Mappings with bounded \((P, Q)\)-distortion on Carnot groups

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Abstract

We study mappings with bounded \((p, q)\)-distortion associated to Sobolev spaces on Carnot groups. Mappings of such type have applications to the Sobolev type embedding theory and classification of manifolds. For this class of mappings, we obtain estimates of linear distortion, and a geometrical description. We prove also Liouville type theorems and give some sufficient conditions for removability of sets.

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Keywords: Mappings with bounded distortion; Sobolev spaces; Carnot groups

Résumé

Nous étudions les applications à \((p, q)\)-distorsion bornée qui sont associées aux espaces de Sobolev sur les groupes de Carnot. Les applications de ce type sont utilisées dans la théorie de plongement de type Sobolev et dans la classification des variétés. Pour cette classe d’applications nous obtenons des estimations de la distorsion linéaire ainsi qu’une description géométrique. Nous démontrons également des théorèmes de type Liouville et donnons des conditions suffisantes pour qu’un ensemble soit effacable.

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0. Introduction

Let $\Omega$ be an open set on Carnot group $\mathbb{G}$. We study Sobolev mappings $f : \Omega \to \mathbb{G}$ of the class $W_{q, \text{loc}}^1(\Omega)$ under the following condition: the local $p$-distortion, $p \geq q \geq 1$,

$$K_p(x, f) = \inf \{k(x) : |D_H f|(x) \leq k(x)J(x, f)^{\frac{1}{p}}\}$$

is integrable in the power $\kappa$, $1/\kappa = 1/q - 1/p$.

The modern approach to the Sobolev mappings theory is based on relations between these mappings, Sobolev spaces theory and the nonlinear potential theory. In [32] Yu.G. Reshetnyak proved that a nonconstant mapping $f : \Omega \to \mathbb{R}^n$, $\Omega \subset \mathbb{R}^n$, belonging to Sobolev space $W_{n, \text{loc}}^1(\Omega)$ and possessing bounded distortion

$$K(x) = \inf \{k(x) : |Df|^n(x) \leq k(x)J(x, f)\} \in L_\infty(\Omega),$$

is continuous discrete and open. Note, that the continuity of mappings with bounded distortion follows only from finiteness of the distortion [51] (see [24] for another proof of this property). The necessity of study of topological properties of Sobolev mappings arises in the nonlinear elasticity theory. In some problems of this theory the uniformly boundedness of distortion of a mapping is too restrictive [2,3]. Typically, we have only integrability of distortion for Sobolev mappings. Modern development of the theory of mappings with integrable distortion shows, that we have no topological properties without additional analytical assumptions (see, for example [14–21,23,25,42]). Typically, it is required that the change of variable equality in the Lebesgue integral holds and that the connected components of the inverse image $f^{-1}(y)$ are compact for each point $y \in f(\Omega)$ (quasilight mappings). Therefore we assume some topological properties for mappings with integrable $p$-distortion.

Another approach to the geometric function theory is based on investigation of the distortion function in the space of mappings with bounded mean oscillation. In the case it is assumed that the distortion is majorized by some $BMO$-function (see, for example [1,35,36]).

The mappings with integrable distortion which are connected with Sobolev spaces $(p, q)$-quasiconformal mappings (see, for example [38,39,46,55–58]).

Mappings of such type have applications in the Sobolev type embedding theory [8–10,58] and classification of manifolds [37].

The aim of this paper is study basic analytical and geometrical properties of mappings with bounded $(p, q)$-distortion, the validity of Liouville type theorems and description of the removable sets on Carnot groups in terms of capacity.

We call a continuous mapping $f : \Omega \to \mathbb{G}$, where $\Omega$ is an open set in Carnot group $\mathbb{G}$, the mapping with bounded $(p, q)$-distortion, $1 \leq q \leq p < \infty$, if

(1) $f$ is open and discrete;
(2) mapping $f$ possesses Luzin condition $\mathcal{N}$ (the image of a set of measure zero is a set of measure zero);
(3) $f$ belongs to the Sobolev class $W^1_{1, \text{loc}}(\Omega)$;
(4) $J(x, f) \geq 0$ a.e. and $J(x, f) \in L_{1, \text{loc}}(\Omega)$;
(5) the local $p$-distortion

$$K_p(x, f) = \inf \{k(x) : |D_H f|(x) \leq k(x)J(x, f)^{\frac{1}{p}}\}$$

belongs to $L_\kappa(\Omega)$, where the number $\kappa$ is defined from the relation $1/\kappa = 1/q - 1/p$ ($\kappa = \infty$ if $q = p$).
The value \( K_{p,q}(f; \Omega) = \| K_p(x, f) \|_{L^\kappa(\Omega)} \) is called the coefficient of distortion of the mapping \( f \) in the open set \( \Omega \).

We note that in the case \( \mathbb{G} = \mathbb{R}^n, n \geq 2, \) and \( p = q = n \), this class of mappings coincide with the space mappings with bounded distortion (quasiregular mappings) [32,34]. In this case a mapping belongs to the Sobolev space \( W^{1,\kappa}_{n,\text{loc}}(\Omega) \) and topological properties follows only from the boundedness of the distortion.

The main results of the article are the Liouville type theorem

**Theorem A.** Let \( f : \mathbb{G} \to \mathbb{G} \) be a mapping with bounded \((p, q)\)-distortion, \( \nu - 1 < q \leq p \leq \nu \). Then \( \text{cap}(\mathbb{G} \setminus f(\mathbb{G}); W^1_{s}(\mathbb{G})) = 0 \) where \( s = p/(p - (\nu - 1)). \)

And the theorem about removable sets

**Theorem B.** Let \( f : \Omega \setminus F \to \mathbb{G} \) be a mapping with bounded \((p, q)\)-distortion, \( p \geq q \geq \nu \), and \( F \) be a closed set in the domain \( \Omega \), \( \text{cap}(F; W^1_s(\mathbb{G})) = 0, s = p/(p - (\nu - 1)). \) Then

1. in the case \( p \geq q > \nu \): the mapping \( f \) extends to a continuous mapping \( \tilde{f} : \Omega \to \mathbb{G}; \)
2. in the case \( p = q = \nu \): if \( \text{cap}(\mathbb{G} \setminus (\text{supp} f); W^1_{s}(\mathbb{G})) > 0 \) then the mapping \( f \) extends to a continuous mapping \( \tilde{f} : \Omega \to \mathbb{G} \cup \{\infty\}. \) (Hereafter \( \mathbb{G} \cup \{\infty\} \) is the one-point compactification \( \mathbb{G} \) with the standard topology.)

Note that removability sets for quasiregular mappings (the case \( \mathbb{G} = \mathbb{R}^n \) and \( p = q = n \)) and mappings with bounded \((p, p)\)-distortion (the case \( n - 1 < p < n \)) was considered in [34] and [53], respectively.

The main technical tool of the article is

**Theorem C.** Let \( \Omega \) be an open set in \( \mathbb{G} \) and \( f : \Omega \to \mathbb{G} \) be a mapping with bounded \((p, q)\)-distortion. For a positive real number \( \Lambda \) the push-forward function \( v = f_*u : f(\Omega) \to \mathbb{R} \), defined by the function \( u \in C_0 \cap W^{1,p}(\Omega) \) as

\[
v(x) = \begin{cases} 
\Lambda \sum_{z \in f^{-1}(x)} i(z, f)u(z), & x \in f(\text{supp } u), \\
0, & x \notin f(\text{supp } u),
\end{cases}
\]

has following properties:

1. \( \text{supp } v \) is a compact set and \( f(\text{supp } u) = \text{supp } v; \)
2. \( v \) is a continuous function;
3. \( v \in \text{ACL}(f(\Omega)) \) while \( q > \nu - 1; \)
4. in every compact embedded subdomain \( D \subset \Omega \) the inequality

\[
\left\| f_*u \right\|_{L^s_f(D)} \leq \Lambda N(f, D) \left( K_{p,q}(f; D) \right)^{\nu-1} \left\| u \right\|_{L^r(D)}
\]

holds with \( s = p/(p - (\nu - 1)), r = q/(q - (\nu - 1)) > 0 \). Here \( N(f, D) \) is the multiplicity function of a mappings \( f \) on the subdomain \( D \subset \Omega \) defined by the rule

\[
N(f, D) = \sup_{x \in \mathbb{G}} N(x, f, D) = \sup_{x \in \mathbb{G}} \sharp \{ f^{-1}(x) \cap D \}.
\]

Recall that a stratified homogeneous group [7], or, in other terminology, a Carnot group [31] is a connected simply connected nilpotent Lie group \( \mathbb{G} \) whose Lie algebra \( V \) is decomposed
into the direct sum \( V_1 \oplus \cdots \oplus V_m \) of vector spaces such that \( \dim V_1 \geq 2 \), \( [V_1, V_k] = V_{k+1} \) for \( 1 \leq k \leq m-1 \) and \( [V_1, V_m] = [0] \). Let \( X_{11}, \ldots , X_{1n_1} \) be left-invariant basis vector fields of \( V_1 \). Since they generate \( V \), for each \( i, 1 < i \leq m \), one can choose a basis \( X_{ij} \) in \( V_i \), where \( 1 \leq j \leq n_i = \dim V_i \), consisting of commutators of order \( i-1 \) of fields \( X_{ik} \in V_1 \). We identify elements \( g \) of \( G \) with vectors \( x \in \mathbb{R}^N \), \( N = \sum_{i=1}^m n_i \), \( x = (x_{ij}) \), \( 1 \leq i \leq m \), \( 1 \leq j \leq n_i \) by means of exponential map \( \exp(\sum x_{ij} X_{ij}) = g \). Dilations \( \delta_t \) defined by the formula \( \delta_t x = (t^\nu x_{ij})_{1 \leq i \leq m, 1 \leq j \leq n_j} \), are automorphisms of \( G \) for each \( t > 0 \). Lebesgue measure \( dx \) on \( \mathbb{R}^N \) is the bi-invariant Haar measure on \( G \) (which is generated by the Lebesgue measure by means of the exponential map), and \( d(\delta_t x) = t^\nu dx \), where the number \( \nu = \sum_{i=1}^m n_i \) is called the homogeneous dimension of the group \( G \). The Lebesgue measure \( \int_E dx \) of a measurable subset \( E \) of \( G \) is \( \int_E dx \).

Euclidean space \( \mathbb{R}^n \) with the standard structure is an example of an abelian group: the vector fields \( \partial / \partial x_i \), \( i = 1, \ldots , n \), have no non-trivial commutation relations and form a basis in the corresponding Lie algebra. One example of a non-abelian stratified group is the Heisenberg group \( \mathbb{H}^n \). Its Lie algebra has dimension \( 2n + 1 \) and its center is one-dimensional. If \( X_1, \ldots , X_n, Y_1, \ldots , Y_n, T \) is a basis in the Heisenberg algebra, then the only non-trivial commutation relations are \([X_i, Y_i] = -4T\), \( i = 1, \ldots , n \); all other vanish.

A homogeneous norm on the group \( G \) is a continuous function \( \rho : G \to [0, \infty) \) that is \( C^\infty \)-smooth on \( G \setminus \{0\} \) and has the following properties:

(a) \( \rho(x) = \rho(x^{-1}) \) and \( \rho(\delta_t(x)) = t^\nu \rho(x) \);
(b) \( \rho(x) = 0 \) if and only if \( x = 0 \);
(c) there exists a constant \( c > 0 \) such that \( \rho(x_1 \cdot x_2) \leq c(\rho(x_1) + \rho(x_2)) \) for all \( x_1, x_2 \in G \).

The system of basis vectors \( X_1, X_2, \ldots , X_n \) of the space \( V_1 \) (here and throughout we set \( n_1 = n \) and \( X_{i1} = X_i \), where \( i = 1, \ldots , n \)) satisfies Hörmander’s hypoellipticity condition. The Carnot–Carathéodory distance \( d(x, y) \) between two points \( x, y \in G \) is defined as the greatest lower bound of lengths of all horizontal curves with endpoints \( x \) and \( y \), where the length is measured in the Riemannian metric with respect to which the vector fields \( X_1, \ldots , X_n \) are orthonormal and a horizontal curve is thought of as a piecewise smooth path whose tangent vectors belong to \( V_1 \). It can be shown that \( d(x, y) \) is a left-invariant finite metric with respect to which the group of the automorphisms \( \delta_t \) is a dilatation group with a coefficient \( t : d(\delta_t x, \delta_t y) = td(x, y) \). By definition, we put \( d(x) = d(0, x) \).

