Set-Theoretic Aspects of Periodic FC-Groups— Extraspecial p-Groups and Kurepa Trees*

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When generalizing a characterization of centre-by-finite groups due to B. H. Neumann, M. J. Tomkinson asked the following question. Is there an FC-group G with $|G/Z(G)| = \kappa$ but $[G : N_G(U)] < \kappa$ for all (abelian) subgroups U of G, where κ is an uncountable cardinal [16, Question 7A, p. 149]. We consider this question for $\kappa = \omega_1$ and $\kappa = \omega_2$. It turns out that the answer is largely independent of ZFC (the usual axioms of set theory), and that it differs greatly in the two cases. © 1993 Academic Press, Inc.

INTRODUCTION

The Problem and its History. The purpose of this paper is to give some independence proofs concerning the existence of FC-groups having some additional properties. Recall that a group G is FC iff every element $g \in G$ has finitely many conjugates; i.e., iff $[G : C_G(g)]$ is finite for any $g \in G$. We shall mostly be concerned with periodic FC-groups.

In the fifties, B. H. Neumann gave the following characterization of *centre-by-finite groups*; i.e., groups with $|G/Z(G)| < \omega$.

(I) *The following are equivalent for any group G.*

(i) *G is centre-by-finite.*

(ii) *Each subgroup of G has only finitely many conjugates; i.e.,* $[G : N_G(U)] < \omega$ *for all* $U \leqslant G$.

If G is an FC-group both are equivalent to

(iii) *U/U_G is finite for all $U \leqslant G$.*

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Here U_G denotes the largest normal subgroup of G contained in U, i.e., $U_G := \bigcap_{g \in G} g^{-1}Ug$; it is called the *core* of U in G. It was indicated by Eremin that in both (ii) and (iii) above it suffices to consider abelian subgroups (cf.[16, 7.12(a) and 7.20]; also note that a group satisfying (iii) above is in general not FC [16, p. 142]).

Following M. J. Tomkinson [15] (see also [4]), for an infinite cardinal κ, let $\mathscr{X}_\kappa$ denote the class of groups G in which $[G : C_G(U)] < \kappa$ whenever $U \leqslant G$ is generated by fewer than κ elements (for $\kappa > \omega$, this is equivalent to saying that $[G : C_G(U)] < \kappa$ for $U \leqslant G$ of size less than κ). Clearly $\mathscr{X}_\omega$ is just the class of FC-groups. Generalizing Neumann's result Tomkinson proved in [15] (see also [16, Theorem 7.20]).

   (II)   *Let κ be an infinite cardinal. The following are equivalent for any FC-group G in $\mathscr{X}_\kappa$.*

   (i) $|G/Z(G)| < \kappa$.

   (ii) $[G : N_G(U)] < \kappa$ *for all* $U \leqslant G$.

   (iii) $[G : N_G(A)] < \kappa$ *for all abelian* $A \leqslant G$.

   (iv) $|U/U_G| < \kappa$ *for all* $U \leqslant G$.

   (v) $|A/A_G| < \kappa$ *for all abelian* $A \leqslant G$.

It was later shown by Faber and Tomkinson in [4] that the condition that G is FC can be dropped in (II).

On the other hand one might ask whether the condition that G is $\mathscr{X}_\kappa$ is necessary to prove the equivalence of (i) through (v) in (II). Clearly the following implications always hold.

$$(i) \Rightarrow (ii) \Rightarrow (iii)$$

$$(i) \Rightarrow (iv) \Rightarrow (v)$$

But what about the others?

One answer to this question was indicated by Tomkinson himself [16, p. 149]. Recall that a p-group E is called *extraspecial* iff $\Phi(E) = E' = Z(E) \cong \mathbb{Z}_p$. (Here $\Phi(E)$ is the *Frattini subgroup* of E; i.e., the intersection of all maximal subgroups of E.) This implies that E/E' is elementary abelian. Let E be an extraspecial p-group of size ω_1 all of whose abelian subgroups are countable (the existence of such groups was proved by S. Shelah and J. Steprāns in [13] improving on earlier work of A. Ehrenfeucht and V. Faber [16, Theorem 3.12] who obtained the same result under the additional assumption of the continuum hypothesis (CH)). Note that if $U \leqslant E$ then U is either normal and so $U = U_E$ or U is abelian and so $|U/U_E| \leqslant |U| \leqslant \omega$. So for $\kappa = \omega_1$ and $G = E$, (iv) and (v) in (II) are true, whereas (i) is not. Also, if $A \leqslant E$ is maximal (abelian) with respect to

$A \cap E' = 1$, then $\langle E', A \rangle = C_E(A) = N_E(A)$. Hence $|A| = |N_E(A)| = \omega$ and $[E : N_E(A)] = \omega_1$. Thus (ii) and (iii) do not hold either.

Why is this so? To obtain (iv) and (v) but not (i) we used an extra-special p-group such that all (maximal) abelian subgroups are small. Dually, to obtain (ii) and (iii) but not (i) we *should* use an extraspecial p-group such that all maximal abelian subgroups are large in the sense that their indices are small. It will be one of our goals to discuss the existence of such groups (see Theorems D and E below and Section 5). Tomkinson proved already in [14] that there are no such groups of size ω_1 (this also follows from our more general Theorem C).

The Main Results. For $\kappa = \omega_1$ our results are as follows.

THEOREM A. *Under CH there is an FC-group G with $|G/Z(G)| = \omega_1$ but $[G : N_G(A)] \leqslant \omega$ for all abelian subgroups $A \leqslant G$.*

THEOREM B. *It is consistent (assuming the consistency of ZFC) that there is no FC-group G with $|G/Z(G)| = \omega_1$ but $[G : N_G(A)] \leqslant \omega$ for all abelian subgroups $A \leqslant G$.*

Theorems A and B show that the question whether (i) and (iii) in (II) are equivalent for $\kappa = \omega_1$ is not decided by the axioms of set theory alone. The example used to prove Theorem A has a countable subgroup U with $[G : N_G(U)] = \omega_1$. So it does not answer the following.

QUESTION 1. *Let $\kappa = \omega_1$. Are* (i) *and* (ii) *in* (II) *equivalent for all FC-groups G?*

We conjecture that the answer is yes. Our reason for believing this is the following partial result.

THEOREM C. *Let $\kappa = \omega_1$. Then* (i)–(iii) *in* (II) *are equivalent for all finite-by-abelian groups G.*

Here, a group G is called *finite-by-abelian* iff G' is finite.

Question 1 and Theorem C are very closely related to another problem of Tomkinson [16, Question 3F, p. 60]. Following [14] (see also [16, Chap. 3]) let $\mathscr{X}$ be the class of locally finite groups G satisfying: for all cardinals κ and all $H \leqslant G$ of size $< \kappa$, $[G : C_G(H)] < \kappa$. So $\mathscr{X}$ is the class of periodic groups in the intersection of the $\mathscr{X}_\kappa$. And $\mathscr{Y}$ is the class of locally finite groups G satisfying: for all cardinals κ and all $H \leqslant G$ of size $< \kappa$, $[G : N_G(H)] < \kappa$. Clearly $\mathscr{X} \subseteq \mathscr{Y}$. Tomkinson asked whether there are $\mathscr{Y}$-groups which are not in $\mathscr{X}$. He proved in [14, Theorem D(i)] that any extraspecial p-group in $\mathscr{Y}$ of size ω_1 lies in $\mathscr{X}$. We generalize this by showing

THEOREM C'. $\mathcal{Y} = \mathcal{Z}$ for finite-by-abelian groups of size ω_1.

THEOREM B'.   *Assuming the consistency of ZFC it is consistent that* $\mathcal{Y} = \mathcal{Z}$ *for FC-groups of size* ω_1.

The proofs of these results use the same ideas as the proofs of Theorems C and B, respectively, and we hope that our argument can be generalized to give a positive answer to

QUESTION 1'.   *Is* $\mathcal{Y} = \mathcal{Z}$ *for FC-groups of size* ω_1?

Note that a positive answer to Question 1' would give a positive answer to Question 1, too. For suppose there is a counterexample G. Then $|G/Z(G)| = \omega_1$ (and without loss we may assume that $|G| = \omega_1$) but $[G : N_G(U)] \leqslant \omega$ for all $U \leqslant G$. By Tomkinson's result (II), $G \notin \mathcal{Z}_{\omega_1}$, so $G \notin \mathcal{Z}$, hence $G \notin \mathcal{Y}$; i.e., there is a countable $U \leqslant G$ such that $[G : N_G(U)] = \omega_1$, a contradiction. The problem seems to be of group theoretical character, and might involve a better understanding of countable periodic FC-groups.

For $\kappa = \omega_2$ the picture changes considerably. Recall that an uncountable cardinal κ is *(strongly) inaccessible* iff it is regular (i.e., it is not the union of $< \kappa$ sets of size $< \kappa$) and all cardinals $\lambda < \kappa$ satisfy $2^\lambda < \kappa$ (especially, κ is a limit cardinal). The existence of inaccessible cardinals cannot be proved in ZFC; in fact, something much stronger is true. Let I denote the sentence *there is an inaccessible cardinal.* Then the consistency of ZFC can be proved in the system ZFC + I (see [9, Chap. IV, Theorem 6.6, p. 145]). Hence, by Gödel's Incompleteness Theorem, the consistency of ZFC + I cannot be proved from the consistency of ZFC alone. A *weak Kurepa tree* is a tree of height ω_1 with ω_2 uncountable branches such that all levels have size $\leqslant \omega_1$. A *Kurepa tree* is a weak Kurepa tree with countable levels (a more formal definition will be given in Section 1). The existence of Kurepa trees is consistent (assuming the consistency of ZFC), and the non-existence of Kurepa trees is equiconsistent with the existence of an inaccessible (see, again, our Section 1 for details).

THEOREM D.   *Assume there is a Kurepa tree. Then there is an extra-special p-group of size* ω_2 *such that* $[G : A] \leqslant \omega_1$ *for all maximal abelian subgroups* $A \leqslant G$.

On the other hand, one can show (Theorem 5.4) that the existence of such a group implies the existence of a weak Kurepa tree. In fact, we can prove the following much stronger result.

THEOREM E. *Assuming the consistency of $ZFC + I$ it is consistent that for both $\kappa = \omega_1$ and $\kappa = \omega_2$ and any FC-group G, (i)–(iii) in (II) are equivalent.*

We thus obtain

COROLLARY. *The following theories are equiconsistent.*

(a) $ZFC + I$.

(b) $ZFC + for$ $\kappa = \omega_1$ and $\kappa = \omega_2$ and for any FC-group: if $|G/Z(G)| = \kappa$ then there is an abelian subgroup $A \leqslant G$ with $[G : N_G(A)] = \kappa$.

(c) $ZFC + any$ extraspecial p-group of size ω_2 has an (abelian) subgroup with $[G : N_G(A)] = \omega_2$.

(a) $\Rightarrow$ (b) is Theorem E; (b) $\Rightarrow$ (c) is trivial; and (c) $\Rightarrow$ (a) follows from Theorem D using the equiconsistency concerning the non-existence of Kurepa trees mentioned above. It should be pointed out that we cannot prove the *equivalence* of (b) and (c); namely, it is consistent (assuming again the consistency of $ZFC + I$) that (c) holds but (b) does not (see Theorem 5.8). Still our corollary shows again (cf. [14] or [16, Chap. 3, especially 3.15]) the importance of the extraspecial p-groups in the class of periodic FC-groups.

The Organization of the Paper. Our results use mainly classical (modern) set theory. For algebraists who might not be familiar with this material, we give a short introduction to this subject in Section 1. We hope that this makes our work more intelligible. The reader who has seen forcing etc. before should skip the entire Section 1.

In the second section we show that a countable finite-by-abelian group is generated by finitely many abelian subgroups (Theorem 2.2). We also discuss what goes wrong when *countable* is dropped from the assumption of the theorem.

In the third section we prove a result on automorphisms of countable periodic abelian groups which turns out to be crucial for our arguments (Theorem 3.2); we will apply it in the proofs of Theorems B, C, and E. If we could generalize this result to countable periodic FC-groups, we would obtain a positive answer to Questions 1 and 1'. Our original proof involved a fragment of Ulm's classification theorem. Since then, M. J. Tomkinson has found a much shorter and more elegant proof which we reproduce with his permission. We close Section 3 with the proof of Theorem C (and C').

Section 4 is devoted to the proofs of Theorems A and B (and B'); i.e., to the case $\kappa = \omega_1$. It turns out that the knowledge of maximal abelian subgroups of countable periodic FC-groups is essential.

