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Multivariate Reciprocal Stationary Gaussian Processes*

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In this paper we examine the characterization of multivariate reciprocal stationary Gaussian processes in terms of their covariance matrix function. As an illustration, we identify all second-order reciprocal Gaussian processes. © 1987 Academic Press, Inc.

INTRODUCTION

In several previous papers (see, e.g., [6, 7, 9]), one-dimensional reciprocal stationary Gaussian processes were studied, essentially from the point of view of their characterization in terms of covariance functions; a complete answer to this problem is given in [4]. The property of being reciprocal can also be found in the literature under the name of quasi-Markov property (see, e.g., [6]). A related slightly more general property is that of being conditionally Markov (see, e.g., [2, 9]). In the more general setting of Gaussian fields, characterization of fields with the Markov property in terms of Hilbert space concepts were given, e.g., in [11, 12, 14].

Our aim is to characterize multivariate reciprocal stationary Gaussian processes with continuous parameter in terms of their covariance matrix function. A matrix differential equation is derived whose solutions are, up to parameter restrictions, the covariance matrix functions of such processes. Instrumental in obtaining these results is a "factorization" property of the covariance matrix function.

As an illustration we apply our results to the special class of second-order reciprocal stationary Gaussian processes. Miroshin [10] studied this

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class but gave an incomplete characterization, as a result of his use of an incorrect differential equation satisfied by the covariance function.

1. Auxiliary Results

Let $-\infty \leqslant a < b \leqslant \infty$ and the integer $m \geqslant 1$ be given. In what follows we shall always consider $m \times m$ or $m \times 1$ real matrices, O and I being the zero and the unit matrix, respectively. All random processes we are dealing with are $\mathbb{R}^m$-valued; without any loss of generality we shall assume that they are centered. E denotes expectation with respect to a probability measure P.

Lemma 1.1. *Suppose that*

$$K(s, t) = \Phi(s)\, \Psi^{\mathrm{T}}(t), \qquad a < s \leqslant t < b, \tag{1.1}$$

("T" *denotes transposition*), *where* Φ *and* Ψ *are two matrix functions defined on* (a, b) *such that*: (1) Φ *and* Ψ *are continuous on* (a, b); (2) $\Phi\Psi^{\mathrm{T}}$ *is invertible on* (a, b). *Then* $K(s, t)$ *is a nonnegative definite kernel*, $a < s \leqslant t < b$, *if and only if*: (i) $A(t) = \Psi^{-1}(t)\,\Phi(t)$ *is positive definite and symmetric*, $t \in (a, b)$; (ii) $\lim_{t \downarrow a} A(t)$ *exists, say* $A(a)$, *and is nonnegative definite*; (iii) $A(t) - A(s)$ *is nonnegative definite*, $a \leqslant s \leqslant t < b$.

Proof. Since $K(t, t)$ is positive definite, $t \in (a, b)$, and $A(t) = \Psi^{-1}(t)\, K(t, t)(\Psi^{-1}(t))^{\mathrm{T}}$, $t \in (a, b)$, we obtain (i). Observe now that there exists a continuous in q.m. Gaussian process $\mathscr{Y} = \{Y(t) \colon t \in (a, b)\}$ whose covariance matrix function is given by K. Furthermore $A(t) - A(s) = \Psi^{-1}(t)[K(t, t) - K^{\mathrm{T}}(s, t)\, K^{-1}(s, s)\, K(s, t)](\Psi^{-1}(t))^{\mathrm{T}}$, $a < s \leqslant t < b$, where the bracketed factor is also nonnegative, being equal to $E\{[Y(t) - E(Y(t) | Y(s))][Y(t) - E(Y(t) | Y(s))]^{\mathrm{T}}\}$, $a < s \leqslant t < b$, which implies (iii) on (a, b). For fixed $x \in \mathbb{R}^m$, we infer that $x^{\mathrm{T}} A(t)\, x$, $t \in (a, b)$, is a nonnegative increasing function and thus $\lim_{t \downarrow a} x^{\mathrm{T}} A(t)\, x$ exists. Since for any $x, y \in \mathbb{R}^m$,

$$y^{\mathrm{T}} A(t)\, x = \left(\frac{x+y}{2}\right)^{\mathrm{T}} A(t) \left(\frac{x+y}{2}\right) - \left(\frac{x-y}{2}\right)^{\mathrm{T}} A(t) \left(\frac{x-y}{2}\right),$$

we obtain that $\lim_{t \downarrow a} y^{\mathrm{T}} A(t)\, x = L(x, y)$, where L is bilinear and continuous; thus L can be represented as $L(x, y) = y^{\mathrm{T}} A x$. It is now obvious that $A = \lim_{t \downarrow a} A(t) \equiv A(a)$ that (ii) holds. Finally, (iii) on $[a, b)$ follows by a limit argument.

Conversely, assume that (i), (ii), and (iii) hold and consider a process $\tilde{\mathscr{W}} = \{\tilde{W}(t) \colon t \in [a, b)\}$ such that ($\tilde{W}$1) $\tilde{W}(a)$ is normally distributed $N(O, A(a))$; ($\tilde{W}$2) $\tilde{W}(t) - \tilde{W}(s)$ is normally distributed $N(O, A(t) - A(s))$,

$a \leqslant s \leqslant t < b$; ($\tilde{W}3$) for $a \leqslant t_0 < t_1 < \cdots < t_l < b$, $\{\tilde{W}(a), \tilde{W}(t_k) - \tilde{W}(t_{k-1}), 1 \leqslant k \leqslant l\}$ is a set of independent random vectors. Then $E[\Psi(s)\, W(s)\, W^{\mathrm{T}}(t)\, \Psi^{\mathrm{T}}(t)] = K(s, t)$, $a < s \leqslant t < b$.

The factorization appearing on the right-hand side of (1.1) is unique up to an invertible constant matrix factor and is equivalent to

$$K(s, t) = K(s, u)\, K^{-1}(u, u)\, K(u, t), \qquad a < s \leqslant u \leqslant t < b. \tag{1.2}$$

Let $\mathscr{X} = \{X(t)\colon t \in (a, b)\}$ be a random process.

LEMMA 1.2. *Suppose that $\mathscr{X}$ is a Gaussian process. Then $\mathscr{X}$ has a covariance matrix function of the form* (1.1) *if and only if $\mathscr{X}$ is continuous in quadratic mean, Markovian, and* det $E(X(t)\, X^{\mathrm{T}}(t)) \neq 0$, $t \in (a, b)$.

Proof. See [13, pp. 54–55].

DEFINITION 1.3. Let $a \leqslant c < d \leqslant b$. $\mathscr{X}$ is called a (c, d)-reciprocal process if and only if, for each $c < s < t < d$, $\mathscr{E}(s, t)$ and $\mathscr{F}(s, t)$ are conditionally independent given $X(s)$ and $X(t)$, where $\mathscr{E}(s, t)$ is the σ-field generated by $X(r)$, $r \in (c, d) \backslash (s, t)$, and $\mathscr{F}(s, t)$ is the σ-field generated by $X(r)$, $r \in (s, t)$.

It is easily seen that Markov processes are (c, d)-reciprocal for any (c, d) (see [7, Lemma 2, p. 1626]). Let us also remark that stationarity implies that a (c, d)-reciprocal process is also (c', d')-reciprocal for any interval $(c', d') \subset (a, b)$ such that $d' - c' = d - c = T$. For the sake of simplicity we shall assume that $(c, d) \equiv (0, T)$ and we shall always refer to $(0, T)$-reciprocal processes.

From now on we shall only consider stationary random processes $\mathscr{X}$ whose covariance matrix function Γ belongs to the class $\mathscr{G}$ of functions G defined on (T, T) with values in the set of $m \times m$ matrices such that G is continuous, $G(0) = I$, $I - G^{\mathrm{T}}(t)\, G(t)$ is positive definite and $G(-t) = G^{\mathrm{T}}(t)$, $t \in (-T, T)$.