Let \( G \) be a Carnot group with one-parameter dilatation group \( \delta_t \), \( t > 0 \), and homogeneous norm \( \rho \), and let \( E \) be a measurable subset of \( G \). As usual, we denote by \( L_p(E) \), \( p \in [1, \infty] \), the space of \( p \)-th-power integrable functions with the standard norm

\[
\|u\|_{L_p(E)} = \left( \int_E |u(x)|^p dx \right)^{\frac{1}{p}},
\]

if \( p \in [1, \infty) \), and \( \|u\|_{L_\infty(E)} = \text{ess sup}_E |u(x)| \) for \( p = \infty \). We denote by \( L_{p, \text{loc}}(E) \) the space of functions \( f : \Omega \to \mathbb{R} \) such that \( f \in L_q(K) \) for each compact subset \( K \) of \( E \).

Let \( \Omega \) be an open set in \( \mathbb{G} \). The Sobolev space \( W^1_p(\Omega) \), \( 1 \leq p \leq \infty \), \( (L^1_p(\Omega), 1 \leq p \leq \infty) \) consists of the functions \( u : \Omega \to \mathbb{R} \) locally integrable in \( \Omega \), having a weak derivatives \( X_i u \) along the vector fields \( X_i, i = 1, \ldots , n \), and a finite (semi)norm

\[
\|u\|_{W^1_p(\Omega)} = \|u\|_{L_p(\Omega)} + \|\nabla_H u\|_{L_p(\Omega)} \quad (\|u\|_{L^1_p(\Omega)} = \|\nabla_H u\|_{L^1(\Omega)}),
\]
where $\nabla_H u = (X_1 u, \ldots, X_n u)$ is the horizontal subgradient of $u$. Recall that a locally integrable function $v_i : \Omega \to \mathbb{R}$ is called the weak derivative of $u$ along the vector fields $X_i, i = 1, \ldots, n$, if
\[
\int_{\Omega} v_i \varphi \, dx = - \int_{\Omega} u X_i \varphi \, dx
\]
for any test function $\varphi \in C^\infty_0(\Omega)$. If $u \in W^1_p(U)$ for each bounded open set $U$ such that $\overline{U} \subset \Omega$ then we say that $u$ belongs to the class $W^1_p, \text{loc}(\Omega)$.

We say that a function $u : \Omega \to \mathbb{R}$ is absolutely continuous on lines ($u \in \text{ACL}(\Omega)$) if for each domain $U$ such that $U \subset \Omega$ and each foliation $\Gamma_i$ defined by a left-invariant vector field $X_i$, $i = 1, \ldots, n$, $u$ is absolutely continuous on $\gamma \cap U$ with respect to one-dimensional Hausdorff measure for $d\gamma$-almost every curve $\gamma \in \Gamma_i$. Recall that the measure $d\gamma$ on the foliation $\Gamma_i$ equals the inner product $i(X_i)$ of the vector field $X_i$ and the bi-invariant volume $dx$; see, for instance [6,54].

Let $(X, r)$ be a complete metric space, $r$ be a metric in $X$, and $\Omega$ be an open set in Carnot group $G$. We say that a mapping $f : \Omega \to X$ is in the class $W^1_p, \text{loc}(\Omega; X)$, if the following conditions fulfill:

(A) for each $z \in X$ the function $[f]_z : x \in \Omega \mapsto r(f(x), z)$ belongs to the class $W^1_p, \text{loc}(\Omega)$;
(B) the family of functions $(\nabla_H[f]_z)_{z \in X}$ has a majorant in the class $L^p, \text{loc}(\Omega)$, that is, there exists a function $g \in L^p, \text{loc}(\Omega)$ independent of $z$ such that $|\nabla_H[f]_z(x)| \leq g(x)$ for almost all $x \in \Omega$ and all $z \in X$.

If $X = \mathbb{R}$, then this definition of mappings with values in $\mathbb{R}$ is equivalent to those given above. For $G = \mathbb{R}^n$ one obtains Reshetnyak’s definition of Sobolev mappings [33]. If $X = \widetilde{G}$ is another stratified group with one-parameter dilation group $\delta$, a homogeneous norm $\rho$, and so on, then we obtain the definition of Sobolev mappings between two groups and denote this class by $W^1_p, \text{loc}(\Omega; \widetilde{G})$. In this case it is convenient to use an equivalent description of Sobolev classes [41,43] involving only coordinate functions.

We say that a mapping $f : \Omega \to \widetilde{G}$, $\Omega \subset G$, is in the Sobolev class $HW^1_p, \text{loc}(\Omega; \widetilde{G}), 1 \leq p < \infty$, if

1. $\tilde{\rho}(f(x)) \in L^p, \text{loc}(\Omega)$;
2. the coordinate functions $f_{ij}$ belong to ACL$(\Omega)$ for all $i, j$ and $f_{1j}$ belongs to $W^1_p, \text{loc}(\Omega)$ for all $j = 1, \ldots, n$;
3. the vector
\[
X_k f(x) = \sum_{1 \leq i \leq \tilde{m}, 1 \leq j \leq \tilde{n}_i} X_k f_{ij}(x) \frac{\partial}{\partial x_{ij}}
\]
belongs to $\widetilde{V}_1$ for all $x \in \Omega, k = 1, \ldots, n$.

In the next proposition we list the equivalent descriptions of Sobolev classes [41,43]. Let $\Omega$ be a domain in a stratified group $G$. Then the following assertions hold.

1. A mapping $f : \Omega \rightarrow \widetilde{G}$ belongs to $W^1_p, \text{loc}(\Omega; \widetilde{G})$ if and only if $f$ can be redefined on a set of measure zero to belong to $HW^1_p, \text{loc}(\Omega; \widetilde{G})$. 

(2) The mapping \( f : \Omega \to \tilde{G} \) belongs to \( W^{1,p,\text{loc}}(\Omega; \tilde{G}) \) if and only if for each function \( u \in \text{Lip}(\tilde{G}) \) the composition \( u \circ f \) belongs to \( W^{1,p,\text{loc}}(\Omega) \) and \( |\nabla_{H}(u \circ f)|(x) \leq \text{Lip} \cdot g(x) \), where \( g \in L^{p,\text{loc}}(\Omega) \) is independent of \( f \).

Since \( X_{i}f(x) \in \tilde{V}_{1} \) for almost all \( x \in \Omega \) \([31]\), \( i = 1, \ldots, n \), the linear mapping \( D_{H}f(x) \) with matrix \( (X_{ij}(x)) \), \( i, j = 1, \ldots, n \), takes the horizontal subspace \( V_{1} \) to the horizontal subspace \( \tilde{V}_{1} \) and is called the formal horizontal differential of the mapping \( f \) at \( x \). Let \( |D_{H}f(x)| \) be its norm

\[
|D_{H}f(x)| = \sup_{\xi \in V_{1}, \rho(\xi) = 1} \tilde{\rho}(D_{H}f(x)(\xi)).
\]

Smooth mappings with differentials respecting the horizontal structure are said to be contact. For this reason one could say that mappings in the class \( W^{1,p,\text{loc}}(\Omega; \tilde{G}) \) are (weakly) contact. It is proved in \([43,54]\) that a formal horizontal differential \( D_{H} : V_{1} \to \tilde{V}_{1} \) induces a homomorphism \( Df : V \to \tilde{V} \) of the Lie algebras \( V \) and \( \tilde{V} \), which is called the formal differential. Hence \( |Df(x)| \leq C|D_{H}f(x)| \) \([43]\) for almost all \( x \in \Omega \) with constant \( C \) depending only on the group and the homogeneous norm. Here we set

\[
|Df(x)| = \sup_{\xi \in V, \rho(\xi) = 1} \tilde{\rho}(Df(x)(\xi)).
\]

If \( G = \tilde{G} \), then the determinant of the matrix \( Df(x) \) is called the (formal) Jacobian of the mapping \( f \), it is denoted by \( J(x,f) \).

A mapping with bounded \((p,q)\)-distortion has a finite distortion: \( Df = 0 \) almost everywhere on the set \( Z = \{x : J(x,f) = 0\} \), and in accordance with Lemma 1, which is formulated below, the mapping \( f \) belongs to the class \( W^{1,q,\text{loc}}(\Omega) \). Therefore, in the case \( G = \mathbb{R}^{n} \) and \( p = q = n \geq 2 \), the mapping \( f \) is a mapping with bounded distortion:

\[
\frac{|Df(x)|^{n}}{J(x,f)} \leq K < +\infty \quad \text{a.e. in } \Omega.
\]

For the first time quasiconformal mappings in non-Riemannian spaces were introduced by G.D. Mostow in \([29]\). In the proof of the rigidity theorem he used quasiconformal transformations of the ideal boundary of some symmetric space. M. Gromov \([11]\) showed that the geometry of such ideal boundary is modelled by a nilpotent Lie group equipped with Carnot–Caratheodory metric, which is non-Riemannian. These works stimulated an interest to the study of quasiconformal mappings on Carnot Lie groups and Carnot–Caratheodory spaces.

For studying quasiconformal mappings on Carnot groups, P. Pansu introduced the notion of the differential (\( P \)-differential) on Carnot groups as a homomorphism of the corresponding Lie algebra \([31]\). Using this notion A. Koranyi and H.M. Reimann constructed foundations for the theory of quasiconformal mappings on the Heisenberg groups \([22]\). The problems of the theory of quasiconformal mappings on Carnot groups were considered in \([40,52]\).

The theory of mappings with bounded distortion on Carnot groups was developed in articles \([40,41,44,47,53]\) (see an alternative approach in \([4,5,12]\)). Quasimeromorphic mappings on homogeneous groups and a corresponding value distribution theory properties were studied in \([26]\).

Note, that in papers \([30,59,60]\) were constructed non-trivial examples of contact mappings and quasiconformal mappings on Carnot groups and classes of Carnot groups with an infinite-dimensional family of contact maps.

If \( f \) is a homeomorphism with bounded distortion, then we obtain an analytic definition of the quasiconformal mappings on stratified group from \([40]\). In this paper, the weakest analytical
definition of quasiconformal mappings was proved to be equivalent to the different descriptions of the quasiconformality from papers [31] and [22].

When \( v - 1 < q = p < \infty \), then the mappings under consideration have bounded \( p \)-distortion and its properties on Riemannian spaces (stratified groups) are studied in [37,48] (also [53]).

Note, that if \( q \geq v \), then Luzin condition \( N \) follows from analytical assumptions [45,47].

Recall that the multiplicity function of a mappings \( f \) on the set \( A \subset \Omega \) is defined by

\[
N(f, A) = \sup_{x \in \mathbb{G}} N(f^{-1}(x) \cap A).
\]

**Lemma 1.** Let \( f : \Omega \to \mathbb{G} \) be a mapping with bounded \((p, q)\)-distortion. Fix a compact domain \( D \subset \Omega \) and an arbitrary bounded domain \( D' \subset f(D) \). Then for every function \( u \in W^1_{\infty}(D') \) the composition \( u \circ f \) belongs to \( W^1_q(D) \) and the inequality

\[
\| u \circ f \|_{L^q(D)} \leq K_{p,q}(f; D) \left( \int_{D'} |\nabla_H u(y)|^p N(y, f, D') dy \right)^{\frac{1}{p}}
\]

holds. In particular, \( f \) belongs to \( W^1_{\infty, \text{loc}}(\Omega) \).

**Proof.** In the case \( \mathbb{G} = \mathbb{R}^n \) the lemma was proved in [42]. Let \( u \) belongs to \( W^1_{\infty}(D') \), \( D' \subset \mathbb{G} \). Then the composition \( u \circ f \) belongs to the class ACL(\( D \)) and has derivatives \( X_i(u \circ f) \), \( i = 1, \ldots, n \) along horizontal vector fields \( X_i \) almost everywhere in \( D \). Since \( f \) has finite distortion, we have

\[
\| u \circ f \|_{L^q(D)} \leq \left( \int_{D} |\nabla_H u(f(x))| |J(x, f)|^q dx \right)^{\frac{1}{q}}.
\]

Using the Hölder inequality we derive

\[
\| u \circ f \|_{L^q(D)} \leq \left( \int_{D} \left( \frac{|D_H f|}{|J(x, f)|} \right)^{\frac{q}{p-q}} dx \right)^{\frac{p-q}{p}} \left( \int_{D} |\nabla_H u(f(x))|^p |J(x, f)|^q dx \right)^{\frac{1}{p}}.
\]

Now applying the change of variable formula we obtain the required estimate of the norm. \( \square \)

In the next assertion we formulate some basic properties of mappings with bounded \((p, q)\)-distortion. One can consider these properties as a generalization of the corresponding properties of mappings with bounded distortion.