In Section 5 we deal with the case $\kappa = \omega_2$ and the relationship between Kurepa trees and extraspecial p-groups; we prove Theorems D and E. We

think that those results are the most interesting and most beautiful of our work.

We close with some generalizations in Section 6.

Finally note that we obtain most of the *main results* mentioned in the preceding subsection as corollaries to more technical theorems and constructions, and we hope that the ideas involved in the latter might be useful when dealing with other problems as well. They are Theorems 2.2, 3.2, and 4.4 (with its elaboration in Section 5.7) and the easy Theorem 5.4 (see also Section 5.7) and the constructions in Section 4.2 (modified in Sections 4.6 and 4.7) and Section 5.3 (modified in Section 5.9).

Group-Theoretic Notation and Basic Facts on FC-Groups. Our group-theoretic notation is standard. Good references are [12] for general group theory, [5, 6] for abelian groups (which will be written additively), and [16] for FC-groups.

For completeness' sake we give our extension-theoretic notation. Let A, G be groups. If $G \leqslant \mathrm{Aut}(A)$, we let $A \rtimes G$ denote the semidirect product of A and G. If $\tau: G^2 \to A$ is a factor system (i.e., $\forall g \in G(\tau(g, 1) = \tau(1, g) = 1)$ and $\forall f, g, h \in G(\tau(fg, h) \tau(f, g) = \tau(f, gh) \tau(g, h)))$, we let $E(\tau)$ denote the corresponding extension (where the operation of G on A is trivial). In the latter case, group multiplication is given by the formula

$$\forall (a, g), (b, h) \in E(\tau), \qquad (a, g) * (b, h) = (\tau(g, h) ab, gh).$$

More details can be found in [12, Chap. 11].

We note that in all of our results (in particular, in Theorems B, C, and E) it suffices to consider *periodic* FC-groups. The reason for this is as follows. By a result of Černikov [16, Theorem 1.7], any FC-group can be embedded in a direct product of a periodic FC-group and a torsion-free abelian group. Now suppose G is an (arbitrary) counterexample to one of our results; i.e., $|G/Z(G)| = \kappa$, but $[G : N_G(A)] < \kappa$ for all (abelian) $A \leqslant G$. Assume $G \leqslant P \times T$ and $\pi(G) = P$, $\rho(G) = T$, where P is a periodic FC-group and T is torsion-free abelian, and π and ρ are the projections. Clearly $|P/Z(P)| = \kappa$, and also $[P : N_P(A)] < \kappa$ for all (abelian) $A \leqslant P$. This gives us a periodic counterexample.

1. SET-THEORETIC PRELIMINARIES

Set-Theoretic Notation. If X is a set, $[X]^\kappa$ denotes the set of subsets of X of size κ; $[X]^{<\kappa}$ is the set of subsets of X of size $< \kappa$; $[X]^{\leqslant \kappa}$, etc. are defined similarly. If $X \in [\kappa]^n$ for some $n \in \omega$, then $X(i)$ $(i < n)$ denotes the ith element of X under the inherited ordering. Further set-theoretic notation can be found in [9] or [7].

Delta-Systems and Almost Disjoint Sets. A family $\mathscr{A}$ of sets is called a *delta-system* (Δ-*system*) if there is an R (called the *root* of $\mathscr{A}$) such that

$$\forall A, B \in \mathscr{A} \qquad (A \neq B \Rightarrow A \cap B = R).$$

The *delta-system lemma* [9, Chap. II, Theorem 1.6] asserts that given a collection $\mathscr{A}$ of sets size $< \kappa$ with $|\mathscr{A}| \geq \theta$, where $\theta > \kappa$ is regular and satisfies $\forall \alpha < \theta (|\alpha^{<\kappa}| < \theta)$, there is a $\mathscr{B} \subseteq \mathscr{A}$ of size θ which forms a Δ-system. We shall use it most often in case $\kappa = \omega$.

If κ is a cardinal, a family of sets $\mathscr{A} \subseteq \mathscr{P}(\kappa)$ is called *almost disjoint* (a.d.) iff

$$\forall A \in \mathscr{A} \; (|A| = \kappa) \text{ and } \forall A, B \in \mathscr{A} \qquad (A \neq B \Rightarrow |A \cap B| < \kappa).$$

Trees. A *tree* is a partial order $\langle T, \leq \rangle$ such that for each $x \in T$, $\{y \in T; y < x\}$ is wellordered by $<$. Let T be a tree. For $x \in T$, the *height* of x in T (ht(x, T)) is the order type of $\{y \in T; y < x\}$. For each ordinal α, the αth *level* of T is Lev$_\alpha(T) = \{x \in T; \text{ht}(x, T) = \alpha\}$. The *height* of T (ht(T)) is the least ordinal α such that Lev$_\alpha(T) = \varnothing$.

A *branch* of T is a maximal totally ordered subset of T.

A *weak Kurepa tree* is a tree T of height ω_1 with at least ω_2 uncountable branches such that $\forall \alpha < \omega_1$ ($|\text{Lev}_\alpha(T)| \leq \omega_1$). Clearly, if CH holds, the complete binary tree of height ω_1 is a weak Kurepa tree. A *Kurepa tree* is a weak Kurepa tree T satisfying $\forall \alpha < \omega_1$ ($|\text{Lev}_\alpha(T)| \leq \omega$). A *Kurepa family* is an $\mathscr{F} \subseteq \mathscr{P}(\omega_1)$ such that $|\mathscr{F}| \geq \omega_2$ and $\forall \alpha < \omega_1$ ($|\{A \cap \alpha; A \in \mathscr{F}\}| \leq \omega$). It is easy to see [9, Chap. II, Theorem 5.18] that there is a Kurepa family iff there is a Kurepa tree.

Partial Orders and Forcing. Forcing was created by Cohen in the early sixties to solve Cantor's famous continuum problem; i.e., to show that for any cardinal κ of cofinality $> \omega$ it is consistent that $2^\omega = \kappa$—assuming the consistency of ZFC. Since then many other independence problems have been solved by the same method. As forcing will occupy a central position in our work, we briefly define its main notions. For a (very nicely written) introduction to this subject, we refer the reader to [9].

Let $\langle \mathbb{P}, \leq \rangle$ be a partial order (p.o. for short; sometimes, $\mathbb{P}$ will be referred to as *forcing notion*). The elements of $\mathbb{P}$ are called *conditions*. If $p, q \in \mathbb{P}$ and $p \leq q$, then p is *stronger than* q (or p is said to *extend* q). p and q are *compatible* iff $\exists r \in \mathbb{P}$ ($r \leq p \wedge r \leq q$); otherwise they are *incompatible* ($p \perp q$). A set $D \subseteq \mathbb{P}$ is called *dense* iff $\forall p \in \mathbb{P} \; \exists q \leq p$ ($q \in D$); D is *open dense* iff it is dense and $\forall p \in \mathbb{P} \; \forall q \in D$ ($p \leq q \Rightarrow p \in D$). $\mathscr{G} \subseteq \mathbb{P}$ is called a *filter* iff $\forall p, q \in \mathscr{G} \; \exists r \in \mathscr{G}$ ($r \leq p \wedge r \leq q$) and $\forall p \in \mathscr{G} \; \forall q \in \mathbb{P}$ ($p \leq q \Rightarrow q \in \mathscr{G}$). Now suppose $\mathscr{M}$ is a countable transitive model for ZFC (called the *ground model*), and $\mathbb{P} \in \mathscr{M}$. A filter $\mathscr{G} \subseteq \mathbb{P}$ is called $\mathbb{P}$-*generic* over $\mathscr{M}$ iff for all dense $D \in \mathscr{M}$, $\mathscr{G} \cap D \neq \varnothing$. The countability of $\mathscr{M}$ implies that there exist

always $\mathbb{P}$-generic $\mathscr{G}$; also, if $\mathbb{P}$ is *non-trivial* in the sense that $\forall p \in \mathbb{P}$ $\exists q, r \in \mathbb{P}$ ($q \leqslant p, r \leqslant p \wedge q \perp r$), then a $\mathbb{P}$-generic $\mathscr{G}$ cannot lie in $\mathscr{M}$, and the *generic extension* $\mathscr{M}[\mathscr{G}]$ (the smallest countable transitive model of ZFC containing $\mathscr{M}$ and $\mathscr{G}$) will be strictly larger than $\mathscr{M}$. The properties of $\mathscr{M}[\mathscr{G}]$ can be described inside $\mathscr{M}$ using the *forcing relation* ($\Vdash$) as follows. For any object in $\mathscr{M}[\mathscr{G}]$ there is a $\mathbb{P}$-*name* in $\mathscr{M}$; we shall use symbols like $\tilde{A}$, $\dot{A}$, ... to denote such names. A sentence of the *forcing language* is a ZFC-formula ψ with all free variables replaced by names. For such ψ and $p \in \mathbb{P}$ we write $p \Vdash_{\mathbb{P}} \psi$ (p *forces* ψ) iff for $\mathscr{G}$ which are $\mathbb{P}$-generic over $\mathscr{M}$, if $p \in \mathscr{G}$, then ψ is true in $\mathscr{M}[\mathscr{G}]$. The relation $\Vdash$ is definable in the ground model $\mathscr{M}$. Furthermore, if $\mathscr{G}$ is $\mathbb{P}$-generic over $\mathscr{M}$ and ψ is true in $\mathscr{M}[\mathscr{G}]$, then for some $p \in \mathscr{G}$, $p \Vdash_{\mathbb{P}} \psi$.

An *antichain* in a p.o. $\mathbb{P}$ is a pairwise incompatible set. $\mathbb{P}$ is said to satisfy the κ-cc (κ-*chain condition*, κ an uncountable cardinal) iff every antichain $A \subseteq \mathbb{P}$ has size $< \kappa$. ccc (countable chain condition) is the same as ω_1-cc. $\mathbb{P}$ is κ-closed iff whenever $\lambda < \kappa$ and $\{p_\xi ; \xi < \lambda\}$ is a decreasing sequence of elements in $\mathbb{P}$ (i.e., $\xi < \eta \Rightarrow p_\xi \geqslant p_\eta$), then $\exists q \in \mathbb{P}$ $\forall \xi < \lambda$ ($q \leqslant p_\xi$). A p.o. $\mathbb{P}$ *preserves cardinals* $\geqslant \kappa$ ($\leqslant \kappa$) iff whenever $\mathscr{G}$ is $\mathbb{P}$-generic over $\mathscr{M}$, and $\lambda \geqslant \kappa$ ($\lambda \leqslant \kappa$, respectively) is a cardinal in the sense of $\mathscr{M}$, it is also a cardinal of $\mathscr{M}[\mathscr{G}]$. Cardinals which are not preserved are *collapsed*. If $\mathbb{P}$ has the κ-cc, then it preserves cardinals $\geqslant \kappa$, if it is κ-closed, it preserves cardinals $\leqslant \kappa$.

A map $e: \mathbb{P} \to \mathbb{Q}$ (where $\mathbb{P}$ and $\mathbb{Q}$ are p.o.) is a *dense embedding* iff $\forall p, p' \in \mathbb{P}$ ($p' \leqslant p \Rightarrow e(p') \leqslant e(p)$), $\forall p, p' \in \mathbb{P}$ ($p \perp p' \Rightarrow e(p) \perp e(p')$), and $e(\mathbb{P})$ is dense in $\mathbb{Q}$. If $e: \mathbb{P} \to \mathbb{Q}$ is dense, $\mathbb{P}$ and $\mathbb{Q}$ are equivalent in the sense that they determine the same generic extensions. Any p.o. can be embedded densely in a (unique) complete Boolean algebra $\mathbb{B}(\mathbb{P})$ (the Boolean algebra *associated with* $\mathbb{P}$).

Sometimes we want to repeat the generic extension process. This leads to the technique of *iterated forcing* (see [1] or [8, Chap. 2] for details). We are mainly concerned with *two-step iterations* which we shall denote by $\mathbb{P} * \check{\mathbb{Q}}$.

We set $Fn(\kappa, \lambda, \mu) := \{ p; p$ is a function, $|p| < \mu$, $\mathrm{dom}(p) \subseteq \kappa$, $\mathrm{ran}(p) \subseteq \lambda \}$; $Fn(\kappa, \lambda, \mu)$ is ordered by $p \leqslant q$ iff $p \supseteq q$. $Fn(\kappa, 2, \lambda)$ is called the ordering for adding κ *Cohen subsets* of λ; for $\lambda = \omega$, the Cohen subsets are referred to as *Cohen reals*. Assume that $2^{<\lambda} = \lambda$, λ regular; then $Fn(\kappa, 2, \lambda)$ is λ-closed, has the λ^+-cc, and so preserves cardinals. Furthermore, if $\kappa^\lambda = \kappa$ (in the ground model), then $2^\lambda = \kappa$ in the generic extension. Cohen extensions can be split and thought of as a two-step iteration (cf. [9, Chap. VIII, Theorem 2.1] for the case $\lambda = \omega$).