LEMMA 1.4. *$\mathscr{X}$ is a $(0, T)$-reciprocal Gaussian process if and only if for $t_0 \in (0, T)$ the processes $\mathscr{Y}^+ = \{Y^+(t)\colon t \in (t_0, T)\}$ and $\mathscr{Y}^- = \{Y^-(t)\colon t \in (0, t_0)\}$ are Markovian, where*

$$Y^+(t) = X(t) - \Gamma^{\mathrm{T}}(t - t_0)\, X(t_0), \qquad t \in (t_0, T),$$

$$Y^-(t) = X(t) - \Gamma(t_0 - t)\, X(t_0), \qquad t \in (0, t_0).$$

Proof. For the proof, see relations (3), (4), and (5) [4, p. 292].

The following result plays an important part in the sequel.

THEOREM 1.5. *$\Gamma \in \mathscr{G}$ is the covariance matrix function of a $(0, T)$-reciprocal Gaussian process if and only if there exist two couples Φ, Ψ and Φ^*, Ψ^* satisfying conditions* (1) *and* (2) *of Lemma* 1.1 *and such that*

$$\begin{aligned} \Gamma(t-s) - \Gamma^{\mathrm{T}}(s)\,\Gamma(t) &= \Phi(s)\,\Psi^{\mathrm{T}}(t), \\ \Gamma^{\mathrm{T}}(t-s) - \Gamma(s)\,\Gamma^{\mathrm{T}}(t) &= \Phi^*(s)\,\Psi^{*\mathrm{T}}(t), \\ 0 < s \leqslant t &< T. \end{aligned} \tag{1.3}$$

Proof. Set

$$\begin{aligned} K(s,t) = \Gamma(t-s) - \Gamma^{\mathrm{T}}(s)\,\Gamma(t) &= \Phi(s)\,\Psi^{\mathrm{T}}(t), \\ 0 < s \leqslant t &< T, \end{aligned}$$

and let us show that conditions (i), (ii), and (iii) of Lemma 1.1 are satisfied. Indeed, $\Lambda(\cdot) = \Psi^{-1}(\cdot)\,\Phi(\cdot) = \Psi^{-1}(\cdot)[1 - \Gamma^{\mathrm{T}}(\cdot)\,\Gamma(\cdot)](\Psi^{-1}(\cdot))^{\mathrm{T}}$ is positive definite and symmetric on $(0, T)$; thus (i) holds. Further, since Λ is differentiable, for any $x \in \mathbb{R}^m$ we have $x^{\mathrm{T}}[\Lambda(t) - \Lambda(s)]\,x = (t-s)\,x^{\mathrm{T}}\Lambda'(\theta_x)\,x$ for a convenient θ_x, $0 < s \leqslant \theta_x \leqslant t$, depending on x; hence, to conclude that $\Lambda(t) - \Lambda(s)$ is nonnegative definite, it suffices to prove that $\Lambda'(\cdot)$ is nonnegative definite on $(0, T)$. From (1.3) and the fact that

$$I - \Gamma^{\mathrm{T}}(t)\,\Gamma(t) = \Phi(t)\,\Psi^{\mathrm{T}}(t) = \Psi(t)\,\Phi^{\mathrm{T}}(t), \qquad t \in (0, T),$$

we obtain

$$\begin{aligned} \Lambda'(u) &= [\Psi^{-1}(u)\,\Phi(u)]' \\ &= \Psi^{-1}(u)\,\Phi'(u) - \Psi^{-1}(u)\,\Psi'(u)\,\Psi^{-1}(u)\,\Phi(u) \\ &= \Psi^{-1}(u)[\Phi'(u)\,\Psi^{\mathrm{T}}(u) \\ &\qquad - \Psi'(u)\,\Phi^{\mathrm{T}}(u)(\Phi^{\mathrm{T}}(u))^{-1}\,\Psi^{-1}(u)\,\Phi(u)\,\Psi^{\mathrm{T}}(u)](\Psi^{-1}(u))^{\mathrm{T}} \\ &= \Psi^{-1}(u)[\Phi'(u)\,\Psi^{\mathrm{T}}(u) - \Psi'(u)\,\Phi^{\mathrm{T}}(u)](\Psi^{-1}(u))^{\mathrm{T}} \\ &= \Psi^{-1}(u)[-\Gamma'(0+) - \Gamma'^{\mathrm{T}}(0+)](\Psi^{-1}(u))^{\mathrm{T}}, \end{aligned}$$

the existence of $\Gamma'(0+)$ following from (1.3). Now

$$-\Gamma'(0+) - (\Gamma'(0+))^{\mathrm{T}} = \lim_{t \downarrow 0} (1/t)[I - \Gamma^{\mathrm{T}}(t)\,\Gamma(t)];$$

thus $-\Gamma'(0+) - (\Gamma'(0+))^{\mathrm{T}}$ is nonnegative definite on $(0, T)$ and consequently (iii) holds. (ii) follows from (iii) as in Lemma 1.1. According to Lemma 1.2, there exists a continuous in quadratic mean Gaussian Markov process $\mathscr{U} = \{U(t) \colon t \in (0, t)\}$ whose covariance matrix function is K. In the same way and using the same arguments, we are led to the same conclusion for

$$K^*(s, t) = \Gamma^{\mathrm{T}}(t-s) - \Gamma(s)\,\Gamma^{\mathrm{T}}(t) = \Phi^*(s)\,\Psi^{*\mathrm{T}}(t),$$
$$0 < s \leqslant t < T.$$

Further, set $X(t) = U(t) + \Gamma^{\mathrm{T}}(t)\,Z$, $t \in (0, T)$, with Z independent of $\mathscr{U}$ and normally distributed $N(O, I)$. It is easy to show that $\mathscr{X} = \{X(t) \colon t \in (0, T)\}$ is a stationary Gaussian process whose covariance matrix function is Γ. It remains to verify that $\mathscr{X}$ is $(0, T)$-reciprocal but this follows from Lemma 1.4.

Conversely, let us suppose that Γ is the covariance matrix function of a $(0, T)$-reciprocal Gaussian process. According to Lemmas 1.4 and 1.2 for any fixed $t_0 \in (0, T)$ there exist on (t_0, T) Φ_{t_0} and Ψ_{t_0} satisfying conditions (1) and (2) of Lemma 1.1 such that

$$\Gamma(t-s) - \Gamma^{\mathrm{T}}(s-t_0)\,\Gamma(t-t_0) = \Phi_{t_0}(s)\,\Psi^{\mathrm{T}}_{t_0}(t), \qquad t_0 < s \leqslant t < T.$$

Next it is easy to extend Φ_{t_0} and Ψ_{t_0} on $(t_0, T + t_0)$ such that conditions (1) and (2) of Lemma 1.1 are satisfied. Since these extensions do not depend on t_0, set $\Phi(u) = \Phi_{t_0}(u + t_0)$ and $\Psi(u) = \Psi_{t_0}(u + t_0)$ for $u \in (0, T)$; clearly these matrix functions satisfy the preceding two conditions as well as the first equation of (1.3). The same argument can be used for the construction of Φ^* and Ψ^*.

Remark 1.6. Given the covariance matrix function of a $(0, T)$-reciprocal process Γ, and setting

$$K(s, t) = \Gamma(t-s) - \Gamma^{\mathrm{T}}(s)\,\Gamma(t), \qquad 0 < s \leqslant t < T,$$

we can always [13, p. 55] exhibit Φ and Ψ. For a fixed $\tau \in (0, T)$, take

$$\Phi(s) = \begin{cases} K(s, \tau) & \text{for } s \leqslant \tau, \\ K(s, s)(K(\tau, s))^{-1}\,K(\tau, \tau) & \text{for } s > \tau, \end{cases}$$

and

$$\Psi(t) = \begin{cases} K(t, t)(K^{\mathrm{T}}(t, c))^{-1} & \text{for } t \leqslant \tau, \\ K^{\mathrm{T}}(\tau, t)(K(\tau, \tau))^{-1} & \text{for } t > \tau. \end{cases}$$

The same argument holds for $\Gamma^* \equiv \Gamma^{\mathrm{T}}$.