Let a continuous mapping \( f : \Omega \to \mathbb{G} \) be open and discrete. The set of points near which \( f \) is not locally homeomorphic, is called the branch set, and is denoted by \( B_f \).

**Proposition 1.** Let \( f : \Omega \to \mathbb{G} \) is a mapping with bounded \((p, q)\)-distortion. Then

1. if \( q > v - 1 \) then \( f \) is \( \mathcal{P} \)-differentiable almost everywhere in \( \Omega \);
2. if \( q > v - 1 \) then \( |f(B_f)| = 0 \);
if \( x \in \Omega \setminus B_f \) then in some neighborhood \( W \) of \( x \), the restriction \( f : W \to V = f(W) \) is a homeomorphism on \( W \), and in the case \( q > v - 1 \) the inverse mapping \( g = f^{-1} : V \to W \) belongs to the class \( W^{1,s,\text{loc}}(V) \) where \( s = p/(p - v + 1) \), and the norm \( |D_H g(y)| \) meets the inequality

\[
|D_H g(y)| \leq K_p(x, f)^{v-1} J(y, g)^{\frac{1}{v}} \tag{1}
\]

at almost all point \( y = f(x) \in V \);

(4) if \( 1 \leq q \leq p \leq v \) then \( f \) has \( N^{-1} \)-property: the condition \( |A| = 0 \) implies \( |f^{-1}(A)| = 0 \);

(5) if \( v - 1 < q \leq p \leq v \) then Jacobian \( J(x, f) > 0 \) almost everywhere in \( \Omega \);

(6) if \( \nu < v-1 \leq q \leq p \leq \nu \) then measure \( |B_f| \) of the branch set vanishes;

(7) for every measurable set \( A \subset \Omega \) and every measurable function \( u \) the function \( u \mapsto u(y)N(y, f, A) \) is integrable in \( G \) if and only if the function \( (u \circ f)(x)J(x, f) \) is integrable on \( A \), and the equality

\[
\int_A (u \circ f)(x)J(x, f) \, dx = \int_G u(y)N(y, f, A) \, dy
\]

holds.

**Proof.** Every continuous, open and discrete mapping is a quasi-monotone mapping in the sense of [40]. In this work, it was proved that every quasi-monotone mapping, belonging to \( W^{1,\text{loc}}(\Omega) \), \( q > v - 1 \), is \( \mathcal{P} \)-differentiable almost everywhere. Assertion 7 was proved in [54] (see also [43]).

It is known that if \( \mathcal{P} \)-differential does not vanish at a point \( x \in \Omega \), then \( x \) can not be a branch point of the mapping \( f \). Therefore \( \mathcal{P} \)-differential vanishes almost everywhere on the set \( B_f \). So, from the change of variable formula in the Lebesgue integral, it follows that measure of the image of the branch set equals zero. Assertion 2 is proved.

Assertion 3 was proved in [55, Theorem 9]. We give here a short proof of inequality (1) only. Since the inverse mapping \( g \) belongs to Sobolev class \( W^{1,\text{loc}}(V) \), then, for almost all \( x \), the derivatives \( X_i g, i = 1, \ldots, n \), exist at the point \( y = f(x) \), and composition \( D_H g(f(x)) \circ D_H f(x) \) is the identical map. In order to prove (1), we note that

\[
|D_H g(y)| \leq \frac{|D_H f(x)|^{v-1}}{J(x, f)}.
\]

Hence we have

\[
|D_H g(y)| \leq \left( \frac{|D_H f(x)|}{J(x, f)^{\frac{1}{p}}} \right)^{v-1} J(x, f)^{\frac{v-p+1}{p}} = K_p(x, f)^{v-1} J(y, g)^{\frac{1}{v}}.
\]

Assertion 4 can be proved by taking into account the methods of the papers [41] where it was proved for mappings with bounded distortion, [55] where it was proved in the case \( 1 \leq q \leq p \leq v \).
and $f$ to be a homeomorphism, and [42] where it was proved in the case $\mathbb{G} = \mathbb{R}^n$. We use the sketch of [48]. Let $A$ be a set of a measure zero. We will show that the set $f^{-1}(A)$ has a measure zero. It is sufficient to prove that $|U(x, f, t_0) \cap f^{-1}(A)| = 0$ for every point $x \in f^{-1}(A)$ and some number $t_0 \in (0, t_x)$ depends on $x$.

Fix a point $x \in f^{-1}(A)$ and a neighborhood $U(x, f, t_0)$ which is a component of $f^{-1}(B(f(x), t_0))$. For arbitrary $\varepsilon > 0$ there exists an open set $U \supset A \cap B(f(x), t_0)$ such that $U \subset B(f(x), t_0)$ and $|U| < \varepsilon$. Fix a cut function $\eta \in C^\infty_0(\mathbb{G})$ which is equal to one on $B(0, 1)$ and zero outside $B(0, 2)$. Putting in Lemma 1 $u(z) = \eta(\frac{|z|}{r})$, $D' = B(y, 2r)$, $D = f^{-1}(B(y, 2r)) \cap U(x, f, t_0)$, where $B(y, 2r) \subset U$ we obtain

$$\|u \circ f \|_{L^2_q(D)} \leq C_1 K_{p,q}(f; D)^{\frac{1}{p}} \left(\int_{D'} |\nabla H u(y)|^p dy \right)^{\frac{1}{p}} \leq C_2 \psi(r),$$

where $\psi(r) \equiv 1$ when $p = v$ and $\psi(r) = |B(y, r)|^{\frac{v-p}{p}}$ when $p < v$. From the other side the function $u \circ f$ is equal to one on the set $f^{-1}(B(y, r))$ and zero on the set $U(x, f, t_0) \setminus f^{-1}(B(y, 2r))$. Using the embedding theorem $i : W^1_q \rightarrow L^q$ we have

$$|f^{-1}(B(y, r)) \cap U(x, f, t_0)| \leq C_2 K_{p,q}(f; D)^{\frac{v}{q}} \psi(r)^q$$

for any ball $B(y, r) \subset U$ such that $B(y, 2r) \subset U$. Applying this inequality to a family of balls $\{B(y_i, r_i)\}$ generates a covering of the set $A \cap U$ and such that $B(y_i, 2r_i) \subset U$ and the multiplicity of a covering $\{B(2y_i, r_i)\}$ is finite.

Using the equality $\frac{1}{q^*} = \frac{v-q}{vq}$ and the inclusion $f^{-1}(A) \cap U(x, f, t_0) \subset \bigcup_j f^{-1}(B(y_i, r_i)) \cap U(x, f, t_0)$ we obtain

$$|f^{-1}(A) \cap U(x, f, t_0)| \leq C_3 K_{v,q}(f^{-1}(U) \cap U(x, f, t_0))^{\frac{v}{v-q}}, \quad p = v,$$

$$C_3 K_{p,q}(f^{-1}(U))^{\frac{(p-q)u}{p(v-q)}} |U|^{\frac{v-p}{p(v-q)}}, \quad p < v.$$

We obtain from here the Lusin $N^{-1}$ property in the case $p < v$ because the measure of $U$ can be choose sufficiently small. If $p = v$ we will use the absolutely continuity of the set function $W \mapsto |f^{-1}(W)|$ outside the set $f(Z)$, $Z = \{x \in \Omega : J(x, f) = 0\}$ (see [48] for details).

Assertion 5 follows from the previous statements and from the change of variable formula.

For proving assertion 6, we note that as long as $J(x, f) > 0$ almost everywhere in $\Omega$ then the branch set has measure zero. $\square$

Let $f : \Omega \rightarrow \mathbb{G}$ be a continuous open and discrete mapping. A domain $D \subset \Omega$ is called normal, if $f(\partial D) = \partial f(D)$. A normal neighborhood of a point $x \in \Omega$ is called a neighborhood $U \subset \Omega$ of the point $x$ such that

1. $U$ is a normal domain,
2. $U \cap f^{-1}(f(x)) = \{x\}$.

We denote by $\mathbb{C}D$ the complement for the set $D$. A connection between the multiplicity function, the degree of a mapping, and the index of a mapping is formulated in the following lemma.
Lemma 2. (See [28].) Let \( f \) be a continuous open discrete and sense-preserving mapping.

1. If \( U \) is an open domain such that \( \overline{U} \subset \Omega \), then for all points \( x \in \mathbb{C}_f(\partial U) \) the inequality \( N(x, f, U) \geq \mu(x, f, U) \) holds, and for all points \( x \in \mathbb{C}(f(\partial U) \cup (U \cap B_f)) \) the equality \( N(x, f, U) = \mu(x, f, U) \) holds.

2. If \( U \) is a normal domain then for all points \( x \in f(U \setminus B_f) \) the equality \( N(x, f, U) = N(f, U) \) holds.

3. If \( U \) is a normal domain then for all points \( x \in f(U \setminus B_f) \) the equality \( \mu(x, f, U) = \sum_{i=1}^{k} i(x_i, f) \) holds, where \( k = N(x, f, U) \) and \( \{x_1, \ldots, x_k\} = f^{-1}(x) \cap U \).

4. If \( U \) is a normal domain then for all points \( x \in f(U) \) the equality \( N(f, U) = \mu(x, f, U) = \mu(f, U) \) holds.

5. For all points \( x \in U \), \( i(x, f) = N(f, U) \) if and only if \( U \) is the normal neighborhood of the point \( x \).

6. \( x \in B_f \) if and only if \( i(x, f) \geq 2 \).

In the present work we will use the following notation. Symbol \( U(x, f, s) \) denotes the \( x \)-component of the set \( f^{-1}(B(f(x), s)) \). If \( x \in \Omega \), \( 0 < r < d(x, \partial \Omega) \) and \( 0 < s < d(f(x), \partial f(\Omega)) \), then

\[
I(x, r) = \inf_{d(x, w) = r} d(f(w), f(x)), \quad L(x, r) = \sup_{d(x, w) = r} d(f(w), f(x)).
\]

The value

\[
H(x, f) = \lim_{r \to 0} \frac{L(x, r)}{I(x, r)}
\]

is called the linear distortion of the mapping \( f \) at \( x \in \Omega \).

The following lemma is true for a wide class of metric spaces. The proof of this lemma in Euclidean space [13,34] is carried on a general situation with obvious modifications.

Lemma 3. Let \( f : \Omega \to \mathbb{G} \) be a continuous, open and discrete mapping. Then for every point \( x \in \Omega \) a number \( \sigma_x > 0 \) exists, such that for every \( 0 < r < \sigma_x \), the next properties are fulfilled:

1. If \( U(x, r) = \mathbb{U}_x(\Omega) \), then \( \overline{U}(x, r) = \mathbb{U}_x(\Omega) \cap f^{-1}(B(f(x), r)) \).
2. If \( r < \sigma_x \), then \( \partial U(x, r) = \mathbb{U}_x(\Omega) \cap f^{-1}(S(f(x), r)) \).
3. The sets \( \mathbb{C}_U(x, r) \) and \( \mathbb{C}_U(x, r) \) are connected.
4. The domain \( U(x, r) \setminus \mathbb{U}(x, t) \) is a ring for every numbers \( 0 < t < r \leq \sigma_x \).

1. Estimates of the linear distortion

A well-ordered triple \( (F_0, F_1; D) \) of nonempty sets, where \( D \) is an open set in \( \mathbb{G} \), and \( F_0, F_1 \) are compact subsets of \( \overline{D} \), is called a condenser in the group \( \mathbb{G} \).

The value

\[
\operatorname{cap}_p(E) = \operatorname{cap}_p(F_0, F_1; D) = \inf \int_D |\nabla_H v|^p dx,
\]
where the infimum is taken over all nonnegative functions \( v \in C(F_0 \cup F_1 \cup D) \cap L^1_p(D) \), such that \( v = 0 \) in a neighborhood of the set \( F_0 \), and \( v \geq 1 \) in a neighborhood of the set \( F_1 \), is called the \( p \)-capacity of the condenser \( E = (F_0, F_1; D) \). If \( U \subset \mathbb{G} \) is an open set, and \( C \subset U \) is a compact set in \( U \), then the condenser \( E = (\partial U, C; U) \) will be denoted by \( E = (U, C) \). Properties of \( p \)-capacity in the geometry of vector fields satisfying Hörmander hypoellipticity condition, can be found in [49,50].