For simplicity, we think of forcing as taking place over the whole universe $\mathscr{V}$ instead of over a countable model $\mathscr{M}$ (though this is not correct from the formal point of view—see [9] for a discussion of this).

Finally we come to *internal forcing axioms*. Those are combinatorial

principles proved consistent via iterated forcing; their statement captures much of this iteration. The easiest is *Martin's Axiom* MA.

(MA) *For all ccc p.o.* $\mathbb{P}$ *and any family* $\mathscr{D}$ *of* $< 2^\omega$ *dense subsets of* $\mathbb{P}$, *there is a filter* $\mathscr{G}$ *in* $\mathbb{P}$ *such that* $\forall D \in \mathscr{D}$ $(\mathscr{G} \cap D \neq \varnothing)$.

For the (rather involved) statement of the *proper forcing axiom* PFA we refer the reader to [2] or [8, Chap. 3].

Forcing and Inner Models. Sometimes the consistency of ZFC is not sufficient for proving the consistency of some combinatorial statement (C) via forcing, and one has to start with a stronger theory (in general some large cardinal assumption)—e.g., the existence of an inaccessible (ZFC + I). In those cases we also want to show that the large cardinal assumption was really necessary; e.g., that $Con(\text{ZFC} + \text{C})$ implies $Con(\text{ZFC} + \text{I})$. The way this is usually done is by showing that if C holds in the universe $\mathscr{V}$, then some cardinal is large (e.g., inaccessible) in a *sub-universe* $\mathscr{U}$ (a transitive class model $\mathscr{U} \subseteq \mathscr{V}$ satisfying ZFC); such sub-universes are called *inner models*. The most important is the *constructible universe* $\mathscr{L}$, invented by Gödel.

To show the consistency of the non-existence of weak Kurepa trees, an inaccessible is collapsed to ω_2 (more correctly, the cardinals between ω_1 and the inaccessible are collapsed)—see [11] or [1]. On the other hand, the non-existence of Kurepa trees in $\mathscr{V}$ implies that ω_2 is an inaccessible cardinal in the sense of $\mathscr{L}$ (see [9, Chap. VII, Exercise (B9)]). The consistency of the existence of Kurepa trees can be proved by forcing or by showing that they exist in $\mathscr{L}$.

2. THE INVARIANT $g(G)$

2.1. For any group G let $g(G)$—the *generating number*—denote the minimum number of abelian subgroups of G needed to generate G. The following result should be thought of as generalizing the fact that any countable extraspecial p-group is a central sum of extraspecial p-groups of order p^3 [16, Corollary 3.10]—and so can be generated by two abelian subgroups.

2.2. THEOREM. *For any countable finite-by-abelian group G, $g(G) < \omega$.*

Proof. We make induction on $|G'|$. The case $|G'| = 1$ is trivial. So suppose $|G'| > 1$. We set $H := C_G(G')$. As G is an FC-group, $|G : H| < \omega$; so it suffices to show that H is generated by finitely many abelian subgroups. H' is a finite abelian group; i.e., it is a direct sum of finite cyclic groups of prime power order: $H' = \langle a_0 \rangle \oplus \cdots \oplus \langle a_n \rangle$. There is a prime p and a

natural number l such that $o(a_n) = p^l$. Let $A := \langle a_0, ..., a_{n-1}, a_n^p \rangle$. We shall define (recursively) two subgroups $H_0, H_1 \leqslant H$ such that $\langle H_0, H_1 \rangle = H$ and $H_k' \leqslant A < H'$ for $k \in 2$. Then the result follows by induction.

Suppose $H = \{b_m; m \in \omega\}$. Let m_0 be minimal with the property that there is an m such that $[b_{m_0}, b_m] \notin A$. Put $b_0, ..., b_{m_0}$ into H_0. Let $c_0 := b_{m_0}$ and $c_1 := b_{m_1}$, where m_1 is minimal such that $[c_0, b_{m_1}] \notin A$, and put c_1 into H_1. For $m > m_0$, $m \neq m_1$ let d_m^0 be a product of b_m and powers of c_0 and c_1 such that $[d_m^0, b_k] \in A$ for any $k \in m_0 + 1 \cup \{m_1\}$. We continue this construction recursively. Suppose we are at step i; i.e., $m_{2i}, m_{2i+1}, c_{2i}, c_{2i+1}$ and d_m^i $(m > m_{2i}, m \neq m_{2j+1}$ for $j \leqslant i)$ have been defined. Then let m_{2i+2} be minimal with the property that there is an m such that $[d_{m_{2i+2}}^i, d_m^i] \notin A$. Put $d_{m_{2i+1}}^i, ..., d_{m_{2i+2}}^i$ into H_0. Let $c_{2i+2} := d_{m_{2i+2}}^i$ and $c_{2i+3} := d_{m_{2i+3}}^i$, where m_{2i+3} is minimal such that $[c_{2i+2}, d_{m_{2i+3}}^i] \notin A$, and put c_{2i+3} into H_1. For $m > m_{2i+2}$, $m \neq m_{2j+1}$ for $j \leqslant i+1$, let d_m^{i+1} be a product of d_m^i and powers of c_{2i+2}, c_{2i+3} such that $[d_m^{i+1}, d_k^i] \in A$ for any $k \in ((m_{2i}, m_{2i+2}] \cup \{m_{2i+3}\}) - \{m_{2j+1}; j \leqslant i\}$.

In the end $H_1 := \langle c_{2j+1}; j \in \omega \rangle$; and H_0 is the group generated by the elements which have been put into H_0. It is easy to see that H_0 and H_1 satisfy the requirements. ∎

(*Remark.* The proof of this result is in two steps. The first shows that finite-by-abelian groups are nilpotent of class 2-by-finite, and does not require countability.)

This property of countable finite-by-abelian groups should be seen as corresponding to an old result of Baer's, that a group G is centre-by-finite iff $\chi(G) < \omega$ [16, Theorem 7.4], where $\chi(G)$ denotes the minimum number of abelian subgroups needed to cover G. Nevertheless there are two drawbacks. First, it is easy to construct a (countable) FC-group G with $|G'| = \omega$ but $g(G) = 2$. Second, our result does not generalize to higher cardinalities. The important example of Shelah and Steprāns [13] shows that there are finite-by-abelian (even extraspecial) groups of size ω_1 all of whose abelian subgroups are countable. But even for nicer classes of groups there is nothing corresponding to the theorem as is shown by the following.

2.3. EXAMPLE. Let E be the group generated by elements $a, a_\alpha, \alpha < \omega_1$, satisfying the relations $a^p = a_\alpha^p = [a, a_\alpha] = 1$ and $[a_\alpha, a_\beta] = a$ for $\alpha < \beta$. E is easily seen to be an extraspecial $\mathcal{Z}$-group of exponent p. We will show that $g(E) = \omega_1$.

For suppose that $g(E) \leqslant \omega$. Then there are abelian subgroups A_n $(n \in \omega)$ such that E is generated by the A_n. Choose $\Gamma \in [\omega_1]^{\omega_1}$ and $n \in \omega$ such that for all $\alpha \in \Gamma$ $a_\alpha \in \langle A_k; k < n \rangle$. For each such α and any $k < n$ we can find $b_{k,\alpha} \in A_k$ such that $a_\alpha = \prod_{k=0}^{n-1} b_{k,\alpha}$ (at least modulo a factor which is a power of a and which is irrelevant for our calculation). Now let $B_{k,\alpha}$

consist of the β so that a_β appears as a factor in $b_{k,\alpha}$. We may assume that the $B_{k,\alpha}$ form a delta-system with root R_k for any fixed k. Let $C_{k,\alpha} := B_{k,\alpha} - R_k$. We can suppose that there is a j_k such that $|C_{k,\alpha}| = j_k$, that for all $\alpha \in \Gamma$ sup $R_k < \min C_{k,\alpha}$, that for $\alpha < \beta$ (both in Γ) sup $C_{k,\alpha} < \min C_{k,\beta}$, and that the multiplicities with which the a_β appear in the $b_{k,\alpha}$ depend only on $\gamma \in R_k$ or $i \in j_k$ (and not on the specific α). Then

$$b_{k,\alpha} = \prod_{\beta \in R_k} a_\beta^{l_\beta} \prod_{i \in j_k} a_{C_{k,\alpha}(i)}^{m_i},$$

where l_β, $m_i \in p$. An easy commutator calculation shows that the commutativity of A_k implies that $\sum_{i \in j_k} m_i \equiv O \pmod{p}$. On the other hand,

$$a_\alpha = \prod_{k=0}^{n-1} b_{k,\alpha} = \prod_{k=0}^{n-1} \left(\prod_{\beta \in R_k} a_\beta^{l_\beta} \prod_{i \in j_k} a_{C_{k,\alpha}(i)}^{m_i} \right).$$

This equation cannot hold for any α with $(\{\alpha\} \cup \bigcup_{k<n} C_{k,\alpha}) \cap (\bigcup_{k<n} R_k) = \varnothing$, thus giving a contradiction.

Note. It is easy to see that E can be embedded in an extraspecial p-group F with $g(F) = 2$. Namely, let F be the group generated by a, a_α, b_α ($\alpha < \omega_1$) satisfying—in addition to the above relations—$b_\alpha^p = [a, b_\alpha] = [b_\alpha, b_\beta] = 1$ and

$$[a_\alpha, b_\beta] = \begin{cases} a^{-1} & \text{if } \alpha < \beta, \\ 1 & \text{otherwise.} \end{cases}$$

Then $F = \langle A_0, A_1 \rangle$, where $A_0 = \langle a_\alpha b_\alpha; \alpha < \omega_1 \rangle$ and $A_1 = \langle b_\alpha; \alpha < \omega_1 \rangle$. In fact, F is a semidirect extension of E.

So the inequality $g(G) \leqslant \kappa$ is *not* necessarily preserved when taking subgroups. It is preserved, however, when taking factor groups. This suggests that instead of dealing with g, one should consider the *hereditary generating number* $\mathrm{hg}(G) := \sup\{g(U); U \leqslant G\}$.

(A much easier example for this is the direct sum E_ω of countably many extraspecial p-groups of size p^3 (of exponent p for $p > 2$). $g(E_\omega) = 2$, but E_ω contains the tree group C of Section 4.2 which has $g(C) = \omega$.)

2.4. Let $\mathscr{QSDF}$ be the $\mathscr{QSD}$-closure of the class of finite groups; i.e., $G \in \mathscr{QSDF}$ iff it is a factor group of a subgroup of a direct sum of finite groups. $\mathscr{QSDF}$ is a subclass of $\mathscr{X}$ [16, Lemma 3.7]. Tomkinson asked [16, Question 3F] whether $\mathscr{X} \neq \mathscr{QSDF}$. This was shown to be true rather indirectly by Tomkinson and L. A. Kurdačenko; namely Kurdačenko [10, Theorem 4] proved that any extraspecial $\mathscr{QSDF}$-group can be embedded in a direct sum of groups of order p^3 with amalgamated centre, and Tomkinson gave a (rather complicated) example [16, Example 3.16] for

an extraspecial $\mathscr{Z}$-group which cannot be embedded in a direct sum of groups of order p^3 with amalgamated centre.

We shall show that the group of E of Example 2.3 does not lie in $\mathscr{Z}\mathscr{S}\mathscr{D}\mathscr{F}$, thus providing an easier example. To this end, for any group G, let $P(G)$ be the least cardinal κ such that any set of pairwise non-commuting elements of G has size less than κ. A canonical Δ-system argument shows that $G \in \mathscr{Z}\mathscr{S}\mathscr{D}\mathscr{F}$ implies $P(G) \leqslant \omega_1$ (this is a special instance of [3, Theorem 6]). On the other hand, the definition of E in Example 2.3 shows that $P(E) = \omega_2$. Hence $E \in \mathscr{Z} \setminus \mathscr{Z}\mathscr{S}\mathscr{D}\mathscr{F}$.

3. FC-AUTOMORPHISMS OF COUNTABLE PERIODIC ABELIAN GROUPS

3.1. Let G be an FC-group. An automorphism ϕ of G is called FC-*automorphism* iff $|\{g(g^{-1})\phi; g \in G\}| < \omega$; i.e., iff the semidirect extension of G by the group generated by ϕ is still an FC-group. For our discussion the following is important.

