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ROUGH FUNCTIONS: *p*-VARIATION, CALCULUS, AND INDEX ESTIMATION

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Abstract. In this paper, we give an overview of several dissipated results on the p-variation property of a function presented in a suitable way. More specifically, we attempt to show: (1) usefulness of this property in a calculus of rough functions; (2) a relatively thorough knowledge of the p-variation property of a sample function of basic stochastic processes; and (3) an almost unexplored area of statistical analysis seeking to estimate the p-variation index.

Keywords: Φ-variation, interval functions, refinement Young–Stieltjes integral, Kolmogorov integral, product integral, Nemytskii operator, integral equations, sample functions of stochastic processes, p-variation index, Orey index, oscillation summing index.

1. INTRODUCTION

The p-variation is a generalization of the total variation of a function. More specifically, for $0 < p < \infty$ and $-\infty < a < b < +\infty$, the *p-variation* of a real-valued function f on an interval $[a, b]$ is

$$v_p(f) := v_p\big(f; [a, b]\big) := \sup\left\{\sum_{i=1}^{n} \big|f(t_i) - f(t_{i-1})\big|^p\colon\ a = t_0 < t_1 < \cdots < t_n = b,\ n \geqslant 1\right\}. \tag{1}$$

We say that f has *bounded p-variation* if $v_p(f) < \infty$. Thus, a function has bounded total variation if it has bounded 1-variation. A regulated function f is called *rough* if it has infinite 1-variation, that is, if $v_1(f) = +\infty$. The p-variation gives a convenient way to measure a degree of roughness.

More importantly, the p-variation appears to be a useful tool in calculus of rough functions. In Section 2 below, we give few selected results from [16] illustrating both the calculus and importance of the p-variation property. To apply this calculus we need to know those values of p for which the p-variation of a given function is bounded. Since $v_p(f)^{1/p}$ is a nonincreasing function of p, there is a minimal value p_0 such that $v_p(f) < \infty$ for each $p > p_0$. Thus, for a function f on an interval $[a, b]$, we define the *p-variation index* of f by

$$\upsilon\big(f; [a, b]\big) := \begin{cases} \inf\{p > 0\colon\ v_p(f; [a, b]) < \infty\} & \text{if the set is nonempty,} \\ +\infty & \text{otherwise.} \end{cases} \tag{2}$$

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Best known examples of rough functions are sample functions of suitable classes of stochastic processes. In Section 3 below, we review boundedness of p-variation results for basic classes of stochastic processes. In the last section, we discuss two methods of estimating the p-variation index of a function given by finitely many values.

We continue the introduction with more details about the p-variation for point functions as well as for interval functions.

Classes of rough functions. As examples in Section 3 show, one obtains more exact information about roughness of a function when the power function $u \mapsto u^p$ of increments $|f(t_i) - f(t_{i-1})|$ in (1) is replaced by a more general function $u \mapsto \Phi(u)$, which gives rise to what is called the Φ-variation or a generalized variation. Let Φ be a strictly increasing continuous unbounded function on $[0, \infty)$ such that $\Phi(0) = 0$. The class of all such functions is denoted by $\mathcal{V}$. Let $J = [a, b]$ be a closed interval of real numbers with $-\infty < a \leqslant b < +\infty$. If $a < b$, an ordered set $\kappa = \{t_i\}_{i=0}^n$ of points in $[a, b]$ such that $a = t_0 < t_1 < \cdots < t_n = b$ is called a (point) partition. The set of all point partitions of $[a, b]$ is denoted by $\mathfrak{P}[a, b]$.

Let $X = (X, \|\cdot\|)$ be a Banach space, and let f be a function from an interval $J = [a, b]$ into X. If $a < b$, for $\kappa = \{t_i\}_{i=0}^n$ in $\mathfrak{P}[a, b]$, the Φ-variation sum is

$$s_\Phi(f; \kappa) := \sum_{i=1}^n \Phi\big(\|f(t_i) - f(t_{i-1})\|\big).$$

Thus, the Φ-*variation* of f over $[a, b]$ is 0 if $a = b$ and otherwise

$$v_\Phi(f) := v_\Phi(f; J) := \sup\big\{s_\Phi(f; \kappa) \colon \kappa \in \mathfrak{P}(J)\big\}.$$

We say that f has *bounded* Φ-*variation* if $v_\Phi(f) < \infty$. The class of all functions from J into X with bounded Φ-variation is denoted by $\mathcal{W}_\Phi(J; X)$, and $\mathcal{W}_\Phi(J) := \mathcal{W}_\Phi(J; \mathbb{R})$ if $X = \mathbb{R}$. The class $\mathcal{W}_\Phi(J; X)$ need not be a vector space in general but can be enlarged to such a space, which also can be made a Banach space. Since we do not use it in this paper, we settle for the following special case.

Taking $\Phi_p(u) := u^p$, $u \geqslant 0$, with $0 < p < \infty$, let $v_p(f; J) := v_{\Phi_p}(f; J)$, which agree with (1) when $X = \mathbb{R}$. In this case, we write $\mathcal{W}_p := \mathcal{W}_{\Phi_p}$. For $1 \leqslant p < \infty$, let

$$\|f\|_{J,(p)} := v_p(f; J)^{1/p} \quad \text{and} \quad \|f\|_{J,[p]} := \|f\|_{\sup} + \|f\|_{J,(p)},$$

where $\|f\|_{\sup} = \sup_{t \in J} \|f(t)\|$ is the sup norm of f. By Hölder's and Minkowski's inequalities, it follows that $\|\cdot\|_{J,(p)}$ is a seminorm and $\|\cdot\|_{J,[p]}$ is a norm on $\mathcal{W}_p(J; X)$. In fact, $\mathcal{W}_p(J; X)$ is a Banach space with norm $\|\cdot\|_{J,[p]}$.

A function f from an interval $J \subset \mathbb{R}$ into X is called *regulated* on J or $f \in \mathcal{R}(J; X)$ if it has left and right limits in X everywhere. In particular, $f \in \mathcal{R}([a, b]; X)$ with $a < b$ if f has a limit $f(t-) := \lim_{s \uparrow t} f(s)$ for each $t \in (a, b]$ and a limit $f(t+) := \lim_{s \downarrow t} f(s)$ for each $t \in [a, b)$. We call $\Delta^- f(t) := f(t) - f(t-)$ the left jump of f at $t \in (a, b]$, and $\Delta^+ f(t) := f(t+) - f(t)$ the right jump of f at $t \in [a, b)$. For $J = [a, b]$, let $\Delta_J^- f(t) := \Delta^- f(t)$ if $t \in (a, b]$ or 0 if $t = a$, let $\Delta_J^+ f(t) := \Delta^+ f(t)$ if $t \in [a, b)$ or 0 if $t = b$, and let $\Delta_J^\pm f := \Delta_J^- f + \Delta_J^+ f$. A regulated function is bounded and has at most countably many nonzero jumps. A function is regulated if and only if it is a uniform limit of step functions. Also, a function is regulated on $[a, b]$ if and only if it has bounded Φ-variation on $[a, b]$ for some convex function $\Phi \in \mathcal{V}$ (see [10] or [15, Part III]).

Again let $X = (X, \|\cdot\|)$ be a Banach space, and let S be a set in $\mathbb{R}$. A function $f \colon S \to X$ is said to be Hölder of order α, or simply α-Hölder if

$$\|f\|_{S,(\mathcal{H}_\alpha)} := \sup\big\{\|f(t) - f(s)\|/|t - s|^\alpha \colon s, t \in S,\ s \neq t\big\} < +\infty.$$

The class of all α-Hölder functions from S to X is denoted by $\mathcal{H}_\alpha(S; X)$, and $\mathcal{H}_\alpha(S) := \mathcal{H}_\alpha(S; \mathbb{R})$ if $X = \mathbb{R}$. If f has the α-Hölder property, then f has bounded p-variation with $p = 1/\alpha$. A continuous function of bounded p-variation for some $1 \leqslant p < \infty$ need not have the α-Hölder property. For example, the series

$$\sum_{k=1}^\infty \frac{\sin kt}{k \log k}, \quad 0 \leqslant t \leqslant 1,$$

converges uniformly to the sum g, which is absolutely continuous and, hence, has bounded p-variation for each $1 \leqslant p < \infty$. However, this g satisfies no Hölder property of order $\alpha > 0$ (see [6, Section 10.6.1 (2)] for details).

Let $f\colon [a,b] \to X$ have bounded p-variation on $[a,b]$. For any $a < u < b$, we have

$$v_p\big(f;[a,u]\big) + v_p\big(f;[u,b]\big) \leqslant v_p\big(f;[a,b]\big).$$

Thus, the function $[a,b] \ni t \mapsto v_p(f;[a,t]) := w(t)$ is nondecreasing and, for any $a \leqslant s \leqslant t \leqslant b$,

$$\big\| f(t) - f(s) \big\| \leqslant \big[w(t) - w(s)\big]^{1/p}. \tag{3}$$

Conversely, if this holds for some nondecreasing w, then f has bounded p-variation. Thus, f has bounded p-variation on $[a,b]$ if and only if (3) holds for some nondecreasing function w.

Local p-variation. As before, let X be a Banach space, $J = [a,b]$, and $0 < p < \infty$. For a function $f\colon J \to X$, its *local p-variation* is defined by

$$v_p^*(f) := v_p^*(f;J) := \inf_{\lambda \in \mathfrak{P}(J)} \sup\big\{s_p(f;\kappa)\colon \kappa \supset \lambda\big\}.$$

For a regulated function $f\colon J \to X$, let

$$\sigma_p(f) := \sigma_p(f;J) := \sum_{a<t\leqslant b} \big\|\Delta^- f(t)\big\|^p + \sum_{a\leqslant t<b} \big\|\Delta^+ f(t)\big\|^p.$$

Then we have $\sigma_p(f) \leqslant v_p^*(f) \leqslant v_p(f)$. We say that a function $f\colon J \to X$ has *p^*-variation* on J and write $f \in \mathcal{W}_p^*(J;X)$ if $\sigma_p(f) = v_p^*(f) < +\infty$.

