A new existence result for nonlinear first-order anti-periodic boundary value problems

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Abstract

In this paper we consider the existence of solutions to boundary value problems (BVPs) involving systems of nonlinear first-order ordinary differential equations and two-point, anti-periodic boundary conditions. A new existence result for the BVPs above is obtained by using fixed-point theory.

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1. Introduction

In this article we will consider the existence of solutions to the nonlinear first-order system of anti-periodic boundary value problems

\[ x' = f(t, x), \quad t \in [0, T], \]
\[ x(0) = -x(T), \]

where \( f : [0, T] \times \mathbb{R}^n \rightarrow \mathbb{R}^n \) is continuous, and \( T \) is a positive constant.

The main tools employed herein are fixed-point methods; homotopy theory; and Leray–Schauder degree.

Anti-periodic problems have been studied extensively in the last ten years. For example, for first-order ordinary differential equations, a Massera’s type criterion is presented in [11] and in [17,27,29] the validity of the monotone iterative technique is shown. Also for higher-order ordinary differential equations existence and uniqueness results based on a Leray–Schauder type argument are presented in [1,2]. Anti-periodic boundary conditions for partial differential equations and abstract differential equations are considered in [4–6,8,12,18,21,22,24,28]. For recent developments involving the existence of anti-periodic solutions of differential equations, inequalities, and other interesting results on anti-periodic boundary value problems, the reader is referred to [3,7,9,10,13–16,19,20,23,25,26].

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2. Main results

Note that we may rewrite the BVPs (1.1) and (1.2) in the following form

\[ x' - x = f(t, x) - x, \quad t \in [0, T], \]  
\[ x(0) = -x(T). \]  
(2.1)

(2.2)

We may also regard the BVPs (2.1) and (2.2) as a special case of the following problem

\[ x' + a(t)x = \varphi(t, x), \quad t \in [0, T], \]  
\[ x(0) = -x(T), \]  
(2.3)

(2.4)

where \( a : [0, T] \to \mathbb{R}^1 \) and \( \varphi : [0, T] \times \mathbb{R}^n \to \mathbb{R}^n \) are both continuous.

Therefore, attention is now turned to (2.3) and (2.4). First, a lemma will be given for later use.

**Lemma 2.1.** The BVPs (2.3) and (2.4) are equivalent to the integral equation

\[ x(t) = e^{-\int_0^t a(u)du} \left( \frac{\int_0^T \varphi(s, x(s))e^{\int_0^s a(u)du} ds}{1 + e^{\int_0^s a(u)du}} + \int_0^t \varphi(s, x(s))e^{\int_0^s a(u)du} ds \right) \]  
(2.5)

for \( t \in [0, T] \).

**Proof.** The result can be obtained by direct computation. \( \square \)

**Remark 2.2.** Indeed, (2.5) implies the expression of Green’s function for the linear anti-periodic boundary value problem

\[ x'(t) + a(t)x = b(t), \quad x(0) = -x(T). \]

And (2.5) can be written in the following form:

\[ x(t) = \int_0^T G(s, t)b(s)ds, \]

where \( G \) is Green’s function

\[ G(s, t) = \begin{cases} 
1 + \frac{1}{1 + e^{\int_0^s a(u)du}} e^{\int_0^t a(u)du}, & 0 \leq s < t \leq T, \\
1 - \frac{1}{1 + e^{\int_0^s a(u)du}} e^{\int_0^t a(u)du}, & 0 \leq t \leq s \leq T.
\end{cases} \]

**Theorem 2.3.** If there exist a nonnegative constant \( \gamma \) and an integrable function \( j : [0, T] \to \mathbb{R}^1 \) with \( \int_0^T j(t)dt > 0 \), and a \( C^1 \) function \( W : \mathbb{R}^n \to [0, +\infty) \) with \( W(p) = W(-p) \) for all \( p \in \mathbb{R}^n \) such that, for each \( \lambda \in [0, 1] \),

\[ \lambda \|\varphi(t, p)\|e^{\int_0^t a(u)du} \leq \gamma (\lambda W(p), \lambda \varphi(t, p) - a(t)p) + j(t), \]  
(2.6)

for all \( (t, p) \in [0, T] \times \mathbb{R}^n \), where \( \langle \cdot, \cdot \rangle \) denotes the usual inner product and \( \| \cdot \| \) denotes the Euclidean norm on \( \mathbb{R}^n \), then the BVPs (2.3) and (2.4) have at least one solution.

