at the density function of the χ^2 distribution that $\varphi(\cdot)$ is finite and increasing on $(-\infty, \frac{1}{2})$, which implies that $\varphi(2\lambda\,\mathrm{tr}(V(u)))$ is bounded if

$$2\lambda\ \mathrm{tr}(V(u)) < \tfrac{1}{2} - \varepsilon \tag{A.9}$$

for some positive constant ε. However, by Eq. (A.5),

$$\begin{aligned} \mathrm{tr}(V(u)) &= E[\mathrm{tr}(\mathrm{e}^{-\alpha u}(\mu_0 - m_0)(\mu_0 - m_0)^*(\mathrm{e}^{-\alpha u})^*)] \\ &\quad + \int_0^u \mathrm{tr}(\mathrm{e}^{-\alpha(u-s)}\beta\beta^*(\mathrm{e}^{-\alpha(u-s)})^*)\,\mathrm{d}s \\ &= E\|\mathrm{e}^{-\alpha u}(\mu_0 - m_0)\|^2 + \int_0^u \|\mathrm{e}^{-\alpha(u-s)}\beta\|^2\,\mathrm{d}s \leqslant K(\mathrm{tr}(\gamma_0) + T\|\beta\|^2), \end{aligned} \tag{A.10}$$

where the constant K is given in Eq. (4.4). Our condition (A.3) implies that for some $\varepsilon > 0$

$$\lambda < \left(\frac{1}{2} - \varepsilon\right) \frac{1}{2K(\mathrm{tr}(\gamma_0) + T\|\beta\|^2)} \tag{A.11}$$

and Eqs. (A.11) and (A.10) now yield Eq. (A.9), which completes the proof.

Proof of Lemma 4.1. The finiteness of EZ_T^5 and EZ_T^{-4} follows from the previous two lemmas and condition (4.2). For $\theta = 4$ or $\theta = -5$, by Jensen's inequality we have

$$\tilde{E}\zeta_T^\theta = E\zeta_T^{\theta+1} = E(E[Z_T \mid \mathscr{F}_T^S])^{\theta+1} \leqslant EZ_T^{\theta+1} < \infty$$

which proves Eq. (4.5) entirely. We still need to show is that Z is a P-martingale for which it suffices to prove that

$$EZ_T = 1. \tag{A.12}$$

We define partition $\{A_n;\ n = 0, 1, \ldots\}$ of Ω by

$$A_n = \{\omega \in \Omega\colon \sup_{t \leqslant T} \|\mu_t\| \in [n, n+1)\}, \quad n = 0, 1 \ldots \tag{A.13}$$

and the index set

$$J = \{n\colon P(A_n) > 0\}. \tag{A.14}$$

We also define the probability measure $P_n \ll P$ for every $n \in J$ as

$$P_n(A) = P(A \mid A_n). \tag{A.15}$$

By the independence of μ and $w^{(1)}$ under P, the process $w^{(1)}$ is still a Brownian motion under the probability measure P_n and the same filtration $\mathscr{F}$. We introduce the random variables

$$Z_T^{(n)} = \exp\left\{-\int_0^T (\mu_u - r\mathbf{1})^*(\sigma^*)^{-1}\,\mathrm{d}w_u^{(1)} - \frac{1}{2}\int_0^T \|\sigma^{-1}(\mu_u - r\mathbf{1})\|^2\,\mathrm{d}u\right\}, \tag{A.16}$$

where the stochastic integral on the right-hand side of the above formula is computed under the probability measure P_n. The absolute continuity of P_n with respect to P implies that

$$Z_T^{(n)} = Z_T, \quad P_n - \text{a.s.},\ n \in J \tag{A.17}$$

(Protter, 1990, Theorem II.5.14). Under P_n the drift process μ is almost surely bounded by $n+1$ thus

$$E_n Z_T^{(n)} = E_n Z_T = 1, \quad n \in J, \tag{A.18}$$

where E_n is the expectation corresponding to P_n. This implies that

$$E[Z_T \mid A_n] = 1, \quad n \in J \tag{A.19}$$

and Eq. (A.12) follows.