By Propositions 2.12 and 2.13 of [15, Part II], $\mathcal{W}_p^* = \mathcal{W}_p$ if $p < 1$ and $v_1^*(f) = v_1(f)$. If $1 \leqslant q < p < \infty$, then $\mathcal{W}_q \subset \mathcal{W}_p^* \subset \mathcal{W}_p$. For $1 < p < \infty$ and an even integer $M > 1$, the series

$$\sum_{k=1}^{\infty} M^{-k/p} \sin\big(2\pi M^k t\big), \quad 0 \leqslant t \leqslant 1,$$

converges uniformly to the sum g, which is $(1/p)$-Hölder continuous, and so $g \in \mathcal{W}_p[0,1]$. However, if $M^{1-(1/p)} > 4$, then $g \notin \mathcal{W}_p^*[0,1]$.

For $1 < p < \infty$ and a regulated function $f\colon J \to X$, one can show that f has p^*-variation on J if and only if f is a limit in $(\mathcal{W}_p, \|\cdot\|_{[p]})$ of step functions. Also, f has p^*-variation on J if and only if the finite limit $\lim_\kappa s_p(f;\kappa)$ exists under refinements of partitions κ of J.

Interval functions. Each nondecreasing and right-continuous function h on a real line $\mathbb{R}$ defines the Lebesgue–Stieltjes measure λ_h on $\mathbb{R}$ by letting $\lambda_h((u,v]) := h(v) - h(u)$ for $-\infty < u \leqslant v < +\infty$ and then extending λ_h to the Borel σ-algebra. Certainly, a σ-finite extension is not possible if h is a rough function. It is known (see, e.g., Appendix B.1 of [33]) that an additive function λ on an algebra extends to a measure if and only if λ is exhaustive and upper-continuous at $\varnothing$. We will see that functions defined and upper-continuous at $\varnothing$ *only* on intervals have a close relation with regulated point functions, similar to the one between the Lebesgue–Stieltjes measure λ_h and the function h.

Let J be a nonempty interval in a real line $\mathbb{R}$, let $\mathfrak{I}(J)$ be the class of all subintervals of J which may be closed or open at either end, including empty set, and let $X = (X, \|\cdot\|)$ be a Banach space. Any function $\mu\colon \mathfrak{I}(J) \to X$ is called an *interval function on J*. An interval function μ on J is called *additive* if $\mu(A \cup B) = \mu(A) + \mu(B)$ whenever $A, B \in \mathfrak{I}(J)$ are disjoint and $A \cup B \in \mathfrak{I}(J)$. If μ is an additive interval function, then $\mu(\varnothing) = 0$. For intervals, $A_n \downarrow A$ means $A_1 \supset A_2 \supset \cdots$ and $\bigcap_{n=1}^\infty A_n = A$. An interval function μ on J is called *upper-continuous* if $\mu(A_n) \to \mu(A)$ for any $A_1, A_2, \ldots \in \mathfrak{I}(J)$ such that $A_n \downarrow A$. If the latter property holds only with $A = \varnothing$, then μ is called *upper-continuous at $\varnothing$*. The class of all additive and upper-continuous interval functions on J with values in X is denoted by $\mathcal{AI}(J;X)$.

Given a regulated point function, one can define an additive interval function as follows. Let h be a regulated function on $[a,b]$ with values in X. Define the interval function $\mu_h \equiv \mu_{h,[a,b]}$ on $[a,b]$ by

$$\begin{aligned} \mu_h((u,v)) &:= h(v-) - h(u+) \quad \text{for } a \leqslant u < v \leqslant b, \\ \mu_h(\{u\}) &:= \Delta^{\pm}_{[a,b]} h(u) \quad \text{for } u \in [a,b], \end{aligned} \tag{4}$$

and extend μ_h to all elements of $\mathfrak{I}[a,b]$ by additivity. Note that μ_h does not depend on the values of h at its jump points in (a,b) and $\mu_h((u,v]) = h(v) - h(u)$ for $a \leqslant u < v \leqslant b$ if h is right-continuous.

In a converse direction, given any interval function μ on $[a,b]$, define a point function $R_{\mu,a}$ on $[a,b]$ by

$$R_{\mu,a}(t) := \begin{cases} \mu(\varnothing) & \text{if } t = a, \\ \mu([a,t]) & \text{if } t \in (a,b]. \end{cases} \tag{5}$$

Now the upper-continuity property of an additive interval function can be characterized as follows.

THEOREM 1. *For an additive interval function μ: $\mathfrak{I}[a,b] \to X$, the following are equivalent*:
(a) $\mu \in \mathcal{AI}([a,b]; X)$;
(b) μ *is upper-continuous at* $\varnothing$;
(c) $\mu = \mu_h$ *on* $\mathfrak{I}[a,b]$ *for some regulated function* h *such that* $h(a) = 0$;
(d) $R_{\mu,a}$ *is regulated,* $R_{\mu,a}(t-) = \mu([a,t))$ *for* $t \in (a,b]$, *and* $R_{\mu,a}(t+) = \mu([a,t])$ *for* $t \in [a,b)$.

Using (d), it follows that, for an additive and upper-continuous interval function μ on $[a,b]$, we have $\mu = \mu_h$ if $h = R_{\mu,a}$.

p-Variation of interval functions. Let J be a nonempty interval, open or closed at either end. An ordered collection $\{A_i\}_{i=1}^n$ of disjoint nonempty subintervals A_i of J is called an *interval partition* of J if their union is J and $s < t$ for all $s \in A_i$ and $t \in A_j$ whenever $i < j$. The class of all interval partitions of J is denoted by $\mathfrak{D}(J)$. An interval partition $\mathcal{A}$ is a *refinement* of an interval partition $\mathcal{B}$, written $\mathcal{A} \sqsupset \mathcal{B}$, if each interval in $\mathcal{A}$ is a subinterval of an interval in $\mathcal{B}$.

Let J be a nonempty interval, let $X = (X, \|\cdot\|)$ be a Banach space, and let μ be an interval function on J with values in X. For $\mathcal{A} = \{A_i\}_{i=1}^n \in \mathfrak{D}(J)$, the p-variation sum is

$$s_p(\mu; \mathcal{A}) := \sum_{i=1}^n \left\| \mu\left(\bigcup_{j=1}^i A_j\right) - \mu\left(\bigcup_{j=1}^{i-1} A_j\right) \right\|^p,$$

where the union over the empty set of indices is defined as the empty set. If μ is additive, then

$$s_p(\mu; \mathcal{A}) = \sum_{i=1}^n \left\| \mu(A_i) \right\|^p.$$

The *p-variation* of μ on J is the interval function $v_p(\mu) := v_p(\mu; \cdot)$ on J defined by

$$v_p(\mu; A) := \sup\left\{ s_p(\mu; \mathcal{A}) \colon \mathcal{A} \in \mathfrak{D}(A) \right\}$$

if $A \in \mathfrak{I}(J)$ is nonempty, or as 0 if $A = \varnothing$. We say that μ has *bounded p-variation* if $v_p(\mu; J) < \infty$. The class of all interval functions μ: $\mathfrak{I}(J) \to X$ with bounded p-variation is denoted by $\mathcal{I}_p(J; X)$. For $1 \leqslant p < \infty$, let

$$\|\mu\|_{J,(p)} := v_p(\mu; J)^{1/p} \quad \text{and} \quad \|\mu\|_{J,[p]} := \|\mu\|_{\sup} + \|\mu\|_{(p)},$$

where $\|\mu\|_{\sup} = \sup\{\|\mu(A)\colon A \in \mathfrak{I}(J)\}$ is the sup norm of μ. As for point functions, we have that, on $\mathcal{I}_p(J; X)$, $\|\cdot\|_{J,(p)}$ is a seminorm and $\|\cdot\|_{J,[p]}$ is a norm, and $(\mathcal{I}_p(J; X), \|\cdot\|_{J,[p]})$ is a Banach space.

Let J be a nonempty interval, and let $\mu \in \mathcal{AI}(J;X)$. For $t \in J$, the singleton $\{t\}$ is an atom of μ if $\mu(\{t\}) \neq 0$. The set of all atoms of μ is at most countable, and so the following is well defined. For $0 < p < \infty$, if $A \in \mathfrak{I}(J)$ is nonempty, let

$$\sigma_p(\mu;A) := \sum_{t\in A} \left\|\mu(\{t\})\right\|^p \quad \text{and} \quad v_p^*(\mu;A) := \inf_{\mathcal{B}\in\mathfrak{D}(A)} \sup\left\{s_p(\mu;\mathcal{A})\colon \mathcal{A} \sqsupset \mathcal{B}\right\},$$

and if $A = \varnothing$, we let the two quantities to be equal to 0. Then $\sigma_p(\mu) = \sigma_p(\mu;\cdot)$ and $v_p^*(\mu) = v_p^*(\mu;\cdot)$ are interval functions on J. In general, we have the following relations:

$$\sigma_p(\mu;A) \leqslant v_p^*(\mu;A) \leqslant v_p(\mu;A)$$

for each $A \in \mathfrak{I}(J)$. For $0 < p < \infty$, we say that μ has *p^*-variation* and write $\mu \in \mathcal{AI}_p^*(J;X)$ if $\sigma_p(\mu;J) = v_p^*(\mu;J) < +\infty$.

It is not hard to prove that if $\mu \in \mathcal{AI}_p^*(J;X)$, then, for each $A \in \mathfrak{I}(J)$, the limit $\lim_{\mathcal{A}} s_p(\mu;\mathcal{A})$ exists under refinements of interval partitions $\mathcal{A}$ of A and is finite. Also, for $\mu \in \mathcal{AI}(J;X)$, if $1 \leqslant q < p < \infty$, then $\mathcal{I}_q(J;X) \subset \mathcal{AI}_p^*(J;X)$.