3.2. THEOREM. *Let A be a countable periodic abelian group. Suppose Φ is a group of FC-automorphisms of A with $cf(|\Phi|) > \omega$. Then there is a subgroup $B \leqslant A$ such that $|\{B\phi; \phi \in \Phi\}| = |\Phi|$.*

Proof (Tomkinson). First, for $\phi \in \Phi$, let $A_\phi := \langle a - a\phi; a \in A \rangle$. There are only countably many finite subgroups $C \leqslant A$. If $\Phi_C := \{\phi \in \Phi; A_\phi \leqslant C\}$, then $\Phi = \bigcup_C \Phi_C$. Since $cf(|\Phi|) > \omega$, there is a $C = C(A) \leqslant A$ such that $|\Phi_C| = |\Phi|$. We make induction on $|C|$.

Second, we can restrict our attention to p-groups (for some fixed prime p). The general result follows easily (as any periodic abelian group is the direct sum of its p-components which are characteristic subgroups).

Now let $m :=$ exponent of C; i.e., m is the smallest integer such that $p^m C = 0$. Then for all $a \in A$ of height $\geqslant m$ and all $\phi \in \Phi$, $a\phi = a$. (To see this let $a \in A$ be of height $\geqslant m$. Choose $\hat{a} \in A$ such that $p^m \hat{a} = a$. Let $\hat{b} := \hat{a}\phi - \hat{a} \in C$. Then $a\phi = (p^m \hat{a})\phi = p^m(\hat{a} + \hat{b}) = p^m \hat{a} = a$.) Especially it suffices to consider reduced p-groups.

As usual let A^1 denote the subgroup of all elements of infinite height in A. Prüfer's Theorem [5, Theorem 17.3] says that A/A^1 is a direct sum of cyclic p-groups. By the preceding paragraph, $\phi \lceil A^1 = \text{id}$ for any $\phi \in \Phi$. Suppose $A^1 \cap C < C$. Choose $B < A$ containing A^1 such that $B/A^1 \cap (C + A^1)/A^1 = 0$ and $[A : B] < \omega$ (this is possible because A/A^1 is a direct sum of finite groups). Then either B satisfies the requirements of the Theorem, or B has as many automorphisms as A. In the latter case we are done by induction because $C(B) < C(A) = C$.

This shows that we may assume $C \leqslant A^1$ (in particular, $A^1 \neq 0$). Now let $D < C$ such that $|C/D| = p$. Each $\phi \in \Phi$ leaves D fixed and so induces an

automorphism of A/D. Let $\Phi_{A/D}$ be the group of induced automorphisms. If $|\Phi_{A/D}| < |\Phi|$, then $|\Phi_D| = |\Phi|$, and we are done by induction.

So we may assume that $|\Phi_{A/D}| = |\Phi|$ and consider $\bar{A} = A/D$. There is an $E \leqslant A$ such that $E \cap C = D$ and $A/E \cong C_{p^\infty}$ (for this use [5, Theorem 32.3] to get $\bar{E}^* \leqslant \bar{A}$, a basic subgroup (this implies $\bar{E}^* \cap \bar{C} = \bar{0}$ and $\bar{A}/\bar{E}^*$ is divisible), and then [5, Theorem 23.1] to get $\bar{E} \geqslant \bar{E}^*$ so that $\bar{E} \cap \bar{C} = \bar{0}$ and $\bar{A}/\bar{E} \cong C_{p^\infty}$; let E be the subgroup of A corresponding to $\bar{E}$). For each $\bar{\phi} \in \Phi_{A/D}$, $\mathrm{id} \lceil p\bar{A} = \bar{\phi} \lceil p\bar{A}$. So $\bar{A} = \bar{E} + p\bar{A}$ implies that if $\bar{\phi} \neq \bar{\psi}$ $(\bar{\phi}, \bar{\psi} \in \Phi_{A/D})$ then $\bar{\phi} \lceil \bar{E} \neq \bar{\psi} \lceil \bar{E}$. Hence $\bar{E} \cap \bar{C} = \bar{0}$ gives us $\bar{E}\phi \neq \bar{E}\psi$. Therefore E has $|\Phi_{A/D}|$ images under Φ. This proves the theorem. ∎

3.3. *Proof of Theorems* C *and* C'. We have to show that for any finite-by-abelian group G of size ω_1,

(i) $G \in \mathscr{X}$ iff $G \in \mathscr{Y}$;

(ii) $|G/Z(G)| \leqslant \omega$ iff $[G : N_G(A)] \leqslant \omega$ for all abelian $A \leqslant G$.

For suppose not. Then there is a finite-by-abelian group G which is not $\mathscr{X}$ such that

in case (i): $G \in \mathscr{Y}$;

in case (ii): $[G : N_G(A)] \leqslant \omega$ for all abelian $A \leqslant G$.

(In case (ii), the fact that G is not in $\mathscr{X}$ follows from Tomkinson's result (II) mentioned in the Introduction.) Then G has a countable subgroup U with $[G : C_G(U)] = \omega_1$. Let $V := U^G := \langle g^{-1}Ug; g \in G \rangle$. As G is FC, $V \trianglelefteq G$ is countable. By Theorem 2.2, $g(V) < \omega$, so there are $n \in \omega$ and $A_i \leqslant V$ abelian such that $\langle A_i; i < n \rangle = V$. Clearly $C_G(V) = \bigcap_{i<n} C_G(A_i)$; thus there is an $i \in n$ such that $[G : C_G(A_i)] = \omega_1$. So either $[G : N_G(A_i)] = \omega_1$ in which case we are done, or $[N_G(A_i) : C_G(A_i)] = \omega_1$. In that case, we may assume $G = N_G(A_i)$, and $G/C_G(A_i)$ can be thought of as a group of FC-automorphisms of A_i, and we are in the situation of Theorem 3.2; i.e., we obtain subgroup $B \leqslant A_i$ such that $[G : N_G(B)] = \omega_1$, a contradiction. ∎

3.4. The argument in Section 3.3 shows that if one could prove the analogue of Theorem 3.2 under the weaker assumption that A is FC instead of abelian, this would solve Questions 1 and 1' in the Introduction. So we should ask

QUESTION 1″. *Suppose G is a countable periodic FC-group, and Φ is a group of FC-automorphisms of G with $\mathrm{cf}(|\Phi|) > \omega$. Is there a subgroup $U \leqslant G$ such that $|\{U\phi; \phi \in \Phi\}| = |\Phi|$?*

4. Maximal Abelian Subgroups of FC-Groups

4.1.   Maximal abelian subgroups (of FC-groups) are important for our discussion, especially those of countable periodic FC-groups in case $\kappa = \omega_1$. For Theorem A, we want to construct a countable FC-group having an uncountable set of automorphisms such that on each (maximal) abelian subgroup only countably many act differently (Sections 4.2 and 4.3). To prove Theorem B, we shall try to shoot a *new* abelian subgroup through an *old* set of automorphisms so that many of these automorphisms act differently on this group (Sections 4.4 and 4.5). These two procedures should be seen as being dual to each other (cf. especially Section 4.6). Therefore we pause for an instant to look at the lattice of abelian subgroups itself.

LEMMA.   *If G is a $\mathscr{L}_\kappa$-group with $|G/Z(G)| \geqslant \kappa$, then G has at least 2^κ maximal abelian subgroups.*

Proof.   We construct recursively a tree $\{A_\sigma; \sigma \in 2^{<\kappa}\}$ of subgroups of G with $Z(G) \leqslant A_\sigma$ and $|A_\sigma/Z(G)| < \kappa$ ($A_\sigma/Z(G)$ is finitely generated in case $\kappa = \omega$) as follows. Let $A_{\langle\rangle} := Z(G)$. If $\alpha \in \kappa$ is a limit ordinal and $\sigma \in 2^\alpha$, let $A_\sigma := \bigcup_{\beta \in \alpha} A_{\sigma\restriction\beta}$. So assume $\alpha = \beta + 1$ for some $\beta \in \kappa$ and $\sigma \in 2^\beta$. Suppose $C_G(A_\sigma)$ is abelian. Choose $B \leqslant G$ such that $\langle B, C_G(A_\sigma) \rangle = G$ and $|B| < \kappa$ (or B is finitely generated if $\kappa = \omega$). Then $[G : C_G(B)] < \kappa$. So $[G : C_G(B) \cap C_G(A_\sigma)] < \kappa$ which contradicts $|G/Z(G)| \geqslant \kappa$. So $C_G(A_\sigma)$ is non-abelian and there are $g, h \in C_G(A_\sigma)$ such that $[g, h] \neq 1$. Then set $A_{\sigma^\wedge\langle 0\rangle} := \langle A_\sigma, g \rangle$ and $A_{\sigma^\wedge\langle 1\rangle} := \langle A_\sigma, h \rangle$.

In the end, for each $f \in 2^\kappa$, extend $\bigcup_{\sigma \subseteq f} A_\sigma$ to a maximal abelian subgroup A_f. By construction, $g \neq f$ implies $A_g \neq A_f$.   ∎

In fact, the proof of the lemma shows that any abelian subgroup A with $|AZ(G)/Z(G)| < \kappa$ is contained in at least 2^κ distinct maximal abelian subgroups; and that it is contained in at least κ subgroups B_α, $\alpha < \kappa$, with $Z(G) \leqslant B_\alpha$ and $|B_\alpha/Z(G)| < \kappa$ and which are pairwise *incompatible* in the sense that $\langle B_\alpha, B_\beta \rangle$ is not abelian for $\alpha \neq \beta$—this fact will be used in the proof of Theorem 4.4. below!

As a consequence in case $\kappa = \omega$ we obtain

COROLLARY.   *An FC-group has either finitely many or at least 2^ω maximal abelian subgroups. It has finitely many iff it is centre-by-finite.*

4.2.   As mentioned earlier we are concerned with the following problem. Suppose G is an FC-group. Under which circumstances is there a set S of κ automorphisms of G such that for all abelian $A \leqslant G$, $|\{\phi \restriction A; \phi \in S\}| < \kappa$? An easy necessary condition is $g(G) \geqslant \omega$. We begin with the following.

EXAMPLE. For each $n \in \omega$ we introduce a finite group C_n as follows. Let A_n be an elementary abelian p-group of size p^n, and B_n an elementary abelian p-group of size $p^{\binom{n}{2}}$. We extend B_n by A_n with factor system τ_n as follows:

$$\tau_n(a_i, a_j) = \begin{cases} 0 & \text{if } i \geqslant j, \\ b_{h(i,j)} & \text{otherwise,} \end{cases}$$

where $h: [n]^2 \to \binom{n}{2}$ is a bijection and the a_i (b_j, respectively) are generators of A_n (B_n). Let C_n be the extension (i.e., $C_n = E(\tau_n)$). Note that C_n is the free object on n generators in the variety of two-step nilpotent groups of exponent p ($p > 2$); and that it is a special p-group with $C'_n = \Phi(C_n) = Z(C_n) = B_n$. Let C be the direct sum of the C_n. If g is any function from ω to $\bigcup A_n$ with $g(n) \in A_n$, then g defines a maximal abelian subgroup $M_g := \langle B_n, g(n); n \in \omega \rangle$. On the other hand each maximal abelian subgroup of C is of this form. So the maximal abelian subgroups can be thought of as branches through a tree. For later reference we shall therefore call C the *tree group*.

Now assume CH. Let $\{M_\alpha; \alpha < \omega_1\}$ be an enumeration of the M_g. We introduce (recursively) a set of automorphisms $\{\phi_\alpha; \alpha < \omega_1\}$ of $G := C \oplus D$, where $D = \langle d \rangle$ is a group of order p as follows: fix α; let $\{N_n; n \in \omega\}$ be an enumeration of $\{M_\beta; \beta < \alpha\}$; and let $\{\psi_n; n \in \omega\}$ be an enumeration of $\{\phi_\beta; \beta < \alpha\}$. We define ϕ_α and an auxiliary function $f: \omega \to \omega$ recursively. Suppose $f \restriction (n+1)$ and $\phi_\alpha \restriction (\bigoplus_{i < f(n)} C_i \oplus D)$ have been defined. We choose $f(n+1)$ so large that we can extend ϕ_α to $(\bigoplus_{i < f(n+1)} C_i \oplus D)$ so that

(i) $\forall c \in \bigoplus_{i < f(n+1)} C_i \oplus D \; \exists k \in p \; (c\phi_\alpha = c + kd)$;

(ii) $\phi_\alpha \restriction (\bigoplus_{i < f(n+1)} B_i \oplus D) = \text{id}$;

(iii) $\forall k < n(\phi_\alpha \restriction (\bigoplus_{i < f(n+1)} C_i \oplus D) \neq \psi_k \restriction (\bigoplus_{i < f(n+1)} C_i \oplus D))$;

(iv) $\phi_\alpha \restriction ((\bigcup_{k < n} N_k) \cap (\bigoplus_{f(n) \leqslant i < f(n+1)} C_i \oplus D)) = \text{id}$.