2. A Differential Equation Satisfied by the Covariance Matrix Function

Let Γ be the covariance matrix function of a $(0, T)$-reciprocal Gaussian process.

PROPOSITION 2.1. *Suppose that Γ is continuous at 0. Then Γ is infinitely differentiable on $(0, T)$. Moreover, $\Gamma^{(n)}(0+)$ exists for all $n \geqslant 1$.*

Proof. For the first part, see [8, p. 10]. The existence of $\Gamma^{(n)}(0+)$, $n \geqslant 1$, follows by letting $y \downarrow x > 0$ in

$$\Gamma^{(n)}(y) = C(x)\,\Gamma^{(n)}(y-x) + D(x)\,\Gamma^{(n)}(y+x),$$

where

$$C(x) = [\Gamma(x) - \Gamma^{\mathrm{T}}(x)\,\Gamma(2x)][I - \Gamma^{\mathrm{T}}(2x)\,\Gamma(2x)]^{-1},$$
$$D(x) = \Gamma^{\mathrm{T}}(x) - C(x)\,\Gamma^{\mathrm{T}}(2x),$$

with $0 < x < x + y < T$.

We are now in a position to give a differential equation which is necessarily satisfied by Γ.

THEOREM 2.2. *Suppose that Γ is continuous at 0. Then there exist an integer n, $0 \leqslant n \leqslant m$ (the smallest) and $n+1$ constant matrices $M_{n,i}$, $0 \leqslant i \leqslant n$, such that Γ satisfies the following differential equation:*

$$\Gamma^{(n+1)}(t) + \sum_{i=0}^{n} M_{n,i}\,\Gamma^{(i)}(t) = 0, \qquad t \in (0, T). \tag{2.1}$$

Proof. Let $r \geqslant 1$ and, set

$$\gamma_r(t) = (-1)^r\,\Gamma^{(r)}(t) - (\Gamma^{(r)}(0+))^{\mathrm{T}}\,\Gamma(t) = \Phi^{(r)}(0+)\,\Psi^{\mathrm{T}}(t).$$

Further let d_r be the dimension of the linear subspace spanned by the rows of $\Phi^{(i)}(0+)$, $1 \leqslant i \leqslant r$.

If $d_1 = 0$, we obtain (2.1) from

$$\gamma_1(t) = 0, \qquad t \in (0, T).$$

If $d_1 > 0$ and if there exists $n < m$ such that $d_n = d_{n+1}$, then there exists n (not necessarily unique) constant matrices $A_{n,i}$, $1 \leqslant i \leqslant n$, such that

$$\Phi^{(n+1)}(0+) + \sum^{n} A_{n,i}\,\Phi^{(i)}(0+) = 0 \tag{2.2}$$

and we obtain (2.1) from

$$\gamma_{n+1}(t) + \sum_{i=1}^{n} A_{n,i}\gamma_i(t) = 0, \qquad t \in (0, T),$$

with

$$M_{n,0} = (-1)^n \sum_{i=1}^{n+1} A_{n,i}\,(\Gamma^{(i)}(0+))^{\mathrm{T}}, \qquad A_{n,n+1} = I,$$

$$M_{n,i} = (-1)^{n+i+1} A_{n,i}, \qquad 1 \leqslant i \leqslant n.$$

Finally if $d_1 > 0$ and d_i, $1 \leqslant i \leqslant m$, are all distinct, then $d_m = m$ and (2.2) holds with $n = m$.

Remark 2.3. Suppose that there exists a smallest integer $n \geqslant 1$ such that rank $\Phi^{(n)}(0+) = m$ (which is equivalent to the invertibility of $\gamma_n(\cdot)$ on $(0, T)$). Then $d_{n+1} = d_n = m$ and there exists two unique constant matrices A_n and B_n such that Γ satisfies the following differential equation:

$$\Gamma^{(n+1)}(t) + A_n\,\Gamma^{(n)}(t) + B_n\,\Gamma(t) = 0, \qquad t \in (0, T), \tag{2.3}$$

with

$$\begin{aligned} A_n &= \phi^{(n+1)}(0+)(\phi^{(n)}(0+))^{-1} = \gamma_{n+1}(t)\,\gamma_n^{-1}(t), \qquad t \in (0, T), \\ B_n &= (-1)^{n+1}[A_n(\Gamma^{(n)}(0+))^{\mathrm{T}} - (\Gamma^{(n+1)}(0+))^{\mathrm{T}}]. \end{aligned} \tag{2.4}$$

If $n \leqslant m$ we obtain (2.3) from (2.1) by taking

$$M_{n,0} = B_n,$$

$$M_{n,i} = 0, \qquad 1 \leqslant i < n,$$

$$M_{n,n} = A_n.$$

Although (2.3) has a simpler form than (2.1) it may be of higher order.

Let us now examine the following problem. Given the differential equation (2.1) (or (2.3)) with the initial conditions $\Gamma^{(i)}(0+)$, $0 \leqslant i < n$, find those solutions which are matrix covariance functions of a $(0, T)$-reciprocal Gaussian process.

We begin by finding the general solution of (2.1) (or (2.3)). This is, at least theoretically, always possible. Indeed (2.1) (or (2.3)) can be reduced to m linear differential systems each with $m(n+1)$ unknown functions: the elements of the columns of Γ and their derivatives up to the order n. With the general solutions of these systems in the form of exponential matrices, we can find the roots of the characteristic equations of the exponent matrices. Given the initial conditions we are then led to the unique solution of (2.1) (or (2.3)).

Given the initial conditions, we must now verify whether the unique solution of (2.1) (or (2.3)) is the covariance matrix function of a $(0, T)$-reciprocal Gaussian process. We begin by checking that $\Gamma \in \mathscr{G}$. This verification leads to the restriction of the domain for the parameters (the initial conditions) appearing in the solution if $T>0$ is given; if the parameters are given then this verification leads to the restriction of the domain of T. Further if $\gamma_n(\cdot)$ is invertible on $(0, T)$, we can normalize $\Phi^{(n)}(0+)=I$ and take $\gamma_n(\cdot)$ and $[I-\Gamma^{\mathrm{T}}(\cdot)\,\Gamma(\cdot)]\,\gamma_n^{-1}(\cdot)$ for $\Psi^{\mathrm{T}}(\cdot)$ and $\Phi(\cdot)$ respectively; these $\Phi(\cdot)$ and $\Psi(\cdot)$ satisfy conditions (1) and (2) of Lemma 1.1, and it remains to check (1.3) to be sure that Theorem 1.5 applies. If $\gamma_n(\cdot)$ is not invertible on $(0, T)$, we refer to Remark 1.6 for obtaining candidates for $\Psi^{\mathrm{T}}(\cdot)$ and $\Phi(\cdot)$. It goes without saying that according to Theorem 1.5 a similar verification must be made for Γ^*.

3. Examples

A. *The Markov Case*

We know that a continuous in quadratic mean Gaussian Markov process on $(0, \infty)$ has the covariance matrix function $\Gamma(t)=\exp(-At)$, $t \geqslant 0$, satisfying the differential equation

$$\Gamma'(t)+A\Gamma(t)=0, \qquad t \in (0, \infty), \tag{3.1}$$

where $A=-\Gamma'(0+)$ and $A+A^{\mathrm{T}}$ is nonnegative definite [5, p. 12].