Proof of Proposition 4.6. We need to modify the proof in Section 5 because for utility functions of the form in Eq. (4.28) with $0<\lambda<1$, conditions (4.18) and (4.19) are not satisfied. However, those conditions were used only in the step of showing the two formulas in Eq. (5.20), thus we have to show those formulas for this utility function separately. We observe that Lemmas A.1, A.2, and conditions (4.33)–(4.34) now guarantee that

$$E[Z_T^{(\theta+3)/(\theta-1)}] < \infty,$$

thus by Jensen's inequality

$$E[\zeta_T^{(\theta+3)/(\theta-1)}] = \tilde{E}[\zeta_T^{4/(\theta-1)}] < \infty. \tag{A.20}$$

The second inequality in Eq. (5.20) is a straightforward consequence of Eq. (A.20). In order to show the first, we observe that its left-hand side is

$$\tilde{E}\left[|I'(x\zeta_T)|^2 \zeta_T^2 \int_0^T \|D_t V_1 + D_t V_2\|^2 \,\mathrm{d}t\right]$$
$$= \text{const} \times \tilde{E}\left[\zeta_T^{2/(\lambda-1)} \int_0^T \|D_t V_1 + D_t V_2\|^2 \,\mathrm{d}t\right],$$

and by Holder's inequality, Eqs. (A.20) and (5.15) this last expression is finite.

References

Arnold, L., 1973. Stochastic Differential Equations: Theory and Applications. Wiley-Inter-science, New York.

Browne, S., Whitt, W., 1996. Portfolio choice and the Bayesian Kelly criterion. Adv. Appl. Probab. 28, 1145–1176.

Cox, J.C., and Huang, C.F., 1989. Optimal consumption and portfolio policies when asset prices follow a diffusion process. J. Economic Theory 49, 33–83.

Cox, J.C., Ingersoll, J.E., Ross, S.A., 1985. An intertemporal general equilibrium model of asset prices. Econometrica 53, 363–384.

Detemple, J., 1991. Further results on asset pricing with incomplete information. J. Economic Dyn. Control 15, 425–453.

Dothan, Feldman, 1986. Equilibrium interest rates and multiperiod bonds in a partially observable economy. J. Finance 41, 369–382.

Duffie, D., Zame, W., 1989. The consumption-based capital asset pricing model. Econometrica 57, 1279–1297.

Gennote, 1986. Optimal portfolio choice under incomplete information. J. Finance 41, 733–746.

He, H., Pearson, N., 1991. Consumption and portfolio policies with incomplete markets: the infinite dimensional case. J. Economic Theory 54, 259–305.

Honda, T., 1998. Optimal portfolio choice for unobservable and regime-switching mean returns. Unpublished manuscript.

Karatzas, I., Lakner, P., Lehoczky, J.P., Shreve, S.E., 1991. Equilibrium in a simplified dynamic, stochastic economy with heterogeneous agents. In: Mayer-Wolf, E., Merzbach, E., Schwartz, A. (Eds.), Stochastic Analysis. Academic Press, New York, pp. 245–272.

Karatzas, I., Lehoczky, J.P., Shreve, S.E., 1987. Optimal portfolio and consumption decisions for a "small investor" on a finite horizon. SIAM J. Control Optim. 25, 1557–1586.

Karatzas, I., Lehoczky, J.P., Shreve, S.E., Xu, 1991. Martingale and duality methods for utility maximization in incomplete markets. SIAM J. Control Optim. 29, 702–730.

Karatzas, I., Ocone, D.L., Li, J., 1991. An Extension of Clark's formula. Stochastics Stochastic Reports 37, 127–131

Karatzas, I., Shreve, S.E., 1988. Brownian Motion and Stochastic Calculus. Springer, New York.

Lakner, P., 1994. Optimal investment processes for utility maximization problems with restricted information. Manuscript.

Lakner, P., 1995. Utility maximization with partial information. Stochastic Processes Appl. 56, 247–273.

Liptser, R.S., Shiryayev, A.N., 1977. Statistics of Random Processes I. Springer, Berlin.

Liptser, R.S., Shiryayev, A.N., 1978. Statistics of Random Processes II. Springer, Berlin.

Ocone, D.L., Karatzas, I., 1991. A generalized Clark representation formula, with application to optimal portfolios. Stochastics and Stochastic Reports 34, 187–220.

Protter, P., 1990. Stochastic Integration and Differential Equations. Springer, New York.

Shigekawa, I., 1980. Derivatives of Wiener functionals and absolute continuity of induced measures. J. Math. Kyoto University 20, 263–289.