We will say that μ is a *rough interval function* if $v_1(\mu) = +\infty$.

Historical notes. The notion of Φ-variation of a (point) function is suggested by Young [63]. He attributed the introduction of the power case $\Phi(u) = u^p$ to Norbert Wiener. Wiener [59], in fact, defined

$$\bar{v}_p(f) := \lim_{\varepsilon\downarrow 0} \sup\left\{s_p(f;\kappa)\colon |\kappa| \leqslant \varepsilon\right\}.$$

It is is different, in general, from the local p-variation $v_p^*(f)$ introduced by Love and Young [35]. Before Young [61], the p-variation was used by Lévy [34] in connection with stable stochastic processes and by Marcinkiewicz [41] in connection with Fourier series.

2. CALCULUS OF ROUGH FUNCTIONS

Integration. We wish to be able to integrate one rough function with respect to another rough function or with respect to a rough interval function. Certainly, we cannot use the Lebesgue–Stieltjes integral for this purpose. The Riemann–Stieltjes and the refinement Riemann–Stieltjes integrals have defects when an integrand and an integrator have a jump at the same point, even if both functions have bounded 1-variation. In this section, we survey few selected results from the calculus based on the Kolmogorov integral used to integrate with respect to a rough interval function. Its superiority in the present context over other integrals, including the refinement Young–Stieltjes integral, is acknowledged.

We recall definitions and discuss some properties of the integrals just mentioned referring to [16] for details. For simplicity, we restrict our presentation to the case where functions take values in a Banach algebra $\mathbb{B}$ with norm $\|\cdot\|$ (the case of a Banach space-valued functions is treated in [16]). Let f and h be $\mathbb{B}$-valued functions defined on $[a,b]$ for some $-\infty < a \leqslant b < +\infty$. If $\kappa = \{t_i\}_{i=0}^n$ is a partition of $[a,b]$ with $a < b$, then $|\kappa| := \max_i |t_i - t_{i-1}|$ is its mesh and an ordered pair $\tau = (\kappa, \{s_i\}_{i=1}^n)$ is called a tagged partition of $[a,b]$ if $s_i \in [t_{i-1}, t_i]$ for $i = 1,\dots,n$. The tagged partition τ is a refinement of a partition λ of $[a,b]$ if $\lambda \subset \kappa$. For a tagged partition $\tau = (\{t_i\}_{i=0}^n, \{s_i\}_{i=1}^n)$ of $[a,b]$, the Riemann–Stieltjes sum based on τ is

$$S_{RS}(f, \mathrm{d}h;\tau) := \sum_{i=1}^n f(s_i)\left[h(t_i) - h(t_{i-1})\right].$$

The *Riemann–Stieltjes* or RS integral $(RS)\int_a^b f\,\mathrm{d}h$ is defined to be 0 in $\mathbb{B}$ if $a = b$ and otherwise exists and equals $A \in \mathbb{B}$ if, for a given $\varepsilon > 0$, there is a $\delta > 0$ such that

$$\left\|A - S_{RS}(f, \mathrm{d}h;\tau)\right\| < \varepsilon \tag{6}$$

for each tagged partition $\tau = (\{t_i\}_{i=0}^n, \{s_i\}_{i=1}^n)$ of $[a,b]$ with mesh $|\{t_i\}_{i=0}^n| < \delta$. The *refinement Riemann–Stieltjes* or RRS integral $(RRS)\int_a^b f\,\mathrm{d}h$ is defined to be 0 in $\mathbb{B}$ if $a=b$ and otherwise exists and equals $A \in \mathbb{B}$ if, for each $\varepsilon > 0$, there is a point partition λ of $[a,b]$ such that (6) holds for each tagged partition τ of $[a,b]$ which is a refinement of λ. The RRS integral extends the RS integral, since, for any partition λ, a refinement κ of λ has mesh less than that of λ.

It is easy to check that an integral of an indicator function with respect to itself does not exist in the RRS sense and, hence, in the RS sense. More generally, if $(RS)\int_a^b f\,\mathrm{d}h$ exists, then f and h have no common discontinuities, that is, for each $t \in [a,b]$, at least one of f and h is continuous at t. Also, if $(RRS)\int_a^b f\,\mathrm{d}h$ exists, then f and h have no common one-sided discontinuities, that is, for each $t \in (a,b]$, at least one of f and h is left-continuous at t and, for each $t \in [a,b)$, at least one of f and h is right-continuous at t.

We prefer integrals which have no restrictions on jumps of an integrand and integrator. Among several such classical integrals, we discuss the refinement Young–Stieltjes integral defined as follows. Suppose, in addition, that h is a regulated function on $[a,b]$. For a point partition $\{t_i\}_{i=0}^n$ of $[a,b]$ with $a<b$, a tagged partition $(\{t_i\}_{i=0}^n, \{s_i\}_{i=1}^n)$ is called a *Young tagged point partition* if $t_{i-1} < s_i < t_i$ for each $i = 1, \ldots, n$. Given a Young tagged point partition $\tau = (\{t_i\}_{i=0}^n, \{s_i\}_{i=1}^n)$, define the *Young–Stieltjes sum* $S_{YS}(f, \mathrm{d}h; \tau)$ based on τ by

$$S_{YS}(f, \mathrm{d}h; \tau) := \sum_{i=1}^n \Big\{[f\,\Delta^+_{[a,b]}h](t_{i-1}) + f(s_i)\big[h(t_i-) - h(t_{i-1}+)\big] + [f\,\Delta^-_{[a,b]}h](t_i)\Big\}. \tag{7}$$

The *refinement Young–Stieltjes* or RYS integral $(RYS)\int_a^b f\,\mathrm{d}h$ is defined to be 0 in $\mathbb{B}$ if $a=b$ and otherwise exists and equals $A \in \mathbb{B}$ if, for each $\varepsilon > 0$, there is a point partition λ of $[a,b]$ such that

$$\big\|A - S_{YS}(f, \mathrm{d}h; \tau)\big\| < \varepsilon \tag{8}$$

for each Young tagged point partition τ of $[a,b]$ which is a refinement of λ.

We say that the *full Stieltjes* integral $(S)\int_a^b f\,\mathrm{d}h$ exists if (a), (b), and (c) hold, where

(a) $(RYS)\int_a^b f\,\mathrm{d}h$ exists;

(b) if f and h have no common one-sided discontinuities, then $(RRS)\int_a^b f\,\mathrm{d}h$ exists;

(c) if f and h have no common discontinuities, then $(RS)\int_a^b f\,\mathrm{d}h$ exists.

Then let

$$(S)\int_a^b f\,\mathrm{d}h := (RYS)\int_a^b f\,\mathrm{d}h.$$

Now we turn to defining an integral of a point function with respect to an interval function. For a nonempty interval J, a *Young interval partition* of J is any interval partition consisting of singletons and open intervals. A *tagged Young interval partition* of J is a Young interval partition of J together with a member (tag) of each interval in the partition. We skip singletons with tags from the notation of a tagged Young interval partition. Let J be an interval, and let μ be an additive interval function on J with values in $\mathbb{B}$. If J is nonempty, for a tagged Young interval partition

$$\sigma = \Big(\big\{(t_{i-1}, t_i)\big\}_{i=1}^n, \{s_i\}_{i=1}^n\Big)$$

of J, let

$$S_{YS}(f, \mathrm{d}\mu; J, \sigma) := \sum_{i=1}^n f(s_i)\,\mu((t_{i-1}, t_i)) + \sum_{i=0}^n f(t_i)\,\mu(\{t_i\} \cap J). \tag{9}$$

The *Kolmogorov* integral $\int_J f\,\mathrm{d}\mu$ is defined to be 0 in $\mathbb{B}$ if J is empty and otherwise exists and equals $A \in \mathbb{B}$ if, for each $\varepsilon > 0$, there is an interval partition $\mathcal{A}$ of J such that

$$\big\|A - S_{YS}(f, \mathrm{d}\mu; J, \sigma)\big\| < \varepsilon \tag{10}$$

for each tagged Young interval partition σ of J which is a refinement of $\mathcal{A}$. This is a special case of the integral introduced by Kolmogorov [26].

Let μ be an additive and upper-continuous interval function on J, and let f be a bounded function on J such that the Kolmogorov integral $\int_J f\,\mathrm{d}\mu$ exists. Then

$$\Im(J) \ni A \mapsto \int_A f\,\mathrm{d}\mu$$

is an additive and upper-continuous interval function on J. Thus, in the case $J = [a,b]$, for any $a \leqslant c \leqslant b$, we have

$$\int_{[a,b]} f\,\mathrm{d}\mu = \int_{[a,c)} f\,\mathrm{d}\mu + \int_{[c,b]} f\,\mathrm{d}\mu = \int_{[a,c]} f\,\mathrm{d}\mu + \int_{(c,b]} f\,\mathrm{d}\mu. \tag{11}$$

The additivity property of the RYS integral means that, for each $a \leqslant c \leqslant b$,

$$(RYS)\int_a^b f\,\mathrm{d}h = (RYS)\int_a^c f\,\mathrm{d}h + (RYS)\int_c^b f\,\mathrm{d}h. \tag{12}$$

One can see a difference between the two additivity properties if one takes h defined on $[a,b] = [0,2]$ by $h(t) := 0$ for $t \in [0,1) \cup (1,2]$ and $h(1) := 1$, $\mu := \mu_{h,[0,2]} \equiv 0$ defined by (4), and $c = 1$. Then each integral in (11) is zero, while the two integrals on the right of (12) are equal to $f(1)\Delta^- h(1)$ and $f(1)\Delta^+ h(1)$, respectively. This example shows that the two integration methods define two different families of indefinite integrals

$$\left\{\int_{[0,t]} f\,\mathrm{d}\mu\colon\ t \in [0,2]\right\} \neq \left\{(RYS)\int_0^t f\,\mathrm{d}h\colon\ t \in [0,2]\right\},$$

and so give rise to two different ways to model evolutions.