As already noted, Markov processes satisfy Definition 1.3; however $I-\Gamma^{\mathrm{T}}(\cdot)\,\Gamma(\cdot)$ is not necessarily positive definite on $(0, \infty)$. According to [3], $I-\Gamma^{\mathrm{T}}(\cdot)\,\Gamma(\cdot)$ is positive definite on $(0, \infty)$ if and only if $A+A^{\mathrm{T}}$ is nonnegative definite and the real parts of the eigenvalues of A are all positive. Next the factorization (1.3) holds for $\Gamma(t)=\exp(-At)$, $t \geqslant 0$, with $\Phi(t)=\exp(At)-\exp(-A^{\mathrm{T}}t)=\Phi^{*\mathrm{T}}(t)$ and $\Psi^{\mathrm{T}}(t)=\exp(-At)=\Psi^*(t)$, $t \geqslant 0$. Hence, by Theorem 1.5, we conclude that $\Gamma(t)=\exp(-At)$, $t \geqslant 0$, is the covariance matrix function of a $(0, \infty)$-reciprocal Gaussian process and as such it is the unique solution of a differential equation of type (2.1) which, in this case, can be taken of the form (3.1).

The question remains open as to whether Markov processes are the only $(0, \infty)$-reciprocal Gaussian processes; for the univariate case [4] as well as for second-order processes (see Section 4, p. 37) the answer is positive.

An example of a 2×2 matrix A satisfying the above conditions is $A=\left(\begin{smallmatrix} 0 & 1 \\ -1 & w \end{smallmatrix}\right)$, $w>0$, corresponding to a second-order stationary Markov process [10] (see also Section 4, p. 65).

B. *The Slepian Case*

Let us consider the case when Γ satisfies (2.3) with $n=1$ and $A_1=B_1=0$:

$$\Gamma''(t)=0, \qquad t\in(0, T). \tag{3.2}$$

We immediately obtain the unique solution of (3.2),

$$\Gamma(t)=I+\Gamma'(0+)\,t, \qquad t\in(0, T). \tag{3.3}$$

The factorization of Γ holds: $\Phi(t)=It=\Phi^*(t)$, $\Psi^{\mathrm{T}}(t)=\gamma_1(t)=A+A^{\mathrm{T}}-A^{\mathrm{T}}At$, $\Psi^{*\mathrm{T}}(t)=A+A^{\mathrm{T}}-AA^{\mathrm{T}}t$, $t\in(0, T)$, where $A=-\Gamma'(0+)$. Then $\Gamma(\cdot)$ given by (3.3) corresponds to a $(0, T)$-reciprocal Gaussian process if and only if $\det(A+A^{\mathrm{T}})>0$ and $A+A^{\mathrm{T}}-A^{\mathrm{T}}AT$ is nonnegative definite. For $m=2$ we obtain $\det(A+A^{\mathrm{T}})>0$, $\operatorname{tr} A>0$, and

$$T\leqslant\{\operatorname{tr} A-[(\operatorname{tr} A)^2-\det(A+A^{\mathrm{T}})]^{1/2}\}/\det A;$$

for $m=1$ we obtain $0<A\leqslant 2/T$. In analogy with the one-dimensional case (see [7, p. 1630]) such a process may be referred to as a multivariate Slepian process.

C. *The Blockwise Independent Case*

Let us suppose that Γ satisfies (2.3) with $n=1$ and $A_1=0$:

$$\Gamma''(t)+B_1\Gamma(t)=0, \qquad t\in(0, T), \tag{3.4}$$

with the initial condition $\Gamma'(0+)$ assumed to be a diagonal matrix. If $B_1=-\Gamma''(0+)$ is also a diagonal matrix with diagonal elements b_i, $1\leqslant i\leqslant m$, then the solution $\Gamma(\cdot)=(p_{ij}(\cdot))$ of (3.4) is given by

$$p_{ij}(t)=\delta_{ij}\rho_i(t), \qquad 1\leqslant i, j\leqslant m, \qquad t\in(0, T),$$

where δ_{ij} is the Kronecker symbol and ρ_i is the solution to the differential equation

$$\rho_i''(t)+b_i\rho_i(t)=0, \qquad i\leqslant 1\leqslant m, \qquad t\in(0, T) \tag{3.5}$$

Since Γ is diagonal the process involved has independent components and each component is a one-dimensional $(0, T)$-reciprocal Gaussian process. In view of [4, p. 292] each of the unique solution ρ_i of (3.5), $1\leqslant i\leqslant m$, say ρ may have only one of the following three forms, $t\in(-T, T)$, depending on whether b_i, $1\leqslant i\leqslant m$, say b, is <0, >0, or $=0$:

$$\rho(t) = \cosh at + a^{-1}\rho'(0+)\sinh a|t|,$$

$$a > 0, \qquad -a\coth(aT/2) \leqslant \rho'(0+) \leqslant a\tanh(aT/2),$$

$$\rho(t) = \cos at + a^{-1}\rho'(0+)\sin a|t|,$$

$$0 < a \leqslant \pi/T, \qquad -a\cot(aT/2) \leqslant \rho'(0+) \leqslant 0,$$

$$\rho(t) = 1 + \rho'(0+)|t|, \qquad -2/T \leqslant \rho'(0+) < 0,$$

where $a = |b|^{1/2}$. If the elements of $\Gamma'(0+)$, i.e., $\rho_i'(0+)$, $1 \leqslant i \leqslant m$, lie within the admissible domains indicated above, then Γ corresponds to a multivariate $(0, T)$-reciprocal Gaussian process.

With slight modifications, the above reasoning applies when B_1 in (3.4) is a block-diagonal matrix. If $\Gamma'(0+)$ is also block-diagonal with the same structure and its block-diagonal elements are $\Gamma_l'(0+)$, $1 \leqslant l \leqslant r$, then we are led to a block-diagonal solution Γ of (3.4) whose block-diagonal elements Γ_l, $1 \leqslant l \leqslant r$, correspond to multivariate $(0, T)$-reciprocal Gaussian processes of lesser dimension insofar as their $\Gamma_l'(0+)$, $1 \leqslant l \leqslant r$, lie whithin the admissible domains.

From blockwise independent multivariate $(0, T)$-reciprocal Gaussian processes we can construct reciprocal processes with dependent components by using orthogonal matrices. For these processes the admissible domains are easily described.

D. *Reciprocity on Small Intervals*

Let us consider the case $m = 2$, i.e.,

$$\Gamma(\cdot) = \begin{pmatrix} p_{11}(\cdot) & p_{12}(\cdot) \\ p_{21}(\cdot) & p_{22}(\cdot) \end{pmatrix},$$

and set $h(\cdot) = \det M(\cdot)$, $M(\cdot) = I - \Gamma^{\mathrm{T}}(\cdot)\,\Gamma(\cdot)$. If $M(\cdot)$ is positive definite on $(0, T)$, then necessarily $M'(0+) = -\Gamma'(0+) - (\Gamma'(0+))^{\mathrm{T}}$ is nonnegative definite and this holds if and only if

$$\begin{aligned} &\max\{p_{11}'(0+), p_{22}'(0+)\} \leqslant 0, \\ &\det(\Gamma'(0) + (\Gamma'(0+))^{\mathrm{T}} \geqslant 0. \end{aligned} \tag{3.6}$$

PROPOSITION 3.1. *Let $\Gamma(\cdot)$ be the unique solution of* (2.1) *with given initial conditions and suppose that*: (a) $\operatorname{tr}\Gamma'(0+) < 0$; (b) *the condition of Theorem* 1.5 *is satisfied*; (c) *there exists an integer $k \geqslant 2$ such that $h^{(k)}(0+) > 0$ and $h^{(j)}(0+) = 0$, $0 \leqslant j < k$. Then there exists an $\varepsilon > 0$ such that $\Gamma(\cdot)$ is the covariance matrix function of a $(0, \varepsilon)$-reciprocal Gaussian process.*

Proof. If, in addition to (3.6), $\operatorname{tr}\Gamma'(0+)<0$, then $M(\cdot)$ is positive definite on $(0,\varepsilon)$, $\varepsilon>0$, if and only if $h(\cdot)>0$ on $(0,\varepsilon)$. Further

$$h(\cdot)=1+[p_{11}(\cdot)\,p_{22}(\cdot)-p_{12}(\cdot)\,p_{21}(\cdot)]^2 \\ -p_{11}^2(\cdot)-p_{12}^2(\cdot)-p_{21}^2(\cdot)-p_{22}^2(\cdot)$$

and $h(0)=0$ and $h'(0+)=0$. In view of (c) and then (b) we are led to our result.

