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Nuclear Physics B 901 (2015) 461–479

www.elsevier.com/locate/nuclphysb

Wilson loop invariants from W_N conformal blocks

Oleg Alekseev, Fábio Novaes*

International Institute of Physics, Federal University of Rio Grande do Norte, Av. Odilon Gomes de Lima 1722, Capim Macio, Natal-RN 59078-400, Brazil

Received 28 September 2015; accepted 2 November 2015

Available online 10 November 2015

Editor: Hubert Saleur

Abstract

Knot and link polynomials are topological invariants calculated from the expectation value of loop operators in topological field theories. In 3D Chern–Simons theory, these invariants can be found from crossing and braiding matrices of four-point conformal blocks of the boundary 2D CFT. We calculate crossing and braiding matrices for W_N conformal blocks with one component in the fundamental representation and another component in a rectangular representation of $SU(N)$, which can be used to obtain HOMFLY knot and link invariants for these cases. We also discuss how our approach can be generalized to invariants in higher-representations of W_N algebra.

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1. Introduction

In the late 80s, the important connection between quantum field theory and Jones polynomials [1] was uncovered by Edward Witten [2] showing that knot theory is deeply connected to topological QFTs. The particular example studied by Witten was $SU(N)$ Chern–Simons theory (CS) in a 3D compact manifold M . Non-trivial states in CS are topological as there are no dynamical degrees of freedom in this theory. Therefore, Wilson loops are natural elementary states of CS. To construct a non-trivial state in this sense, we must create knotted loops or link several loops together. The expectation value of these composite objects gives us certain topological invariants

* Corresponding author.

E-mail addresses: alekseev@itp.ac.ru (O. Alekseev), fabio.nsantos@gmail.com (F. Novaes).

called *knot (or link) polynomials*. For the $SU(2)$ case, Witten has shown that one obtains Jones polynomials from the Wilson loop expectation values. This relation between Chern–Simons and knot theory is an important example of integrability in quantum field theories, which serves as a tool to organize and construct physical theories.

The explicit construction of these invariants starts with a partition of M into two manifolds with a boundary with Wilson lines making punctures on those boundaries. With each boundary is associated a WZNW theory whose Hilbert space is the space of conformal blocks. Braiding matrices of the boundary CFT can then be used to construct the original knot or link in M [2–5]. Here we call this the *crossing-matrix method*. This approach can be related to quantum groups, as the crossing transformations of conformal blocks are directly related to $SU(N)_q$ quantum Racah matrices [6–9]. For the most recent and broad discussion of this method, see [10]. When combined with the evolution method [11] and cabling procedure [12], the Racah matrices can be extrapolated to give explicit formulas for many families of knots and links [13]. This combination of techniques gives not only Jones polynomials $J_R^C(q)$ [14] but can also be uplifted to calculate HOMFLY polynomials $H_R^C(q, A)$ [15,16] and *superpolynomials* $P_R^C(q, A, t)$.

Superpolynomials appeared in physics in the connection between topological string theories, M-theory and Chern–Simons [17–19]. Understanding better how HOMFLY polynomials come about in W_N models might also shed some light on the nature of superpolynomials. The crossing-matrix method has also been used to construct loop operators of $\mathcal{N} = 2$ gauge theories via its AGT relation with Liouville [20,21] and Toda field theory [22–24], so the study of W_N conformal blocks is also interesting for this AGT approach. Finally, the calculation of crossing matrices for W_N theories might also be interesting to understand higher-spin gravity/CFT duality of Gaberdiel and Gopakumar [25].

Going back to the relation between knots and conformal blocks, the crossing-matrix method is well-understood in the Virasoro case (W_2 algebra), but has not been explicitly developed before for W_N algebras to the authors knowledge. Here we develop the crossing-matrix approach for W_N minimal models directly from the CFT point of view in detail. Evidence has been put forward in [26] that knot and link invariants in W_N models should factorize in terms of $SU(N)_q$ invariants, but the crossing matrices have not been calculated. Much more is known about $SU(N)_q$ Racah matrices and topological invariants constructed with it [27,28], including representations with non-trivial multiplicities [29]. Therefore, as proposed in [26], we expect that W_N invariants should reduce to $SU(N)_q$ invariants and indeed that is what we find in the cases studied below. However, we still have limited information about higher-representations and it is not clear if the crossing-matrices will factorize in general for W_N correlators.

Four-point Virasoro conformal blocks need only one completely degenerate field to obey a hypergeometric differential equation, the BPZ equation [30]. However, for higher W_N algebras ($N > 2$), we need one more constraint to find a differential equation and to obtain explicit crossing S and braiding T matrices [31,32]. If we also set some other field to be semi-degenerate, the conformal blocks obey a generalized hypergeometric equation. In this note, we construct S and T matrices with two fields in the fundamental and anti-fundamental representations of $SU(N)$ and the other two in a rectangular representation and its conjugate. These cases are somewhat degenerate with respect to higher-representations of W_N primary fields, as the dimension of the space of conformal blocks is two-dimensional, but are the first step to obtain more general S-matrices for higher-representations [33,14]. In our particular case, the generalized hypergeometric equation reduces to a Gauss hypergeometric equation, for which the connection formulas are explicitly known and, thus, the crossing matrices. Finally, we also discuss how to obtain link

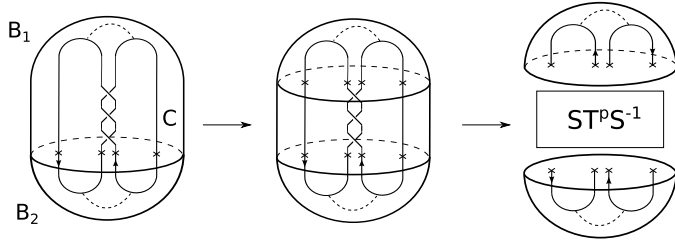


Fig. 1. Splitting a two-bridge knot C into an inner product of conformal blocks. The braiding and fusion matrices allow for a reconstruction of C from basic conformal blocks with zero weight intermediate states.

invariants with one fundamental component and another in an arbitrary representation using the generalized hypergeometric monodromy group.

In section 2, we revise the relation between Wilson loop operators in 3D Chern–Simons, knot invariants and conformal blocks. In section 3, we set up our notation by reviewing how to obtain knot invariants in Virasoro models. In section 4, we discuss how to calculate knot and link invariants from W_N conformal blocks. Finally, in section 5 we present our conclusions and discuss further developments.

2. Knot and link invariants from conformal blocks

Following Witten’s construction [2], we are interested in calculating the expectation value of non-trivial Wilson loops forming a knot or link C embedded in a closed three-dimensional manifold M in Chern–Simons theory. For simplicity, we take M isomorphic to a 3-sphere. We can then cut C into two bounded parts, B_1 and B_2 , by slicing M with a 2-dimensional surface (see Fig. 1). To each B_k we relate a state ψ_k of the WZNW CFT defined on its boundary ∂B_k . In this interpretation, a Wilson loop invariant is given by the inner product between these two states

$$Z_R(C) = \left\langle \text{Tr}_R \mathcal{P} \exp \left(\oint_C A \right) \right\rangle_{CS} = \langle \psi_1 | \psi_2 \rangle. \tag{1}$$

The Hilbert space \mathcal{H}_k of each B_k is isomorphic to the space of conformal blocks of the boundary CFT. These blocks have extra proportionality parameters coming from the braiding and crossing operations to build up C , as explained below. Here and in the rest of the paper we restrict our attention to invariants build up from four-point conformal blocks, also called *two-bridge* states.