**Proposition 2.** Let \( f : \Omega \to \mathbb{G} \) be a mapping with bounded \((p, q)\)-distortion. Suppose that \( A \subset \Omega \) is a normal domain for which \( N(f, A) < \infty \). Then for a condenser \( E = (F_0, F_1; A) \) the inequality

\[
\cap_1^q (F_0, F_1; A) \leq K_{p, q}(f, A) N(f, A)^{\frac{1}{p}} \cap_1^p \left( f(F_0), f(F_1); f(A) \right)
\]

holds.

**Proof.** Let \( u \) be an admissible function for condenser \((f(F_0), f(F_1); f(A))\). Then \( u \circ f \) is an admissible function for the condenser \((F_0, F_1; A)\). Applying the change of variable formula (Proposition 1) we have

\[
\begin{align*}
\cap_1^q (F_0, F_1; A) & \leq \left( \int_A |\nabla_H (u \circ f)|^q dx \right)^{\frac{1}{q}} \\
& \leq \left( \int_A |\nabla_H u|^q (f(x)) |D_H f|^q dx \right)^{\frac{1}{q}} \\
& \leq K_{p, q}(f, A) \left( \int_A |\nabla_H u|^p (f(x)) J(x, f) dx \right)^{\frac{1}{p}} \\
& \leq K_{p, q}(f, A) N(f, A)^{\frac{1}{p}} \left( \int_{f(A)} |\nabla_H u|^p (y) dy \right)^{\frac{1}{p}}.
\end{align*}
\]

Since \( u \) is an arbitrary admissible function, then the proposition is proved. \( \square \)

In the next theorem we give a geometric description of a mapping with bounded \((p, q)\)-distortion. This result generalizes the corresponding results proved for \((p, q)\)-quasiconformal mappings in [55], and for mappings with bounded \( p \)-distortion in [53].

**Theorem 1.** Let \( f : \Omega \to \mathbb{G} \) be a mapping with bounded \((p, q)\)-distortion, \( v - 1 < q \leq p < \infty \). Then there exists a number \( r_0 > 0 \) such that, for all \( 0 < r < \lambda r < r_0 \) where \( \lambda > 1 \) is a fix constant, the inequality

\[
\lim_{r \to 0} \frac{L(x, r)^{1/(v-q)/q}}{f(B(x, \lambda r))} \leq c \cap_i (x, f)^{1/p}
\]

holds at all points \( x \in \Omega \).
Proof. We choose a point \( x \in \Omega \) and let \( \sigma_x > 0 \) be a number from Lemma 3. Suppose that a number \( t > 0 \) is such that \( L(x, t) < \sigma_x \), and \( r_0 > 0 \) is such that the inclusion \( U(x, f, s) \subset B(x, t) \) holds for all \( 0 < s \leq r_0 \).

In the domain \( \Omega \), we choose balls \( B(x, r) \subset B(x, \lambda r) \subset B(x, t) \), \( \lambda > 0 \). Let \( y_1 \in f(S(x, \lambda r)) \) be a point such that \( d(f(x), y_1) = L(x, r) \). Take a point \( y_2 \in f(S(x, \lambda r)) \) which is the most remote from the point \( y_1 \), and a point \( y_3 \in f(S(x, \lambda r)) \) which is the least remote from the point \( y_1 \). We will use the notations \( d_0 = d(y_2, y_1) \) and \( d_1 = d(y_2, y_3) \). In the domain \( f(\Omega) \), consider the continua

\[
F'_0 = CB(y_2, d_0) \cap f(B(x, \lambda r)) \quad \text{and} \quad F'_1 = CB(y_2, d_1) \cap f(B(x, \lambda r)).
\]

It is clear that the function

\[
\eta(y) = \frac{1}{cL(x, r)} \min(\text{dist}(y, F'_0), cL(x, r))
\]

is admissible for the condenser \( (F'_0, F'_1; f(B(x, \lambda r))) \). Here a constant \( c \) is defined from the condition \( cL(x, r) = \text{dist}(F'_0, F'_1) \) and \( F_0 = B(x, \lambda r) \cap f^{-1}(F'_0) \) and \( F_1 = B(x, \lambda r) \cap f^{-1}(F'_1) \) intersect spheres \( S(x, t) \) where \( r < t < \lambda r \). Indeed, if we consider an arbitrary horizontal curve in \( F'_i, i = 0, 1 \), connecting \( f(S(x, r)) \) and \( f(S(x, \lambda r)) \), then by Lemma 3 there exists a curve in \( f^{-1}(F'_i), i = 0, 1 \), connecting the sets \( S(x, r) \) and \( S(x, \lambda r) \). In this notation the condenser \( (F'_0, F'_1; f(B(x, \lambda r))) \) is the image of the condenser \( (F_0, F_1; B(x, \lambda r) \setminus B(x, r)) \) under the mapping \( f \).

In work [40,50] the following estimate for Teichmüller capacity of the condenser \( (F_0, F_1; B(x, \lambda r) \setminus B(x, r)) \) was obtained

\[
\text{cap}_q(F_0, F_1; B(x, \lambda r) \setminus B(x, r)) \geq c_1 r^{\nu - q}.
\]

Applying Proposition 2, we have

\[
c_1 r^{\nu - q} \leq \text{cap}_q(F_0, F_1; B(x, \lambda r) \setminus B(x, r)) \leq \text{cap}_q(F_0, F_1; B(x, \lambda r) \setminus B(x, r)) \leq K_{p, q}(f; B(x, \lambda r)) i(x, f)^{\frac{1}{p}} |f(B(x, \lambda r))|^{\frac{1}{p}}.
\]

Hence

\[
\frac{L(x, r)r^{\nu - q}}{|f(B(x, \lambda r))|^{\frac{1}{p}}} \leq c_2 K_{p, q}(f; B(x, \lambda r)) i(x, f)^{\frac{1}{p}}.
\]

Passing to the limit as \( r \to 0 \) we obtain the desired inequality. The theorem is proved.

Corollary 1. If \( p \geq q \geq \nu \), then we can take \( \lambda = 1 \).

In the case \( q > \nu \), the statement can be proved along the line of the proof of the theorem with taking into account the positivity of the capacity of the two one-point sets in \( L_1^q(\mathbb{G}) \). In the case \( p = q = \nu \) it is follows from following proved in [53].
Lemma 4. Let \( f : \Omega \to \mathbb{G} \) be a nonconstant mapping with bounded \( \nu \)-distortion and with the distortion coefficient \( K(f) \). Then for every point \( x \in \Omega \) the estimate

\[
H(x, f) \leq C(v, K_{\nu,\nu}(f, \Omega) i(x, f)) < \infty
\]

holds, where the constant \( K_{\nu,\nu}(f, \Omega) i(x, f) \) depends on the homogeneous dimension \( \nu \) and the product \( K_{\nu,\nu}(f, \Omega) i(x, f) \) only.

2. Capacity estimates

Let \( f : \Omega \to \mathbb{G} \) be a continuous, open, discrete, and sense-preserving mapping. Assume that \( x \) belongs to \( \Omega \). Consider a horizontal curve \( \beta : [a, b] \to \mathbb{G} \) such that \( \beta(a) = f(x) \). A curve \( \alpha : \Delta_c \to \Omega \), where \( \Delta_c = [a, c] \) or \( \Delta_c = [a, b] \), is called the lifting of the curve \( \beta \) with the origin at the point \( x \), if \( \alpha(a) = x \) and \( f \circ \alpha = \beta | [a, c] \). A curve \( \alpha \) is called full (maximal) lifting of the curve \( \beta \), if the domain of \( \alpha \) coincide with \([a, b]\).

The next assertion holds.

Lemma 5. Suppose that \( D \) is a normal domain for a continuous open discrete and sense-preserving mapping \( f \), and \( y \in f(D) \). Let \( f^{-1}(y) \cap D = \{x_1, \ldots, x_k\} \), where \( k = N(f, D) \), and every point is counted according to the index \( i(x, f) \). If \( \beta : [a, b] \to f(D) \) is a horizontal curve, \( \beta(a) = y \), then

1. there exists full liftings \( \alpha_1, \ldots, \alpha_k \) of the path \( \beta \) such that \( \alpha_i \) begins at the point \( x_i \), \( i = 1, \ldots, k \);
2. \#\{\alpha_i(t) = \alpha_j(t)\} = i(\alpha_j(t), f) \) for every point \( t \in [a, b] \) and every number \( 1 \leq j \leq k \);
3. \( f^{-1}(\beta(t)) \cap D = \{\alpha_1(t), \ldots, \alpha_k(t)\} \) for every point \( t \in [a, b] \).

Proof of Lemma 5 repeats almost verbatim the proof of the corresponding assertion in Euclidean space \([27,34]\).

Let \( f : \Omega \to \mathbb{G} \) be a mapping with bounded \( (p, q) \)-distortion. For a positive real number \( \Lambda \) the push-forward function \( v = f_* u : f(\Omega) \to \mathbb{R} \), defined by the function \( u \in C_0 \cap W^{1,p}_r(\Omega) \) as

\[
v(x) = \begin{cases} 
\Lambda \sum_{z \in f^{-1}(x)} i(z, f)u(z), & x \in f(\text{supp } u), \\
0, & x \notin f(\text{supp } u).
\end{cases}
\]

Since \( \text{supp } u \) is a compact set in \( \Omega \) and the mapping \( f \) is discrete then the sum contains finitely many items only.

Theorem 2. Let \( \Omega \) be an open set in \( \mathbb{G} \) and \( f : \Omega \to \mathbb{G} \) be a mapping with bounded \( (p, q) \)-distortion. The push-forward function \( v = f_* u : f(\Omega) \to \mathbb{R} \) has following properties:

1. \( \text{supp } v \) is a compact set and \( f(\text{supp } u) = \text{supp } v \);
2. \( v \) is a continuous function;
3. \( v \in \text{ACL}(f(\Omega)) \) while \( q > \nu - 1 \);
4. in every compact embedded subdomain \( D \subset \Omega \) the inequality

\[
\|f_* u \|_{L^1_s(f(D))} \leq \Lambda N(f, D)^{\frac{\nu - 1}{\nu}} (K_{p,q}(f; D))^{\nu - 1}\|u\|_{L^1_s(D)}
\]
holds with \( s = p/(p - (\nu - 1)) \), \( r = q/(q - (\nu - 1)) > 0 \). Here \( N(f, D) \) is the multiplicity function of a mapping \( f \) on the subdomain \( D \subset \Omega \) defined by the rule

\[
N(f, D) = \sup_{x \in G} N(x, f, D) = \sup_{x \in G} \sharp \{ f^{-1}(x) \cap D \}.
\]

In the next lemmas regularity properties of pushed forward functions will be proved.

**Lemma 6.** The function \( v \) has the following properties:

1. \( \text{supp} v \) is a compact set and \( f(\text{supp} u) = \text{supp} v \);
2. \( v \) is a continuous function.

We will prove differential properties of the function \( v \).

**Lemma 7.** The function \( v \) is an ACL-function in the open set \( f(\Omega) \).

**Proof.** It is enough to show that \( v \) is an ACL-function in some neighborhood of every point of \( \text{supp} v \). Fix a point \( x_0 \in \text{supp} v \), and let \( f^{-1}(x_0) \cap \text{supp} u = \{ q_1, \ldots, q_s \} \). By Lemma 5, there exists a number \( r_0 : 0 < r_0 < d(x_0, \partial(f(\Omega))) \) such that the normal neighborhoods \( U(q_i, f, r_0), i = 1, \ldots, s \), are disjoint. We choose a number \( r_1 \leq r_0 \) such that \( B(x_0, r_1) \cap f(\text{supp} u) \setminus \bigcup_{i=1}^s U(q_i, f, r_0) = \emptyset \). Then the components \( f^{-1}(B(x_0, r_1)) \), which intersect \( \text{supp} u \), are sets \( U(q_i, f, r_1), i = 1, \ldots, s \). We put \( U = \bigcup_{i=1}^s U(q_i, f, r_1) = \bigcup_{i=1}^s U_i \) where \( U_i = U(q_i, f, r_1) \).

Let \( x_0 = e \). The general case reduces to the previous one by applying left translations. Fix a horizontal vector field \( X_\tau \), and let \( Y \) be a fibration generated by this field. We choose a cube \( Q = S\beta_0 \), where \( \beta_0 = \exp_s X_\tau, |s| \leq M \) and \( S \) is a hyperplane transversal to \( X_\tau \):

\[
S = \{(a; b) \mid x_1 = 0, |a| \leq M, |b| \leq M \}
\]

(here \( a = (x_{1j}), 1 \leq j \leq n_1; b = (x_{ij}), 1 < i \leq m, 1 \leq j \leq n_i \), and \( M \) is a number such that \( Q \subset B(e, r_1) \)).