Consider, for example, a two-dimensional Slepian process. If $\operatorname{tr} A>0$ and $\det(A+A^{\mathrm{T}})>0$, then according to Proposition 3.1 we are sure that the solution Γ corresponds to a $(0,\varepsilon)$-reciprocal process with a sufficiently small $\varepsilon>0$.

4. Second-Order Reciprocal Processes

For the remainder of this section we shall assume $m=2$.

Definition 4.1. Let $\mathscr{Z}=\{Z(t)\colon t\in(a,b)\}$ be a real-valued differentiable in quadratic mean process and let $\mathscr{Z}'$ denote its derivative. $\mathscr{Z}$ is called a *second-order (c,d)-reciprocal process* if and only if $\mathscr{X}=\{X(t)\colon t\in(a,b)\}$, $X(t)=(\mathscr{Z}(t),\mathscr{Z}'(t))^{\mathrm{T}}$, $t\in(a,b)$, is a (c,d)-reciprocal process.

In what follows we shall consider only second-order processes $\mathscr{Z}$ such that its associated $\mathscr{X}$ satisfies the conditions already assumed in the previous sections.

Set $p(t)=E(Z(\tau)\,Z(\tau+t))$, $0\leqslant\tau\leqslant\tau+t<T$. Then the covariance matrix function Γ of $\mathscr{X}$ is given by

$$\Gamma(t)=\begin{pmatrix} p(t) & p'(t)\\ -p'(t) & -p''(t)\end{pmatrix},\qquad t\in(-T,T).$$

According to Proposition 2.1, Γ is infinitely differentiable on $(0,T)$ and $\Gamma^{(r)}(0+)$ exists, $r\geqslant 1$.

For $r\geqslant 1$ let us set

$$\gamma_r=\gamma_r(0+).$$

Remark 4.2. If γ_n is invertible, then $\gamma_n(\cdot)$ is also invertible on $(0,T)$ and in this case (cf. (2.4))

$$A_n=\gamma_{n+1}\,\gamma_n^{-1}. \tag{4.1}$$

Remark 4.3. Set $p^{(k)}(0+)=w_k$, $k\geqslant 0$; then $w_0=1$, $w_1=0$, $w_2=-1$, and $w_3\geqslant 0$. We obtain

$$\gamma_{2j}=\begin{pmatrix} 0 & 2w_{2j+1} \\ -2w_{2j+1} & 0 \end{pmatrix},\qquad \gamma_{2j+1}=\begin{pmatrix} -2w_{2j+1} & 0 \\ 0 & 2w_{2j+3} \end{pmatrix},\quad j\geqslant 0. \tag{4.2}$$

The first w_{2j+1}, $j\geqslant 1$, which eventually does not vanish is w_3. Then, for $w_3>0$, (2.3) becomes

$$\Gamma^{(3)}(t)+A_2\Gamma^{(2)}(t)+B_2\Gamma(t)=0,\qquad t\in(0,T), \tag{4.3}$$

and in view of (4.1) and (4.2), we obtain from (2.4),

$$A_2=\begin{pmatrix} 0 & 1 \\ w_3^{-1}w_5 & 0 \end{pmatrix},\qquad B_2=\begin{pmatrix} 0 & 0 \\ w_4+w_3^{-1}w_5 & 0 \end{pmatrix}.$$

Due to the special form of Γ, (4.3) is equivalent to the following differential equation satisfied by $p(\cdot)$:

$$p^{(4)}(t)+\alpha p''(t)+\beta p(t)=0,\qquad t\in(0,T), \tag{4.4}$$

with $\alpha=-w_3^{-1}w_5$ and $\beta=-w_4-w_3^{-1}w_5$. This differential equation corrects that proposed by Miroshin in his Lemma 5 [10, p. 849], which, by the way, is not satisfied by Markov processes as noted in [1, p. 190].

Further, for the case under consideration, the matrices Φ, Ψ and Φ^*, Ψ^* in Theorem 1.5 are of the form

$$\Phi(t)=\begin{pmatrix} f_1(t) & f_2(t) \\ f_1'(t) & f_2'(t) \end{pmatrix},\qquad \Psi(t)=\begin{pmatrix} g_1(t) & g_2(t) \\ g_1'(t) & g_2'(t) \end{pmatrix},\quad t\in(0,T),$$

and

$$\Phi^*=L\Phi L,\qquad \Psi^*=L\Psi L,\qquad L=\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

If instead of Γ we use p then (1.3) is equivalent to

$$p(t-s)-p(s)\,p(t)-p'(s)\,p'(t)=f_1(s)\,g_1(t)+f_2(s)\,g_2(t),\qquad 0<s\leqslant t<T. \tag{4.5}$$

We are now interested in identifying all solutions p of (4.4) which are covariance functions of second-order $(0,T)$-reciprocal Gaussian processes by making use of Theorem 1.5. Note that in what follows α and β are not necessarily of the form $\alpha=-w_3^{-1}w_5$ and $\beta=-w_4-w_3^{-1}w_5$.

THEOREM 4.4. *Let* $(\alpha, \beta) \in \mathbb{R}^2$ *and let* $p(\cdot)$ *be the solution of the differential equation* (4.4) *together with the initial conditions* $w_0 = 1$, $w_1 = 0$, $w_2 = -1$, *and* $w_3 \geqslant 0$ *arbitrary. Then there exist four real-valued functions* f_1, f_2, g_1, *and* g_2 *defind on* $(0, T)$ *such that* $p(\cdot)$ *satisfies* (4.5).

Proof. Let $0 < s \leqslant t < T$ and set $F_t(s) = p(t-s) - p(s)\,p(t) - p'(s)\,p'(t)$. It is easily seen that $F_t(\cdot)$ is a solution of (4.4) and

$$\begin{aligned} F_t(0) &= 0, \\ F_t'(0) &= 0, \\ F_t''(0+) &= p''(t) + p(t) - p^{(3)}(0+)\,p'(t), \\ F_t^{(3)}(0+) &= -p^{(3)}(t) - p^{(3)}(0+)\,p(t) - p^{(4)}(0+)\,p'(t). \end{aligned}$$

Further, take two solutions f_1 and f_2 of (4.4) with the following initial conditions:

$$\begin{aligned} f_1(0) &= f_1'(0) = 0, & f_2(0) &= f_2'(0) = 0, \\ f_1''(0+) &= 1, & f_2''(0+) &= 0, \\ f_1^{(3)}(0+) &= 0, & f_2^{(3)}(0+) &= 1 \end{aligned}$$

and set

$$G_t(s) = f_1(s)\,F_t''(0+) + f_2(s)\,F_t^{(3)}(0+).$$

Clearly $G_t(\cdot)$ is a solution of (4.4) and

$$\begin{aligned} G_t(0) &= F_t(0), & G_t'(0) &= F_0'(0), \\ G_t''(0+) &= F_t''(0+), & G_t^{(3)}(0+) &= F_t^{(3)}(0+). \end{aligned}$$

Since the initial conditions for $F_t(\cdot)$ and $G_t(\cdot)$, are identical, we conclude that $F_t(\cdot) = G_t(\cdot)$. Thus, setting $g_1(t) = F_t''(0+)$ and $g_2(t) = F_t^{(3)}(0+)$, we obtain (4.5).

Remark 4.5. The construction of f_1 and f_2 in the proof of Theorem 4.4 shows that we can always assume $\Phi''(0+) = I$. Note also that g_1 and g_2 are solutions of (4.4).