The two-bridge knot invariants can be constructed via braiding and closure of a j -channel conformal block \mathcal{F}_0^j with zero weight¹ intermediate state, where the index j represents either the s , t or u -channel. Each puncture represented in Fig. 1 corresponds to a field insertion of the conformal block in some representation R of $SU(N)$, as shown in Fig. 2. We fuse the relevant fields via a crossing matrix S and then braid several times with a diagonal half-monodromy matrix² T . General S and T matrices depend on the field representations

¹ As the total charge of a closed manifold must be zero.

² These matrices are also called fusion matrix F and braiding matrix B in the literature. S and T usually refer to modular transformations on the torus.

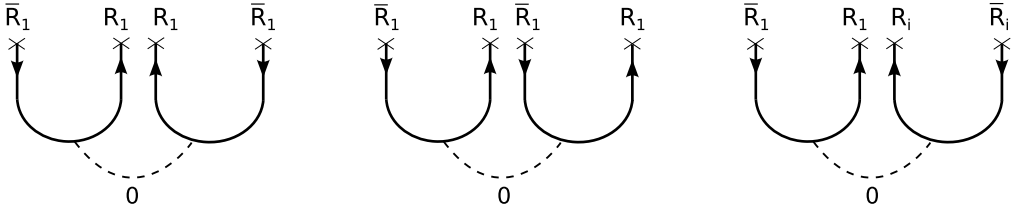


Fig. 2. Different possibilities of basic conformal blocks. From left to right: parallel, antiparallel and mixed case.

$$S_{i_1 i_2} \begin{bmatrix} R_2 & R_3 \\ R_1 & R_4 \end{bmatrix}, \quad T_{j_1 j_2} [R_1 R_2], \tag{2}$$

with the internal indices being labeled by the result of the fusion of appropriate representations, i.e., $i_1, i_2 \in (R_1 \otimes R_2) \cap (R_3 \otimes R_4)$ and $j_1, j_2 \in [R_k]$, where $[R_k]$ represents the intermediate states in the appropriate channel. Non-trivial multiplicities might also appear in the fusion rules of certain fields but we do not consider those here. In the following, we omit the matrix dependence on representations.

Back to Fig. 2, lines going up in representation R_j must close with lines going down in the conjugate representation \bar{R}_j after the braiding evolution in the last step of Fig. 1. The first two cases in Fig. 2 have parallel and anti-parallel fusing strands, respectively. The third case has one of the bridges in a different representation. Only in the first two cases we can close the strands to form a knot³ or a link and in the third case we have only links. In this paper, we are going to consider the first two cases with R_1 in the fundamental representation of $SU(N)$ and the third case with R_i in a rectangular representation of $SU(N)$.

For different sequences made up of S and T matrices, we can construct several types of knots [14,10]. The simplest examples of two-bridge invariants are described by the following formula

$$Z_R^{j,p}(C) \equiv \langle \mathcal{F}_0^j | ST^p S^{-1} | \mathcal{F}_0^j \rangle = (ST^p S^{-1})_{00}, \quad p \in \mathbb{Z}^+, \tag{3}$$

where the last equality represents the singlet diagonal component of the matrix. When p is odd, we have a knot, and when it is even, we have a link. In the $SU(N)_q$ case, these invariants are proportional to HOMFLY polynomials depending on the variables $q = e^{\pi i/(k+N)}$ and $A = q^N$ [27–29]. The proportionality factor depends on the choice of framing for the Wilson loops, but is canonically chosen to not depend explicitly on N , except through A [29]. When $N = 2$, we get Jones polynomials as a special case of the HOMFLY ones.

In this paper, we look for the appropriate S and T matrices for fields labeled by representations of W_N algebra. Explicit calculation shows that these matrices are not properly normalized to give the usual quantum Racah matrices [34,35,29]. In order to have an explicit representation of the braid group, we have to find a conformal block normalization such that S is an unitary hermitian matrix, that is,

$$SS^\dagger = \mathbb{1}, \quad S = S^\dagger \quad \Rightarrow \quad S^2 = \mathbb{1}. \tag{4}$$

This property will allow us to fix the normalization. The S matrix is not hermitian in general, but it will be valid in our particular two-dimensional case. For knots, we have two types of crossing and braiding matrices corresponding to the parallel case (S, T) and to the antiparallel

³ In the second case, we can have twist knots [14].

case (\bar{S}, \bar{T}) . We also need that the Yang–Baxter equation be satisfied for certain 3-strand moves. These matrices must obey the unknot constraint [10]

$$S\bar{T}\bar{S} = T^{-1}S\bar{T}^{-1}, \tag{5}$$

and the Yang–Baxter equation

$$S\bar{T}\bar{S}\bar{T}STS = TS\bar{T}\bar{S}\bar{T}. \tag{6}$$

These two equations will then allow us to choose a correct framing for the T matrices below.

2.1. Normalization of conformal blocks

In general, a four-point correlation function of primary fields in a CFT with symmetry algebra g can be written as

$$\langle V_{\alpha_1}(z)V_{\alpha_2}(1)V_{\alpha_3}(0)V_{\alpha_4}(\infty) \rangle = (\mathcal{G}^s)^\dagger \mathcal{M}^s \mathcal{G}^s = (\mathcal{G}^t)^\dagger \mathcal{M}^t \mathcal{G}^t, \tag{7}$$

with α_i being g -valued vectors labeling the primaries, $\mathcal{G}^k = \mathcal{G}^k(z)$ are the conformal blocks in the $k = s, t$ -channels and \mathcal{M}^k are constant matrices (more generally, bilinear transformations) formed by the product of structure constants of each channel. The space of conformal blocks \mathcal{H}^k is finite dimensional depending if one or more fields in (7) are degenerate. This is always the case for rational conformal field theories. In the following, we suppose that the matrices \mathcal{M}^k are diagonal, which is not necessarily true in general because of non-trivial monodromy properties of higher-spin conformal blocks [36]. We shall review the calculation of W_N correlators in the next section.

We want to change the normalization of the conformal blocks in such a way that the S matrix is an unitary hermitean matrix. First, let us define the new blocks as $\mathcal{F}^k = N_k \mathcal{G}^k$ (with no index summation), where $k = s, t$ denotes the respective channels and N_k are diagonal normalization matrices. Also, we have that $\mathcal{G}^s = S\mathcal{G}^t$ and, using this in (7), we get

$$S^\dagger \mathcal{M}^s S = \mathcal{M}^t. \tag{8}$$

Changing the S matrix to \tilde{S} in the new normalization, we write $S = N_s^{-1} \tilde{S} N_t$. If we plug this into (8) we obtain

$$\mathcal{M}^k = \alpha N_k^\dagger N_k, \quad \alpha \in \mathbb{R}. \tag{9}$$

In the following, we set $\alpha = 1$ as it depends only on the overall normalization of the correlation function.