The following shows that the question of existence of the Kolmogorov integral can be reduced to the question of existence of the RYS integral. Let μ be an additive and upper-continuous interval function on $[a,b]$ with values in $\mathbb{B}$, and let $h := R_{\mu,a}$ be defined by (5). Then, for any function $f\colon [a,b] \to \mathbb{B}$, we have

$$\int_{[a,b]} f\,\mathrm{d}\mu = (RYS)\int_a^b f\,\mathrm{d}h$$

if either side is defined. This follows from the fact that $\mu = \mu_h$ noted following Theorem 1 and since

$$S_{YS}\big(f,\,\mathrm{d}\mu_h; [a,b], \sigma\big) = S_{YS}(f,\,\mathrm{d}h; \tau)$$

for each

$$\sigma = \big(\big\{(t_{i-1}, t_i)\big\}_{i=1}^n, \{s_i\}_{i=1}^n\big) \quad \text{and} \quad \tau = \big(\{t_i\}_{i=0}^n, \{s_i\}_{i=1}^n\big).$$

It appears simpler to treat the existence question for the RYS integral described in the next subsection.

Existence of the full Stieltjes integral. Let $f \in \mathcal{W}_q[a,b]$, $h \in \mathcal{W}_p[a,b]$, and $p^{-1} + q^{-1} > 1$. In 1936, L.C. Young [61] proved that $\int_a^b f\,\mathrm{d}h$ exists (1) as the RS integral if the two functions have no common discontinuities, (2) as the RRS integral if the two functions have no common one-sided discontinuities, and

(3) in the sense defined by L.C. Young (not defined here) for any two such functions. Also, in which ever sense the integral exists, the inequality

$$\left| \int_a^b f \, \mathrm{d}h \right| \leqslant K \|f\|_{[q]} \|h\|_{(p)}$$

holds with a finite constant $K = K(p, q)$. L.C. Young acknowledged that E.R. Love had contributed to finding a proof of this inequality, and so we call it the *Love–Young inequality*.

The p- and q-variation conditions are no longer sufficient for the stated integrability fact if $p^{-1} + q^{-1} = 1$. But one can replace these conditions by the more general Φ- and Ψ-variation condition as follows. Let $\Phi, \Psi \in \mathcal{V}$, and let ϕ, ψ be their inverses, respectively, such that

$$\sum_{k=1}^{\infty} \phi(k^{-1}) \psi(k^{-1}) < +\infty. \tag{13}$$

If $f \in \mathcal{W}_\Phi[a, b]$ and $h \in \mathcal{W}_\Psi[a, b]$, then the full Stieltjes integral $\int_a^b f \, \mathrm{d}h$ exists and, for any $s \in [a, b]$, we have

$$\left| \int_a^b \bigl[f - f(s)\bigr] \mathrm{d}h \right| \leqslant K \sum_{k=1}^{\infty} \phi\Bigl(\frac{v_\Phi(f)}{k}\Bigr) \psi\Bigl(\frac{v_\Psi(h)}{k}\Bigr)$$

for some finite constant K. This result follows from Theorems 5.1 and 5.5 of Young [64].

D'yačkov [17] found necessary and sufficient conditions for a pair of convex functions Φ and Ψ in $\mathcal{V}$ such that the Riemann–Stieltjes integral $(RS) \int_a^b f \, \mathrm{d}h$ exists for any $f \in \mathcal{W}_\Phi[a, b]$ and $h \in \mathcal{W}_\Psi[a, b]$, provided that f and h have no common discontinuities. More generally, we will say that a pair of functions $\Phi, \Psi \in \mathcal{V}$ form a *Stieltjes pair* if the full Stieltjes integral $(S) \int_a^b f \, \mathrm{d}h$ exists for any $f \in \mathcal{W}_\Phi([a, b]; \mathbb{B})$ and $h \in \mathcal{W}_\Psi([a, b]; \mathbb{B})$, where $\mathbb{B}$ is a Banach algebra.

The necessary and sufficient integrability conditions are formulated in terms of increasing sequences $\{n_k\} = \{n_1, n_2, \ldots\}$ of positive integers which may be finite or infinite. Also, an inverse of a convex function in $\mathcal{V}$ will be a continuous nondecreasing function from $[0, \infty)$ onto itself, which is subadditive and called a modulus of continuity. Let ξ be a modulus of continuity. A sequence $\{n_k\}$ is called *ξ-lacunary* if, for some $C < \infty$, the inequalities

$$\sum_{k=1}^{m} n_k \xi\bigl(n_k^{-1}\bigr) \leqslant C n_m \xi\bigl(n_m^{-1}\bigr) \quad \text{and} \quad \sum_{k=m}^{\infty} \xi\bigl(n_k^{-1}\bigr) \leqslant C \xi\bigl(n_m^{-1}\bigr)$$

hold for each $m = 1, 2, \ldots$. There exists an infinite ξ-lacunary sequence $\{n_k\}$ if and only if $v = \mathrm{o}(\xi(v))$ as $v \downarrow 0$. If a convex function $\Phi \in \mathcal{V}$ is such that, for some $a > 0$ and $1 < \alpha < \beta < \infty$, the inequalities $u^\beta \leqslant \Phi(u) \leqslant u^\alpha$ hold for each $u \in [0, a]$, then the sequence $\{2^k\}$ is Φ^{-1}-lacunary.

Let ϕ and ψ be moduli of continuity. We say that a sequence $\eta = \{n_k\}$ is a *$W(\phi, \psi)$-sequence* if

$$\inf_{k \geqslant 1} \max \left\{ \frac{\phi(n_{k+1}^{-1})}{\phi(n_k^{-1})}, \frac{\psi(n_{k+1}^{-1})}{\psi(n_k^{-1})}, \frac{n_k \phi(n_k^{-1})}{n_{k+1} \phi(n_{k+1}^{-1})}, \frac{n_k \psi(n_k^{-1})}{n_{k+1} \psi(n_{k+1}^{-1})} \right\} > 0.$$

The sequence $\{2^k\}$ is $W(\phi, \psi)$-sequence for ϕ and ψ. For moduli of continuity ϕ, ψ and for a sequence $\eta = \{n_k\}$, let

$$\Lambda(\phi, \psi; \eta) := \sum_{k=1}^{\infty} n_k \phi\bigl(n_k^{-1}\bigr) \psi\bigl(n_k^{-1}\bigr) \leqslant +\infty.$$

For any moduli of continuity ϕ and ψ, one can show that $\Lambda(\phi, \psi; \{2^k\}) < +\infty$ if and only if (13) holds.

The next statement is a special case of a theorem proved in [16], which extends Theorem 2 of D'yačkov [17].

THEOREM 2. *The following conditions about convex functions Φ and Ψ in $\mathcal{V}$ with inverses ϕ and ψ, respectively, are equivalent*:

(a) *Φ and Ψ form a Stieltjes pair*;
(b) *$\Lambda(\phi, \psi; \eta) < +\infty$ for any ϕ- and ψ-lacunary sequence η*;
(c) *$\Lambda(\phi, \psi; \eta) < +\infty$ for some $W(\phi, \psi)$-sequence η.*

Since (c) holds for the sequence $\{2^k\}$ if and only if (13) holds, the preceding theorem implies the result of L.C. Young. Also, using Theorem 2 one can show (see [16]) that Beurling's condition is sufficient for the integrability: convex functions Φ and Ψ in $\mathcal{V}$ form a Stieltjes pair if

$$(RS)\int_{0+}^{1} \frac{\phi(u)}{u}\,\mathrm{d}\psi(u) < +\infty.$$

Product integration. To give an heuristic idea of a product integral, consider the Riemann integral equation

$$f(t) = f(a) + \int_a^t f(s)A(s)\,\mathrm{d}s, \quad a \leqslant t \leqslant b,$$

with respect to a vector-valued function f, where A is a matrix-valued continuous function on $[a, b]$. Let $t \in (a, b]$, and let $\kappa = \{t_i\}_{i=0}^n$ be an interval partition of $[a, t]$. Letting $\Delta_i := t_i - t_{i-1}$ for $i = 1, \ldots, n$, if f is a solution, then we approximately have

$$\begin{aligned} f(t) &\approx f(t_{n-1}) + f(t_{n-1})A(t_{n-1})\Delta_n = f(t_{n-1})\bigl[I + A(t_{n-1})\Delta_n\bigr] \\ &= f(a)\bigl[I + A(t_0)\Delta_1\bigr]\cdots\bigl[I + A(t_{n-1})\Delta_n\bigr], \end{aligned}$$

where I is the identity matrix. Taking a limit of the right side as the mesh $|\kappa| = \max_i \Delta_i$ of the partition κ goes to 0, if it exists, we get the matrix

$$\prod_a^t \bigl(I + A(s)\,\mathrm{d}s\bigr) := \lim_{|\kappa|\downarrow 0}\bigl[I + A(t_0)\Delta_1\bigr]\cdots\bigl[I + A(t_{n-1})\Delta_n\bigr], \tag{14}$$

called the product integral of A over $[a, t]$. If this holds for each $t \in (a, b]$, the product integral gives the solution

$$f(t) = f(a)\prod_a^t \bigl(I + A(s)\,\mathrm{d}s\bigr), \quad a \leqslant t \leqslant b.$$

This representation is originated by Volterra [57]. Almost at the same time, Peano [47] suggested a representation of the solution as a series of iterated integrals of the form

$$f(t) = f(a) + \sum_{k=1}^{\infty} \int_{a<s_1<s_2<\cdots<s_k<t} f(a)A(s_1)A(s_2)\cdots A(s_k)\,\mathrm{d}s_1\,\mathrm{d}s_2\cdots\mathrm{d}s_k$$

for each $t \in (a, b]$, which is often considered as a series representation of the product integral (14). Similarly, if h is a matrix-valued C^1 function on $[a, b]$, the unique solution f of the Riemann–Stieltjes integral equation

$$f(t) = f(a) + \int_a^t f\,\mathrm{d}h, \quad a \leqslant t \leqslant b,$$

can be represented using the product integral with respect to h,

$$\prod_a^t (I + \mathrm{d}h) := \lim_{|\kappa|\downarrow 0} [I + \Delta_1 h] \cdots [I + \Delta_n h], \tag{15}$$

where $\Delta_i h := h(t_i) - h(t_{i-1})$ for $i = 1, \dots, n$. Apparently, the existence and some properties of the product integral with respect to a rough function h were treated only in [15, Part II], where one can find further related references.