For every point \( z \in S \), we denote by \( \beta_z \) the element of the horizontal fibration \( z\beta_0 \) started at \( z \). Thus, a cube \( Q \) is the union of all such segments of integral lines. We consider the tubular neighborhood of the fiber \( \beta_z \) with radius \( r \):

\[
E(z, r) = \beta_z B(e, r) \cap Q = \left( \bigcup_{\tau \in \beta_z} B(\tau, r) \right) \cap Q.
\]

We recall that a mapping \( \Phi \) defined on open subsets from \( D \) and taking nonnegative values is called a finitely quasiadditive set function [57] if

1. for any point \( x \in D \), exists \( \delta, 0 < \delta < \text{dist}(x, \partial D) \), such that \( 0 \leq \Phi(B(x, \delta)) < \infty \) (here and in what follows \( B(x, \delta) = \{ y \in \mathbb{R}^n : |y - x| < \delta \} \));
2. for any finite collection \( U_i \subset U \subset D, i = 1, \ldots, k \), of mutually disjoint open sets the following inequality \( \sum_{i=1}^k \Phi(U_i) \leq \Phi(U) \) takes place.
Obviously, the inequality in the second condition of this definition can be extended to a countable collection of mutually disjoint open sets from $D$, so a finitely quasiadditive set function is also countable quasiadditive.

If instead of the second condition we suppose that for any finite collection $U_i \subset D$, $i = 1, \ldots, k$, of mutually disjoint open sets the equality

$$ \sum_{i=1}^{k} \Phi(U_i) = \Phi(U) $$

takes place, then such a function is said to be finitely additive. If the equality in this condition can be extended to a countable collection of mutually disjoint open sets from $D$, then such a function is said to be countably additive.

A mapping $\Phi$ defined on open subsets of $D$ and taking nonnegative values is called a monotone set function [57] if $\Phi(U_1) \leq \Phi(U_2)$ under the condition that $U_1 \subset U_2 \subset D$ are open sets.

Let us formulate a result from [57] in a form convenient for us.

**Proposition 3.** (See [57].) Let a finitely quasiadditive set function $\Phi$ be defined on open subsets of the domain $D \subset \mathbb{R}^n$. Then for almost all points $x \in D$ the finite derivative

$$ \Phi'(x) = \lim_{\delta \to 0} \frac{\Phi(B_\delta)}{|B_\delta|} $$

exists and for any open set $U \subset D$, the inequality

$$ \int_U \Phi'(x) \, dx \leq \Phi(U) $$

is valid.

A nonnegative function $\Phi$ defined on a certain collection of measurable subsets from the open set $D$ and taking finite values is said to be absolutely continuous if, for every number $\varepsilon > 0$, a number $\delta > 0$ can be found such that $\Phi(A) < \varepsilon$ for any measurable set $A \subset D$ from the domain of definition, which satisfies the condition $|A| < \delta$.

Let the set function $\Phi$ be defined by the rule $\Phi(V) = K_{pq}(f; f^{-1}(V))^{pq/(p-q)}$, $V \subset \Omega'$. We define also a Borel function $\Psi$ as $\Psi(V) = |U \cap f^{-1}(V \cap Q)|$. Fix a point $z \in S$ such that the upper volume derivatives $\Phi'(z)$ and $\Psi'(z)$ are finite. It is sufficiently to prove that the function $v$ is absolutely continuous on $\beta_z$. We consider a lifting $\alpha : [a, b] \to U$ of the curve $\beta_z : [a, b] \to Q$ such, that if $t_0 \in [a, b]$ and $x_0 \in U$, $f(x_0) = \beta(t_0)$, then $\alpha(t_0) = x_0$ and $f \circ \alpha = \beta$. Such lifting exists by Lemma 5. For verifying the absolute continuity of $v$ on $\beta_z$ we need the following assertion. For proving this assertion, we apply methods of paper [55].

We recall, that the Hausdorff $\alpha$-measure of a set $A$ is said to be the value

$$ \mathcal{H}^\alpha(A) = \lim_{\varepsilon \to 0} \left\{ \inf \sum_i r_i^\alpha \right\} $$

where the infimum is taken over all coverings of the set $A$ by balls $B_i$ with radii $r_i$ less then $\varepsilon$.

It is known [52, Proposition 1] that the additive function $\Psi$, defined on Borel subsets of $\mathbb{G}$, possesses the next property: the upper volume derivative

$$ \overline{\Psi}'(z) = \lim_{r \to 0} \frac{\Psi(E(z, r))}{r^{v-1}} < +\infty $$

exists for almost all $z \in S$. 
Lemma 8. Fix a point \( z \in S \) in which the functions \( \Psi \) and \( \Phi \) have a finite upper derivative
\[
\lim_{r\to 0} \frac{\Psi(E(z,r))}{r^{\nu-1}} < \infty, \quad \lim_{r\to 0} \frac{\Phi(E(z,r))}{r^{\nu-1}} < \infty.
\]
Let \( \alpha(t) : [a, b] \to U_k \) be a lifting of the horizontal curve \( \beta_z(t) : [a, b] \to Q \) (here \( U_k \subset U \) is a fix normal neighborhood). Then \( \alpha \) is an absolutely continuous curve (with respect to the Hausdorff 1-measure).

Proof. The proof of Lemma 8 uses the following result.

Lemma 9. (See \([55, \text{Proposition 5}]\).) Let \( E \subset G \) be a connected set, and \( G = \{ x : d(x, E) \leq c_0 \text{diam} \, E \} \) where \( c_0 \) is a small number depending on the constant in the generalized triangle inequality. Then
\[
\text{cap}_{p}^{\nu-1}(E, G) \geq c \frac{(\text{diam} \, E)^p}{|G|^{p-(\nu-1)}}
\]
for \( \nu - 1 < p < \infty \) where a constant \( c \) depends only on \( \nu \) and \( p \).

On a horizontal curve \( \beta_z \) we choose mutually disjoint closed arcs \([\delta_1, \delta_1], \ldots, [\delta_l, \delta_l]\) with lengths \( \Delta_1, \ldots, \Delta_l \) respectively such that
\[
\sum_{i=1}^{l} \Delta_i < \delta.
\]
By symbol \( R_i \), we denote the union of balls \( B_{c}(\beta_z(\tau), r) \) where \( \beta_z(\tau) \in [\delta_i, \delta_i] \) (balls are considered in Carnot–Carathéodory metric \( d_c \)). In this case the set \((R_i, [\delta_i, \delta_i])\) is a condenser. We choose a small number \( r > 0 \) such that the following properties are fulfilled: for some constant \( c_1 \) we have \( r < c_1 \Delta_i \), the sets \( R_i, i = 1, \ldots, l \), are disjoint, and the conditions of Lemma 9 hold.

Let \([a_i, b_i] = \beta_z^{-1}([\delta_i, \delta_i]) \subset [a, b]\). Then \( \alpha([a_i, b_i]) \subset f^{-1}(R_i) \cap U_k \) and the pair \((f^{-1}(R_i) \cap U_k, \alpha([a_i, b_i]))\) is also a condenser, since
\[
f^{-1}(R_i) \cap U_k = \bigcup_{\tau \in [a_i, b_i]} U(\alpha(\tau), f, r)
\]
is an open connected set. We note, that the image of the condenser \( E = (f^{-1}(R_i) \cap U_k, \alpha([a_i, b_i])) \) is the condenser \( f(E) = (R_i, [\delta_i, \delta_i]) \), since \( f \circ (\alpha([a_i, b_i])) = [\delta_i, \delta_i] \) and
\[
f(f^{-1}(R_i) \cap U_k) = f \left( \bigcup_{\tau \in [a_i, b_i]} U(\alpha(\tau), f, r) \right) = \bigcup_{\tau \in [a_i, b_i]} B_{c}(\beta_z(\tau), r) = R_i.
\]
We note that the function
\[
w(q) = \frac{d(q, \partial R_i)}{r}
\]
is an admissible function for the condenser \((R_i, [\delta_i, \delta_i])\) and \(|\nabla H u(q)| \leq 1/r\). Hence, we obtain the inequalities
\[
\text{cap}_{p}(R_i, [\delta_i, \delta_i]) \leq \int_{R_i} |\nabla H u(x)|^p \, dx \leq \frac{|R_i|}{r^p} \leq \frac{c_1 \Delta_i r^{\nu-1}}{r^p}.
\]
(2)
On the other hand, by Lemma 9, we have

$$\text{cap}_q \left( f^{-1}(R_i) \cap U_k, \alpha([a_i, b_i]) \right) \geq c_2 \frac{\text{diam}^{\frac{p}{p-1}}(\alpha([a_i, b_i]))}{|f^{-1}(R_i) \cap U_k|^\frac{q}{q-p+1}}. \quad (3)$$

Using Proposition 2 and the inequalities (2), (3), we have

$$\frac{1}{c_2^\frac{1}{q}} \frac{\text{diam}^{\frac{1}{v-1}}(\alpha([a_i, b_i]))}{|f^{-1}(R_i) \cap U_k|^\frac{q}{q-v+1}} \leq K_{pq} \left( f; R_i \right) \Phi(E(z, \lambda^r))^{\frac{1}{p}} c_1^{\frac{1}{q}} \nu_r^{\frac{1-v+1}{p}} r^{\frac{v-1-p}{p}}. \quad (4)$$

Hence,

$$\text{diam}(\alpha([a_i, b_i])) \leq c_3 \left( \frac{\Phi(R_i)}{\nu_r^{v-1}} \right)^{\frac{1-(q)v-1}{p}} \left( \frac{|f^{-1}(R_i) \cap U_k|}{r^{v-1}} \right)^{\frac{q-v+1}{q}} \Delta_i^{\frac{v-1}{p}} \nu_r^{\frac{v-1}{p}}. \quad (5)$$

where $c_3 = (c_1 \nu_r^{v-1} / c_2^\frac{1}{q})$.

We note that the tubular neighborhood $E(z, \lambda^r) = \{ x \in Q | d(x, \beta z) < \lambda^r \}$ where $\lambda^r$ is a constant, depending on the geometry of the group $G$ only, contains the union $\bigcup_{i=1}^l R_i$. So, we have $\bigcup_{i=1}^l f^{-1}(R_i) \cap U_k \subset f^{-1}(E(z, \lambda^r))$. Summing the last inequality over $i = 1, \ldots, l$ and applying Hölder inequality, we obtain

$$\sum_{i=1}^l \text{diam}(\alpha([a_i, b_i])) \leq c_4 \left( \frac{\Phi(E(z, \lambda^r))}{\nu_r^{v-1}} \right)^{\frac{1-(q)v-1}{p}} \left( \frac{|f^{-1}(E(z, \lambda^r))|}{\nu_r^{v-1}} \right)^{\frac{q-v+1}{q}} \left( \sum_{i=1}^l \Delta_i \right)^{-\frac{v-1}{p}}. \quad (6)$$

Letting $r$ go to 0, and using the condition of the lemma at the point $z$, we have

$$\sum_{i=1}^l \text{diam}(\alpha([a_i, b_i])) \leq c_5 \left( \sum_{i=1}^l \Delta_i \right)^{-\frac{v-1}{p}}. \quad (7)$$

Thus, the curve $\alpha$ is absolutely continuous. Lemma 8 follows. □

For proof of Lemma 7 it is sufficiently to verify that the function $v$ is absolutely continuous on $\beta z$, where $z \in S$ and $\beta z$ are the same as in Lemma 8. Let $U_i = U(q_i, f, r_1)$ be normal neighborhoods defined at the beginning of the proof of Lemma 7, and $\beta = \beta z$. By Lemma 2, the relation

$$\sum_{x \in f^{-1}(\beta)(t) \cap U_i} i(x, f) = N(f, U_i) = i(q_i, f) = k(i)$$

holds. For every $i = 1, \ldots, s$, we choose full liftings $\alpha_{i,j}$, $j = 1, \ldots, k(i)$, of the curve $\beta$ in $U_i$ according to Lemma 5. By Lemma 8, all curves $\alpha_{i,j}$ are absolutely continuous. For every $j$, $j = 1, \ldots, k(i)$, we know number of liftings passing through a point belonging to $f^{-1}(\beta(t)) \cap U_i$: Lemma 5 implies that

$$\sum_{x \in f^{-1}(\beta(t)) \cap U_i} i(x, f) u(x) = \sum_{j=1}^{k(i)} u(\alpha_{i,j}(t)).$$
From here we come to
\[ v(\beta(t)) = \sum_{i=1}^{s} A \sum_{x \in f^{-1}(\beta(t)) \cap U_i} i(x, f) u(x) = A \sum_{i=1}^{s} \sum_{j=1}^{k(i)} u(\alpha_{i,j}(t)). \] (4)

According to (4), it is sufficient to prove that \( u(\alpha_{i,j}(t)) \) is absolutely continuous on \([a, b]\) for every \(i, j\). The last property follows from the absolute continuity of the curve \( \alpha_{i,j} \) and from Lipschitz continuity of the function \( u \) in the domain \( U \):
\[ |u(z_1) - u(z_2)| \leq L d(z_1, z_2), \quad z_1, z_2 \in U, \]

since \( u \in C_0^1 \). Lemma 7 is proved. □

**Lemma 10.** Suppose that \( x_0 \in \text{supp} v \setminus f(\text{supp} u \cap B_f) \). Then there exists a neighborhood \( V_0 \) of \( x_0 \) such that, for every connected neighborhood \( V \subset V_0 \) of the point \( x_0 \), the following conditions hold:

1. \( V \cap f(\text{supp} u \cap B_f) = \emptyset \);
2. the number of components \( f^{-1}(V) \) intersecting \( \text{supp} u \) is finite; we denote them by \( D_1, \ldots, D_k \);
3. the restriction \( f|_{D_i} = f_i : D_i \to V, \ i = 1, \ldots, k, \) is a \((p, q)\)-quasiconformal homeomorphism;
4. if \( g_i = f_i^{-1} \), then \( |\nabla_H v(z)| \leq A \sum_{i=1}^{k} |\nabla_H u(g_i(z))||D_H g_i(z)| \) for almost all points \( z \in V \).