Now, as will be clear below, we suppose that \mathcal{H}^k is two-dimensional. Then we can parametrize the normalization matrices as

$$N_k = \delta_k \begin{pmatrix} \zeta_k^{-1} & 0 \\ 0 & \zeta_k \end{pmatrix}. \tag{10}$$

Setting $\mathcal{M}^k = \text{diag}(C_1^k, C_2^k)$, we get

$$\frac{C_1^k}{C_2^k} = \frac{1}{|\zeta_k|^4}, \tag{11}$$

where C_j^k corresponds to the products of structure constants in the k -channel appearing in (7). We determine δ_k up to a phase by

$$\text{Tr } \mathcal{M}_k = C_1^k + C_2^k = |\delta_k|^2 (|\zeta_k|^{-2} + |\zeta_k|^2). \tag{12}$$

From equations (11) and (12) we thus get

$$|\zeta_k| = \left(\frac{C_2^k}{C_1^k}\right)^{1/4}, \quad |\delta_k| = (C_1^k C_2^k)^{1/4}. \tag{13}$$

Therefore, if we can find the products of structure constants C_j^k , we can fix the normalization. When we know the S matrix, we can go the other way around and use it to find the structure constants, which is at the core of the bootstrap approach.

The discussion presented here should be compared with [35], where its authors define the proper normalization of conformal blocks to obtain the Racah–Wigner b - $6j$ symbols associated to the modular double of $\mathcal{U}_q(\mathfrak{sl}(2, \mathbb{R}))$ with $q = e^{i\pi b^2}$. The representations of this quantum group will explicitly appear below in our approach of loop invariants in the Virasoro case.

2.2. Knot and link invariants from Virasoro representations

Here we briefly review how to calculate Jones polynomials from S and T matrices related to fields in Virasoro representations. We start with a few definitions following [37], for example. Chiral vertex operators V_α are labeled by charge vectors $\alpha \equiv \alpha_{r,s}$, which also label the conformal dimension of V_α

$$\Delta_{(r,s)} = \alpha_{r,s}(Q - \alpha_{r,s}), \quad \alpha_{r,s} = \frac{1}{2}\{(1-r)b + (1-s)b^{-1}\}, \tag{14}$$

where $Q = b + 1/b$. The integers r, s label unitary irreducible Verma modules with central charge c less than one, appearing in the minimal models. Let us calculate the conformal blocks of the 4-point correlation function holomorphic part

$$F_{\hat{\alpha}}(z, z_1, z_2, z_3) \equiv \langle V_\alpha(z) V_{\alpha_1}(z_1) V_{\alpha_2}(z_2) V_{\alpha_3}(z_3) \rangle, \tag{15}$$

where $\hat{\alpha} = (\alpha, \alpha_1, \alpha_2, \alpha_3)$ and the field V_α is degenerate at level 2, i.e., $(r, s) = (1, 2)$ or $(2, 1)$. The charge vectors must obey the neutrality condition

$$\alpha + \sum_{i=1}^3 \alpha_i + mb + nb^{-1} = Q, \quad m, n \in \mathbb{Z}_+. \tag{16}$$

One of the fields being degenerate at level 2 implies either one of the null vector conditions below

$$(L_{-2} + b^2 L_{-1}^2) V_\alpha = 0 \quad \text{for } (r, s) = (1, 2), \tag{17}$$

$$(L_{-2} + \frac{1}{b^2} L_{-1}^2) V_\alpha = 0 \quad \text{for } (r, s) = (2, 1). \tag{18}$$

Let us focus on the choice $(r, s) = (2, 1)$, such that $\alpha = -\frac{b}{2}$. Using the conformal Ward identity, we find the action of the Virasoro operators on the correlators, which then implies in the BPZ equation

$$\left(\frac{1}{b^2} \frac{\partial^2}{\partial z^2} + \sum_{i=1}^3 \left[\frac{1}{z-z_i} \frac{\partial}{\partial z_i} + \frac{\Delta_i}{(z-z_i)^2}\right]\right) F_{\hat{\alpha}}(z, z_1, z_2, z_3) = 0. \tag{19}$$

Using $SL(2, \mathbb{C})$ invariance, we set $(z_1, z_2, z_3) \rightarrow (0, \infty, 1)$ and then get

$$\frac{1}{b^2} F_{\hat{\alpha}}''(z) + \frac{2z-1}{z(1-z)} F_{\hat{\alpha}}'(z) + \left(\frac{\Delta_1}{z^2} + \frac{\Delta_3}{(1-z)^2} + \frac{\Delta_{(2,1)} + \Delta_1 - \Delta_2 + \Delta_3}{z(1-z)}\right) F_{\hat{\alpha}}(z) = 0. \tag{20}$$

Finally, we put this equation into hypergeometric form by making $F(z) = z^{b\alpha_1}(1-z)^{b\alpha_3}G(z)$

$$\{z(1-z)\frac{\partial^2}{\partial z^2} + [C - (A + B + 1)z]\frac{\partial}{\partial z} - AB\}G(z) = 0, \tag{21}$$

where

$$A = \frac{1}{2} + b(\alpha_1 + \alpha_3 - Q) + b(\alpha_2 - \frac{Q}{2}), \tag{22}$$

$$B = \frac{1}{2} + b(\alpha_1 + \alpha_3 - Q) - b(\alpha_2 - \frac{Q}{2}), \tag{23}$$

$$C = 1 + b(2\alpha_1 - Q). \tag{24}$$

The solutions of (21) are given by hypergeometric functions ${}_2F_1(A, B; C|z)$. If we label the conformal blocks as $\mathcal{F}^k = (F_1^k, F_2^k)$, where $k = s, t$ denotes the channels, we have the s-channel conformal blocks

$$F_1^s(z) = z^{b\alpha_1}(1-z)^{b\alpha_3} {}_2F_1(A, B; C|z), \tag{25a}$$

$$F_2^s(z) = z^{b(Q-\alpha_1)}(1-z)^{b\alpha_3} {}_2F_1(A - C + 1, B - C + 1; 2 - C|z), \tag{25b}$$

and the t-channel conformal blocks

$$F_1^t(z) = z^{b\alpha_1}(1-z)^{b\alpha_3} {}_2F_1(A, B; A + B - C + 1|1-z), \tag{26a}$$

$$F_2^t(z) = z^{b\alpha_1}(1-z)^{b(Q-\alpha_3)} {}_2F_1(C - A, C - B; C - A - B + 1|1-z). \tag{26b}$$

The crossing matrix S is given by $\mathcal{F}^s = S\mathcal{F}^t$ where

$$S = \begin{pmatrix} \frac{\Gamma(C)\Gamma(C-A-B)}{\Gamma(C-A)\Gamma(C-B)} & \frac{\Gamma(C)\Gamma(A+B-C)}{\Gamma(A)\Gamma(B)} \\ \frac{\Gamma(2-C)\Gamma(C-A-B)}{\Gamma(1-A)\Gamma(1-B)} & \frac{\Gamma(2-C)\Gamma(A+B-C)}{\Gamma(A-C+1)\Gamma(B-C+1)} \end{pmatrix}. \tag{27}$$