In [16], the product integral is defined and treated with respect to an interval function, and so it is a multiplicative interval function. In this section, we consider interval functions with values in a Banach algebra. Let $\mathbb{B}$ be a Banach algebra with norm $\|\cdot\|$ and identity $1\mathbb{I}$. The set of all interval functions on $J = [a, b]$ with values in $\mathbb{B}$ having bounded p-variation is denoted by $\mathcal{I}_p(J; \mathbb{B})$. The subset $\mathcal{AI}_p(J; \mathbb{B})$ consists of all elements in $\mathcal{I}_p(J; \mathbb{B})$ which are additive and upper-continuous. Endowed with the norm $\|\cdot\|_{J,[p]}$ and $1 \leqslant p < \infty$, the two sets are Banach spaces.

Let μ be a $\mathbb{B}$-valued additive interval function on $J = [a, b]$ with $a < b$. For a nonempty interval $A \in \mathfrak{I}(J)$ and for an interval partition $\mathcal{A} = \{A_i\}_{i=1}^n$ of A, let

$$P(\mu; A, \mathcal{A}) := \prod_{i=1}^n \bigl(1\mathbb{I} + \mu(A_i)\bigr) = \bigl(1\mathbb{I} + \mu(A_1)\bigr) \cdots \bigl(1\mathbb{I} + \mu(A_n)\bigr).$$

Since $\mathbb{B}$ may not be commutative, the product sign is used with the prescribed order, different from the one in [16]. Also in this paper, the *product integral* with respect to μ is the interval function

$$\mathcal{P} := \prod (1\mathbb{I} + \mathrm{d}\mu)\colon\ A \mapsto \prod_A (1\mathbb{I} + \mathrm{d}\mu) =: \mathcal{P}(\mu; A), \quad A \in \mathfrak{I}(J),$$

defined by $\prod_A (1\mathbb{I} + \mathrm{d}\mu) := \lim_{\mathcal{A}} P(\mu; A, \mathcal{A})$ if A is nonempty and the limit exists under refinements of partitions $\mathcal{A}$ of A, and $\prod_{\varnothing} (1\mathbb{I} + \mathrm{d}\mu) := 1\mathbb{I}$.

The product integral $\prod (1\mathbb{I} + \mathrm{d}\mu)$, if it exists, is $*$-multiplicative, that is, for any $A, B \in \mathfrak{I}(J)$ such that $A \cup B \in \mathfrak{I}(J)$ and $s < t$ for $s \in A$, $t \in B$ if both intervals are nonempty, we have

$$\mathcal{P}(\mu; A \cup B) = \mathcal{P}(\mu; A)\mathcal{P}(\mu; B).$$

The product integral with the opposite order of multiplication is called multiplicative.

For a real-valued interval function, we have the following equivalent statements:

THEOREM 3. *Let μ be a real-valued additive upper-continuous interval function on J. $\prod (1\mathbb{I} + \mathrm{d}\mu)$ exists and is nonzero on J if and only if $\mu \in \mathcal{AI}_2^*(J)$ and $\mu(\{t\}) \neq -1$ for all $t \in J$. If either of the two equivalent statements hold, then, for each $A \in \mathfrak{I}(J)$,*

$$\prod_A (1 + \mathrm{d}\mu) = \mathrm{e}^{\mu(A)} \prod_A (1 + \mu)\mathrm{e}^{-\mu},$$

where the product converges absolutely for nonempty A and equals 1 for $A = \varnothing$.

The situation is not as clear when $\mathbb{B}$ is a general Banach algebra. There are examples (discussed in [16]) of Banach algebras $\mathbb{B}$ and interval functions μ on J with values in $\mathbb{B}$ such that $\mu \in \mathcal{AI}_p(J; \mathbb{B})$ for all $p \in (2, \infty)$ but the product integral with respect to μ does not exist. For an interval function μ on J with values in an arbitrary Banach algebra $\mathbb{B}$, the following holds.

THEOREM 4. *If $\mathbb{B}$ is a Banach algebra and $\mu \in \mathcal{AI}_p(J; \mathbb{B})$ for some $p < 2$, then $\prod (1\mathbb{I} + \mathrm{d}\mu)$ exists and the nonlinear operator $\mathcal{P}$*

$$\mathcal{AI}_p(J; \mathbb{B}) \ni \mu \mapsto \mathcal{P}(\mu) = \prod (1\mathbb{I} + \mathrm{d}\mu) \in \mathcal{I}_p(J; \mathbb{B}) \tag{16}$$

acts between the two Banach spaces.

We call $\mathcal{P}$ the *product integral operator*. Due to the earlier result in [15, Part II] for the product integral operator acting between *point functions*, it is not surprising that $\mathcal{P}$ has a Taylor series expansion around each element $\mu \in \mathcal{AI}_p(J;\mathbb{B})$, and the series has infinite radius of uniform convergence. The kth term of the Taylor series is defined by the interval function $Q_\mu^k(\nu) = \{Q_\mu^k(A)\colon\ A \in \mathfrak{I}(J)\}$, where

$$Q_\mu^k(\nu)(A) = \begin{cases} \int_A \hat{\mu}((s,b]\cap A)\nu(\mathrm{d}s)\hat{\mu}([a,s)\cap A) & \text{if } k=1,\\ \int_A \hat{\mu}((s_1,b]\cap A)\nu(\mathrm{d}s_1)\int_{[a,s_1)\cap A}\hat{\mu}((s_2,b]\cap A)\nu(\mathrm{d}s_2) & \\ \qquad \cdots \int_{[a,s_{k-1})\cap A}\hat{\mu}((s_k,b]\cap A)\nu(\mathrm{d}s_k)\hat{\mu}([a,s_k)\cap A) & \text{if } k \geqslant 2. \end{cases} \tag{17}$$

More specifically, the following holds.

THEOREM 5. *Let* $\mathbb{B}$ *be a Banach algebra and* $1 \leqslant p < 2$. *The product integral operator* (16) *is a uniformly entire mapping on* $\mathcal{AI}_p(J;\mathbb{B})$. *More specifically, for* $\mu \in \mathcal{AI}_p(J;\mathbb{B})$, *the following hold*:

(a) *for each integer* $k \geqslant 1$ *and* $\nu \in \mathcal{AI}_p(J;\mathbb{B})$, *relation* (17) *defines the interval function*

$$Q_\mu^k(\nu) = \left\{Q_\mu^k(\nu)(A)\colon\ A \in \mathfrak{I}(J)\right\}$$

of bounded p*-variation*;

(b) *for each integer* $k \geqslant 1$, *the mapping*

$$\mathcal{AI}_p(J;\mathbb{B}) \ni \nu \mapsto Q_\mu^k(\nu) \in \mathcal{I}_p(J;\mathbb{B})$$

is a k*-homogeneous polynomial*;

(c) *the power series*

$$\sum_{k\geqslant 1} Q_\mu^k(\nu-\mu) \quad \textit{from } \mathcal{AI}_p(J;\mathbb{B}) \textit{ to } \mathcal{I}_p(J;\mathbb{B})$$

around μ *has infinite radius of uniform convergence and its sum is equal to* $\mathcal{P}(\nu) - \mathcal{P}(\mu)$.

The following is a special case of the preceding theorem: a Taylor series expansion of the product integral operator around 0.

COROLLARY 6. *Let* $\mu \in \mathcal{AI}_p([a,b];\mathbb{B})$ *with* $1 \leqslant p < 2$ *and* $a < b$. *Then, for any interval* $A \subset J$,

$$\prod_A (1\mathbb{I} + \mathrm{d}\mu) = 1\mathbb{I} + \mu(A) + \sum_{k\geqslant 2}\int_A \mu(\mathrm{d}s_1)\int_{[a,s_1)\cap A}\mu(\mathrm{d}s_2)\cdots\int_{[a,s_{k-1})\cap A}\mu(\mathrm{d}s_k). \tag{18}$$

This Taylor expansion has a form analogous to Peano's series mentioned earlier.

The product integral with respect to an interval function also gives solutions of suitable linear integral equations. First, consider the most straightforward candidate of such a linear Kolmogorov integral equation:

$$f(t) = x + \int_{[a,t]} f\,\mathrm{d}\mu, \quad a \leqslant t \leqslant b.$$

Letting $\mu = \delta_c$ be a unit mass at $c \in (a,b)$ yields the equality $f(c) = x + f(c)$, and so there is no solution to this equation, unless $x = 0$.

Now consider two other linear integral equations for Kolmogorov integrals

$$f(t) = x + \int_{(a,t]} f_- \, \mathrm{d}\mu, \quad a \leqslant t \leqslant b, \tag{19}$$

$$g(t) = x + \int_{[a,t)} g \, \mathrm{d}\mu, \quad a \leqslant t \leqslant b. \tag{20}$$

We will see that the two equations have nontrivial solutions. But the solutions can be different if μ has an atom. For example, let $\mu = \delta_c$ again be a unit mass at $c \in (a,b)$. Then we have two linear equations, and their solutions are $f(t) = x(1 + 1_{[c,b]}(t))$ and $g(t) = x(1 + 1_{(c,b]}(t))$, respectively. Thus, $f(c) = 2x \neq x = g(c)$, unless $x = 0$.

