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The volume flux group and nonpositive curvature

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Abstract We show that every closed nonpositively curved manifold with non-trivial volume flux group has zero minimal volume, and admits a finite covering with circle actions whose orbits are homologically essential. This proves a conjecture of Kedra–Kotschick–Morita for this class of manifolds.

Keywords Volume flux group · Minimal volume · Nonpositive curvature

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Let M be a closed smooth manifold and μ a volume form on M. Denote by Diff^{μ} the group of μ-preserving diffeomorphisms of M, and by Diff_0^{μ} its identity component. The μ-flux homomorphism Flux_{μ}, from the universal covering $\widetilde{\mathrm{Diff}_0^{\mu}}$ to the $(n-1)$-cohomology group $H^{n-1}(M;\mathbf{R})$, is defined by the formula

$$\mathrm{Flux}_{\mu}([\varphi_t]) = \int_0^1 [i_{\dot{\varphi}_t}\mu]\mathrm{d}t.$$

It induces a homomorphism

$$\mathrm{Flux}_{\mu}\colon \pi_1(\mathrm{Diff}_0^{\mu}) \to H^{n-1}(M;\mathbf{R})$$

whose image is the volume flux group $\Gamma_{\mu} \subset H^{n-1}(M;\mathbf{R})$. The μ-flux homomorphism descends to a homomorphism

$$\mathrm{Flux}_{\mu}\colon \mathrm{Diff}_0^{\mu} \to H^{n-1}(M;\mathbf{R})/\Gamma_{\mu}.$$

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For a closed connected smooth Riemannian manifold (M, g), let $\mathrm{Vol}(M, g)$ denote the volume of g, and let K_g be its sectional curvature. We define the minimal volume of M following Cheeger and Gromov [2]:

$$\mathrm{MinVol}(M) := \inf_g \{\mathrm{Vol}(M, g) \ : \ |K_g| \leq 1\}$$

The minimal volume is a very sensitive invariant; it was first observed by Bessières [1] that its value may depend on the differentiable structure of (M, g). Indeed, Kotschick [6] has shown that even the vanishing of $\mathrm{MinVol}(M)$ can detect changes in the smooth structure of M.

The investigation of relationships between the volume flux group of M and various invariants which bound $\mathrm{MinVol}(M)$ from below was set in motion by Kedra et al.[5]. They put forward the idea that if a closed manifold has non-trivial volume flux group, then its minimal volume should vanish. The aim of this report is to verify that statemtent for closed manifolds which carry a metric of nonpositive sectional curvature.

The volume flux group Γ_μ is independent of the form μ—see Sect. 3 of [5]—so it can be considered as an invariant of the manifold M itself.

Theorem 1 *Every closed nonpositively curved manifold with non-trivial volume flux group has zero minimal volume.*

The technique we will use to show that the minimal volume vanishes is an $\mathcal{F}$-structure, which was introduced by Gromov as a generalisation of an S^1-action.

Definition 2 An $\mathcal{F}$-structure on a closed manifold M is given by the following conditions.

(1) A finite open cover $\{U_1, \ldots, U_N\}$
(2) $\pi_i : \widetilde{U}_i \to U_i$ a finite Galois covering with group of deck transformations Γ_i, $1 \leq i \leq N$
(3) A smooth torus action with finite kernel of the k_i-dimensional torus,