In the particular case of the other fields also in the fundamental representation, $\alpha_1 = \alpha_3 = -b/2$ and $\alpha_2 = Q - \alpha_1$. Thus, we get

$$\begin{aligned} S &= \begin{pmatrix} \frac{1}{[2]} & \frac{\Gamma(-2b^2)\Gamma(-2b^2-1)}{\Gamma(-3b^2-1)\Gamma(-b^2)} \\ \frac{\Gamma(2b^2+1)\Gamma(2b^2+2)}{\Gamma(b^2+1)\Gamma(3b^2+2)} & -\frac{1}{[2]} \end{pmatrix} \\ &= U \begin{pmatrix} \frac{1}{[2]} & \frac{\sqrt{[3]}}{[2]} \\ \frac{\sqrt{[3]}}{[2]} & -\frac{1}{[2]} \end{pmatrix} U^{-1} \end{aligned} \tag{28}$$

where $q = e^{i\pi b^2}$, $[N] = \frac{q^N - q^{-N}}{q - q^{-1}}$ and

$$U = \begin{pmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{pmatrix}, \quad \zeta^2 = \frac{\sqrt{[3]} \Gamma(b^2 + 1) \Gamma(3b^2 + 2)}{[2] \Gamma(2b^2 + 1) \Gamma(2b^2 + 2)}. \tag{29}$$

It is now easy to check that $S^2 = \mathbb{1}$, as required by (4). The s-channel braiding matrix T is obtained by making a half turn around zero in (25)

$$T = N(q) \begin{pmatrix} q^{-1/2} & 0 \\ 0 & -q^{3/2} \end{pmatrix}, \tag{30}$$

where $N(q)$ is an overall normalization. The Yang–Baxter equation in this case is $(ST)^3 = \mathbb{1}$ and this let us choose $N(q) = -q^{-1/2}$. Summing up, the matrices representing the braid group B_3 colored with $SU(2)$ representations are

$$S = \begin{pmatrix} \frac{1}{[2]} & \frac{\sqrt{[3]}}{[2]} \\ \frac{\sqrt{[3]}}{[2]} & -\frac{1}{[2]} \end{pmatrix}, \quad T = \begin{pmatrix} -\frac{1}{q} & 0 \\ 0 & q \end{pmatrix}. \tag{31}$$

We can construct knot and link polynomials by starting with the t -channel conformal blocks, braiding representations in the s -channel k number of times and then going back to the t -channel, like in Fig. 1. In this way, we get

$$ST^p S = \frac{1}{[2]^2} \begin{pmatrix} q^p [3] + (-1)^p q^{-p} & -(q^p + (-1)^{p+1} q^{-p}) \sqrt{[3]} \\ -(q^p + (-1)^{p+1} q^{-p}) \sqrt{[3]} & q^p + (-1)^p q^{-p} [3] \end{pmatrix}. \tag{32}$$

According to the fusion rules, the conformal block with zero intermediate weight is (25b) and, thus, the knot/link invariant of interest is the second diagonal component of (32), which is an *unreduced* Jones invariant. The reduced Jones knot polynomial is defined as the ratio of the unreduced polynomial and the *unknot* in the same representation

$$J_R^{C,3}(q) = \frac{Z'_R(3|C)}{Z'_R(1|C)} = \frac{(ST^3 S)_{22}}{(STS)_{22}} = -q^4 + q^2 + q^{-2}. \tag{33}$$

Notice that the second diagonal component can be recovered from the first by making $q \rightarrow -1/q$. A more extensive discussion of the types of knots and link invariants calculated in this way is given in [14,10].

3. Toda field theory and W_N conformal blocks

In this section, we review the machinery of Toda field theory, discussed in [38], in order to generalize the construction above of S and T matrices for W_N models. Specifically, we fix two fields in the four-point function to be in the fundamental representation of $SU(N)$ paired with its conjugate representation and deduce some results for the other fields in an more general representation. This section follows mostly the definitions and conventions of [38]. Other relevant references about correlators in W_N models are [39,36].

The generalization of Liouville theory extending Virasoro to W_N algebra is called Toda field theory (TFT). The basic field is a scalar field $\varphi = \sum_{i=1}^{N-1} \varphi_i e_i$, where $e_i, i = 1, \dots, N - 1$, are the simple roots of $su(N)$ algebra. The most important information about the algebra is contained in the *Cartan matrix*, K_{ij} , defined by the inner product of the simple roots, $K_{ij} = (e_i, e_j)$. From the inner product, one can define the dual weight space in terms the *fundamental weights* ω_k by $(\omega_k, e_j) = \delta_{kj}$ and the *quadratic form* of the algebra by $(\omega_i, \omega_j) = K_{ij}^{-1}$. A highest weight λ takes the form

$$\lambda = \sum_{i=1}^{N-1} \lambda_i \omega_i, \tag{34}$$

where $(\lambda_1, \dots, \lambda_{N-1}) \in \mathbb{Z}_{\geq 0}^{N-1}$ are called Dynkin labels. In particular, the conjugate representation of λ is represented by $\bar{\lambda} = \sum_{i=1}^{N-1} \lambda_{N-i} \omega_i$.

To each highest weight we can associate a *partition* $\lambda = \{\ell_1; \ell_2; \dots; \ell_{N-1}\}$ where $\ell_i = \lambda_i + \lambda_{i+1} + \dots + \lambda_{N-1}$. We then associate a *Young tableau* to the partition by assigning ℓ_i boxes to the i -th row of the tableau. Some simple examples are

$$\begin{aligned}
 F_1 &= (1, 0, 0, \dots, 0) \sim \square \\
 F_2 &= (2, 0, 0, \dots, 0) \sim \square \square \\
 A_2 &= (0, 1, 0, \dots, 0) \sim \begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \end{array}
 \end{aligned}$$

where the first example is the fundamental representation, the second a symmetric representation and the third an antisymmetric representation. Young diagrams are useful to build up tensor product representations and thus analyze possible states of fusion rules. For more details, see [40], for example. To find all the states in an irreducible module with highest weight λ , we subtract all possible combinations of simple roots e_i up to $\lambda_i e_i$ for each positive λ_i . Then we repeat the process with the new weights until there is no way to produce a new weight with positive Dynkin label. In the case of the fundamental representation, the weights are expressed as

$$h_k = \omega_1 - \sum_{i=1}^{k-1} e_i, \quad k = 1, \dots, N. \tag{35}$$

The TFT action on a Riemann surface with reference metric \hat{g}_{ab} and scalar curvature \hat{R} is given by

$$S_{TFT} = \int \left(\frac{1}{8\pi} \hat{g}^{ab} (\partial_a \varphi, \partial_b \varphi) + \frac{(Q, \varphi)}{4\pi} \hat{R} + \mu \sum_{k=1}^{N-1} e^{b(e_k, \varphi)} \right) \sqrt{\hat{g}} d^2 x, \tag{36}$$

where μ is the cosmological constant and Q is the background charge. To ensure conformal invariance, we must set the charge to be

$$Q = (b + 1/b)\rho, \quad \rho = \sum_{k=1}^{N-1} \omega_k, \tag{37}$$

where ρ is the Weyl vector of the algebra. The theory is invariant under symmetries generated by the currents $W^k(z)$ with spins $k = 2, 3, 4, \dots, N$ and its antiholomorphic counterparts. The current $W^2(z) \equiv T(z)$ is equal to the energy–momentum tensor and the other are higher-spin currents. Together, these currents generate the W_N algebra containing the Virasoro algebra with central charge $c = N - 1 + 12Q^2$.