**Proof.** We chose neighborhoods \( U_1, U_2, \ldots, U_k \) of points \( f^{-1}(x_0) \cap \text{supp} u \) such that \( \overline{U}_i \subset \Omega \) and the restriction \( f|_{\overline{U}_i} \) is an injective mapping. We have to show that
\[ V_0 = \left( \bigcap_{i=1}^{k} f(U_i) \right) \setminus f \left( \text{supp} u \setminus \bigcup_{i=1}^{k} U_i \right) \]

is the desired neighborhood \( V_0 \) of the point \( x_0 \).

Let \( V \subset V_0 \) be a connected neighborhood of the point \( x_0 \). The first assertion is valid since \( U_i \) does not intersect \( B_f \) for every \( i = 1, \ldots, k \). If a connected component \( D \) of the preimage \( f^{-1}(V) \) intersects \( \text{supp} u \), then it intersects one of the neighborhoods \( U_i \). Since the restriction \( f|_{\overline{U}_i} \) is injective then \( V_0 \cap f(\partial U_i) = \emptyset \), and hence \( D \cap \partial U_i = \emptyset \). From here it follows that \( D \subset U_i \). Thus, the second and the third assertions of Lemma 10 are proved.

Since the mappings \( g_i = f_i^{-1} \) are \((r, s)\)-quasiconformal homeomorphisms [55], then they are \( \mathcal{P} \)-differentiable almost everywhere in \( V \). In view of \( i(q, f) = 1 \) at points \( q \in \Omega \setminus B_f \) we have
\[ v(z) = A \sum_{i=1}^{k} u(g_i(z)) \]

for every points \( z \in V \). Then at every point \( z \in V \) of simultaneous \( \mathcal{P} \)-differentiability of all \( g_i \), we have
\[ |\nabla_H v(z)| \leq A \sum_{i=1}^{k} |\nabla_H u(g_i(z))||D_H g_i(z)|. \]

Hence, the last assertion of the lemma is also proved. □
**Proof of Theorem 2.** Let \( u \in C_0^1(\Omega) \) be an arbitrary function. Applying Vitali covering theorem, we obtain a countable collection of disjoint balls \( \{B_1, B_2, \ldots\} \), covering \( f(D) \setminus f(\text{supp } u \cap B_f) \) up to a set of measure zero, such that for balls \( B_j \) intersecting \( \text{supp } v \) the conditions (1)--(4) of the previous lemma hold. Since in view of Proposition 1 \( |f(B_f)| = 0 \), we have

\[
\int_{f(D)} |\nabla_H v|^s \, dz \leq \sum_{j=1}^{\infty} \int_{B_j} |\nabla_H v|^s \, dx,
\]

where \( s = \frac{p}{p-(\nu-1)}. \) Fix some index \( j \). If the ball \( B_j \) does not intersect \( \text{supp } v \), then

\[
\int_{B_j} |\nabla_H v|^s \, dx = 0.
\]

If \( B_j \) intersects \( \text{supp } v \), let \( g_i : B_j \to W_i, i = 1, \ldots, k(j) \), denote the inverse mapping, defined by property 3 of Lemma 10. From the Minkowski inequality and from relation (1) it follows

\[
\left( \int_{B_j} |\nabla_H v|^s \, dz \right)^{\frac{1}{s}} \leq \left( \int_{B_j} \left| \Lambda \sum_{i=1}^{l} |\nabla_H u(g_i(z))| |D_h g_i(z)|^s \right| \, dz \right)^{\frac{1}{s}} \leq \Lambda \sum_{i=1}^{l} \left( \int_{B_j} |\nabla_H u(g_i(z))|^s |D_h g_i(z)|^s \, dz \right)^{\frac{1}{s}} \leq \Lambda \sum_{i=1}^{l} \left( \int_{B_j} |\nabla_H u(g_i(z))|^s K_p(x; f) \right)^{\frac{1}{s}} ( f_i(B_j) )^\frac{1}{s} \leq \begin{cases} \Lambda K_{p,p}^{v-1}(f; D) \sum_{i=1}^{l} (\int_{B_i} |\nabla_H u|^s \, dx)^{\frac{1}{s}}, & q = p, \\ \Lambda \sum_{i=1}^{l} (\int_{g_i(B_j)} K_p(x; f) \frac{p}{q-1} \, dx)^{\frac{r}{s-1}} ( f_i(B_j) )^\frac{1}{s} , & q < p. \end{cases}
\]

In the last inequality we have used the \((r, s)\)-quasiconformality of the mappings \( g_i \). We can also assume that \( \text{diam } W_i < d(\text{supp } u, \partial D) \). Then \( l \leq N(f, D) \) and, in the case \( p = q \), the Hölder inequality implies

\[
\left( \int_{B_j} |\nabla_H v|^s \, dz \right)^{\frac{1}{s}} \leq \Lambda K_{p,p}^{v-1}(f; D) N^{(s-1)/s}(f, D) \left( \int_{f^{-1}(B_j)} |\nabla_H u|^s \, dx \right)^{\frac{1}{s}}.
\]
If \( q < p \) then applying the Hölder inequality twice we have

\[
\left( \int_{B_j} |\nabla_H v|^s \, dz \right)^{\frac{1}{s}} \leq A^{l(s-1)/s} \left[ \sum_{i=1}^l \left( \int_{g_i(B_j)} K_p(x; f)^{\frac{pq}{p-q}} \, dx \right)^{\frac{r-s}{r}} \cdot \left( \int_{g_i(B_j)} |\nabla_H u|^r \, dx \right)^{\frac{s}{r}} \right]^{\frac{1}{s}}
\]

\[
\leq A^{l(s-1)/s} \left[ \left( \sum_{i=1}^l \int_{g_i(B_j)} K_p(x; f)^{\frac{pq}{p-q}} \, dx \right)^{\frac{r-s}{r}} \cdot \left( \sum_{i=1}^l \int_{g_i(B_j)} |\nabla_H u|^r \, dx \right)^{\frac{s}{r}} \right]^{\frac{1}{s}}
\]

\[
\leq A^{l(s-1)/s} \| K_p(x; f) \|_{L_{\frac{pq}{p-q}}(f^{-1}(B_j))} \|_{s(v-1)} \left( \int_{f^{-1}(B_j)} |\nabla_H u|^r \, dx \right)^{\frac{s}{r}}
\]

where the equality \( \frac{rs}{r-s} = \frac{pq}{p-q(v-1)} \) was used.

Since \( l \leq N(f, D) \), we obtain

\[
\int_{f(D)} |\nabla_H v|^s \, dz
\]

\[
\leq \sum_{j=1}^\infty \int_{B_j} |\nabla_H v|^s \, dx
\]

\[
\leq \sum_{j=1}^\infty A^s(N(f, D))^{(s-1)} \| K_p(x; f) \|_{L_{\frac{pq}{p-q}}(f^{-1}(B_j))} \|_{s(v-1)} \left( \int_{f^{-1}(B_j)} |\nabla_H u|^r \, dx \right)^{\frac{s}{r}}.
\]

Applying the Hölder inequality we have

\[
\int_{f(D)} |\nabla_H v|^s \, dz
\]

\[
\leq A^s(N(f, D))^{s-1} \left( \sum_{j=1}^\infty \| K_p(x; f) \|_{L_{\frac{pq}{p-q}}(f^{-1}(B_j))} \|_{s(v-1)} \right)^{\frac{r-s}{r}}
\]

\[
\times \left( \sum_{j=1}^\infty \int_{f^{-1}(B_j)} |\nabla_H u|^r \, dx \right)^{\frac{s}{r}}
\]

\[
= A^s(N(f, D))^{s-1} \left( \sum_{j=1}^\infty \| K_p(x; f) \|_{L_{\frac{pq}{p-q}}(f^{-1}(B_j))} \|_{s(v-1)} \right)^{\frac{r-s}{r}}
\]

\[
\times \left( \sum_{j=1}^\infty \int_{f^{-1}(B_j)} |\nabla_H u|^r \, dx \right)^{\frac{s}{r}}
\]
\[
\leq \Lambda^s \left( N(f, D) \right)^{s-1} \left\| K_p(x; f) \left| L_{\frac{p q}{p - q}} \left( \bigcup_{j=1}^{\infty} f^{-1}(B_j) \right) \right| \right\|_{\frac{p q}{p - q}, \frac{r - s}{r}} \times \left( \int_{\bigcup_{j=1}^{\infty} f^{-1}(B_j)} |\nabla H u|^r \, dx \right)^{\frac{s}{r}}
\]

\[
\leq \Lambda^s \left( N(f, D) \right)^{s-1} \left( \left( \frac{p q}{p - q} \right)^{(s-1)/s} \right) \left( \frac{\left( \int_D |\nabla H u|^r \, dx \right)^{\frac{r}{s}}}{M(f, C)} \right)^{\frac{p}{p - (v - 1)}}.
\]

Theorem 2 is proved. \( \Box \)

**Corollary 2.** Let \( f : \Omega \to G \) be a mapping with bounded \((p, q)\)-distortion, \( v - 1 < q \leq p < \infty \). If \( E = (A, C) \) is a condenser in the domain \( \Omega \) such that \( A \subset \Omega, C \) is a compact set in \( A \) and \( N(f, A) < \infty \), then

\[
\left( \text{cap}_s f(E) \right)^{\frac{1}{s}} \leq \frac{(K_{p, q}(f; \Omega))^{v-1}(N(f, A))^{(s-1)/s}}{M(f, C)} \left( \text{cap}_r E \right)^{\frac{1}{r}},
\]

where \( r = \frac{q}{q - (v - 1)} \) and \( s = \frac{p}{p - (v - 1)} \).

**Proof.** In the definition of the push forward function \( v \) choose the value \( \Lambda = (M^{-1}(f, C))^{-1} \), where

\[
M(f, C) = \inf_{x \in f(C)} \sum_{z \in f^{-1}(x) \cap C} i(z, f).
\]

Since \( u \) is an admissible function for condenser \( E(A, C) \), then \( v(x) \geq 1 \) at points \( x \in f(C) \). Indeed, let \( x \in f(C) \) and \( f^{-1}(x) \cap C = \{ z_1, z_2, \ldots, z_k \} \). Then

\[
v(x) = \frac{1}{M(f, C)} \sum_{z \in f^{-1}(x)} i(z, f) u(z) \geq \frac{1}{M(f, C)} \sum_{l=1}^{k} i(z_l, f) u(z_l)
\]

\[
\geq \frac{1}{M(f, C)} \sum_{l=1}^{k} i(z_l, f) \geq 1
\]

since \( u(z_l) \geq 1 \). Properties of the function \( v(x) \), proved in Theorem 2, imply that \( v \) is an admissible function for the condenser \( f(E) = (f(A), f(C)) \). From here we have

\[
\left( \text{cap}_s f(E) \right)^{\frac{1}{s}} \leq \left( \int_{f(A)} |\nabla H v|^s \, dx \right)^{\frac{1}{s}} \leq \frac{(K_{p, q}(f; \Omega))^{v-1}(N(f, A))^{(s-1)/s}}{M(f, C)} \left( \int_A |\nabla H u|^r \, dx \right)^{\frac{1}{r}}.
\]

Since \( u \) is an arbitrary function Corollary 2 is proved. \( \Box \)
Corollary 3. Let \( f : \Omega \to \mathbb{G} \) be a mapping with bounded \((p, q)\)-distortion, \( v - 1 < q \leq p < \infty \), and \( A = U(z, f, r_0) \) be a normal neighborhood of the point \( z \in \Omega \). Then for the condenser \( E = (U(z, f, r_0), \overline{U}(z, f, r)), 0 < r < r_0 \), the estimate
\[
\left( \text{cap}_s f(E) \right)^{\frac{1}{s}} \leq \frac{K_{p,q}(f; \Omega)^{v-1}}{i(z, f)^{v-1}} \left( \text{cap}_r E \right)^{\frac{1}{r}}
\]
holds where \( r = \frac{q}{q-(v-1)} \) and \( s = \frac{p}{p-(v-1)} \).