This is clearly possible. It is easy to see that $\{\phi_\alpha; \alpha < \omega_1\}$ is a set of (distinct) automorphisms of G such that for all maximal abelian $A \leqslant G$, $|\{\phi_\alpha \restriction A; \alpha < \omega_1\}| \leqslant \omega$.

4.3. *Proof of Theorem* A.   Form the semidirect extension E of the group G defined in Section 4.2 by the group H generated by the automorphisms $\{\phi_\alpha; \alpha < \omega_1\}$ (also defined in Section 4.2). Then E is easily seen to be an FC-group with the required properties (in fact, for all abelian $A \leqslant E$, $[E : C_E(A)] \leqslant \omega$).   ∎

4.4.   We now show that CH was necessary in the example above.

THEOREM. *Let $\lambda > \kappa^+$ be cardinals, κ regular. Denote by $Fn(\lambda, 2, \kappa)$ the p.o. for adding λ Cohen subsets of κ. Suppose $\mathscr{V} \models 2^{<\kappa} = \kappa$. Then in $\mathscr{V}[\mathscr{G}]$,*

where $\mathscr{G}$ is $Fn(\lambda, 2, \kappa)$-generic over $\mathscr{V}$, the following holds: for any $\mathscr{L}_\kappa$-group H of size κ and any set of automorphisms Φ of H of size $> \kappa$ there is an abelian subgroup $A \leqslant H$ such that $|\{\phi \lceil A; \phi \in \Phi\}| = |\Phi|$.

Proof. Let H be any $\mathscr{L}_\kappa$-group of size κ and with $|H/Z(H)| = \kappa$. Note that if $|H/Z(H)| < \kappa$, then $Z(H)$ has the required property (as $2^{<\kappa} = \kappa$). We define the ordering $\mathbb{P}_H$ for shooting new abelian subgroups through H as follows: $\mathbb{P}_H := \{A \leqslant H; A$ is abelian and A is generated by $< \kappa$ elements$\}$. $\mathbb{P}_H$ is ordered by reverse inclusion; i.e., $A \leqslant_{\mathbb{P}_H} B$ iff $B \subseteq A$. $\mathbb{P}_H$ is κ-closed, and non-trivial by the discussion in Section 4.1. Since $2^{<\kappa} = \kappa$, $\mathbb{P}_H$ is trivially $\kappa^+ - cc$. So forcing with $\mathbb{P}_H$ preserves cardinals and cofinalities.

We first claim that if Φ is any set of automorphisms of H of size $> \kappa$ in $\mathscr{V}$, then in $\mathscr{V}[\mathscr{G}]$, where $\mathscr{G}$ is $\mathbb{P}_H$-generic over $\mathscr{V}$, there is an abelian subgroup $A \leqslant H$ such that $|\{\phi \lceil A; \phi \in \Phi\}| = |\Phi|$.

For A we take the generic object, i.e., $A = \bigcup \{B \leqslant H; B \in \mathscr{G}\}$. Suppose the claim is false. Let μ be regular with $|\Phi| \geqslant \mu \geqslant \kappa^+$. Then there is a $\Psi \subseteq \Phi$ in $\mathscr{V}[\mathscr{G}]$ of size μ with $\forall \phi, \psi \in \Psi$ ($\phi \lceil A = \psi \lceil A$). So this statement is forced by a condition $B \in \mathbb{P}_H$; i.e., there is a $\mathbb{P}_H$-name $\check{\Psi}$ such that

$$B \Vdash \check{\Psi} \subseteq \Phi \wedge |\check{\Psi}| = \mu \wedge \forall \phi, \psi \in \check{\Psi} (\phi \lceil \check{A} = \psi \lceil \check{A}).$$

As $|\mathbb{P}_H| = 2^{<\kappa} = \kappa$, there is (in $\mathscr{V}$) a $X \in [\Phi]^\mu$ and a $C \leqslant_{\mathbb{P}_H} B$ such that

$$C \Vdash X \subseteq \check{\Psi}.$$

Now, C is an abelian subgroup of the $\mathscr{L}_\kappa$-group H of size less than κ. So $[H : C_H(C)] < \kappa$. As $|X| > \kappa$ and $2^{<\kappa} = \kappa$, $|\{\chi \lceil C_H(C); \chi \in X\}| = \mu$ so that we can find $\psi, \chi \in X$ and $c \in C_H(C) \backslash C$ such that $\psi(c) \neq \chi(c)$. But then the condition $\langle C, c \rangle$ forces contradictory statements. This proves the claim.

Next we remark that for any $\mathscr{L}_\kappa$-group H of size κ, $\mathbb{P}_H$ is equivalent (from the forcing theoretic point of view) to the Cohen forcing $Fn(\kappa, 2, \kappa)$ for adding a single new subset of κ. For $\kappa = \omega$ this follows from the fact that any two non-trivial countable notions of forcing are equivalent [9, p. 242, Chap. VII, Exercise (C4)]. The proof for this generalizes as follows. Let $\{A_\alpha; \alpha < \kappa\}$ enumerate $\mathbb{P}_H$. We construct recursively a dense embedding e from $\{p \in Fn(\kappa, \kappa, \kappa); \text{dom}(p) \in \kappa\}$ into $\mathbb{P}_H$. Let $\alpha < \kappa$ and suppose $e \lceil \{p \in Fn(\kappa, \kappa, \kappa); \text{dom}(p) \in \alpha\}$ has been defined. If α is limit let, for any p with $\text{dom}(p) = \alpha$, $e(p) = \bigcup_{\beta < \alpha} e(p \lceil \beta)$. So suppose $\alpha = \beta + 1$ for some $\beta \in \kappa$. There is by induction (at least) one $p_0 \in Fn(\kappa, \kappa, \kappa)$ with $\text{dom}(p_0) = \beta$ such that A_β is compatible with $e(p_0)$. For each $p \in Fn(\kappa, \kappa, \kappa)$ with $\text{dom}(p) = \beta$ choose a maximal antichain M_p of size κ of conditions below $e(p)$ in $\mathbb{P}_H$ (the existence of such an antichain is guaranteed by the discussion in Section 4.1), such that $\langle A_\beta, e(p_0) \rangle$ is a subgroup of some group in M_{p_0}. Let $e \lceil \{q \in Fn(\kappa, \kappa, \kappa); \text{dom}(q) = \alpha \text{ and } q \lceil \beta = p\}$ be a bijection onto M_p. It is easy to check that e works. The same argument shows

that $\{p \in Fn(\kappa, \kappa, \kappa); \mathrm{dom}(p) \in \kappa\}$ can be densely embedded into $Fn(\kappa, 2, \kappa)$. This gives equivalence (cf. Section 1).

Finally we prove the theorem. Let H and Φ be as in the statement of the theorem. First suppose $|\Phi| < \lambda$. Then Φ is contained in an initial segment of the extension, and any subset which is Cohen over this initial segment produces the required A by the above arguments. So suppose $|\Phi| \geqslant \lambda$. In that case we think of the whole extension as a two-step extension which first adds λ and then μ Cohen subsets of κ, where $\lambda \geqslant \mu \geqslant \kappa^+$ is regular with $cf(|\Phi|) \neq \mu$ and $|\Phi| > \mu$. Then there is a subset $\Psi \in [\Phi]^{|\Phi|}$ which is contained in an initial segment of the second extension, and our argument applies again. ▮

Remark. Note that in the theorem, the assumption $|H| = \kappa$ may be replaced by $|H/Z(H)| = \kappa$. The p.o. $\mathbb{P}_H$ in the proof contains in that case the abelian subgroups A with $Z(H) \leqslant A$ and $|A/Z(H)| < \kappa$.

4.5. *Proof of Theorems* B *and* B'. Let $\mathscr{V} \models$ ZFC. We show that in the model obtained by adding ω_2 Cohen reals to $\mathscr{V}$,

    (i) any $\mathscr{Y}$-group of size ω_1 is a $\mathscr{Z}$-group;

    (ii) there is no FC-group G with $|G/Z(G)| = \omega_1$ but $[G : N_G(A)] \leqslant \omega$ for all abelian subgroups $A \leqslant G$.

For suppose not. Then there is an FC-group G of size ω_1 which is not in $\mathscr{Z}$ such that

    in case (i): $G \in \mathscr{Y}$;

    in case (ii): $[G : N_G(A)] \leqslant \omega$ for all abelian $A \leqslant G$.

We argue as in the proof of Theorems C and C' (Section 3.3) using Theorem 4.4 instead of Theorem 2.2: let $U \leqslant G$ be countable with $[G : C_G(U)] = \omega_1$; let $V := U^G$. Apply Section 4.4 (with $\kappa = \omega$, $\lambda = \omega_2$, $H = V$, and $\Phi = G/C_G(V)$) to obtain an abelian $A \leqslant V$ with $[G : C_G(A)] = \omega_1$. Now finish as in Section 3.3 with Theorem 3.2. ▮

4.6. The proof of Theorem B shows that its statement follows from $MA + 2^\omega > \omega_1$. Still this is not the right way to look at the problem from the combinatorial point of view. Namely, when iterating Cohen forcing one merely goes through one *particular* ccc p.o., whereas MA asserts that generic objects exist for *all* ccc p.o.—not only for those which shoot new abelian subgroups through an FC-group G but also for those which shoot a new automorphism through G (see below). The consequences of this will become clear in Section 5 (see the difference between Theorem E and Theorem 5.8).

Also MA is a weakening of CH, and many statements which are provable in ZFC + CH are still provable in ZFC + MA if we replace ω by $< 2^\omega$. We shall see now that this is the case for our problem as well.

PROPOSITION. *Assume MA. Then there is an FC-group G with $|G/Z(G)| = 2^\omega$ but $[G : N_G(A)] < 2^\omega$ for all abelian subgroups $A \leqslant G$.*

Proof. Let C be again the tree group of Section 4.2. We define the partial order $\mathbb{Q}_C$ for shooting new automorphisms through $C \oplus D$ (where $D = \langle d \rangle$ is again a group of order p). $\mathbb{Q}_C := \{(\phi, \mathcal{A}); \phi$ is a finite partial automorphism of $C \oplus D$ with (i) $\exists n \in \omega$ with $\mathrm{dom}(\phi) = \bigoplus_{i \in n} C_i \oplus D$, (ii) $\forall c \in \mathrm{dom}(\phi)\ \exists k \in p\ (c\phi = c + kd)$, and (iii) $\phi \restriction (\bigoplus_{i \in n} B_i \oplus D) = \mathrm{id}$; and $\mathcal{A}$ is a finite collection of maximal abelian subgroups of $C\}$; $(\phi, \mathcal{A}) \leqslant_{\mathbb{Q}_C} (\psi, \mathcal{B})$ iff $\phi \supseteq \psi$ and $\mathcal{A} \supseteq \mathcal{B}$ and $\forall c \in (\mathrm{dom}(\phi) - \mathrm{dom}(\psi)) \cap (\bigcup \mathcal{B})\ (c\phi = c)$. $\mathbb{Q}_C$ is ccc and generically shoots a *new* automorphism through $C \oplus D$ which equals the identity on all *old* abelian subgroups from some point on.

To prove the proposition let $\{A_\alpha; \alpha < 2^\omega\}$ enumerate the maximal abelian subgroups of C. We construct recursively a set of automorphisms $\{\phi_\alpha; \alpha < 2^\omega\}$. Let $\mathscr{D}_\alpha := \{D_\beta; \beta < \alpha\}$, where $D_\beta := \{(\phi, \mathcal{A}) \in \mathbb{Q}_C; A_\beta \in \mathcal{A}$ and $c\phi \neq c\phi_\beta$ for some $c \in \mathrm{dom}(\phi)\}$. Each D_β is dense in $\mathbb{Q}_C$; hence, by MA, there is a $\mathscr{D}_\alpha$-generic filter G_α. Let $\phi_\alpha := \bigcup \{\phi; \exists \mathcal{A}(\phi, \mathcal{A}) \in G_\alpha\}$. Then for all maximal abelian $A \leqslant C \oplus D$, $|\{\phi_\alpha \restriction A; \alpha < 2^\omega\}| < 2^\omega$. Now let $G := (C \oplus D) \rtimes \langle \phi_\alpha; \alpha \in 2^\omega \rangle$. ∎

Certainly one should ask whether MA is necessary at all in the above result; or whether it can be proved in ZFC alone. It turns out that the answer (to the second question) is no, at least if we assume the existence of an inaccessible cardinal—see Section 5 (Theorem E).