The Toda correlators can be calculated in the Coulomb gas formalism by introducing the chiral vertex operators $V_\alpha = e^{(\alpha, \varphi)}$. The OPE with the currents $W^k(z)$ classify the states in terms of the quantum numbers $w^k(\alpha)$, $k = 2, 3, \dots, N - 1$. In particular, the conformal weight is given by

$$w^{(2)}(\alpha) = \Delta(\alpha) = \frac{(\alpha, 2Q - \alpha)}{2}. \tag{38}$$

The $w^k(\alpha)$ are invariant under the $su(N)$ Weyl group. After a Weyl reflection, the field V_α acquires a reflection amplitude [38]. As an example, the conjugate representation $\bar{\alpha}$ is equivalent to $2Q - \alpha$ under the longest Weyl reflection. This changes the correlation function by a multiplicative factor which will not be relevant to calculate reduced polynomials, as overall factors cancel.

All of this corresponds to the general Toda theory. The W_N minimal model can be realized as the coset model $SU(N)_k \oplus SU(N)_1 / SU(N)_{k+1}$. After imposing the constraint in the root lattice,

we find that the primaries are labeled by $(\rho; \nu) \equiv (\Lambda; \tilde{\Lambda})$ (level k and $k + 1$ respectively). The details of this construction can be found in [41,36], for example. As we are going to see below, the W_N conformal blocks can be obtained by taking the residue of Toda conformal blocks, similarly to the Virasoro and Liouville case.

The Toda 3-point function with one semi-degenerate field was first calculated in [32] and a general formula using AGT relation was proposed in [42,43]. In the Virasoro case, knowledge of the two and three-point functions allow us to obtain multipoint correlators by the conformal bootstrap [30]. As we saw in section 2.2, the four-point function is completely determined by setting one of the fields to be completely degenerate. However, for the W_N case this is not enough [31,32]. The structure of the Verma modules is more constrained by the extra higher-spin symmetries and we need to fix another field to be in a semi-degenerate state $\alpha = \kappa \omega_{N-1}$, where κ is an arbitrary constant [38]. Here we shall restrict our discussion to this semi-degenerate case. For more details, see [32,38,39].

Three-point correlators are constrained by conformal invariance to be

$$\langle V_{\alpha_1}(x_1)V_{\alpha_2}(x_2)V_{\alpha_3}(x_3) \rangle = \frac{C(\alpha_1, \alpha_2, \alpha_3)}{|z_{12}|^{2(\Delta_1+\Delta_2-\Delta_3)}|z_{13}|^{2(\Delta_1+\Delta_3-\Delta_2)}|z_{23}|^{2(\Delta_2+\Delta_3-\Delta_1)}}. \tag{39}$$

Analysis of the Coulomb integral for calculating this function shows that the structure constants $C(\alpha_1, \alpha_2, \alpha_3)$ have poles when the screening condition is satisfied

$$(2Q - \sum_{i=1}^3 \alpha_i, \omega_k) = bs_k + b^{-1}\tilde{s}_k, \quad s_k, \tilde{s}_k \in \mathbb{Z}_{\geq 0}. \tag{40}$$

Taking the residues of the correlator in those poles gives the W_N structure constants [38,36]. Those can be expressed in terms of complicated Coulomb integrals, but in some simple cases, like when one of the fields is semi-degenerate, they can be written in terms of known special functions [38]. Defining

$$C_{\alpha_1, \alpha_2}^{\alpha_3} \equiv C(\alpha_1, \alpha_2, 2Q - \alpha_3), \tag{41}$$

our particular case of interest is when one of the fields above is in the fundamental representation

$$C_{-b\omega_1, \alpha_1}^{\alpha_1 - bh_k} = \left(-\frac{\pi\mu}{\gamma(-b^2)} \right)^{k-1} \prod_{i=1}^{k-1} \frac{\gamma(b(\alpha_1 - Q, h_i - h_k))}{\gamma(1 + b^2 + b(\alpha_1 - Q, h_i - h_k))}, \tag{42}$$

where $\gamma(z) = \Gamma(z)/\Gamma(1 - z)$. With this formula, we can show that the fusion rule of $V_{-b\omega_1}$ and $V_{\kappa\omega_{N-1}}$ has only two fields. In particular, we have only two intermediate states in a channel with a fundamental and anti-fundamental field and thus the space of conformal blocks is two-dimensional.

Now let us consider the four-point correlator after fixing three points by global $SL(2, \mathbb{C})$ invariance

$$\langle V_{-b\omega_1}(z, \bar{z})V_{\alpha_1}(0)V_{\alpha_2}(\infty)V_{-b\omega_{N-1}}(1) \rangle = |z|^{2b(\alpha_1, \omega_1)}|1 - z|^{-\frac{2b^2}{N}}G(z, \bar{z}), \tag{43}$$

where $\alpha_2 = 2Q - \alpha_1$ and, in the s -channel expansion,

$$G(z, \bar{z}) = \sum_{j=1}^N C_{-b\omega_1, \alpha_1}^{\alpha_1 - bh_j} C(\alpha_1 - bh_j, \alpha_2, -b\omega_{N-1})G_j(z)G_j(\bar{z}). \tag{44}$$

The summation over intermediate states follows from the fusion rules [44,38]

$$V_{-b\omega_k} V_\alpha = \sum_s C_{-b\omega_k, \alpha}^{\alpha - bh_s^{(k)}} \left[V_{\alpha - bh_s^{(k)}} \right], \tag{45}$$

where $h_s^{(k)}$ are the weights of the representation with highest weight ω_k .