**Proof.** We note that \( E = (U(z, f, r_0), \overline{U}(z, f, r)) \) is a normal condenser satisfying the conditions of Lemma 2. Hence \( N(f, A) = i(z, f) = \mu(f, A) = M(f, C) \). The desired inequality follows from Corollary 2.

Remark 1. In the Euclidean space \( \mathbb{R}^n \) and \( p = q = n \) Corollaries 2 and 3 were established in [27].

Proposition 4. Let \( f : \Omega \to \mathbb{G} \) be a mapping with bounded \((p, q)\)-distortion, \( v - 1 < q \leq p < \infty \). If \( E = (A, C) \) is a condenser in the domain \( \Omega \) such that \( A \subset \Omega \), \( A \) is bounded, and \( C \) is a compact set in \( A \), then
\[
\left( \text{cap}_s f(E) \right)^{\frac{1}{s}} \leq K_{p,q}(f; \Omega)^{v-1} \left( \text{cap}_r E \right)^{\frac{1}{r}}
\]
where \( r = \frac{q}{q-(v-1)} \) and \( s = \frac{p}{p-(v-1)} \).

**Proof.** For \((p, q)\)-quasiconformal mappings the inequality (5) was proved in [55]. In the case \( \mathbb{G} = \mathbb{R}^n \) the inequality was proved in [48]. We describe the basic steps of the proof in our situation. Since the closure \( \overline{A} \) is a compact set then \( N(f, A) < \infty \). For a nonnegative function \( u \in W^1_{\infty}(A) \cap C_0(A) \), we define the push-forward function
\[
(fu)_* (z) = \begin{cases} 
\sup_{y \in f^{-1}(z)} u(y), & z \in f(\text{supp} u), \\
0, & z \notin f(\text{supp} u).
\end{cases}
\]
This operator has the following properties, proofs of which are based on Theorem 3:

1. the function \((fu)_* \) is continuous and \( \text{supp}(fu)_* \subset f(\text{supp} u) \);
2. \((fu)_* : W^1_{\infty}(A) \cap C_0(A) \to W^1_s(\mathbb{G}) \cap C_0(\mathbb{G}) \);
3. \( \int_{\mathbb{G}} |\nabla_H (fu)_*|^s \ dx \leq K_{p,q}(f; \Omega)^{v-1} \left( \int_{\mathbb{G}} |\nabla_H u|^r \ dz \right)^{\frac{1}{r}} \);
4. if \( u \) is an admissible for the condenser \( E = (A, C) \), then \((fu)_* \) is an admissible function for the condenser \( f(E) = (f(A), f(C)) \).

From the last two properties we can to obtain inequality (5). □

The next theorem, giving an estimate for the local distortion of the metrics under the mappings with bounded \((p, q)\)-distortion was proved in [42] for the Euclidean space \( \mathbb{R}^n \). For mappings of Carnot groups, the proof remain the same, taking into account the lower estimate for the \( q \)-capacity, \( v - 1 < q < v \) [55].
Theorem 3. Let \( f : \Omega \to \mathbb{G} \) be a mapping with bounded \((p, q)\)-distortion where \( v - 1 < q < v \). Consider an arbitrary point \( x \in \Omega \) and a neighborhood \( U(x, f, t_0) \), \( t_0 = \min \{t_\delta, 1\} \). Then, for every point \( y \in U(x, f, t) \), \( t \leq t_0^4 \) at \( p = v \), and \( t \leq t_0^4/4 \) at \( p < v \), the following inequalities

\[
d(x, y)^{\frac{v-q}{q}} \leq \begin{cases} 
C^\frac{v}{q} \left( \frac{1}{d(f(x), f(y))} \right)^{\frac{v-q}{p}} & \text{ at } p = v, \\
C^\frac{v}{p} \left( d(f(x), f(y)) \right)^{\frac{v-p}{p}} & \text{ at } p < v,
\end{cases}
\]

hold where \( C = K_{p,q}(f; U(x, f, t_0)) \sup_{y \in \mathbb{G}} N(y, f, U(x, f, t_0)) \).

3. Liouville type theorem and removable sets

Let \( K \subset \Omega \) be a compact set. We define capacity of \( \text{cap}(K; W^1_p(\Omega)) \) in the space \( W^1_p(\Omega) \) as

\[
\inf \left\{ \left\| f \right\|_{L^p(\Omega)}^p + \left\| \nabla f \right\|_{L^p(\Omega)}^p : f \in C_0^\infty(\Omega) \text{ and } f \equiv 0 \text{ on } K \right\}.
\]

The capacity, defined initially on compact sets, extends by a standard way on arbitrary sets (see, for example, [49,50], where properties of the capacity are established).

Theorem 4. Let \( f : \mathbb{G} \to \mathbb{G} \) be a mapping with bounded \((p, q)\)-distortion, \( v - 1 < q \leq p \leq v \). Then \( \text{cap}(\mathbb{G} \setminus f(\mathbb{G}); W^1_p(\mathbb{G})) = 0 \) where \( s = \frac{p}{p-(v-1)} \).

Proof. Indeed, fix a compact set \( C \) in \( \mathbb{G} \) with a nonempty interior, and a sequence of open bounded sets \( A_k \supset C \) exhausting \( \mathbb{G} \). For the condenser \( E_k = (A_k, C) \), the estimate (6)

\[
\left( \text{cap}_{A_k} f(E_k) \right)^\frac{1}{2} \leq K_{p,q}(f; \mathbb{G}) \left( \text{cap}_C E_k \right)^\frac{1}{2}
\]

holds. Since the right-hand side goes to zero as \( k \to \infty \), then the left-hand side does the same. Note that

\[
\lim_{k \to \infty} \text{cap}_{A_k} f(E_k) = \text{cap}_{A_k} (f(\mathbb{G}), f(C); \mathbb{G}) = 0.
\]

Hence

\[
\text{cap}(\mathbb{G} \setminus f(\mathbb{G}); W^1_s(\mathbb{G})) \leq \text{cap}_{A_k} (\mathbb{G} \setminus f(\mathbb{G}), f(C); \mathbb{G}) = \text{cap}_{A_k} (f(\mathbb{G}), f(C); \mathbb{G}) = 0.
\]

From here it obviously follows, that \( \text{cap}(\mathbb{G} \setminus f(\mathbb{G}); W^1_s(\mathbb{G})) = 0 \).

Corollary 4. Let \( f : \mathbb{G} \to \mathbb{G} \) be a mapping with bounded distortion, and

\[
\text{cap}(\mathbb{G} \setminus f(\mathbb{G}); W^1_v(\mathbb{G})) > 0.
\]

Then \( f \) is a constant mapping.

Corollary 5. Let \( f : \mathbb{G} \to \mathbb{G} \) be a mapping with bounded \((p, q)\)-distortion, \( p, q \in (v - 1, v) \). Then \( f(\mathbb{G}) = \mathbb{G} \).

In order to formulate the next assertion, we recall that the space \( W^1_p(\mathbb{G}) \), \( 1 < p < v \), is defined as the completion of the space \( C_0^\infty(\mathbb{G}) \) with respect to the norm \( L^1_p(\mathbb{G}) \). The capacity of compact \( K \) in the space \( W^1_p(\mathbb{G}) \) is defined in the following way:

\[
\text{cap}(K; W^1_p(\mathbb{G})) = \inf \left\{ \left\| \nabla f \right\|_{L^p(\mathbb{G})}^p : f \in C_0^\infty(\mathbb{G}) \text{ and } f \equiv 1 \text{ on } K \right\}.
\]
It is proved in [49,50], that the families of the sets of capacity zero in the spaces \( w^1_p(G) \) and \( W^1_p(G) \) coincide.

Let \( E \subset G \) be a closed set, \( v \)-capacity of which is positive: \( \text{cap}(E; W^1_v(G)) > 0 \). We say, that set \( E \) has an essentially positive capacity at the point \( x \in E \) if \( \text{cap}(E \cap B(x, r); W^1_v(B(x, 2r))) > 0 \) for every \( r \in (0, 1) \). It is easy to show that if \( \text{cap}(E; W^1_v(G)) > 0 \) then the set \( \bar{E} \) of the points where \( E \) has essentially positive capacity is nonempty. We note that the set \( \bar{E} \) is closed. Hence, there exists a point \( x_0 \in \bar{E} \) which is nearest to zero: \( d(x_0) = \inf \{d(x): x \in \bar{E} \} \). Consider the intersection \( E_0 = \bar{E} \cap B(x_0, 1) \). The capacity \( \text{cap}(E_0; W^1_v(G)) \) is positive.

The following two lemmas were established in [53].

**Lemma 11.** Let \( C \) be a continuum in \( G \) and \( \text{diam} \, C \geq \alpha > 0 \).

1. Case \( v - 1 < p < v \). For every \( \alpha > 0 \), there exists \( \delta > 0 \) such that if \( \text{diam} \, C \geq \alpha > 0 \) then \( \text{cap}(C; w^1_p(G)) > \delta \).
2. Case \( p = v \). If \( E \subset G \) is a compact set and its capacity \( \text{cap}(E; W^1_v(G)) \) is positive, then for every \( \alpha > 0 \) and \( d > 0 \), there exists \( \delta > 0 \) such that \( \text{cap}_v(C \cap E, C) > \delta \) under conditions \( \text{diam} \, C \geq \alpha > 0 \) and \( \text{dist}(C, E_0) \leq d \).

As a corollary, we obtain the next assertion.

**Lemma 12.** Let \( v - 1 < p < v \), \( A \) is a bounded open set in \( G \), and \( C \) is a continuum in \( A \). Then for every \( \alpha > 0 \) there exists a number \( \varepsilon > 0 \) such that if \( \text{diam} \, C \geq \alpha \) then \( \text{cap}_p(A, C) \geq \varepsilon \).

**Proof.** It is enough to apply the inequality \( \text{cap}(C; w^1_p(G)) \leq \text{cap}_p(A, C) \).

**Corollary 6.** Consider the family \( f : \Omega \to G \) of mappings with bounded \((p, q)\)-distortion and with a coefficient of distortion not greater than a fixed number \( K \).

1. If \( p \geq q > v \), then the given family of the mappings is locally uniformly continuous.
2. If \( p = q = v \) and the family of the mappings \( f : \Omega \to G \setminus E \), where \( E \) is a closed set of positive capacity \( \text{cap}(E; W^1_v(G)) \), is locally uniformly bounded, then it is locally equicontinuous.

**Proof.** Fix a compact set \( A \subset \Omega \) and an open bounded set \( U \supset A \) with \( \bar{U} \subset \Omega \). There exists \( R_0 \) such that capacity \( \text{cap}_r(U, B(x, R)) \) is uniformly small with respect to all \( R \in (0, R_0) \) and \( x \in A \), \( r = \frac{q}{q-(v-1)} \). By inequality (6), we have the estimate
\[
(c_{p}(f(U), f(B(x, r))))^{\frac{1}{2}} \leq K_{p,q}(f; \Omega)^{v-1}(\text{cap}_r(U, B(x, r)))^{\frac{1}{2}},
\]
\( s = p/(p - v + 1) \).

Now the required assertions follow from Lemma 11 and Lemma 12.