**Proof.** The BVPs (2.3) and (2.4) are equivalent to the integral equation (2.5) from Lemma 2.1.

Define the map \( A : C([0, T]; \mathbb{R}^n) \to C([0, T]; \mathbb{R}^n) \) by

\[ Ax(t) = e^{-\int_0^t a(u)du} \left( \frac{\int_0^T \varphi(s, x(s))e^{\int_0^s a(u)du} ds}{1 + e^{\int_0^s a(u)du}} + \int_0^t \varphi(s, x(s))e^{\int_0^s a(u)du} ds \right), \]

for each \( t \in [0, T] \).
Thus, our problem is reduced to proving the existence of at least one fixed point of $A$. Let

$$B_{R+1} = \left\{ x \in C([0, T]; \mathbb{R}^n) \mid \max_{t \in [0, T]} \|x(t)\| < R + 1 \right\}.$$  

$$R = \max_{t \in [0, T]} \left( \frac{1}{e^{\int_0^t a(u)du}} \left( 1 + \frac{1}{e^{\int_0^t a(u)du} + 1} \right) \right) \int_0^T j(t)dt.$$  

Set for each $\lambda \in [0, 1]$

$$h_{\lambda}(x) = I - \lambda Ax, \quad x \in C([0, T]; \mathbb{R}^n),$$

where $I$ is the identity.

If we can prove that

$$0 \in C([0, T]; \mathbb{R}^n) \setminus h_{\lambda}(\partial B_{R+1}) \quad (2.7)$$

for each $\lambda \in [0, 1]$, then we have

$$\text{deg}_{\text{LS}}(h_{\lambda}, B_{R+1}, 0) = \text{deg}_{\text{LS}}(I - \lambda A, B_{R+1}, 0) = \text{deg}_{\text{LS}}(I - A, B_{R+1}, 0) = \text{deg}_{\text{LS}}(I, B_{R+1}, 0) = 1,$$

where $\text{deg}_{\text{LS}}$ denotes Leray–Schauder degree. Since $\text{deg}_{\text{LS}}(I - A, B_{R+1}, 0) = 1 \neq 0$, then there exists at least one fixed point of $A$ in $B_{R+1}$ such that

$$x = Ax.$$

Therefore, we only need to prove that (2.7) holds under the assumptions.

Consider the following BVPs

$$x' + a(t)x = \lambda \varphi(t, x), \quad t \in [0, T], \lambda \in [0, 1],$$

$$x(0) = -x(T). \quad (2.8)$$

$$x(0) = x(T) = x. \quad (2.9)$$

Note that if there is a point $x \in C([0, T]; \mathbb{R}^n)$ such that

$$x = \lambda Ax$$

for some $\lambda \in [0, 1]$, then $x$ must be a solution of (2.8) and (2.9). Assume that

$$x = \lambda_0 Ax$$

for some $x \in C([0, T]; \mathbb{R}^n)$ and $\lambda_0 \in [0, 1]$. Now for each $t \in [0, T]$, we have

$$\|x(t)\| = \lambda_0 \|Ax(t)\| \leq \left( \frac{\int_0^T \lambda_0 \|\varphi(s, x(s))\| e^{\int_0^t a(u)du} ds}{e^{\int_0^t a(u)du} \left( 1 + e^{\int_0^t a(u)du} \right)} + \frac{\int_0^T \lambda_0 \|\varphi(s, x(s))\| e^{\int_0^t a(u)du} ds}{e^{\int_0^t a(u)du} \left( 1 + e^{\int_0^t a(u)du} + 1 \right)} \right) \int_0^T \lambda_0 \|\varphi(s, x(s))\| e^{\int_0^t a(u)du} ds \leq \max_{t \in [0, T]} \left( \frac{1}{e^{\int_0^t a(u)du} \left( 1 + e^{\int_0^t a(u)du} + 1 \right)} \right) \int_0^T (\gamma(DW(x(s)), \lambda_0 \varphi(s, x(s)) - a(s)x(s)) + j(s))ds \leq \frac{R}{\int_0^T j(s)ds} \int_0^T (\gamma(DW(x(s)), x'(s)) + j(s))ds.$$
Thus, we have
\[
\max_{t \in [0, T]} \|x(t)\| \leq R < R + 1,
\]
and so (2.7) holds. \[\square\]

**Example 2.4.** Consider
\[x'(t) - t^2x = \varphi(t, x),\]
where
\[
\varphi(t, x) = (\varphi_1(t, x), \ldots, \varphi_n(t, x)), \quad \varphi_i(t, x) = \sin \left(t \prod_{j \neq i, j=1}^n x_j\right) \text{ for } 1 \leq i \leq n.
\]
Take \(W(p) \equiv 0\) and \(j(t) = \sqrt{\kappa t}\) for \(p \in \mathbb{R}^n\) and \(t \in [0, T]\), respectively. Then it is easy to see that (2.6) holds.