More generally, the product integral with respect to an interval function gives solutions of the nonhomogeneous linear Kolmogorov integral equations as stated next.

THEOREM 7. *Let $\mu, \nu \in \mathcal{AI}_p(J;\mathbb{B})$ for some $1 \leqslant p < 2$, let $q \geqslant p$ be such that $p^{-1} + q^{-1} > 1$, and let $x \in \mathbb{B}$. Then the following two statements hold*:

(a) *The integral equation*

$$f(t) = x + \int_{(a,t]} f_- \, \mathrm{d}\mu + \nu((a,t]), \quad a \leqslant t \leqslant b,$$

has a solution f in $\mathcal{W}_p(J;\mathbb{B})$, which is unique in $\mathcal{W}_q(J;\mathbb{B})$,

$$f(t) := x\mathcal{P}\big(\mu; (a,t]\big) + \int_{(a,t]} \mathcal{P}\big(\mu; (\cdot,t)\big) \, \mathrm{d}\nu, \quad a \leqslant t \leqslant b.$$

(b) *The integral equation*

$$g(t) = x + \int_{[a,t)} g \, \mathrm{d}\mu + \nu\big([a,t)\big), \quad a \leqslant t \leqslant b,$$

has a solution g in $\mathcal{W}_p(J;\mathbb{B})$, which is unique in $\mathcal{W}_q(J;\mathbb{B})$,

$$g(t) := x\mathcal{P}\big(\mu; [a,t)\big) + \int_{[a,t)} \mathcal{P}\big(\mu; (\cdot,t)\big) \, \mathrm{d}\nu, \quad a \leqslant t \leqslant b.$$

Nonlinear integral equations. To simplify a formulation we state further results for real-valued point functions and interval functions. Given a function $(u,s) \mapsto \psi(u,s)$ from $\mathbb{R} \times J$ into $\mathbb{R}$ and a function $f\colon J \to \mathbb{R}$, let

$$(N_\psi f)(s) := \psi\big(f(s), s\big), \quad s \in J.$$

Then N_ψ takes any function f into another function $N_\psi f$ on J and is called the *Nemytskii operator* (or the superposition operator).

Consider two nonlinear integral equations for Kolmogorov integrals

$$f(t) = x + \int_{(a,t]} (N_\psi f)_- \, \mathrm{d}\mu, \quad a \leqslant t \leqslant b, \tag{21}$$

$$g(t) = x + \int_{[a,t)} (N_\psi g) \, \mathrm{d}\mu, \quad a \leqslant t \leqslant b, \tag{22}$$

which agree with Eqs. (19) and (20), respectively, when $\psi(u,s)\equiv u$. Clearly, the two nonlinear equations can have different solutions if μ has an atom as well. Next we formulate conditions on ψ sufficient to prove the existence of a unique solution for each of the two equations (21) and (22).

Definition 8. Let $0<\gamma\leqslant 1$, $0<q<\infty$, $J=[a,b]$ with $a<b$, and let $\psi\equiv\psi(u,s)\colon\ \mathbb{R}\times J\to\mathbb{R}$.

1. For a nonempty set $B\subset\mathbb{R}$, we say that ψ is s-uniformly γ-Hölder on $B\times J$ or $\psi\in\mathcal{U}H_\gamma(B\times J)$, if there exists a finite constant H such that
$$\left|\psi(u,s)-\psi(v,s)\right|\leqslant H|u-v|^\gamma$$
for all $u,v\in B$ and all $s\in J$.
2. For any $0<p<\infty$ and $\mu=\{u_i\}_{i=1}^n\subset\mathbb{R}$, let
$$w_p(\mu):=\max_i|u_i|+\max\left\{\sum_{j=1}^m|u_{\theta(j)}-u_{\theta(j-1)}|^p\colon\ \{\theta(j)\}_{j=0}^m\subset\{1,\dots,n\}\right\}.$$
For a nonempty set $B\subset\mathbb{R}$, we say that ψ is in the class $\mathcal{W}_{\gamma,q}(B\times J)$ if, for each $0\leqslant K<\infty$, there is a finite constant $W=W(K)$ such that
$$\sum_{i=1}^n\left|\psi(u_i,s_i)-\psi(u_i,s_{i-1})\right|^q\leqslant W^q \tag{23}$$
for each partition $\{s_i\}_{i=0}^n$ of J and for each finite set $\mu=\{u_i\}_{i=1}^n\subset B$ such that $w_{\gamma q}(\mu)\leqslant K$. Let $W_{\gamma,q}(\psi,K;B\times J)$ be the minimal $W\geqslant 0$ such that (23) holds for a given K.
3. Let
$$\mathcal{H}W_{\gamma,q}(B\times J):=\mathcal{U}H_\gamma(B\times J)\cap\mathcal{W}_{\gamma,q}(B\times J)\quad\text{for any } B\subset\mathbb{R}.$$
We say that ψ is in the class $\mathcal{H}W^{\mathrm{loc}}_{\gamma,q}(\mathbb{R}\times J)$ if, for each positive integer m, $B_m:=\{u\in\mathbb{R}\colon\ |u|\leqslant m\}$ and ψ is in the class $\mathcal{W}_{\gamma,q}(B_m\times J)$.
4. We say that ψ is in the class $\mathcal{W}G_{\gamma,q}(\mathbb{R}\times J)$ if there is a finite constant C such that, for each $0\leqslant K<\infty$,
$$W_{\gamma,q}(\psi,K;\mathbb{R}\times J)\leqslant C(1+K^\gamma). \tag{24}$$
5. Finally, let
$$\mathcal{H}WG_{\gamma,q}(\mathbb{R}\times J):=\mathcal{U}H_\gamma(\mathbb{R}\times J)\cap\mathcal{W}G_{\gamma,q}(\mathbb{R}\times J).$$

To give some feeling about the stated conditions on ψ we make some remarks. Let $\ell^\infty(S)$ be the Banach space of real-valued bounded functions defined on a set S with the supnorm. First, it is clear that $\psi\in\mathcal{U}H_\gamma(\mathbb{R}\times J)$ if and only if
$$\mathbb{R}\ni u\mapsto\psi(u,\cdot)\in\mathcal{H}_\gamma\bigl(\mathbb{R};\ell^\infty(J)\bigr).$$
Second, if $\psi\in\mathcal{W}_{1,q}(\mathbb{R}\times J)$, then $\psi(u,\cdot)\in\mathcal{W}_q(J)$ for each $u\in\mathbb{R}$. Also, if
$$J\ni s\mapsto\psi(\cdot,s)\in\mathcal{W}_q\bigl(J;\ell^\infty(\mathbb{R})\bigr),$$
then $\psi\in\mathcal{W}_{\gamma,q}(\mathbb{R}\times J)$ for each $\gamma\in(0,1]$. Finally, $\psi\in\mathcal{H}WG_{\gamma,q}(\mathbb{R}\times J)$ if and only if $\psi\in\mathcal{U}H_\gamma(\mathbb{R}\times J)$ and there is a finite constant D such that, for each $g\in\mathcal{W}_{\gamma q}(J)$ and any nondegenerate interval $A\subset J$,
$$\left\|N_\psi g\right\|_{A,[q]}\leqslant D\bigl(1+\|g\|^\gamma_{A,[\gamma q]}\bigr).$$

Now we are ready to formulate conditions for the existence of solutions of nonlinear Kolmogorov integral equations.

THEOREM 9. *Let $J=[a,b]$, $\alpha\in(0,1]$, $1\leqslant p<1+\alpha$, and $\mu\in\mathcal{AI}_p(J)$. Suppose that $\psi\in\mathcal{H}WG_{1,p}(\mathbb{R}\times J)$ is u-differentiable on $\mathbb{R}$ with derivative $\psi_u^{(1)}\in\mathcal{H}W^{\mathrm{loc}}_{\alpha,p/\alpha}(\mathbb{R}\times J)$. Then each of the two integral equations* (21) *and* (22) *has a unique solution in $\mathcal{W}_p(J)$.*

For functions $f,F\colon\mathbb{R}\to\mathbb{R}$, N_F defined by $N_F f:=F\circ f$ is called the autonomous Nemytskii operator. If $\psi(u,s)\equiv F(u)$, this theorem implies:

COROLLARY 10. *Let $J=[a,b]$, $\alpha\in(0,1]$, $1\leqslant p<1+\alpha$, and $\mu\in\mathcal{AI}_p(J)$. Suppose that F is Lipschitz on $\mathbb{R}$ with derivative $F'\in\mathcal{H}^{\mathrm{loc}}_\alpha$. Then the conclusion of Theorem* 9 *holds with the Nemytskii operator N_ψ replaced by the autonomous Nemytskii operator N_F.*

If, in this corollary, the function F is locally Lipschitz, that is, if $F\in\mathcal{H}^{\mathrm{loc}}_{1+\alpha}$, then the existence and uniqueness of solutions hold locally.

Theorem 9 is proved in [16] using two methods. First, the Banach fixed-point theorem is applied, and, second, performing an iteration similar to the classical Picard iteration. Namely, let $f_0:=g_0:=x$, and, for each $n\geqslant1$ and all $t\in[a,b]$, let

$$f_n(t):=x+\int_{(a,t]}(N_\psi f_{n-1})_-\,\mathrm{d}\mu\quad\text{and}\quad g_n(t):=x+\int_{[a,t)}(N_\psi g_{n-1})\,\mathrm{d}\mu.$$

We call $\{f_n\}_{n\geqslant0}$ and $\{g_n\}_{n\geqslant0}$ the *Picard iterate* associated to the integral equations (21) and (22), respectively. Under the hypotheses of Theorem 9, each of the two sequences of Picard iterates converges in $\mathcal{W}_p(J)$ to the unique solution in $\mathcal{W}_p(J)$ of the corresponding integral equation.