4.7. It is quite usual that combinatorial statements are not decided by ¬ CH. Again this is true in our situation.

PROPOSITION. *It is consistent that $2^\omega > \omega_1$ and there is an FC-group G with $|G/Z(G)| = \omega_1$, but $[G : N_G(A)] \leqslant \omega$ for all abelian subgroups $A \leqslant G$.*

Sketch of the Proof. The proof uses the tree group of Section 4.2 as main ingredient. Start with $\mathscr{V} \models 2^\omega > \omega_1$ and make a finite support iteration of length ω_1 of the partial order $\mathbb{Q}_C$ described in Section 4.6. ∎

5. EXTRASPECIAL P-GROUPS AND KUREPA TREES

5.1. The goal of this section is a detailed investigation of extraspecial p-groups, especially of those of size ω_2. The philosophy behind this is that many *bad* things that can happen to (periodic) FC-groups already happen in case of extraspecial p-groups; or even that *bad* periodic FC-groups involve *bad* extraspecial groups—the most surprising example for this is Tomkinson's result that a periodic FC-group G which does not lie in $\mathscr{Y}$ contains $U \trianglelefteq V \leqslant G$ such that V/U is extraspecial and not in $\mathscr{Y}$ (see [14] or

[16, Theorem 3.15]). Our main contribution in this direction is the equi-consistency result mentioned in the Introduction (corollary to Theorems D and E). Another example is the equivalence in Proposition 5.5. On the other hand, because of their simple *algebraic* structure (e.g., the fact that subgroups are either normal or abelian), extraspecial examples are in general the easiest to construct, and such constructions depend only on the underlying *combinatorial* structure—the classical example for this is the existence of a Shelah–Steprāns group [13].

We let $\mathscr{Y}_\kappa$ be the class of groups in which $[G : N_G(U)] < \kappa$ whenever $U \leqslant G$ is generated by fewer than κ elements. So the class $\mathscr{Y}$ is just the class of locally finite groups in the intersection of the $\mathscr{Y}_\kappa$; and also $\mathscr{Y}_\omega = \mathscr{X}_\omega =$ the class of FC-groups. It follows from Theorems B' and C' that $\mathscr{Y}_{\omega_1}$ and $\mathscr{X}_{\omega_1}$ are consistently equal for periodic FC-groups, and that they are equal for periodic finite-by-abelian groups.

By Tomkinson's result mentioned in the Introduction (II), if $\lambda < \kappa$ are cardinals and G is a group with $|G/Z(G)| = \kappa$ and $[G : N_G(U)] \leqslant \lambda$ for all subgroups $U \leqslant G$, then G is $\mathscr{Y}_\mu$ for any $\mu \geqslant \lambda^+$ but not $\mathscr{X}_\kappa$. Furthermore, for extraspecial p-groups G, the following are equivalent (where κ is any cardinal).

(i) $[G : N_G(A)] < \kappa$ for all (abelian) subgroups $A \leqslant G$.

(ii) For all maximal abelian subgroups A of G, $[G : A] < \kappa$.

In particular, an extraspecial p-group of size ω_2 whose maximal abelian subgroups satisfy $[G : A] \leqslant \omega_1$ is $\mathscr{Y}_{\omega_2}$ but not $\mathscr{X}_{\omega_2}$.

This should motivate us to study the three classes $\mathscr{Y}_{\omega_1} = \mathscr{X}_{\omega_1}$, $\mathscr{Y}_{\omega_2}$, and $\mathscr{X}_{\omega_2}$ for extraspecial p-groups more thoroughly. Clearly, there are groups lying in none or in all of these classes, or in $\mathscr{X}_{\omega_2} \setminus \mathscr{X}_{\omega_1}$. The existence of groups which are in $\mathscr{Y}_{\omega_2} \setminus \mathscr{X}_{\omega_2}$ or in $\mathscr{X}_{\omega_1} \setminus \mathscr{X}_{\omega_2}$ will be discussed in the subsequent subsections (up to Section 5.5). Our results can be summarized in the following chart.

	$\mathscr{X}_{\omega_1} = \mathscr{Y}_{\omega_1}$	$\neg\, \mathscr{X}_{\omega_1} = \neg\, \mathscr{Y}_{\omega_1}$
$\mathscr{X}_{\omega_2}$	Easy	Easy
$\neg\, \mathscr{X}_{\omega_2}$ but $\mathscr{Y}_{\omega_2}$	? (cf. Section 5.5)	Sections 5.3 and 5.4 (follows from the existence of Kurepa trees, and implies the existence of weak Kurepa trees)
$\neg\, \mathscr{Y}_{\omega_2}$	Section 5.5 (equivalent to the existence of Kurepa trees)	Easy

5.2. The following is useful for the proof of Theorem D.

LEMMA (Folklore). *Assume there is a Kurepa family. Then there is an a. d. Kurepa family of the same size.*

Proof. Let $\{A_\alpha; \alpha < \kappa\}$ be a Kurepa family (where $\kappa \geqslant \omega_2$). Let f be a bijection between $\{A_\alpha \cap \beta; \alpha < \kappa, \beta < \omega_1\}$ and ω_1. Then $\{\{f(A_\alpha \cap \beta);$ $\beta < \omega_1\}; \alpha < \kappa\}$ is easily seen to be an a. d. Kurepa family. ∎

5.3. *Proof of Theorem* D. Let E be a Shelah–Steprāns group [13] of size ω_1, and let $\mathscr{A} = \{A_\alpha; \alpha < \omega_2\}$ be an a. d. Kurepa family. We extend E semidirectly by an elementary abelian group B of automorphisms using $\mathscr{A}$ as follows: for all $\alpha < \omega_2$ define ϕ_α by

$$a_\beta \phi_\alpha = \begin{cases} a_\beta & \text{if } \beta = 0 \text{ or } \beta \notin A_\alpha, \\ a_\beta a_0 & \text{otherwise,} \end{cases}$$

where a_0 generates $Z(E)$ and $\{a_\beta; 1 \leqslant \beta < \omega_1\}$ generates E. Set $B :=$ $\langle \phi_\alpha; \alpha < \omega_2 \rangle$. This completes the construction. $G := E \rtimes B$ is easily seen to be extraspecial.

Now suppose $A \leqslant G$ is abelian. Let $\pi(A)$ denote the subgroup of E generated by the projection of A on the first coordinate (we think of the semidirect product as a set of tuples). We claim that $\pi(A)$ is countable. For suppose not. Then clearly $a_0 \in \pi(A)$. Let C be a maximal abelian subgroup of $\pi(A)$. C is countable, and $C_{\pi(A)}(C) = C$. Choose a subset $\{(b_\alpha, \psi_\alpha);$ $\alpha < \omega_1\}$ of A such that $C \leqslant \langle b_n; n \in \omega \rangle$ and $b_\alpha \neq b_\beta$ for $\alpha \neq \beta$. Now let B_α consist of the β so that a_β appears as a factor in b_α. We may assume that the B_α $(\alpha \geqslant \omega)$ form a delta-system with root R. Let $C_\alpha := B_\alpha \backslash R$. We can suppose that there is a j such that $|C_\alpha| = j$ for $\alpha \geqslant \omega$, that for $\alpha < \beta$ we have $\sup C_\alpha < \min C_\beta$, and that the multiplicities with which the a_β appear in the b_α depend only on $\gamma \in R$ or $i \in j$. As $\mathscr{A}$ is a. d., we may assume that for each of the (countably many) automorphisms ϕ_δ appearing as a factor in some ψ_n $(n \in \omega)$ and each $i \in j$ either $\forall \alpha \geqslant \omega$ $(a_{C_\alpha(i)} \phi_\delta = a_{C_\alpha(i)})$ or $\forall \alpha \geqslant \omega$ $(a_{C_\alpha(i)} \phi_\delta = a_{C_\alpha(i)} a_0)$ (without loss the corresponding A_δ's are disjoint above $C_\omega(0)$). In particular we have that for fixed $n \in \omega$, $c_n := b_\alpha^{-1}(b_\alpha) \psi_n = b_\beta^{-1}(b_\beta) \psi_n$ for any $\alpha, \beta \geqslant \omega$. As $\mathscr{A}$ is a Kurepa family, we may assume that $\psi_\alpha \upharpoonright \langle b_n; n \in \omega \rangle = \psi_\beta \upharpoonright \langle b_n; n \in \omega \rangle$ for any $\alpha, \beta \geqslant \omega$. But then

$$[(b_n, \psi_n), (b_\alpha, \psi_\alpha)] = ((b_n^{-1} \psi_n^{-1}) \psi_\alpha^{-1}(b_\alpha^{-1}) \psi_\alpha^{-1}, \psi_n^{-1} \psi_\alpha^{-1})$$
$$\times (b_n \psi_\alpha b_\alpha, \psi_n \psi_\alpha)$$
$$= (b_n^{-1}(b_\alpha^{-1}) \psi_n(b_n) \psi_\alpha b_\alpha, 1)$$
$$= (c_n^{-1} d_n [b_n, b_\alpha], 1),$$

where $d_n = b_n \psi_\alpha b_n^{-1}$. As C is maximal abelian in $\pi(A)$, there is certainly an $n \in \omega$ such that $[b_n, b_\alpha] \neq [b_n, b_\beta]$ for some $\alpha, \beta \geqslant \omega$. But then the above calculation shows that A cannot be abelian.

Now the fact that $\pi(A)$ is countable and that $\mathscr{A}$ is a Kurepa family implies $[G : C_G(\pi(A))] \leqslant \omega_1$ (in fact, equality holds unless $\pi(A)$ is finite, because E is a Shelah–Steprāns-group). If $\rho(A)$ is the projection of A on the second coordinate, $[G : C_G(\rho(A))] \leqslant \omega_1$ holds trivially; and $A \leqslant \langle \pi(A), \rho(A) \rangle$ implies $[G : N_G(A)] \leqslant [G : C_G(A)] \leqslant \omega_1$. ∎

5.4. THEOREM.   *If there is an extraspecial p-group of size ω_2 in $\mathscr{Y}_{\omega_2}$ but not in $\mathscr{Z}_{\omega_2}$, then there is a weak Kurepa tree.*

Proof. Let G be such a group. Choose $U \leqslant G$ of size ω_1 such that $[G : C_G(U)] = \omega_2$. Let $\{u_\alpha; \alpha < \omega_1\}$ generate U. Let $\{f_\beta; \beta < \omega_2\}$ be a subset of $G \backslash U$ such that $f_\alpha C_G(U) \neq f_\beta C_G(U)$. Define $g_\beta : \omega_1 \to p$ for $\beta < \omega_2$ by $g_\beta(\alpha) = k$ iff $[u_\alpha, f_\beta] = ka$, where a generates G'. We claim that the g_β form the branches of a weak Kurepa tree.

For suppose not. Then there is an $\alpha \in \omega_1$ such that $|\{g_\beta \lceil \alpha; \beta < \omega_2\}| = \omega_2$. This immediately implies that $[G : C_G(V)] = \omega_2$ for a countable subgroup $V \leqslant U$. V is a direct sum of an extraspecial and an abelian group; especially $g(V) \leqslant 2$ (this follows from the fact that countable extraspecial p-groups are central sums of groups of order p^3). So there is an abelian $A \leqslant V$ such that $[G : C_G(A)] = \omega_2$. Cutting away G' if necessary we may assume that $[G : N_G(A)] = \omega_2$, contradicting the fact that $G \in \mathscr{Y}_{\omega_2}$. ∎

Note that in the hypothesis of the theorem, *extraspecial p-group* can be replaced by (*periodic*) *finite-by-abelian group*. To see that this more general result is true, just apply Theorems 2.2 and 3.2 at the end of the proof. And if Question 1″ had a positive answer, we could prove this for (*periodic*) *FC-groups*.

There is a gap between Theorem D and Theorem 5.4. We feel that it should be possible to make a construction like the one in Section 5.3 using a weak Kurepa tree only.