The conformal blocks $G_j(z)$ satisfy the generalized hypergeometric equation

$$\left[z \prod_{k=1}^N (\theta + A_k) - \theta \prod_{k=1}^{N-1} (\theta + B_k - 1) \right] G_j(z) = 0, \tag{46}$$

where $\theta = z \frac{d}{dz}$ and the coefficients A_k and B_k are given by

$$\begin{aligned} A_k &= -b^2 + b(\alpha_1 - Q, e_1 + \dots + e_{k-1}), \\ B_k &= 1 + b(\alpha_1 - Q, e_1 + \dots + e_k). \end{aligned} \tag{47}$$

In terms of the generalized hypergeometric function,

$$G_1 = {}_N F_{N-1} \left(\begin{matrix} A_1 \dots A_N \\ B_1 \dots B_{N-1} \end{matrix} \middle| z \right), \tag{48}$$

$$G_j = z^{1-B_j} {}_N F_{N-1} \left(\begin{matrix} 1-B_j+A_1 \dots 1-B_j+A_N \\ 1-B_j+B_1 \dots 2-B_j \dots 1-B_j+B_{N-1} \end{matrix} \middle| z \right), \quad 1 < j \leq N. \tag{49}$$

By consistency between s - and u -channel expansions, we find

$$\begin{aligned} & \frac{C_{-b\omega_1, \alpha_1}^{\alpha_1 - bh_1} C(\alpha_1 - bh_1, \alpha_2, -b\omega_{N-1})}{C_{-b\omega_1, \alpha_1}^{\alpha_1 - bh_k} C(\alpha_1 - bh_k, \alpha_2, -b\omega_{N-1})} \\ &= \frac{\prod_{j=1}^N \gamma(A_j) \gamma(B_{k-1} - A_j) \prod_{j \neq k-1} \gamma(1 + B_j - B_{k-1})}{\prod_{j=1}^{N-1} \gamma(B_j) \gamma(B_{k-1} - 1)}, \end{aligned} \tag{50}$$

which follows from connection formulas of generalized hypergeometric functions [45,38]. Those are the basic equations used to find three-point functions with one partially degenerate field.

4. Crossing and braiding matrices in W_N models

Now that we know the W_N conformal blocks, we can analyze particular cases to calculate crossing S and braiding matrices T with one fundamental and one anti-fundamental field, as in (43). Let us start by taking a highest-weight state in the form

$$\alpha_1 = -b\Lambda - b^{-1} \tilde{\Lambda}, \tag{51}$$

related to the pair of representations $(\Lambda; \tilde{\Lambda})$ labelling a W_N primary, where

$$\Lambda = \sum_{i=1}^{N-1} n_i \omega_i, \quad \tilde{\Lambda} = \sum_{i=1}^{N-1} \tilde{n}_i \omega_i, \quad n_i, \tilde{n}_i \in \mathbb{Z}_{\geq 0}, \tag{52}$$

such that

$$\sum_{i=1}^{N-1} n_i \leq k, \quad \sum_{i=1}^{N-1} \tilde{n}_i \leq k + 1, \tag{53}$$

with k corresponding to the WZW level. We are going to see that if a number r of labels n_i or \tilde{n}_i are different from zero, then (46) is reducible to a lower order hypergeometric equation. First, notice that

$$B_k = A_{k+1} + b^2 + 1 = A_k + b(\alpha_1, e_k), \quad k = 1, \dots, N - 1. \tag{54}$$

As $z(\theta + A)f(z) = (\theta + A - 1)zf(z)$, it is easy to see that for each $B_k = A_k$ we can factor out a term $(\theta + B_k - 1)$ from eq. (46), effectively reducing its order. For the particular α_1 we are considering,

$$B_k = A_k - n_k b^2 - \tilde{n}_k. \tag{55}$$

Therefore, $B_k = A_k$ except for $n_k, \tilde{n}_k \neq 0$. If r labels n_i or \tilde{n}_i are different from zero, the $(N, N - 1)$ generalized hypergeometric operator factorizes to a product $D_{N,N-1} = P_{N-r-1} D_{r+1,r}$ of an order $N - r - 1$ operator and a $(r + 1, r)$ hypergeometric operator. This proves our assertion. Finally, we can explicitly write the A_k as

$$A_k = -(\ell_1 - \ell_k + k)b^2 - (\tilde{\ell}_1 - \tilde{\ell}_k + k - 1), \tag{56}$$

where $\ell_k, \tilde{\ell}_k$ are the number of boxes in the k -th row of the representation $\Lambda, \tilde{\Lambda}$, respectively.

4.1. Rectangular representations

The connection matrices between $z = 0$ and $z = 1$ for higher-order hypergeometric equations are not easy to find and we will not consider those in this paper. However, for a reduction to Gauss hypergeometric equation, we have explicit formulas like (50). This correspond to the case of *rectangular* representations $\alpha_1 = -(nb + \tilde{n}b^{-1})\omega_m$. Here m corresponds to the number of rows and n, \tilde{n} the number of columns of the Young diagram of $\Lambda, \tilde{\Lambda}$ respectively. The field $\alpha_2 = 2Q - \alpha_1$ is Weyl equivalent to $\bar{\alpha}_1 = -(nb + \tilde{n}b^{-1})\omega_{N-m}$ and then the correlation function differs from (43) by an overall reflection amplitude, which will not be relevant for us. In this case, eq. (56) becomes

$$A_k = -(nH(k - m - 1) + k)b^2 - (\tilde{n}H(k - m - 1) + k - 1), \tag{57}$$

where $H(k)$ is the step function. We have that $B_k = A_k - (nb^2 + \tilde{n})\delta_k^m$ for $1 \leq k < N$, therefore $B_k = A_k$ for all $k \neq m$ and (46) reduces to a second order hypergeometric equation (21) with parameters

$$A = A_m = -mb^2 - (m - 1), \quad B = A_N = -(N + n)b^2 - (N + \tilde{n} - 1), \tag{58}$$

$$C = B_m = -(m + n)b^2 - (m + \tilde{n} - 1). \tag{59}$$

The S matrix now is

$$S = \begin{pmatrix} \frac{\Gamma(-(n+m)b^2 - (m+\tilde{n}-1))\Gamma(Nb^2+N-1)}{\Gamma(-nb^2-\tilde{n})\Gamma((N-m)(b^2+1))} & \frac{\Gamma(-(n+m)b^2 - (m+\tilde{n}-1))\Gamma(-Nb^2 - (N-1))}{\Gamma(-mb^2 - (m-1))\Gamma(-(N+n)b^2 - (N+\tilde{n}-1))} \\ \frac{\Gamma((m+n)b^2+m+\tilde{n}+1)\Gamma(Nb^2+N-1)}{\Gamma(m(b^2+1))\Gamma((N+n)b^2+N+\tilde{n})} & \frac{\Gamma((m+n)b^2+m+\tilde{n}+1)\Gamma(-Nb^2 - (N-1))}{\Gamma(nb^2+\tilde{n}+1)\Gamma(-(N-m)b^2 - (N-m-1))} \end{pmatrix} \tag{60}$$

and, as $S = N_s^{-1} S_{m,n} N_t$, we have that

$$S = \delta_{m,n} \begin{pmatrix} \xi_{m,n}^2 s_1 & \zeta_{m,n}^2 s_2 \\ \zeta_{m,n}^{-2} s_2 & -\xi_{m,n}^{-2} s_1 \end{pmatrix} = \delta_{m,n} U R S_{m,n} R U^{-1}, \tag{61}$$

where

$$R = \begin{pmatrix} \xi_{m,n} & 0 \\ 0 & \xi_{m,n}^{-1} \end{pmatrix}, \quad U = \begin{pmatrix} \zeta_{m,n} & 0 \\ 0 & \zeta_{m,n}^{-1} \end{pmatrix}. \tag{62}$$