According to Theorem 2.2, we are sure that the differential equation (4.4) together with the initial conditions $w_0 = 1$, $w_1 = 0$, $w_2 = -1$, and $w_3 \geqslant 0$ arbitrary lead to a unique solution $p(\cdot)$ which satisfies (4.5). Since we want that $p(\cdot)$ to be the covariance function of a second-order $(0, T)$-reciprocal Gaussian process, we have to extend $p(\cdot)$ on $(-T, T)$ by setting $p(-t) = p(t)$, $t \in (0, T)$, and find the restriction to be imposed on w_3 such that Γ constructed with this p belongs to the class $\mathscr{G}$. It is easily seen that Γ is continuous at 0, $\Gamma(0)=I$, $\Gamma(-t)=\Gamma^{\mathrm{T}}(t)$, $t\in(-T,T)$. It remains to examine the matrix $M(\cdot)=I-\Gamma^{\mathrm{T}}(\cdot)\,\Gamma(\cdot)$. Then $h(\cdot)=\det M(\cdot)=k_{-}(\cdot)\,k_{+}(\cdot)$, where

$$k_{\pm}(\cdot)=(1\pm p(\cdot))(1\pm p''(\cdot))-(p'(\cdot))^2. \tag{4.6}$$

THEOREM 4.6. *Suppose*: (a) $w_3>0$ *or* (b) $w_3=$ *and* $w_4>1$. *Then* $M(\cdot)=(m_{ij}(\cdot))$, $1\leqslant i,j\leqslant 2$, *is positive definite on* $(0,T)$ *if and only if* $h(\cdot)>0$ *on* $(0,T)$.

Proof. The necessity is obvious. For the sufficiency consider $m_{22}(\cdot)$ given by

$$m_{22}(\cdot)=1-(p'(\cdot))^2-(p''(\cdot))^2.$$

Then $m_{22}'(0+)=2w_3$ and $m_{22}''(0+)=2(w_4-1)-2w_3^2$ which implies in both cases (a) and (b) that there exists an $\varepsilon>0$ such that $m_{22}(t)>0$ for $t\in(0,\varepsilon)$. If $m_{22}(t)=0$, $t_0\in(0,T)$, then $h(t_0)<0$ because $m_{12}=m_{21}$, which is a contradiction.

COROLLARY 4.7. *Let* $p(\cdot)$ *be the unique solution of* (4.4) *together with the initial conditions* $w_0=1$, $w_1=0$, $w_2=-1$, $w_3>0$ *or* $w_3=0$, *and* $w_4>1$. *Then* p *is the covariance function of a second-order* $(0,T)$*-reciprocal Gaussian process if and only if* $h(\cdot)>0$ *on* $(0,T)$.

Remark 4.8. When $w_3=0$ the proof of Theorem 4.6 shows that necessarily $w_4-1\geqslant 0$ for the positive definiteness of $M(\cdot)$ on $(0,T)$.

Remark 4.9. An immediate consequence of Theorems 1.5, 4.4, 4.6, and of the fact that $h^{(4)}(0+)=8w_3^2$ is that (cf. Proposition 3.1) there exists an $\varepsilon>0$ such that the unique solution $p(\cdot)$ of (4.4) together with the initial conditions $w_0=1$, $w_1=0$, $w_2=-1$, and $w_3>0$ is the covariance function of a second-order $(0,\varepsilon)$-reciprocal Gaussian process.

(A) *The Case* $w_3>0$

Set $w=w_3>0$ and let λ_i, $1\leqslant i\leqslant 4$, be the roots of the biquadratic characteristic equation associated with (4.4),

$$\lambda_1=[(-\alpha+\Delta^{1/2})/2]^{1/2},\qquad \lambda_2=-\lambda_1,$$
$$\lambda_3=[(-\alpha-\Delta^{1/2})/2]^{1/2},\qquad \lambda_4=-\lambda_3,$$

where $\Delta=\alpha^2-4\beta$. There are nine regions in the (α,β)-plane corresponding to the nine forms of the unique solutions p of (4.4) which are all dependent upon the initial condition $w>0$ (see Table I). Region Ⓜ where the process

TABLE I

Regions Corresponding to the Unique Solutions of Eq. (4.4)

Region	Parameters α and β	Characteristic roots λ_1 and λ_3
①	$\Delta=0, \alpha=0$	$\lambda_1=\lambda_3=0$
②	$\Delta=0, \alpha<0$	$\lambda_1=\lambda_3=\lambda=(-\alpha/2)^{1/2}$
③	$\Delta=0, \alpha>0$	$\lambda_1=\lambda_3=\lambda=i(\alpha/2)^{1/2}$
④	$\Delta>0, \alpha<0, \beta=0$	$\lambda_1=0, \lambda_3=(-\alpha)^{1/2}$
⑤	$\Delta>0, \alpha>0, \beta=0$	$\lambda_1=0, \lambda_3=i(\alpha)^{1/2}$
⑥	$\Delta>0, \alpha<0, \beta>0$	$\lambda_1\neq\lambda_3$, both real
⑦	$\Delta>0, \alpha>0, \beta>0$	$\lambda_1\neq\lambda_3$, both imaginary
⑧	$\Delta>0, \beta<0$	λ_1 real, λ_3 imaginary
⑨	$\Delta<0$	$\lambda_3=\bar{\lambda}_1$, both complex

is Markovian is defined by the half-line $\alpha<2$, $\beta=1$ going across Regions ⑥, ②, and ⑨.

Further, we list the nine forms of the solution p corresponding to these regions as well as their factorization (4.5) (with $\Phi''(0+)=I$). Since $p(-t)=p(t)$, we consider only the case $t\in[0, T)$.

Region ①:

$$p(t)=1-(\tfrac{1}{2})\,t^2+(\tfrac{1}{6})\,wt^3, \qquad t\in[0, T),$$

$$f_1(s)=(\tfrac{1}{2})\,s^2,$$

$$f_2(s)=(\tfrac{1}{6})\,s^3,$$

$$g_1(t)=2wt-(\tfrac{1}{2})(1+w^2)\,t^2+(\tfrac{1}{6})\,wt^3,$$

$$g_2(t)=-2w+(\tfrac{1}{2})\,wt^2-(\tfrac{1}{6})\,w^2\,t^3.$$

Region ②:

$$p(t)=\cosh\lambda t-(\tfrac{1}{2})\,w\lambda^{-3}\sinh\lambda t$$
$$+(\tfrac{1}{2})\,w\lambda^{-2}\,t\cosh\lambda t-(\tfrac{1}{2})\,\lambda^{-1}(\lambda^2+1)\,t\sinh\lambda t, \qquad t\in[0, T),$$

$$f_1(s)=(\tfrac{1}{2})\,\lambda^{-1}\,s\sinh\lambda s,$$

$$f_2(s)=-(\tfrac{1}{2})\,\lambda^{-3}\sinh\lambda s+(\tfrac{1}{2})\,\lambda^{-2}\,s\cosh\lambda s,$$

$$g_1(t)=-(\tfrac{1}{2})\,w\lambda^{-3}(\lambda^2-1)^2\sinh\lambda t$$
$$-(\tfrac{1}{2})\,\lambda^{-1}[(\lambda^2+1)^2+w^2]\,t\sinh\lambda t$$
$$+(\tfrac{1}{2})\,w\lambda^{-2}(\lambda^2+1)^2\,t\cosh\lambda t.$$

$$g_2(t)=(\tfrac{1}{2})\,\lambda^{-2}[\lambda^4(\lambda^2+1)^2+w^2][\lambda^{-1}\sinh\lambda t-t\cosh\lambda t]$$
$$+(\tfrac{1}{2})\,w\lambda^{-1}(1+\lambda^2)^2\,t\sinh\lambda t-2w\cosh\lambda t.$$

Region ③: p, as well as the corresponding functions f and g, are obtained from those of Region ② by replacing the hyperbolic sines and cosines by trigonometric ones using the identities $\sinh iu = i \sin u$ and $\cosh iu = \cos u$.