**Corollary 2.5.** If there exist a nonnegative constant \(\gamma\) and an integrable function \(j : [0, T] \rightarrow \mathbb{R}^1\) with \(\int_0^T j(t)dt > 0\), and a \(C^1\) function \(W : \mathbb{R}^n \rightarrow [0, +\infty)\) with \(W(p) = W(-p)\) for all \(p \in \mathbb{R}^n\) such that, for each \(\lambda \in [0, 1]\),
\[
\lambda \|f(t, p) - p\| \leq \gamma(DW(p), \lambda f(t, p) + (1 - \lambda)p) + j(t),
\]
for all \((t, p) \in [0, T] \times \mathbb{R}^n\), then the BVPs (1.1) and (1.2) have at least one solution.

**Proof.** This is a special case of Theorem 2.3 with \(a(t) \equiv -1\) and \(\varphi(t, p) = f(t, p) - p\). \[\square\]

### 3. A more concrete condition

In this section we will discuss a more concrete condition than the one in Theorem 2.3.

**Theorem 3.1.** If there exist a nonnegative constant \(\gamma\) and an integrable function \(j : [0, T] \rightarrow \mathbb{R}^1\) with \(\int_0^T j(t)dt > 0\), such that, for each \(\lambda \in [0, 1]\),
\[
\lambda \|\varphi(t, p)\|e^{\int_0^T a(t)dt} \leq 2\gamma(p, \lambda \varphi(t, p) - a(t)p) + j(t),
\]
for all \((t, p) \in [0, T] \times \mathbb{R}^n\), then the BVPs (2.3) and (2.4) have at least one solution.

**Proof.** From the proof of Theorem 2.3, we only need to show that (2.7) holds. Let \(x = \lambda_0 Ax\), for some \(\lambda_0 \in [0, 1]\). Then for each \(t \in [0, T]\), we have
\[
\|x(t)\| = \lambda_0 \|Ax(t)\|
\leq \frac{R}{\int_0^T j(s)ds} \int_0^T \left(\gamma(2x(s), \lambda_0 \varphi(s, x(s)) - a(s)x(s)) + j(s)\right)ds
= \frac{R}{\int_0^T j(s)ds} \int_0^T \left(2\gamma(x(s), x'(s)) + j(s)\right)ds
= \frac{R}{\int_0^T j(s)ds} \int_0^T \left(2\gamma \frac{d}{ds}\|x(s)\|^2 + j(s)\right)ds
= R.
\]
So (2.7) holds. \[\square\]
Remark 3.2. In fact, Theorem 3.1 is a special case of Theorem 2.3 with $W(p) = \|p\|$. However, the condition in Theorem 3.1 is easily verifiable in practice.

Example 3.3. Consider the following ordinary differential equation

$$x'(t) - t^{2k}x = \frac{t x}{T|\|x\| + 1},$$

where $k$ is an arbitrary natural number. (3.1) is immediate if we take $j(t) \equiv n$ for $t \in [0, T]$.

Corollary 3.4. If one of the following two conditions

(1) $a < 0$ and there exist a nonnegative constant $\gamma$ and an integrable function $j(t) : [0, T] \to \mathbb{R}^1$ with $\int_0^T j(t)dt > 0$, such that

$$\|\varphi(t, p)\| \leq 2\gamma(p, \varphi(t, p)) + j(t),$$

for all $(t, p) \in [0, T] \times \mathbb{R}^n$;

(2) $\varphi$ is bounded on $[0, T] \times \mathbb{R}^n$ holds, then the BVPs (2.3) and (2.4) have at least one solution.

Proof. This is a direct consequence of Theorem 3.1. □

Remark 3.5. Let $a(t) \equiv -1$ and $\varphi(t, p) = f(t, p) - p$. From Theorem 3.1 and Corollary 3.4, we can obtain some similar existence results for BVPs (1.1) and (1.2). We will omit them for brevity.

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