Relating with the parametrization (10), we have that $\delta_{m,n} = \delta_t/\delta_s$ and

$$N_t = \delta_t R U^{-1} = \delta_t \begin{pmatrix} \gamma_t^{-1} & 0 \\ 0 & \gamma_t \end{pmatrix}, \quad \gamma_t = \frac{\zeta_{m,n}}{\xi_{m,n}}, \tag{63}$$

$$N_s = \delta_s R^{-1} U^{-1} = \delta_s \begin{pmatrix} \gamma_s^{-1} & 0 \\ 0 & \gamma_s \end{pmatrix}, \quad \gamma_s = \zeta_{m,n} \xi_{m,n}, \tag{64}$$

fixing the normalization in the t - and s -channel conformal blocks, now defined as $\mathcal{F}^k = N_k \mathcal{G}^k$. Choosing $\det S_{m,n} = -1$, the parametrization coefficients are given by

$$\delta_{m,n}^2 = -\det S = \frac{(m+n)b^2 + m + \tilde{n}}{N b^2 + N - 1}, \tag{65}$$

$$\begin{aligned} \zeta_{m,n}^2 &= \left(\frac{S_{12}}{S_{21}} \right)^{\frac{1}{2}} \\ &= \delta_{m,n}^{-1} \sqrt{\frac{[N+n][m]}{[N][n+m]} \frac{\Gamma(m(b^2+1))\Gamma((N+n)b^2+N+\tilde{n})}{\Gamma((N(b^2+1)))\Gamma((m+n)b^2+m+\tilde{n})}}, \end{aligned} \tag{66}$$

$$\begin{aligned} \xi_{m,n}^2 &= \left(-\frac{S_{11}}{S_{22}} \right)^{\frac{1}{2}} \\ &= \delta_{m,n}^{-1} \sqrt{\frac{[N][n]}{[N-m][n+m]} \frac{\Gamma(nb^2+\tilde{n}+1)\Gamma(Nb^2+N-1)}{\Gamma((m+n)b^2+m+\tilde{n})\Gamma((N-m)(b^2+1))}}. \end{aligned} \tag{67}$$

The orthogonal and symmetric S matrix obtained in (61) is thus

$$S_{m,n} = \frac{1}{\sqrt{[m+n][N]}} \begin{pmatrix} (-1)^{m+1} \sqrt{[N-m][n]} & \sqrt{[N+n][m]} \\ \sqrt{[N+n][m]} & (-1)^m \sqrt{[N-m][n]} \end{pmatrix}. \tag{68}$$

This is the main result of our paper. For $N = 2$ and $n = m = 1$, we recover the Virasoro case (28). Therefore, we conclude that $S \sim S_{m,n}$ up to a normalization redefinition of the conformal blocks.

Now, to calculate the half-monodromy matrix, let us set $\tilde{n} = 0$, as its value only changes monodromy signs and does not change the form of (68). The s -channel braiding matrix is obtained from the asymptotics of the conformal blocks near $z = 0$, giving

$$T_{m,n} = N_{m,n}(q) \begin{pmatrix} q^{-n} & 0 \\ 0 & (-1)^m q^m \end{pmatrix}, \tag{69}$$

where we choose a framing in which a $q^{nm/N}$ factor is canceled. Now, consider the case $n = 1$. The two special cases in which we can construct knots are the parallel case ($m = 1$) and the anti-parallel case ($m = N - 1$). The framing normalization matrices $N_{m,n}(q)$ can be chosen using the unknot constraint (5) and Yang–Baxter equation (6)

$$N_{1,1}(q) = (-1)^{N+1} q^{2-N}, \quad N_{N-1,1}(q) = 1. \tag{70}$$

Summing up, the parallel case has the crossing and braiding matrices

$$S = \frac{1}{\sqrt{[2][N]}} \begin{pmatrix} \sqrt{[N-1]} & \sqrt{[N+1]} \\ \sqrt{[N+1]} & -\sqrt{[N-1]} \end{pmatrix}, \quad T = (-1)^N \begin{pmatrix} -q^{1-N} & 0 \\ 0 & q^{3-N} \end{pmatrix}, \quad (71)$$

where $S \equiv S_1, T \equiv T_1$, while the anti-parallel case has

$$\bar{S} = \frac{1}{[N]} \begin{pmatrix} 1 & \sqrt{[N+1][N-1]} \\ \sqrt{[N+1][N-1]} & -1 \end{pmatrix}, \quad \bar{T} = \begin{pmatrix} q^{-1} & 0 \\ 0 & (-1)^{N+1} q^{N-1} \end{pmatrix}, \quad (72)$$

where $\bar{S} \equiv S_{N-1}$ and $\bar{T} \equiv T_{N-1}$. For the case of braiding two parallel strands, we can calculate the following invariants

$$\begin{aligned} ST^p S &= \frac{(-1)^{pN} q^{p(2-N)}}{[2][N]} \\ &\times \begin{pmatrix} (-1)^p [N-1] q^{-p} + [N+1] q^p & ((-1)^p q^{-p} - q^p) \sqrt{[N-1][N+1]} \\ ((-1)^p q^{-p} - q^p) \sqrt{[N-1][N+1]} & (-1)^p [N+1] q^{-p} + [N-1] q^p \end{pmatrix}, \end{aligned} \quad (73)$$

which reduces to (32) when $N = 2$. Therefore, we can find, for example, the knot polynomial for the trefoil knot 3_1 , up to framing redefinition,

$$H_F^{3_1}(q, A) = \frac{(ST^3 S)_{22}}{(STS)_{22}} = q^4 (-1 + A^{-2}(q^2 + q^{-2})). \quad (74)$$

We can now try to compare our results with [27] for linking matrices with one link in the fundamental and the other in an arbitrary symmetric ($m = 1, n$ arbitrary) or antisymmetric representation (m arbitrary, $n = 1$). For appropriate comparison, we note that the quantum dimension of a representation R_λ with partition λ is given by

$$\dim_q R_\lambda = \prod_{(i,j) \in \lambda} \frac{[N+j-i]}{[\ell_i - i + \ell_j^\vee - j + 1]}, \quad (75)$$

where ℓ_j^\vee is the number of boxes in the j -th column of λ . For the antisymmetric case, we get

$$S_{m,1} = \frac{1}{\sqrt{[m+1][N]}} \begin{pmatrix} (-1)^{m+1} \sqrt{[N-m]} & \sqrt{[N+1][m]} \\ \sqrt{[N+1][m]} & (-1)^m \sqrt{[N-m]} \end{pmatrix} \quad (76)$$

and

$$S_{N-m,1} = \frac{1}{\sqrt{[N-m+1][N]}} \begin{pmatrix} (-1)^{N-m+1} \sqrt{[m]} & \sqrt{[N+1][N-m]} \\ \sqrt{[N+1][N-m]} & (-1)^{N-m} \sqrt{[m]} \end{pmatrix}. \quad (77)$$

These match, up to signs and column permutation, the first and third matrices of sec. 4, item 5 of [27]. To obtain the second matrix, necessary to check the Yang–Baxter equation for links, we have to rederive (46) with the fields at z and at $z = 0$ interchanged.