Region ④:

$$p(t) = 1 + \lambda_3^{-2} - w\lambda_3^{-2} t - \lambda_3^{-2} \cosh \lambda_3 t + w\lambda_3^{-3} \sinh \lambda_3 t, \qquad t \in [0, T),$$

$$f_1(s) = -\lambda_3^{-2} + \lambda_3^{-2} \cosh \lambda_3 s,$$

$$f_2(s) = -\lambda_3^{-2} s + \lambda_3^{-3} \sinh \lambda_3 s,$$

$$\begin{aligned} g_1(t) = {} & \lambda_3^{-2} (\lambda_3^2 + 1 + w^2)(1 - \cosh \lambda_3 t) \\ & + w\lambda_3^{-3} (2\lambda_3^2 + 1) \sinh \lambda_3 t + w\lambda_3^{-2} t, \end{aligned}$$

$$g_2(t) = -w\lambda_3^{-2} (2\lambda_3^2 - wt) - w\lambda_3^{-2} (1 - \cosh \lambda_3 t) - w^2 \lambda_3^{-3} \sinh \lambda_3 t.$$

Region ⑤: Same remark as for Region ③ applied to Region ④.

Region ⑥:

$$\begin{aligned} p(t) = {} & (\lambda_1^2 - \lambda_3^2)^{-1} \{ (\lambda_1^2 + 1) \cosh \lambda_3 t - (\lambda_3^2 + 1) \cosh \lambda_1 t \\ & - w\lambda_3^{-1} \sinh \lambda_3 t + w\lambda_1^{-1} \sinh \lambda_1 t \}, \qquad t \in [0, T), \end{aligned}$$

$$f_1(s) = (\lambda_1^2 - \lambda_3^2)^{-1} (\cosh \lambda_1 s - \cosh \lambda_3 s),$$

$$f_2(s) = (\lambda_1^2 - \lambda_3^2)^{-1} (\lambda_1^{-1} \sinh \lambda_1 s - \lambda_3^{-1} \sinh \lambda_3 s),$$

$$\begin{aligned} g_1(t) = {} & (\lambda_1^2 - \lambda_3^2)^{-1} \{ [\lambda_1^2 + 1)(\lambda_3^2 + 1) + w^2] (\cosh \lambda_3 t - \cosh \lambda_1 t) \\ & - w \{ [\lambda_3^2 (\lambda_1^2 + 1) + (\lambda_3^2 + 1)] \lambda_3^{-1} \sinh \lambda_3 t \\ & + [\lambda_1^2 (\lambda_3^2 + 1) + (\lambda_1^2 + 1)] \lambda_1^{-1} \sinh \lambda_1 t \} \}, \end{aligned}$$

$$\begin{aligned} g_2(t) = {} & (\lambda_1^2 - \lambda_3^2)^{-1} \{ [\lambda_1^2 \lambda_3^2 (\lambda_1^2 + 1)(\lambda_3^2 + 1) + w^2] \\ & \times (\lambda_3^{-1} \sinh \lambda_3 t - \lambda_1^{-1} \sinh \lambda_1 t) \\ & + w(\lambda_1^2 + 1)(\lambda_3^2 + 1)(\cosh \lambda_1 t - \cosh \lambda_3 t) \\ & - w(\cosh \lambda_1 t + \cosh \lambda_3 t) \}. \end{aligned}$$

Regions ⑦, ⑧, and ⑨: Same remark as for Region ③ applied to Region ⑥.

Newt we are faced with the following problem:

PROBLEM (I). For each point (α, β) and each $T > 0$, find the domain $D(T)$ for w such that the corresponding p is the covariance function of a second-order $(0, T)$-reciprocal Gaussian process.

Theorem 4.6 is instrumental in solving Problem (I). We want to ensure that $h(\cdot)$ is positive on $(0, T)$. This reduces to the examination of the factors $k_{\pm}(\cdot)$ of $h(\cdot)$ (cf. (4.6)). Due to the form of (4.4), these factors are second-degree polynomials in w whose coefficients depend upon $t \in (0, T)$ and (α, β) (through λ_1 and λ_3):

$$k_{\pm}(t, w, \lambda_1, \lambda_3) = A_{\pm}(t, \lambda_1, \lambda_3)\, w^2 + B_{\pm}(t, \lambda_1, \lambda_3)\, w + C_{\pm}(t, \lambda_1, \lambda_3). \qquad (4.7)$$

For the sake of simplicity we drop, in what follows, any reference to the dependence upon (λ_1, λ_3) in (4.7).

For $t > 0$ let $l_-(t)$, $L_-(t)$ and $l_+(t)$, $L_+(t)$ be the roots of $k_-(t, w) = 0$ and $k_+(t, w) = 0$ respectively and denote by $\Delta_{\pm}(t)$ the discriminant of $k_{\pm}(t, w)$. Set $b_{\pm} = \inf\{t > 0 \colon \Delta_{\pm}(t) = 0\}$; in view of Remark 2.7, $b_{\pm} > 0$; if $\Delta_{\pm}(t) > 0$ for any $t > 0$ we take $b_{\pm} = \infty$.

Remark 4.10. $l_{\pm}$, $L_{\pm}$ are strictly monotonic on $(0, b_{\pm})$ if and only if, for any $w > 0$, the equation in t, $k_{\pm}(t, w) = 0$, has at most one root $t \in (0, b_{\pm})$. We verified this criterion for all the forms of the solution p; this verification required sometimes lengthy and tedious work. For all the forms of p, we also showed that $l_{\pm}$ is an increasing function on $(0, b_{\pm})$ if $l_{\pm} > 0$ and that $L_{\pm}$ is a decreasing function on $(0, b_{\pm})$, $\lim_{t \downarrow 0} l_{\pm}(t) = 0$, and $\lim_{t \downarrow 0} L_{\pm}(t) = \infty$.

Set $l = \max\{0, l_-, l_+\}$, $L = \min\{L_-, L_+\}$, and $c = \inf\{t > 0 \colon l(t) = L(t)\}$; clearly $c > 0$ and if $l(t) < L(t)$ for any $t > 0$ we take $c = \infty$. Moreover $c \leqslant b_{\pm}$.

Remark 4.11. $l(t) < L(t)$, $t \in (0, c)$.

The answer to Problem (I) is summarized as follows: Let (α, β) and $T > 0$ be given; if $T \leqslant c$ then $D(T) = [l(t), L(t)]$ (for $l \equiv 0$ the value $w = 0$ is rejected by our assumption $w > 0$); if $T > c$, $D(T) = \varnothing$.

Another problem related to (I) is the following:

PROBLEM (II). For each point (α, β) and each $w > 0$, find the largest positive real number $\tau(w)$ such that the corresponding p is the covariance function of a second-order $(0, T)$-reciprocal Gaussian process for any $T \leqslant \tau(w)$.

The answer to Problem (II) is summarized as follows: Let (α, β) and $w > 0$ be given and set $\tau_{\pm}(w) = \min\{t > 0 \colon k_{\pm}(t, w) = 0\}$ if this set is non-empty; otherwise, set $\tau_+(w) = \infty$, $\tau_-(w)$ being analogously defined; then $\tau(w) = \min\{\tau_-(w), \tau_+(w)\}$. The relationship between (I) and (II) can be stated as: $D(T) = \{w \colon \tau(w) \geqslant T\}$,

$$\tau(w) = \begin{cases} \infty & \text{if for each } T > 0,\ w \in D(T), \\ \inf\{T > 0 \colon w \notin D(T)\} & \text{otherwise.} \end{cases}$$

Although c always exists, its explicit determination raises serious computational difficulties so that numerical approximations may be needed in a specific situation.