$$\phi_i : T^{k_i} \to \mathrm{Diff}(\widetilde{U}_i),\ 1 \leq i \leq N$$

(4) A homomorphism $\Psi_i : \Gamma_i \to \mathrm{Aut}(T^{k_i})$ such that

$$\gamma(\phi_i(t)(x)) = \phi_i(\Psi_i(\gamma)(t))(\gamma x)$$

for all $\gamma \in \Gamma_i$, $t \in T^{k_i}$ and $x \in \widetilde{U}_i$
(5) For any finite sub-collection $\{U_{i_1}, \ldots, U_{i_l}\}$ such that $U_{i_1 \ldots i_l} := U_{i_1} \cap \ldots \cap U_{i_l} \neq \emptyset$ the following compatibility condition holds: let $\widetilde{U}_{i_1 \ldots i_l}$ be the set of points $(x_{i_1}, \ldots, x_{i_l}) \in \widetilde{U}_{i_1} \times \cdots \times \widetilde{U}_{i_l}$ such that $\pi_{i_1}(x_{i_1}) = \cdots = \pi_{i_l}(x_{i_l})$. The set $\widetilde{U}_{i_1 \ldots i_l}$ covers $\pi_{i_j}^{-1}(U_{i_1 \ldots i_l}) \subset \widetilde{U}_{i_j}$ for all $1 \leq j \leq l$, and we require that ϕ_{i_j} leaves $\pi_{i_j}^{-1}(U_{i_1 \ldots i_l})$ invariant and it lifts to an action on $\widetilde{U}_{i_1 \ldots i_l}$ such that all lifted actions commute

An $\mathcal{F}$-structure is said to be *pure* if all the orbits of all actions at a point, for every point have the same dimension. We will say an $\mathcal{F}$-structure is *polarised* if the smooth torus action ϕ_i above is fixed point free for every U_i. The existence of a polarised $\mathcal{F}$-structure on a manifold M implies the minimal volume $\mathrm{MinVol}(M)$ is zero by the main result of Cheeger and Gromov [2]. The interested reader is invited to consult the illuminating examples found there as well.

The attentive reader will notice that the definition of an $\mathcal{F}$-structure above is different from the sophisticated one found in [2]. Despite this, it is sufficiently practical to be implemented and also satisfies the properties needed in the proof that a polarised $\mathcal{F}$-structure forces the minimal volume to vanish, which can be consulted in [2].

Proof of Theorem 1. Let M be a compact nonpositively curved manifold whose volume flux group is not trivial. The fundamental group $\pi_1(M)$ has non-trivial centre Z, a proof of which can be found in [5, Theorem 15] and compared with work of Fathi [4, Proposition 5.1] and the various references and attributions contained therein. In this case M admits a finite covering space M^* diffeomorphic to $T^k \times N$, where T^k is a flat torus of dimension k and N is a compact nonpositively curved manifold as was shown by Eberlein [3].

Even though it may seem that this already implies $\mathrm{MinVol}(M) = 0$, for completeness will now show how to construct a pure polarised $\mathcal{F}$-structure on M; since the torus T^k splits off M^* smoothly and the action of T^k on itself as a factor of M^* is compatible with the covering transfomation in the required sense. This will also provide an example of a detailed construction of a pure polarised $\mathcal{F}$-structure.

Represent M as H/Γ, where H is simply connected and Γ is a properly discontinuous group of isometries of H which acts freely. The space H decomposes into $H_1 \times H_2$, where $H_1 = \mathbf{R}^k$ is a Euclidean space of dimension $k = \mathrm{rank}(Z)$.

Notice that every element γ of Γ is of the form $\gamma = \gamma_1 \times \gamma_2 \in \mathrm{Iso}(H_1) \times \mathrm{Iso}(H_2)$, here $\mathrm{Iso}(H_i)$ denotes the group of isometries of H_i (see Lemma 1 in [3], and the subsequent discussion). Let $p_i : \Gamma \to \mathrm{Iso}(H_i)$ denote the projection homomorphisms, then $\Gamma_1 = p_1(\Gamma)$ acts by translations on H_1 and $\Gamma_2 = p_2(\Gamma)$ is a discrete subgroup of $\mathrm{Iso}(H_2)$. So $Z \subset \Gamma_1$ and H_1/Z is a compact flat torus T^k. The projection $p : H_1 \to T^k$ allows us to define $\rho : \Gamma_2 \to T^k$ by setting $\rho(p_2\gamma) = p(p_1\gamma)$. The function ρ is well defined since $\ker(p) = \ker(p_2) = Z$. The centre Z can also be thought of as a set of vectors in H_1, as $Z \subset \Gamma_1$. In this way Z acts on H_1 by translations.

Recall that Γ_2 has trivial centre and that M is isometric to $(T^k \times H_2)/\Gamma_2$ [3], here Γ_2 acts on $T^k \times H_2$ by $\psi(s, h) = (\rho(\psi)s, \psi(h))$ with (s, h) in $T^k \times H_2$ and ψ in Γ_2. This can be read from the diagram

$$
\begin{array}{ccc}
H = H_1 \times H_2 & \xrightarrow{p \times Id} & T^k \times H_2 \\
\downarrow & & \downarrow q \\
M = H/\Gamma & \xrightarrow{F} & (T^k \times H_2)/\Gamma_2
\end{array}
$$

where F is defined so that the diagram commutes.