Suppose that $\psi(u,s)\equiv u$. Then the integral equations (21) and (22) are linear. In this case, the nth Picard iterate associated to (21) with $n\geqslant2$ can be recursively written in a closed form as follows: for $a\leqslant t\leqslant b$,

$$\begin{aligned}f_n(t)&=x+x\mu\big((a,t]\big)+\int_{(a,t]}\left[\int_{(a,s)}(f_{n-2})_-\,\mathrm{d}\mu\right]\mu(\mathrm{d}s)=\cdots\\&=x\left\{1+\mu\big((a,t]\big)+\sum_{k=2}^{n}\int_{(a,t]}\mu(\mathrm{d}s_1)\int_{(a,s_1)}\mu(\mathrm{d}s_2)\cdots\int_{(a,s_{k-1})}\mu(\mathrm{d}s_k)\right\}.\end{aligned}$$

By (18), $f_n(t)$ is a partial sum of the Taylor series expansion of the product integral with respect to μ over the interval $A=(a,t]$.

Historical notes. A detailed treatment of the Riemann–Stieltjes and the refinement Riemann–Stieltjes integrals was given by Pollard [48]. W.H. Young [66], in a paper written in detail by his wife G.C. Young, first defined what we call the Young–Stieltjes sum but only for a step function. Their daughter R.C. Young [65] defined the sums more generally but considered limits of such sums only as mesh tends to zero, and the integral so obtained does not go beyond the Riemann–Stieltjes integral (Proposition 6.35 of [14]). Glivenko [21] wrote a book on the Stieltjes integral, where he defined and pointed out the superiority of the RYS integral over the RRS integral when the integrator function has bounded 1-variation. The RYS integral was used by L.C. Young [64] (son of W.H. and G.C. Young) and Gehring [7] and latter was rediscovered by Ross [50]. There are several alternatives to the integrals discussed above. Some of them were suggested by Smith [54], L.C. Young [61], Ward [58], L.C. Young [62], Izumi and Kawata [22], Kurzweil [32], Macaev and Solomjak [39], and Korenblum [28], [29]. Different approaches to defining a contour integral $\int_\Gamma f(z)\,\mathrm{d}z$ over a fractal curve (continuous rough function) Γ are discussed by Kac [24].

An interest of the p-variation property and an integration of rough functions was motivated by demands of a vast area of Fourier analysis. We refer to [15, Part IV] for an annotated bibliography, which includes some of these works. To the best of our knowledge, the first treatment of integral equations with respect to rough functions is due to Kurzweil [32]. A recent interest to such integral equations was motivated by works of Dudley [11], [12] and Lyons [36], [37]. As compared to the calculus above, T. Lyons' analysis uses iterated integral representation of solutions of integral equations with respect to continuous rough functions and algebraic approach (see [38]).

3. STOCHASTIC PROCESSES AND p-VARIATION: EXAMPLES

Let $X = \{X(t)\colon\ t \geqslant 0\}$ be a real-valued stochastic process on a complete probability space $(\Omega, \mathcal{F}, \Pr)$. For each $\omega \in \Omega$, the function $X(\cdot) = X(\cdot, \omega)$ is called a *sample function* of X. In this section, we review results concerning degree of roughness of sample functions of stochastic processes.

Suppose that almost all sample functions of a stochastic process X have bounded p-variation on $[0, t]$ for some $0 < p < \infty$ and $0 < t < \infty$. Since the set of partitions $\mathfrak{P}[0, t]$ is uncountable, the function

$$\omega \mapsto v_p\big(X(\cdot, \omega); [0, t]\big) \equiv \sup\big\{s_p\big(X(\cdot, \omega); \kappa\big)\colon\ \kappa \in \mathfrak{P}[0, t]\big\} \tag{25}$$

need not be measurable. But the measurability of this function is often needed once we are going to apply the p-variation calculus to stochastic processes as in Kubilius [30] and [31]. The idea is to use, instead of (25), its measurable modification defined as follows.

Definition 11. Let $X = \{X(t)\colon\ t \geqslant 0\}$ be a stochastic process, and let $\Phi \in \mathcal{V}$. We say that X has *locally bounded* Φ*-variation* (or locally bounded p-variation if $\Phi(u) \equiv u^p$) and write $X \in \mathcal{W}_\Phi^{\mathrm{loc}}$, if (a) and (b) hold, where

(a) for almost all $\omega \in \Omega$,

$$v_\Phi\big(X(\cdot, \omega); [0, t]\big) < +\infty \quad \forall t > 0; \tag{26}$$

(b) there exists an $\mathbb{F}(X)$-adapted stochastic process $v_\Phi(X) = \{v_\Phi(X; t)\colon\ t \geqslant 0\}$ such that, for a Pr-null set Ω_0, if $\omega \notin \Omega_0$, then

$$v_\Phi\big(X(\cdot, \omega); [0, t]\big) = v_\Phi(X; t)(\omega) \quad \forall t > 0. \tag{27}$$

We call $v_\Phi(X)$ the Φ*-variation stochastic process* (or p-variation stochastic process if $\Phi(u) \equiv u^p$) of X.

For large classes of stochastic processes, condition (b) follows from (a) in Definition 11, and so one needs to worry only about (26). A stochastic process $X = \{X(t)\colon\ t \geqslant 0\}$ has regulated sample functions if $X(\cdot, \omega) \in \mathcal{R}[0, \infty)$ for almost all $\omega \in \Omega$. If almost all sample functions of a regulated stochastic process X are right-continuous, then X is called a *cadlag* stochastic process. The following statement is proved in [16].

THEOREM 12. *Let* $X = \{X(t)\colon\ t \geqslant 0\}$ *be either a separable stochastic process continuous in probability, or a cadlag stochastic process. If, for some* $\Phi \in \mathcal{V}$, (26) *holds for almost all* $\omega \in \Omega$, *then* X *has locally bounded* Φ*-variation.*

The p-variation property has been investigated for many classes of stochastic processes. However, the results are scattered over many different journals. Below we provide some of these results. More information can be found in the annotated reference list on p-variation in [15, Part IV].

Gaussian stochastic processes. A stochastic process $X=\{X(t)\colon\ t\geqslant 0\}$ is said to be *Gaussian* if the vector $\{X(t_1),\dots,X(t_n)\}$ has an $n$dimensional normal distribution for all $t_1,\dots,t_n$ in $[0,\infty)$ and all $n\geqslant 1$.

The boundedness of p-variation of a Gaussian stochastic process X can be estimated using σ_X defined by

$$\sigma_X(s,t):=\Big(E\big[X(t)-X(s)\big]^2\Big)^{1/2}\quad\text{for } s,t\geqslant 0.$$

For $\Psi\in\mathcal{V}$ and $0<t<\infty$, let

$$v_\Psi(\sigma_X):=v_\Psi\big(\sigma_X;[0,t]\big):=\sup\left\{\sum_{i=1}^n\Psi\big(\sigma_X(t_{i-1},t_i)\big)\colon\ \{t_i\}_{i=0}^n\in\mathfrak{P}[0,t]\right\}.$$

Let

$$\Psi_{p,2}(u):=\Big[u\big(\log\log(1/u)\big)^{1/2}\Big]^p\quad\text{for } 0<u\leqslant \mathrm{e}^{-\mathrm{e}},$$

$$\Psi_{p,2}(u):=u^p\quad\text{for } u>\mathrm{e}^{-\mathrm{e}},\quad\text{and}\quad \Psi_{p,2}(0):=0.$$

If $\Psi(u)\equiv u^p$ or $\Psi(u)\equiv\Psi_{p,2}(u)$, then we write $v_p(\sigma_X)$ or $v_{p,2}(\sigma_X)$, respectively, instead of $v_\Psi(\sigma_X)$.

A necessary condition for a separable mean-zero Gaussian stochastic process X to have locally bounded p-variation can be obtained easily as follows. By a theorem of Fernique [13, Lemma 2.2.5] applied to the seminorm $\|\cdot\|=\|\cdot\|_{(p),[0,t]}$ we have

$$\infty>E\|X\|^p\geqslant\sup\big\{Es_p(X;\kappa)\colon\ \kappa\in\mathfrak{P}[0,t]\big\}=K_pv_p\big(\sigma_X;[0,t]\big),$$

where $K_p:=E|Z|^p$ for a standard normal random variable Z. Thus, we have the following:

PROPOSITION 13. *Let X be a separable mean-zero Gaussian stochastic process, and let $1<p<\infty$. If X has locally bounded p-variation, then $v_p(\sigma_X;[0,t])<\infty$ for each $0<t<\infty$.*

This necessary condition is not sufficient as the case of a Brownian motion shows. Let $B=\{B(t)\colon\ t\geqslant 0\}$ be a Brownian motion with $\sigma_B(s,t)=\sqrt{|t-s|}$, $t,s\geqslant 0$. As proved by P. Lévy, for any $0<t<\infty$, we have almost surely

$$v_p(B;[0,t])\begin{cases}<+\infty & \text{if } p>2,\\ =+\infty & \text{if } p\leqslant 2.\end{cases}\tag{28}$$

On the other hand, $v_2(\sigma_B;[0,t])=t$.

Next is a sufficient condition for boundedness of the p-variation of almost all sample functions of a Gaussian process due to Jain and Monrad [23], which is necessary in some cases.