In what follows we give the coefficients $A_\pm$, $B_\pm$, and $C_\pm$ appearing in (4.7) for Regions ①, ②, ④, and ⑥ which enable us to determine c, $D(T)$, and $\tau(w)$. The coefficients for the other regions are determined by replacing the hyperbolic sines and cosines by trigonometric ones:

Region ①:

$$A_- = -(\tfrac{1}{12})\,t^4, \qquad A_+ = A_-,$$
$$B_- = (\tfrac{1}{6})\,t^3, \qquad B_+ = \tfrac{1}{2}(t^3 + 4t),$$
$$C_- = 0, \qquad C_+ = -t^2.$$

Region ②:

$$A_\pm = -(\tfrac{1}{4})\,\lambda^{-4}(\sinh^2 u - u^2),$$
$$B_\pm = (\tfrac{1}{2})\,\lambda^{-3}(\sinh u \pm u)[\lambda^2(\cosh u \pm 1) + \cosh u \mp 1],$$
$$C_\pm = -(\tfrac{1}{4})\,\lambda^{-2}(\lambda^2 + 1)^2(\sinh u \pm u)^2,$$
$$u = \lambda t.$$

Region ④:

$$A_- = -4\lambda^{-4} \sinh u(u \cosh u - \sinh u),$$
$$B_- = 4\lambda^{-3} \cosh u(u \cosh u - \sinh u),$$
$$C_- = 0,$$
$$A_+ = A_-,$$
$$B_+ = 4\lambda^{-3} \sinh u(u \sinh u + \lambda^2 \cosh u),$$
$$C_+ = -4 \sinh^2 u(1 + \lambda^{-2}),$$
$$u = (\tfrac{1}{2})\,\lambda t.$$

Region ⑥:

$$A_- = -4\lambda_1^{-1} \lambda_3^{-1}(\lambda_1^2 - \lambda_3^2)^{-2} \sinh^2 u_1 \sinh^2 u_3(\lambda_1 \coth u_3 - \lambda_3 \coth u_1)$$
$$\times (\lambda_1 \coth u_1 - \lambda_3 \coth u_3),$$
$$B_- = 4\lambda_1^{-1} \lambda_3^{-1}(\lambda_1^2 - \lambda_3^2)^{-1} \sinh^2 u_1 \sinh^2 u_3(\lambda_1 \coth u_1 - \lambda_3 \coth u_3)$$
$$\times (\lambda_1 \lambda_3 + \coth u_1 \coth u_3),$$

$$C_- = -4(\lambda_1^2+1)(\lambda_3^2+1)(\lambda_1^2-\lambda_3^2)^{-2}\sinh^2 u_1 \sinh^2 u_3$$
$$\times(\lambda_1 \coth u_1 - \lambda_3 \coth u_3)^2,$$

$$A_+ = A_-,$$
$$B_+ = 4\lambda_1^{-1}\lambda_3^{-1}(\lambda_1^2-\lambda_3^2)^{-1}\sinh^2 u_1 \sinh^2 u_3(\lambda_1 \coth u_3 - \lambda_3 \coth u_1)$$
$$\times(1+\lambda_1\lambda_3 \coth u_1 \coth u_3),$$
$$C_+ = -4(\lambda_1^2+1)(\lambda_3^2+1)(\lambda_1^2-\lambda_3^2)^{-2}\sinh^2 u_1 \sinh^2 u_3$$
$$\times(\lambda_1 \coth u_3 - \lambda_3 \coth u_1)^2.$$
$$u_1 = (\lambda_1/2)\,t, \qquad u_3 = (\lambda_3/2)\,t.$$

By way of illustration consider Region ①. In this case $l = l_+$, $L = L_-$, where

$$l_+(t) = t^{-3}[3t^2 + 12 - (-3t^4 + 72t^2 + 144)^{1/2}],$$
$$L_-(t) = 2t^{-1}, \qquad t > 0, \qquad \text{and} \qquad c = 2\sqrt{3}.$$

(I) For $0 < T \leqslant 2\sqrt{3}$ the admissible interval for w is $[l_+(T), 2T^{-1}]$.

(II) If $w > 0$ is given then $\tau_-(w) = 2w^{-1}$, $\tau_+(w) =$ the only real (positive) root of the equation in t, $w^2t^3 - 6wt^2 + 12t - 24w = 0$, and $\tau(w) = \min\{2w^{-1}, \tau_+(w)\}$.

The solution p corresponding to this region was also obtained by Miroshin [10, p. 850] but he failed to notice that T and w have to be restricted.

Finally let us examine Region Ⓜ corresponding to the Markov case. There are essentially three covariance functions associated with second-order Markov Gaussian processes as already shown by Miroshin [10, p. 847]. They can be represented as the (1, 1)-entry of the exponential matrix $\exp\{-At\}$, $t > 0$, where $A = \begin{pmatrix} 0 & 1 \\ -1 & w \end{pmatrix}$; namely Region ⑥: $w < 1$, Region ②: $w = 1$, and Region ⑨: $w > 1$. For each of these covariance functions, $c = +\infty$; in fact, the Markov case is the only one with this property. For further details on Markov covariance functions and on the relationship with the nondegeneracy condition (2), see [3].

(B) *The Case* $w_3 = 0$

Assume that $w_3 = 0$. In view of Remark 4.8 we also assume that $w_4 \geqslant 1$. Since $w_3 = 0$ implies $w_{2j+1} = 0$, $j \geqslant 2$, the possibility $w_4 = 1$ must also be rejected. Next, it is easily seen that we are led to a differential equation of the same form as (4.4) with $\alpha \in \mathbb{R}$ and $\beta = -w_4 + \alpha$, $w_4 > 1$. Then $\Delta \geqslant 0$ so that the only admissible solutions are those corresponding to Regions ⑦ or ⑧ by setting $w_3 = 0$.

REFERENCES

[1] ABRAHAMS, J. (1984). On Miroshin's second-order reciprocal processes. *SIAM J. Appl. Math.* **44**, 190–192.

[2] ABRAHAMS, J., AND THOMAS, J. B. (1981). Some comments on conditionally Markov and reciprocal Gaussian processes. *IEEE Trans. Inform. Theory* **IT–27**, 523–525.

[3] AUPETIT, B., AND MASSE, J. C. (1986). Contractions strictes et covariances markoviennes. *C. R. Math. Rep. Acad. Sci. Canada* **8**, 181–184.

[4] CARMICHAEL, J. P., MASSÉ, J. C., AND THEODORESCU, R. (1982). Processus gaussiens stationnaires réciproques sur un intervalle. *C. R. Acad. Sci. Paris Sér. I Math.* **295**, 291–293.

[5] CARMICHAEL, J. P., MASSÉ, J. C, AND THEODORESCU, R. (1984). Remarks on multiple Markov dependence. *Rend. Sem. Mat. Univ. Politec. Torino* **42**, 65–75.

[6] CHAY, S. C. (1972). On quasi-Markov random fields. *J. Multivariate Anal.* **2**, 14–76.

[7] JAMISON, B. (1970). Reciprocal processes: The stationary Gaussian case. *Ann. Math. Statist.* **41**, 1624–1630.

[8] KRENER, A. (1985). Reciprocal processes and the stochastic realization problem for acausal systems. In *Proceedings 7th International Symposium on Math. Theory of Networks and Systems, Stockholm.*

[9] MEHR, C. R., AND MCFADDEN, J. A. (1965). Certain properties of Gaussian processes and their first-passage times. *J. Roy. Statist. Soc. B* **27**, 505–522.

[10] MIROSHIN, R. N. (1979). Second-order Markov and reciprocal stationary Gaussian processes. *Theor. Probab. Appl.* **24**, 845–852.

[11] MOLČAN, G. M. (1971). Characterization of Gaussian fields with Markovian property. *Soviet Math. Dokl.* **12**, 563–567.

[12] MOLČAN, G. M. (1974). *L*-Markov Gaussian fields. *Soviet Math. Dokl.* **15**, 657–662.

[13] NEVEU, J. (1968). *Processus aléatoires gaussiens.* Presses Univ. Montréal, Montreal.

[14] PITT, L. D. (1971). A Markov property for Gaussian processes with a multidimensional parameter. *Arch. Rational Mech. Anal.* **43**, 367–391.