5.5.   Using the same techniques as in Section 5.3 and Theorem 5.4 we obtain

PROPOSITION.   *The following are equivalent.*

   (i)   *There is a Kurepa tree.*

   (ii)   *There is an extraspecial p-group which is $\mathscr{Z}_{\omega_1}$ but not $\mathscr{Z}_{\omega_2}$.*

   (iii)   *There is an FC-group which is $\mathscr{Z}_{\omega_1}$ but not $\mathscr{Z}_{\omega_2}$.*

Proof. To see one direction $((i) \Rightarrow (ii))$ let E be any extraspecial $\mathscr{Z}$-group of size ω_1, and let, as in Section 5.3, $\mathscr{A} = \{A_\alpha; \alpha < \omega_2\}$ be an a.d. Kurepa family. For all $\alpha < \omega_2$ define ϕ_α by

$$a_\beta \phi_\alpha = \begin{cases} a_\beta & \text{if} \quad \beta = 0 \text{ or } \beta \notin A_\alpha, \\ a_\beta a_0 & \text{otherwise,} \end{cases}$$

where a_0 generates $Z(E)$ and $\{a_\beta; 1 \le \beta < \omega_1\}$ generates E. Set $G := E \rtimes \langle \phi_\alpha; \alpha < \omega_2 \rangle$. Clearly G has the required properties.

Conversely, to see $(iii) \Rightarrow (i)$, make the same construction as in Theorem 5.4. ∎

In general, the group constructed in the first part of the proof will not lie in $\mathscr{Y}_{\omega_2}$ either. Hence the only question left open is whether there are extraspecial $\mathscr{Z}_{\omega_1}$-groups in $\mathscr{Y}_{\omega_2} \backslash \mathscr{Z}_{\omega_2}$. We conjecture that they exist in the constructible universe $\mathscr{L}$. Such a group of size ω_2 would lie in $\mathscr{Y} \backslash \mathscr{Z}$ as well and so give an answer to [16, Question 3F].

On the other hand, unlike the other classes considered so far, the consistency of ZFC *alone* implies the consistency of the non-existence of extraspecial $\mathscr{Z}_{\omega_1}$-groups in $\mathscr{Y}_{\omega_2} \backslash \mathscr{Z}_{\omega_2}$. To see this, let $\mathscr{V} \models \text{ZFC} + \text{GCH}$. Add ω_3 Cohen subsets of ω_1. We claim that in the resulting model $\mathscr{V}[\mathscr{G}]$, there are no such groups. For suppose G is such a group. Find $U \le G$ of size ω_1 with $[G : C_G(U)] = \omega_2$, without loss $U \trianglelefteq G$. Apply Theorem 4.4 with $\lambda = \omega_3$, $\kappa = \omega_1$, $H = U$, and $\Phi = G/C_G(U)$ (this can be done as $U \in \mathscr{Z}_{\omega_1}$). Find $A \le U$ abelian such that $[G : C_G(A)] = \omega_2$. Cutting G' away, if necessary, we can assume $C_G(A) = N_G(A)$, a contradiction.

5.6. We now want to turn to the proof of Theorem E. Certainly, in a model where its statement is true, neither of the *bad* situations discussed in Section 4.2 (and Sections 4.3, 4.6, 4.7) and in Section 5.3 can occur. So we had better look for a model where there are no Kurepa trees and where CH is false. The discussion in Sections 5.4 and 4.4 suggests that there should not be weak Kurepa trees either and that there should be reals Cohen over $\mathscr{L}$. One possible attack would be to destroy all weak Kurepa trees by collapsing an inaccessible to ω_2 (as in [11, Sects. 3, 4] or [1, Sect. 8]) and then to add ω_2 Cohen reals (by the last ω_2 we mean, of course, the ω_2 of the intermediate model). Unfortunately, we do not know whether it is true in general that there are no weak Kurepa trees in the final extension; but this is true if the intermediate model is Mitchell's [11]. In fact, it turns out that in this case the second extension is unnecessary, and what we want to show consistent already holds in Mitchell's model. The reason for this is essentially that this model is gotten by first adding κ Cohen reals (where κ is inaccessible) and then collapsing κ to ω_2 using

a forcing which does not add reals—and hence does not destroy the *nice* situation created by the Cohen reals—while killing all weak Kurepa trees.

First, we will review Mitchell's model and some elementary facts about it. Let $\mathscr{V} \models$ "ZFC + GCH + there is an inaccessible." Let κ be inaccessible in $\mathscr{V}$. $\mathbb{P} = Fn(\kappa, 2, \omega)$ is the ordering for adding κ Cohen reals. $\mathbb{B} = \mathbb{B}(\mathbb{P})$ is the Boolean algebra associated with $\mathbb{P}$. Set $\mathbb{P}_\alpha := \{p \in \mathbb{P}; \operatorname{supp}(p) \subseteq \alpha\}$ for $\alpha < \kappa$. $\mathbb{B}_\alpha$ is the Boolean algebra associated with $\mathbb{P}_\alpha$. $f \in \mathscr{V}$ is the set $\mathscr{A}$ of *acceptable functions* iff

(1) $\operatorname{dom}(f) \subseteq \kappa$; $\operatorname{ran}(f) \subseteq \mathbb{B}$;

(2) $|\operatorname{dom}(f)| \leq \omega$;

(3) $f(\gamma) \in \mathbb{B}_{\gamma+\omega}$ for $\gamma \in \operatorname{dom}(f)$.

If $\mathscr{F}$ is $\mathbb{P}$-generic over $\mathscr{V}$, $f \in \mathscr{A}$, then we define $\tilde{f}: \operatorname{dom}(f) \to 2$ in $\mathscr{V}[\mathscr{F}]$ by $\tilde{f}(\gamma) = 1$ iff $f(\gamma) \in \mathscr{F}$. Define $\mathbb{Q}$ in $\mathscr{V}[\mathscr{F}]$ by letting the underlying set of $\mathbb{Q}$ be $\mathscr{A}$ and $f \leq_\mathbb{Q} g$ iff $\tilde{f} \supseteq \tilde{g}$. So we obtain a 2-step iteration $\mathbb{P} * \mathbb{Q}$ with $(p, f) \leq (q, g)$ iff $p \leq q$ and $p \Vdash_\mathbb{P} f \leq_\mathbb{Q} g$. We shall denote the final extension by $\mathscr{V}[\mathscr{F}][\mathscr{G}]$.

Facts (Mitchell [11]). (1) *Suppose* $p \Vdash_\mathbb{P} \text{"}\check{D}$ *is open dense in* $\mathbb{Q}$ *below* f,*" where* $f \in \mathscr{A}$. *Let* $g \in \mathscr{A}$ *such that* $g \supseteq f$. *Then there is* $h \in \mathscr{A}$ *such that* $h \supseteq g$ *and* $p \Vdash_\mathbb{P} h \in \check{D}$.

(2) $\mathbb{Q}$ *does not add new functions with countable domain over* $\mathscr{V}[\mathscr{F}]$; *i.e., if* $t: \omega \to \mathscr{V}[\mathscr{F}]$ *where* $t \in \mathscr{V}[\mathscr{F}][\mathscr{G}]$, *then* $t \in \mathscr{V}[\mathscr{F}]$.

(3) *Let* $\{g_\alpha; \alpha < \kappa\} \subseteq \mathscr{A}$. *Then there are* $X \in [\kappa]^\kappa$ *and* $g \in \mathscr{A}$ *such that* $\forall \alpha, \beta \in X$ $(\operatorname{dom}(g_\alpha) \cap \operatorname{dom}(g_\beta) = \operatorname{dom}(g))$ *and* $\forall \alpha \in X$ $(g_\alpha \restriction \operatorname{dom}(g) = g)$.

(4) $\mathbb{P} * \mathbb{Q}$ *preserves* ω_1 *(this follows from the ccc-ness of* $\mathbb{P}$ *and fact (2)) and cardinals* $\geq \kappa$ *(it follows from (3) that* $\mathbb{P} * \mathbb{Q}$ *is* κ*-cc), but collapses all cardinals in between to* ω_1; *i.e.,* $\kappa^\mathscr{V} = \omega_2^{\mathscr{V}[\mathscr{F}][\mathscr{G}]}$.

(5) *In* $\mathscr{V}[\mathscr{F}][\mathscr{G}]$, $2^\omega = 2^{\omega_1} = \omega_2$.

(6) *Let* $v < \kappa$ *be such that* $v' + \omega \leq v$ *for each* $v' < v$. *Then the generic extension via* $\mathbb{P} * \mathbb{Q}$ *can be split in a 2-step extension, the first of which adds* $|v|$ *Cohen reals and a* $\mathbb{Q} \restriction v$*-generic function from* v *to* 2, *whereas the second adds the remaining* κ *Cohen reals and the remaining part of the* $\mathbb{Q}$*-generic function from* κ *to* 2.

(7) *In* $\mathscr{V}[\mathscr{F}][\mathscr{G}]$, *there are no weak Kurepa trees.*

Proofs. (1)–(5) are (more or less) [11, Sects. 3.1–3.5], concerning (3) we note that it is proved via a straightforward Δ-system-argument. (6) is made more explicit in [11, pp. 29, 30] and proved in Section 3.6. For (7), see [11, Sect. 4.7]. ∎

5.7. *Proof of Theorem* E.   We show that $\mathscr{V}[\mathscr{F}][\mathscr{G}] \models$ "for both ω_1 and ω_2 and any FC-group G, (i) through (iii) in (II) are equivalent," where $\mathscr{V}[\mathscr{F}][\mathscr{G}]$ is Mitchell's model as in the preceding section.

Counterexamples G with $|G/Z(G)| = \omega_1$ are easily excluded. Without loss such G would have size ω_1. By the κ-cc of $\mathbb{P} * \mathbb{Q}$ (which follows from the ccc of $\mathbb{P}$ and fact (3) in Section 5.6) it would lie in an intermediate extension (see fact (6)). Any real Cohen over this intermediate extension shows that the assumption was false (by the argument of Theorem 4.4).

Suppose G is a counterexample with $|G/Z(G)| = \omega_2$; without loss $|G| = \omega_2$; $[G : N_G(A)] \leqslant \omega_1$ for all abelian $A \leqslant G$; and there is a $U \leqslant G$ of size $\leqslant \omega_1$ such that $[G : C_G(U)] = \omega_2$ (because G cannot be a $\mathscr{X}_{\omega_2}$-group by Tomkinson's result (II) in the Introduction). Without loss $|U| = \omega_1$. Let $V := U^G = \langle x^{-1}Ux; x \in G \rangle$. As G is an FC-group, $|V| = \omega_1$; and $V \trianglelefteq G$. Clearly $|G/C_G(V)| = \omega_2$. For any $\bar{g} \in G/C_G(V)$ define a function $f_g : V \to V$ by $f_g(v) := g^{-1}vg^{-1}$, where $g \in \bar{g}$ is arbitrary. Think of $\{f_g; \bar{g} \in G/C_G(V)\}$ as the set of branches through a tree T. As $\mathscr{V}[\mathscr{F}][\mathscr{G}]$ does not contain weak Kurepa trees (fact (7) in Section 5.6), there is a countable $S \subseteq V$ such that $\{f_{\bar{g}} \restriction S; \bar{g} \in G/C_G(V)\}$ has size ω_2. Let $W := \langle S^G \rangle$. W is a countable normal subgroup of G; and $|G/C_G(W)| = \omega_2$ by construction.

We now want to prove that there is an abelian $A \leqslant W$ such that $[G : C_G(A)] = \omega_2$ (*main claim*). The way we do this is an elaboration of the proof of Theorem 4.4. For this argument it is crucial that we use Mitchell's model and not just any model without weak Kurepa trees.