In Γ_2 there exists a finite index subgroup Γ_0—denoted by Γ_2^{**} in [3]—which makes the following diagram commute.

$$
\begin{array}{ccc}
H & \longrightarrow & T^k \times (H_2/\Gamma_0) := M^* \\
& \searrow & \downarrow \\
& & (T^k \times H_2)/\Gamma_2 = M
\end{array}
$$

Define $q^* : T^k \times (H_2/\Gamma_0) = M^* \to M$ as in the previous diagram and denote by Γ^* the group of deck transformations of q^* seen as a covering map. Notice that $\Gamma^* \subset \Gamma_2$.

The function ρ is defined on all of Γ_2, so we can also consider the restriction of ρ to Γ^*.

We are now in a position to verify that this construction gives M a pure polarised $\mathcal{F}$-structure, and hence $\mathrm{MinVol}(M) = 0$ as claimed. Let us check that every condition which guarantees the existence of a polarised $\mathcal{F}$-structure is met.

(1) Take $U = M$, as an open cover with a single set.
(2) Define $\tilde{U} := M^* = T^k \times N$, here $N = H_2/\Gamma_0$. The quotient map

$$q^* : \tilde{U} = T^k \times N \to (T^k \times H_2)/\Gamma_2 = M = U$$

is a finite Galois covering with deck transformation group Γ^*.
(3) The k-torus T^k acts smoothly and without fixed points on itself as a factor of $T^k \times (H_2/\Gamma_0)$. So we have the action

$$\phi : T^k \to \mathrm{Diff}(\tilde{U}) = \mathrm{Diff}(T^k \times N)$$

given by $\phi(t)(s, h) = (s + t, h)$.
(4) We will use the function ρ to define the automorphism $\Psi : \Gamma^* \to \mathrm{Aut}(T^k)$, set $\Psi(\gamma) := \rho(\gamma)$. Let $t \in T^k$ and $x \in \tilde{U} = T^k \times N \Rightarrow x = (s, h) \in T^k \times N$. Notice that for s and t in T^k we have that $\rho(\gamma)(s + t) = (\rho(\gamma)s + \rho(\gamma)t)$ since $\rho(\gamma) \in T^k$.
We plug in this information to obtain the following equalities:

$$\begin{aligned}\gamma(\phi(t)x) &= \gamma(\phi(t)(s, h)) \\ &= \gamma(s + t, h) \\ &= (\rho(\gamma)(s + t), \gamma h) \\ &= (\rho(\gamma)s + \rho(\gamma)t, \gamma h) = \star\end{aligned}$$

$$\begin{aligned}\phi(\Psi(\gamma)t)(\gamma x) &= \phi(\rho(\gamma)t)(\gamma x) \\ &= \phi(\rho(\gamma)t)(\rho(\gamma)s, \gamma h) \\ &= (\rho(\gamma)s + \rho(\gamma)t, \gamma h) = \star\end{aligned}$$

Therefore $\gamma(\phi_i(t)(x)) = \phi_i(\Psi_i(\gamma)(t))(\gamma x)$ and the condition is satisfied.
(5) This condition does not need to be verified, because we only have one covering set.

Since the action of T^k on itself as a factor of $T^k \times (H_2/\Gamma_0)$ is fixed point free, the above construction gives M a pure polarised $\mathcal{F}$-structure. □

We will state the contrapositive statement to the theorem because it strengthens Corollary 17 of [5]: *Let M be a closed nonpositively curved manifold with positive minimal volume. Then the volume flux group Γ_μ is trivial for every volume form μ on M.*

Another noteworthy feature of the covering $M^* \to M$ is that on $M^* \cong T^k \times N$ elements of the volume flux group which come from rotations on the T^k factor are circle actions with homologically essential orbits. It is not yet clear if in the general case a non-trivial volume flux group implies existence of a circle action with homologically essential orbits at least in a multiple cover—compare with Remark 19 in [5]—but it is virtually so for every closed nonpositively curved manifold M, since it is true for M^*.

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