For the symmetric case, we get the first and third matrix of sec. 4, item 3 of [27], up to signs and column permutation,

$$S_{1,n} = \frac{1}{\sqrt{[n+1][N]}} \begin{pmatrix} \sqrt{[N-1][n]} & \sqrt{[N+n]} \\ \sqrt{[N+n]} & -\sqrt{[N-1][n]} \end{pmatrix}, \tag{78}$$

and

$$S_{N-1,n} = \frac{1}{\sqrt{[N][N+n-1]}} \begin{pmatrix} (-1)^N \sqrt{[n]} & \sqrt{[N+n][N-1]} \\ \sqrt{[N+n][N-1]} & (-1)^{N-1} \sqrt{[n]} \end{pmatrix}. \tag{79}$$

This strongly suggests that (68) is the correct crossing matrix for links with one fundamental component and a rectangular component. To calculate the T matrix in the correct framing, we need the other type of matrix mentioned in [27] to apply the constraints.

4.2. Links with higher representations

We finish this session by discussing how it is possible to obtain links with one fundamental component linked to a component in an arbitrary W_N representation. This corresponds to take α_1 above arbitrary in the sense of equations (52) and (53). To find explicit S matrices in this case corresponds to solving the connection problem for generalized hypergeometric functions but we shall pursue another approach. In fact, the braiding operation for links is essentially equivalent to write, for example, the t -channel monodromy matrix in another basis

$$S^{-1}(T_0^{(s)})^{2k} S = S^{-1}(M_0^{(s)})^k S = (M_0^{(t)})^k, \quad k \in \mathbb{Z}_{>0}, \tag{80}$$

where the subscript 0 means we are calculating the analytic continuation around zero. The monodromy group of the generalized hypergeometric equation is well-known [46] and was explicitly used to calculate Wilson loop invariants in Toda field theory in [23]. Here we just need to repeat the steps in [23] to obtain two-bridge link invariants with one fundamental component and another higher representation component.

In our prescription, as explained above, we start in the t -channel conformal block with zero weight intermediate state $\mathcal{G}_N^{(t)} = z^{b(\alpha_1, \omega_1)} (1-z)^{-\frac{b^2}{N}} G_N^{(t)}$, the N -th component of the vector $\mathcal{G}^{(t)}$, which has diagonal monodromy around $z = 1$. A two-bridge link invariant is the result of the following braiding operation

$$\mathcal{L}_{F,R}^k = e^{2\pi i k b(\alpha_1, \omega_1)} (S^{-1}(M_0^{(s)})^k S)_{NN} = e^{2\pi i k b(\alpha_1, \omega_1)} [(M_0^{(t)})^k]_{NN}. \tag{81}$$

The diagonal change of normalization discussed in this paper does not change the diagonal components of the monodromy matrices as, in the new basis $\mathcal{F}^i = N_i \mathcal{G}^i$,

$$M_0^{(i)} \rightarrow N_i M_0^{(i)} N_i^{-1}. \tag{82}$$

In the Mellin–Barnes basis [47], the monodromy group of the hypergeometric equation (46) has a simple form depending directly on the coefficients A_j, B_j of (46). Now we just need to find the similarity transformation that take us to a basis with diagonal monodromy at $z = 1$, i.e., the t -channel basis. This was done in [23] and the final result is

$$(M_0^{(t)})_{NN} = \frac{e^{-2\pi i(\alpha_1, \omega_1)}}{[N]} \sum_{j=1}^N e^{2\pi i b(\alpha_1 - Q, h_j)}, \tag{83}$$

which then implies in

$$\mathcal{L}_{F,R} = \frac{1}{[N]} \sum_{j=1}^N e^{2\pi i b(\alpha_1 - Q, h_j)}, \quad (84)$$

for a link with only one braiding. This result is valid for the parallel case. For the anti-parallel we just need to use the inverse monodromy matrix.

The caveat in the above derivation is that we have to suppose $A_i \neq B_i$ for all $i = 1, \dots, N$, otherwise the monodromy group is reducible and we have to find a new representation for the monodromy matrices. This means that we have to take α_1 to depend explicitly on all ω_k for the calculation to be correct, i.e., every row of the Young diagram must have at least one box. It is easy to check that this formula gives the $N = 2$ result in (32) for $p = 2$. Moreover, for links with k braidings, we need to calculate the k -th power of the monodromy matrix and take the last diagonal component. We leave explicit calculations in this monodromy method for future work.

This monodromy method is very powerful for links as we do not need to calculate the explicit S matrices. For knots, the monodromy matrices are not enough because there is always an odd number of braidings and, thus, we need the half-monodromy in a non-diagonal basis:

$$S^{-1}(T_0^{(s)})^{2k+1}S = (M_0^{(s)})^k T_0^{(t)}. \quad (85)$$

We thus need to know the S matrix to calculate the half-monodromy in a non-diagonal basis.

5. Conclusions

In this work, we obtained crossing and braiding matrices for certain W_N algebra representations. In particular, we can use these braiding matrices to obtain HOMFLY knot invariants in the fundamental representation and two-component HOMFLY link invariants, one component in the fundamental and another in a rectangular representation of $SU(N)$ algebra. To construct generic link invariants in this case, we need three types of matrices [27], and we have explicitly studied two types, linking and anti-linking. The third (mixed) case can be obtained by making a rederivation of (46) with the position of the relevant fields exchanged.

Links with one fundamental component and the other component in an arbitrary W_N representation are related to the generalized hypergeometric function described in this paper. The connection problem in this case is more intricate but the monodromy group is well known [46]. This is enough to find link invariants with one fundamental component linked to a maximal rank higher-representation components. However, knot invariants need the explicit S -matrix for the calculation of the half-monodromy. In principle, the problem of non-trivial multiplicity should be automatically solved by the monodromy properties of the generalized hypergeometric functions.

The case of knot invariants for higher-representations still remains elusive. Higher-representations are also not completely understood in the quantum group approach, whose knowledge is limited to lower representations [29], and research in this direction is under active development [48]. In our approach, we cannot calculate these invariants due to the limited knowledge we have on correlation functions in W_N models, apart from the cases discussed in [38,39]. It is known that correlators more general than the one with a semi-degenerate field do not obey a linear differential equation [32,38]. Something can be said about integral representations of more general correlators [39], although via complicated integrals and the monodromy analysis might be interesting for those results.

The pentagon identity can be used to recursively obtain crossing matrices for knots in higher-representations [27,29,14]. However, this approach is computationally expensive and it is limited

to representations with just a few boxes. Promising results have been recently obtained for Toda 3-point functions and W_4 crossing matrices in [49] with one of the fields in the representation $\alpha = -b\omega_2$ and the other fields in partially degenerate representations $\beta_a = k_a\omega_2b$. Finally, another interesting approach to understand W_N conformal blocks is via the AGT expansion of isomonodromic tau-functions [50–52]. All these are relevant lines of attack for the problem of finding Wilson loop invariants for W_N models. We expect that our results will serve as a basis to further expand the understanding of W_N models and higher-spin topological invariants.

Acknowledgements

The authors thank Dmitry Melnikov, Andrei Mironov and Alexei Morozov for important discussions and valuable suggestions. The work of FN was supported by the CNPq PDJ fellowship of the Science without Borders initiative process 400635/2012-7.

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