We think of $G/C_G(W)$ as a group of automorphisms Φ of W; more explicitly, $\Phi : \omega_2 \to \text{Aut}(W)$. In $\mathscr{V}[\mathscr{F}]$, let $\check{\Phi}$ be a $\mathbb{Q}$-name for Φ. Let D_α $(\alpha < \kappa)$ be the set of conditions deciding $\check{\Phi}(\alpha)$. D_α is open dense by fact (2). Let $\dot{D}_\alpha$ be a $\mathbb{P}$-name for D_α $(\alpha < \kappa)$. Then

$$\Vdash_{\mathbb{P}} \text{``}\dot{D}_\alpha \text{ is open dense.''}$$

By fact (1) there are $g_\alpha \in \mathscr{A}$ such that

$$\Vdash_{\mathbb{P}} g_\alpha \in \dot{D}_\alpha.$$

So in $\mathscr{V}[\mathscr{F}]$, there are ϕ_α such that $g_\alpha \Vdash_{\mathbb{Q}} \check{\Phi}(\alpha) = \phi_\alpha$. Let $\dot{\phi}_\alpha$ $(\alpha < \kappa)$ be $\mathbb{P}$-names for the ϕ_α. Then $\Vdash_{\mathbb{P}} g_\alpha \Vdash_{\mathbb{Q}} \dot{\Phi}(\alpha) = \dot{\phi}_\alpha$, where $\dot{\Phi}$ is a $\mathbb{P}$-name for Φ. Using fact (3) we obtain $X \in [\kappa]^\kappa$ and $g \in \mathscr{A}$ such that $\forall \alpha, \beta \in X$, $\text{dom}(g_\alpha) \cap \text{dom}(g_\beta) = \text{dom}(g)$ and $\forall \alpha \in X$, $g_\alpha \restriction \text{dom}(g) = g$. Now we split $\mathbb{P}$ into two parts (i.e., $\mathbb{P} = \mathbb{P}_1 \times \mathbb{P}_2$) such that

  (1) $\mathbb{P}_1$ adds κ Cohen reals and $\mathbb{P}_2$ adds one Cohen real;

  (2)   there is $Y \in [X]^\kappa$ such that $\forall \alpha \in Y$ $(\phi_\alpha \in \mathscr{V}[\mathscr{F}_1])$, where $\mathscr{F}_1$ is $\mathbb{P}_1$-generic over $\mathscr{V}$.

So in $\mathcal{V}[\mathscr{F}_1]$,

$$\Vdash_{\mathbb{P}_2} g_\alpha \Vdash_\mathbb{Q} \dot{\Phi}(\alpha) = \phi_\alpha,$$

where $\alpha \in Y$. From now on we work in $\mathcal{V}[\mathscr{F}_1]$. As in the proof of Theorem 4.4 we think of $\mathbb{P}_2$ as adding a new abelian subgroup of W. Let $\dot{A}$ be a $\mathbb{P}_2$-name for this generic object.

The rest of the proof of the main claim is by contradiction. Suppose that

$$\Vdash_{\mathbb{P} \ast \mathbb{Q}} \forall B \leqslant W \text{ abelian} \qquad (|\{\dot{\Phi}(\alpha)\restriction B; \alpha < \kappa\}| < \kappa).$$

Especially, in $\mathcal{V}[\mathscr{F}_1]$,

$$\Vdash_{\mathbb{P}_2 \ast \mathbb{Q}} |\{\dot{\Phi}(\alpha)\restriction \dot{A}; \alpha \in Y\}| < \kappa.$$

Hence,

$$\Vdash_{\mathbb{P}_2 \ast \mathbb{Q}} \exists \alpha \, \forall \beta \geqslant \alpha \, (\beta \in Y \Rightarrow \exists \gamma < \alpha(\gamma \in Y \wedge \dot{\Phi}(\beta)\restriction \dot{A} = \dot{\Phi}(\gamma)\restriction \dot{A})).$$

So there are $\alpha < \kappa$ and $C \in \mathbb{P}_2$, $h \supseteq g$, $h \in \mathcal{A}$ such that

$$(C, h) \Vdash_{\mathbb{P}_2 \ast \mathbb{Q}} \forall \beta \geqslant \alpha \, (\beta \in Y \Rightarrow \exists \gamma < \alpha(\gamma \in Y \wedge \dot{\Phi}(\beta)\restriction \dot{A} = \dot{\Phi}(\gamma)\restriction \dot{A})).$$

Choose $Z \in [Y \backslash \alpha]^\kappa$ such that for $\beta \in Z$, g_β and h are compatible (in $\mathcal{V}$). For $\beta \in Z$ let (in $\mathcal{V}[\mathscr{F}]$) D_β be the set of conditions forcing $\check{\Phi}(\beta)\restriction A = \check{\Phi}(\gamma)\restriction A$ for some $\gamma < \alpha$. D_β is open dense below h. Let $\dot{D}_\beta$ be a $\mathbb{P}_2$-name for D_β. By fact (1), there are h_β such that $h_\beta \supseteq g_\beta$ and $h_\beta \supseteq h$ and

$$C \Vdash_{\mathbb{P}_2} h_\beta \in \dot{D}_\beta.$$

I.e.

$$(C, h_\beta) \Vdash_{\mathbb{P}_2 \ast \mathbb{Q}} \dot{\Phi}(\beta)\restriction \dot{A} = \phi_\beta \restriction \dot{A} = \dot{\Phi}(\gamma)\restriction \dot{A}$$

for some $\gamma = \gamma(\beta) < \alpha$. Choose by fact (3) $Z' \in [Z]^\kappa$ and $\tilde{h} \supseteq h$ ($\tilde{h} \in \mathcal{A}$ of course) and γ such that

(1) $\forall \beta_1 \neq \beta_2 \in Z'$ $(\text{dom}(h_{\beta_1}) \cap \text{dom}(h_{\beta_2}) = \text{dom}(\tilde{h}))$; $\forall \beta \in Z'$ $(\tilde{h} = h_\beta \restriction \text{dom}(\tilde{h}))$;

(2) $\gamma(\beta) = \gamma$ for $\beta \in Z'$.

$C \leqslant W$ is a finite abelian subgroup. So $[W : C_W(C)] < \omega$. As $|Z'| = \kappa$, $|\{\phi_\beta \restriction C_W(C); \beta \in Z'\}| = \kappa$ so that we can find $c \in C_W(C) \backslash C$ and $\beta_1, \beta_2 \in Z'$ such that $(c) \phi_{\beta_1} \neq (c) \phi_{\beta_2}$. Then

$$(\langle C, c \rangle, h_{\beta_1} \cup h_{\beta_2}) \Vdash_{\mathbb{P}_2 \ast \mathbb{Q}} \text{``}\dot{\Phi}(\beta_1)\restriction \dot{A} = \phi_{\beta_1}\restriction \dot{A} = \dot{\Phi}(\gamma)\restriction \dot{A}$$
$$= \phi_{\beta_2}\restriction \dot{A} = \dot{\Phi}(\beta_2)\restriction \dot{A} \text{ and } \phi_{\beta_1}\restriction \dot{A} \neq \phi_{\beta_2}\restriction \dot{A},\text{''}$$

which is a contradiction.

This ends the proof of the main claim and shows that there is indeed an abelian $A \leqslant W$ such that $[G : C_G(A)] = \omega_2$. Then either $[G : N_G(A)] = \omega_2$ or we apply Theorem 3.2 — as we did before in the proofs of Theorems B and C — to obtain $B \leqslant A$ such that $[G : N_G(B)] = \omega_2$. This is the final contradiction. ∎

It should be clear that this proof also yields that in Mitchell's model, both $\mathscr{Y}_{\omega_1} = \mathscr{Z}_{\omega_1}$ and $\mathscr{Y}_{\omega_2} = \mathscr{Z}_{\omega_2}$ for periodic FC-groups; especially $\mathscr{Y} = \mathscr{Z}$ for groups of size $\leqslant \omega_2$.

5.8. THEOREM. *The consistency of* $ZFC + I$ *implies the consistency of* $ZFC + the following statements.*

   (i) $\mathscr{Y}_{\omega_1} = \mathscr{Z}_{\omega_1}$ *for periodic FC-groups—and any periodic FC-group G with* $|G/Z(G)| = \omega_1$ *has an abelian subgroup A with* $[G : N_G(A)] = \omega_1$;

   (ii) $\mathscr{Y}_{\omega_2} = \mathscr{Z}_{\omega_2}$ *for periodic finite-by-abelian groups—and any periodic finite-by-abelian group G with* $|G/Z(G)| = \omega_2$ *has an abelian subgroup A with* $[G : N_G(A)] = \omega_2$;

   (iii) *There is an FC-group G with* $|G/Z(G)| = \omega_2$ *but* $[G : N_G(A)] \leqslant \omega_1$ *for all abelian $A \leqslant G$.*

Proof. Let κ be inaccessible in $\mathscr{V}$. By [1, Theorem 8.8], there is a partial order $\mathbb{P}$ such that for $\mathscr{G}$ $\mathbb{P}$-generic over $\mathscr{V}$,

$$\mathscr{V}[\mathscr{G}] \models MA + 2^\omega = \omega_2 + \text{``there are no weak Kurepa trees.''}$$

(This is proved by a countable support iteration of length κ of partial orders which alternately make ω_1-trees special and are ccc.) Now apply Sections 4.4/4.5 (for (i)), 4.6 (for (iii)), and 5.4 (for (ii)). ∎

As both MA and *no weak Kurepa trees* follow from the proper forcing axiom PFA (by [1, Sect. 8]—see also [2, 7.10]), (i)–(iii) in the theorem hold if we assume $ZFC + PFA$.

5.9. PROPOSITION. *For any cardinal $\kappa \geqslant \omega_2$ it is consistent that there is an extraspecial p-group of size κ such that for all maximal abelian subgroups $A \leqslant G$, $[G : A] \leqslant \omega_1$.*

Proof. By the arguments of Sections 5.2 and 5.3 it suffices to generically add a Kurepa tree with κ branches as follows (Folklore). Let $\mathbb{K}_\kappa := \{p; p$ is a function and $\mathrm{dom}(p) = \alpha \times A$, where $\alpha < \omega_1$ and $A \in [\kappa]^\omega$, and $\mathrm{ran}(p) \subseteq 2\}$, ordered by $p \leqslant q$ iff $\alpha(p) \geqslant \alpha(q)$ (where $\alpha(p)$ is the α of the definition of the p.o.), $A(p) \supseteq A(q)$, $p \restriction (\alpha(q) \times A(q)) = q$, and for all $\beta \in A(p) - A(q)$ there is $\gamma \in A(q)$ such that $p \restriction (\alpha(q) \times \{\beta\}) = p \restriction (\alpha(q) \times$

$\{\gamma\}$). $\mathbb{K}_\kappa$ is (if we assume CH in the ground model $\mathscr{V}$) $\omega_2 - $cc and ω_1-closed, and so preserves cardinals. Clearly this works. $\blacksquare$

6. GENERALIZATIONS

6.1. It was mentioned in the Introduction that (II) holds for arbitrary $\mathscr{Z}_\kappa$-groups G [4]. One might ask what goes wrong in this general case (where G is not required to be FC) if we drop the $\mathscr{Z}_\kappa$-condition.

PROPOSITION. *For any cardinal κ there is a group G with $|G/Z(G)| = 2^\kappa$ and $[G : N_G(A)] \leqslant \kappa$ for all abelian $A \leqslant G$.*

Proof. Let A and B be two elementary abelian p-groups of size κ. Let $h: [\kappa]^2 \to \kappa$ be a bijection. We define a factor system τ as follows:

$$\tau(a_\alpha, a_\beta) = \begin{cases} 0 & \text{if } \alpha \geqslant \beta, \\ b_{h(\alpha, \beta)} & \text{otherwise,} \end{cases}$$

where the a_α (b_α, resp.) ($\alpha < \kappa$) generate A (B, resp.); extend τ bilinearly to A^2. Let $C := E(\tau)$ be the extension. (Note that C is the free object on κ generators in the variety of two-step nilpotent groups of exponent p ($p > 2$); and that its maximal abelian subgroups are of the form $\langle B, a \rangle$, where $a \notin B$.) Let D be a group of order p. Let G be the (abelian) subgroup of $\mathrm{Aut}(C \oplus D)$ consisting of all automorphisms ϕ which fix $B \oplus D$ and satisfy

$$\forall c \in C \oplus D \; \exists k \in p \qquad (c\phi = c + kd).$$

Clearly, $|G| = 2^\kappa$. Let E be the semidirect extension of $C \oplus D$ and G (i.e., $E = (C \oplus D) \rtimes G$). We leave it to the reader to verify that $|E/Z(E)| = 2^\kappa$ and $[E : C_E(A)] \leqslant \kappa$ for all abelian $A \leqslant E$. $\blacksquare$

Note. The proof is similar to (but easier than) the proof of Theorem A (see Sections 4.2 and 4.3). Unlike the latter it does not involve any set-theoretic hypotheses. On the other hand, the group E constructed above is not κC but $\kappa^+ C$ (a group G is κC iff every $g \in G$ has less than κ conjugates).

6.2. We restricted our attention to $\kappa = \omega_1$ or ω_2. This is reasonable because the problem seems to be most interesting for small cardinals. Also, the constructions in Section 5 (Sections 5.3 and 5.9) show how to obtain consistency results concerning the existence of pathological groups for larger cardinals (just use λ-Kurepa families instead of (ω_1-)Kurepa families for the appropriate λ). Nevertheless we ignore whether the non-existence of such groups is consistent for $\kappa \geqslant \omega_3$ (cf. Theorem E).

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