**Theorem 5.** Let \( f : \Omega \setminus F \to G \) be a mapping with bounded \((p, q)\)-distortion, \( p \geq q \geq v \), and \( F \) be a closed set in the domain \( \Omega \), \( \text{cap}(F; W^1_s(G)) = 0, s = p/(p - (v - 1)) \). Then

1. in the case \( p \geq q > v \): the mapping \( f \) extends to a continuous mapping \( \tilde{f} : \Omega \to G \);
(2) in the case \( p = q = v \): if \( \operatorname{cap}(\mathbb{C} f(\Omega \setminus F); W^1_v(G)) > 0 \) then the mapping \( f \) extends to a continuous mapping \( \tilde{f} : \Omega \to G \cup \infty \). (Hereafter \( G \cup \{ \infty \} \) is the one-point compactification with the standard topology.)

**Proof.** Case \( p \geq q \geq v \). Let \( x_0 \in F \). Assume that the mapping \( f \) has no limit at the point \( x_0 \). Then in some ball \( B(x_0, R) \subset \Omega \) there exist two sequences of points \( \{x_i\} \) and \( \{x'_i\} \), tending to the point \( x_0 \), but for every index \( i \) the inequality \( d(f(x_i), f(x'_i)) \geq \alpha > 0 \) holds. Since \( \operatorname{cap}_p(F) = 0 \), then the sets \( B(x_0, r_i) \setminus F \) are connected, here \( r_i = 2 \max \{d(x_0, x_i), d(x_0, x'_i)\} \).

The points \( x_i \) and \( x'_i \) can be connected by a horizontal curve \( C_i \), lying in the set \( B(x_0, r_i) \setminus F \). Since \( \text{diam}(f(C_i)) \geq \alpha \), then by Lemma 11, \( \operatorname{cap}_p(f(B(x_0, R) \setminus F), f(C_i)) \geq \varepsilon^q > 0 \) for every index \( i \). Pick open sets \( A_j \), compactly embedded in \( \Omega \), contained in \( B(x_0, R) \setminus F \) and which exhaust \( B(x_0, R) \setminus F \) as \( f \) tends to \( \infty \). Then by inequality (5) we have

\[
(\operatorname{cap}_r(f(A_j), f(C_i)))^{\frac{1}{2}} \leq K_{p,q}(f; \Omega)^{v-1} \left(\operatorname{cap}_r(A_j, C_i)\right)^{\frac{1}{2}}.
\]

Passing to the limit as \( j \to \infty \), we arrive to the relations

\[
\varepsilon \leq \left(\operatorname{cap}_r(f(B(x_0, R) \setminus F), f(C_i))\right)^{\frac{1}{2}} \\
\leq K_{p,q}(f; \Omega)^{v-1} \left(\operatorname{cap}_r(B(x_0, R) \setminus F, C_i)\right)^{\frac{1}{2}} \\
\leq K_{p,q}(f; \Omega)^{v-1} \left(\operatorname{cap}_r(B(x_0, R), C_i)\right)^{\frac{1}{2}}.
\]

(6) Since \( \operatorname{cap}_r(B(x_0, R), C_i) \leq \theta\left(\frac{r-1}{r-v}(R^{\frac{1}{r-v}} - r_i^{\frac{r-1}{r-v}})\right)^{1-r} \), then the capacity in the right-side hand of the inequalities (6) tends to zero, as \( i \to \infty \).

We obtained a contradiction, since the left side of the inequality (6) is bounded away from zero, but the right side tends to zero, as \( i \) tends to \( \infty \).

Case \( p = q = v \), is divided in two subcases depending on whether

\[
\min(\text{dist}(f(x_i), E_0), \text{dist}(f(x'_i), E_0)) \to \infty,
\]

or not (here \( E = \mathbb{C} f(\Omega \setminus F) \), and \( E_0 \) is defined before Lemma 11). If it tends to \( \infty \), then we can assume that \( \lim_{x \to x_0} f(x) = \infty \). If it does not then the proof does not differ from the previous case. Theorem 5 is proved. \( \square \)

**Corollary 7.** Let \( f : \Omega \to G \) be a mapping with bounded \( (p, q) \)-distortion, \( p \geq q \geq v \). Let \( b \) be an isolated point of \( \partial \Omega \). Then

(1) in the case \( p \geq q > v \): the mapping \( f \) has a continuous extension \( \tilde{f} : \Omega \cup \{b\} \to G \);

(2) in the case \( p = q = v \): if \( \operatorname{cap}(\mathbb{C} f(\Omega \setminus \{b\}); W^1_v(G)) > 0 \) then the mapping \( f \) extends to a continuous mapping \( \tilde{f} : \Omega \cup \{b\} \to G \cap \{\infty\} \).

Proof of Corollary 7 is obvious.

**Corollary 8.** Let \( f : \Omega \to G \) be a nonconstant mapping with bounded distortion and let \( b \) be an isolated point of \( \partial \Omega \). Assume that the mapping \( f \) has no limit at the point \( b \). Then \( \operatorname{cap}(\mathbb{C} f(U \setminus \{b\}); W^1_v(G)) = 0 \) for every neighborhood \( U \subset \Omega \cup \{b\} \) of the point \( b \). Furthermore, there exists \( \sigma \)-set \( E \subset G \cup \{\infty\} \), having the capacity zero in the space \( W^1_v(G) \), such that \( N(z, f, U \setminus \{b\}) = \infty \) for every point \( z \in (G \cup \{\infty\}) \setminus E \) and for every above-mentioned neighborhood of the point \( b \).
Proof. By contradiction. Assume that
\[ \text{cap} (\mathcal{C} f (U \setminus \{ b \}); W^1_v (\mathbb{G})) > 0. \]
By Corollary 7, the mapping \( f \) has a limit at the point \( b \). This contradicts the fact that \( f \) is discrete. Indeed, let \( k_0 \) be a number such that \( 1/k_0 < \text{dist}(b, \partial U) \). The set \( \mathcal{C} f (B(b, 1/k) \setminus \{ b \}) \) has capacity zero in the space \( W^1_v (\mathbb{G}) \) for every \( k \geq k_0 \). Then the set
\[ E = \bigcup_{k=k_0}^{\infty} \mathcal{C} f (B(b, 1/k)) \]
in view of subadditivity of capacity, has also capacity zero in the space \( W^1_v (\mathbb{G}) \). Moreover, the preimage of every point \( z \in \mathcal{CE} \) belongs to the balls \( B(b, 1/k) \) for \( k \geq k_0 \). Hence, there exists a sequence of pairwise different points \( x_i \in B(b, 1/k_i) \) such that \( f(x_i) = z \) for every \( i \). Corollary is proved.

Theorem 6. Let \( f : \Omega \to \mathbb{G} \) be a mapping with bounded \((p, q)\)-distortion, \( v - 1 < q \leq p < \infty \). Let \( b \) be an isolated point of \( \partial \Omega \). Assume that \( f \) admits a continuous extension \( \tilde{f} : \Omega \cup \{ b \} \to \mathbb{G} \). Then \( \tilde{f} \) is a mapping with bounded \((p, q)\)-distortion.

Proof. It is obvious that the extension \( \tilde{f} \) belongs to the class ACL, satisfies the relation
\[ |D_H \tilde{f} (x)| \leq K_p (x; f) J(x, \tilde{f})^{1/p} \]
almost everywhere, and has Luzin condition \( \mathcal{N} \). We have to show, that the function \( |D_H \tilde{f}|^q \) is integrable in the neighborhood of the point \( b \). Since the mapping \( f \) is discrete, then there exists a ball \( B(b, r) \subseteq \Omega \) such that its image is bounded and \( \partial B(b, r) \cap \tilde{f}^{-1}(\tilde{f}(b)) = \emptyset \). Let \( U \) be the component of \( f(b) \) in \( \mathcal{C} \tilde{f} (\partial B(b, r)) \), and \( V \) be the component of \( b \) in the set \( \tilde{f}^{-1}(U) \). By change of variable formula (see Proposition 1), we have
\[
\int_V |D_H \tilde{f}(x)|^q \, dx \leq \int_V (K_p^q (x; f) J(x, \tilde{f}))^{q/p} \, dx
\]
\[
\leq \left( \int_V (K_p(x; f))^{pq} \, dx \right)^{p-q} \left( \int_V J(x, \tilde{f}) \, dx \right)^{q/p}
\]
\[
\leq K_{p,q} (f; V)^q \left( \int_U N(z, \tilde{f}, V) \, dz \right)^{q/p} < \infty.
\]
Theorem 6 is proved.

4. Geometrical definition of \((p, q)\)-quasiregular mappings

Boundedness of a linear distortion is an equivalent geometric characteristic for mappings with bounded distortion \((v\text{-distortion})\).

The following description of mappings in the case \( \mathbb{G} = \mathbb{R}^n \) was formulated in [27]. A non-constant mapping \( f : \Omega \to \mathbb{G}, \Omega \subset \mathbb{G} \), is called quasiregular, if

1. \( f \) is continuous, open, discrete, and sense-preserving at the points of a local homeomorphism;
(2) the value $H(x,f) = \lim_{r \to 0} \frac{\max d(x,y) = d(f(x), f(y))}{\min d(x,y) = d(f(x), f(y))}$ is locally bounded in $\Omega$;

(3) for all points $x \in \Omega \setminus B_f$, where $B_f$ is the branch set of $f$, the relation $H(x,f) \leq K$ holds where $K$ is independent of point $x$.

Analytic properties of quasiregular mappings are formulated in the next assertion.

**Theorem 7.** (See [53].) Every nonconstant quasiregular mappings $f : \Omega \to \mathbb{G}$, defined on domain $\Omega \subset \mathbb{G}$, is a mapping with bounded distortion.

**Proof.** ACL-property of the mapping $f$ was proved in [52]. The integrability of the horizontal differential follows from the change of variable formula [45]. □

In the similar manner we can describe geometrically the mappings with bounded $(p,q)$-distortion.

A nonconstant mapping $f : \Omega \to \mathbb{G}$, $\Omega \subset \mathbb{G}$, is called $(p,q)$-quasiregular if

1. $f$ is continuous open discrete and sense-preserving at the points of local homeomorphism;
2. in the domain $\Omega$, there exists a bounded quasiadditive set function $\Phi$ defined on open bounded subsets of $\Omega$ such that the value
   
   $$
   H_{p,q}(x,f) = \lim_{r \to 0} \frac{L_{p_f}(x,r)r^{\nu-p}}{|f(B(x,\lambda r))|} \left( \frac{\Phi(B(x,\lambda r))}{|B(x,r)|} \right)^{\frac{p-q}{q}}
   $$

   is locally bounded in $\Omega$. Here $\lambda > 1$ is a fixed number, and $L_{\Phi}(x,r) = \max\{d(f(y), f(x)) : d(y,x) = r\}$ by the condition for $r < \text{dist}(x, \partial D)$;

3. for all points $x \in \Omega \setminus B_f$, where $B_f$ is the branch set of $f$, the relation $H_{p,q}(x,f) \leq K$ holds, where $K$ is independent of $x$;

4. if $p, q \in (\nu-1, \nu)$ the mapping $f$ has Luzin condition $\mathcal{N}$.

**Theorem 8.** Every $(p,q)$-quasiregular mapping $f : \Omega \to \mathbb{G}$, defined on the domain $\Omega \subset \mathbb{G}$, is a mapping with bounded $(p,q)$-distortion.

**Proof.** Fix a point $x \in D \setminus B_f$, there exists a neighborhood $W$ of $x$ such that the restriction $f : W \to f(W)$ is a homeomorphism. Then $f$ is ACL on $W$ and is differentiable almost everywhere in $W$ (see [55] for details). Therefore $f$ belongs to the class $\text{ACL}(\Omega)$ and is differentiable almost everywhere in $\Omega$. The integrability of the local $p$-distortion follows now from the boundedness of the value $H_{p,q}(x,f)$. Indeed, since $f$ is an ACL mapping differentiable a.e. in $\Omega$, we have

$$
H_{p,q}(x,f) = \frac{|Df|^p}{\lambda^v J(x,f)} / (\lambda^v \Phi'(x))^{\frac{p-q}{q}} \leq K \quad \text{a.e. in } \Omega.
$$

Hence

$$
\frac{|Df|^p}{\lambda^v J(x,f)} \leq K(\lambda^v \Phi'(x))^{\frac{p-q}{q}} \quad \text{a.e. in } \Omega.
$$
Since the volume derivative $\Phi'(x)$ integrable in $\Omega$ [57], we see that the value

$$I_{p,q}(x, f) = \left( \frac{|Df|^p}{J(x, f)} \right)^{\frac{q}{p-q}}$$

is integrable in $\Omega$ and $f$ is a mapping with bounded $(p, q)$ distortion.

References