THEOREM 14. *Let X be a separable mean-zero Gaussian stochastic process, and let $1<p<\infty$. If, $v_{p,2}(\sigma_X;[0,t])<\infty$ for each $0<t<\infty$, then X has locally bounded p-variation.*

Let X be a separable mean-zero Gaussian stochastic process. If $v_p(\sigma_X)<\infty$ for some $1<p<\infty$ and if $p'>p$, then $v_{p',2}(\sigma_X)<\infty$. Thus, by Theorem 14, X has locally bounded p'-variation. On the other hand, if, for some $1<p<\infty$, X has locally bounded p-variation, then $v_p(\sigma_X)<\infty$ by Proposition 13. This observation allows one to find the index of p-variation.

COROLLARY 15. *Let X be a separable mean-zero Gaussian stochastic process. Then, for each $0<t<\infty$, with probability 1*

$$\upsilon\big(X;[0,t]\big)=\inf\big\{p>0\colon\ v_p\big(\sigma_X;[0,t]\big)<\infty\big\}.$$

Fractional Brownian motion. A fractional Brownian motion $B_H = \{B_H(t)\colon t \geqslant 0\}$ with exponent $H \in (0,1)$ is a Gaussian process with mean 0 and covariance function

$$E\{B_H(t)B_H(s)\} = \frac{1}{2}\{t^{2H} + s^{2H} - |t-s|^{2H}\} \quad \text{for } t, s \geqslant 0 \tag{29}$$

and $B_H(0) = 0$ almost surely. Since the right side of (29) is equal to $t \wedge s$ for $H = 1/2$, B_H is a Brownian motion B in this case. From (29) it follows that, for all $t, u \geqslant 0$,

$$\sigma_H(u) := \sigma_{B_H}(t, t+u) = \Big[E\big[B_H(t+u) - B_H(t)\big]^2\Big]^{1/2} = u^H.$$

Let $0 < H < 1$ and $1 < p < \infty$. Then, for any $0 < T < \infty$,

$$v_p(\sigma_{B_H}; [0,T]) = \begin{cases} +\infty & \text{if} \quad pH < 1, \\ T^{pH} & \text{if} \quad pH \geqslant 1. \end{cases}$$

Thus, by Corollary 15, for each $0 < t < \infty$, the p-variation index of a fractional Brownian motion B_H is given by

$$\upsilon(B_H; [0,t]) = 1/H \quad \text{with probability 1.} \tag{30}$$

The next statement provides a more precise description of the p-variation of a fractional Brownian motion. Let

$$\Psi_H(u) := \frac{u^{1/H}}{(2LLu)^{1/(2H)}} \quad \text{for } u > 0,$$

where

$$LLu := \log_e|\log_e u|, \quad \text{and} \quad \Psi_H(0) := 0.$$

THEOREM 16. *Let $0 < H < 1$ and $0 < t < \infty$. Almost all sample functions of a fractional Brownian motion B_H satisfy the relation*

$$\lim_{\delta \downarrow 0} \sup\{s_{\Psi_H}(B_H; \kappa)\colon \ \kappa \in \mathfrak{P}[0,t] \text{ and mesh } |\kappa| \leqslant \delta\} = t. \tag{31}$$

For the case $H = 1/2$, (31) was proved by Taylor [55, Theorem 1]. The general case of (31) with $\leqslant$ instead of $=$ follows from Theorem 3 of Kawada and Kôno [25], since almost all sample functions of B_H are continuous. The converse inequality does not follow from Theorem 4 of Kawada and Kôno [25], since their condition (v) fails to hold for $\sigma^2(u) = \sigma_H^2(u) = u^{2H}$, $u > 0$. A direct proof of this theorem can be found in [16].

Markov processes. Let $\mathcal{B}$ be the Borel σ-algebra on $\mathbb{R}$. A stochastic process X on a probability space $(\Omega, \mathcal{F}, \Pr)$ is called a Markov process if there exists a Markov kernel $P_{s,t}(\cdot,\cdot)$, $0 \leqslant s < t < \infty$, on $(\mathbb{R}, \mathcal{B})$ such that, for all $0 \leqslant s < t < \infty$, $x \in \mathbb{R}$, and $B \in \mathcal{B}$,

$$\Pr\big(X(t) \in B \mid X(s) = x\big) = P_{s,t}(x, B).$$

Markov processes with certain power-type bounds on their Markov kernels can be chosen to have locally bounded p-variation.

Let $P = \{P_{s,t}(\cdot,\cdot)\colon \ 0 \leqslant s < t < \infty\}$ be a Markov kernel on $(\mathbb{R}, \mathcal{B})$. For $0 < T < \infty$, let α_T be the function on $[0,T] \times (0,\infty)$ defined by $\alpha_T(0,r) := 0$ for $r > 0$ and, for $r, h > 0$,

$$\alpha_T(h,r) := \sup\Big\{P_{s,t}\big(x, \{y \in \mathbb{R}\colon \ |x-y| \geqslant r\}\big)\colon \ x \in \mathbb{R}, \ 0 \leqslant s \leqslant t \leqslant (s+h) \wedge T\Big\}.$$

It is well known that a suitable behavior of α_T guarantee the existence of a cadlag or a sample-continuous Markov process X with the Markov kernel P (see, e.g., Theorem 11.1 of [52]). As will be seen, the following property of α_T yields the boundedness of p-variation for some p of sample functions of a Markov process. For $\beta \geqslant 1$ and $\gamma > 0$, the kernel P is said to belong to the class $\mathcal{M}(\beta, \gamma)$ if, given $0 < T < \infty$, there exist constants $r_0 > 0$ and $K > 0$ such that

$$\alpha_T(h, r) \leqslant \frac{K h^\beta}{r^\gamma}$$

for all $h \in [0, T]$ and $r \in (0, r_0]$.

Next is a variant of the result due to Manstavičius [40].

THEOREM 17. *Let $\beta \geqslant 1$ and $\gamma > 0$. For any probability measure ν on $\mathbb{R}$ and any Markov kernel P on $(\mathbb{R}, \mathcal{B})$ in the class $\mathcal{M}(\beta, \gamma)$, there exists a cadlag Markov process with the Markov kernel P and initial distribution ν which has locally bounded p-variation for each $p > \gamma/\beta$.*

We apply this theorem to the Markov kernel of a symmetric α-stable Lévy motion. A stochastic processes $X = \{X(t)\colon\ t \geqslant 0\}$ is called a *symmetric α-stable Lévy motion* if X is cadlag, $X(0) = 0$ almost surely, has independent increments, and the increments $X(t) - X(s)$ have stable distribution $S_\alpha((t-s)^{1/\alpha}, 0, 0)$ for any $0 \leqslant s < t < \infty$ and for some $0 < \alpha < 2$ (see Samorodnitsky and Taqqu [51]). Using the fact that X is $1/\alpha$-self-similar, it follows that

$$P_{s,t}\big(x, \{y \in \mathbb{R}\colon\ |x - y| \geqslant r\}\big) = \Pr\big(|X(t) - X(s)| \geqslant r\big) \leqslant K(t-s) r^{-\alpha}$$

for all $x \in \mathbb{R}$, $0 \leqslant s \leqslant t < \infty$, and $r > 0$, where $K := \sup_{\lambda > 0} \lambda^\alpha \Pr(|X(1)| \geqslant \lambda) < \infty$. It then follows that, for any $0 < T < \infty$,

$$\alpha_T(h, r) \leqslant \frac{K h}{r^\alpha}$$

for all $h \in [0, T]$ and $r > 0$. Then Theorem 17 implies that there is a symmetric α-stable Lévy motion which has locally bounded p-variation for each $p > \alpha$.

In Theorem 17, the hypothesis $p > \gamma/\beta$, in general, cannot be replaced by $p \geqslant \gamma/\beta$ as follows from Theorem 18 or Theorem 19 below.

Homogeneous Lévy process. Let $X = \{X(t)\colon\ t \geqslant 0\}$ be a separable and continuous in probability stochastic process with independent increments. Then X is called a *Lévy process* if almost all its sample functions are cadlag and $X(0) = 0$ almost surely. A Lévy process X is called *homogeneous* if, for all $t, s \geqslant 0$, the distribution of the increment $X(t+s) - X(t)$ does not depend on t. Given a real number a, a positive number b, and a σ-finite measure L_X on $\mathbb{R} \setminus \{0\}$ such that

$$L_X\big(\{x \in \mathbb{R}\colon\ |x| > \delta\}\big) < \infty \quad \text{for each } \delta > 0 \quad \text{and} \quad \int_{\mathbb{R}\setminus\{0\}} (1 \wedge x^2)\, L_X(\mathrm{d}x) < \infty,$$

let

$$\Phi(u) := iau - bu^2 + \int_{\mathbb{R}\setminus\{0\}} \big(\mathrm{e}^{iux} - 1 + iuh(x)\big)\, L_X(\mathrm{d}x) \tag{32}$$

for $u \in \mathbb{R}$, where h is a bounded Borel function. The function Φ is called the *characteristic exponent*, and the measure L_X is called the *Lévy measure*. Then the characteristic function of a homogeneous Lévy process X is given by

$$E \exp\big\{iuX(t)\big\} = \exp\big\{t\Phi(u)\big\} \quad \text{for all } t \geqslant 0 \quad \text{and} \quad u \in \mathbb{R}.$$

It is well known that sample functions of a homogeneous Lévy process X with the characteristic exponent (32) have bounded 1-variation if and only if $b \equiv 0$ and

$$\int_{\mathbb{R}\setminus\{0\}} \big(1 \wedge |x|\big) L_X(\mathrm{d}x) < \infty \tag{33}$$

(cf., e.g., Theorem 3 on p.279 of Gikhman and Skorokhod [8]). The following result of Bretagnolle [4, Théorème III b] is less well known.

THEOREM 18. *Let* $1 < p < 2$, *and let* $X = \{X(t): t \geqslant 0\}$ *be a mean-zero homogeneous Lévy process with characteristic exponent* (32) *such that* $b \equiv 0$. *Then* X *has locally bounded p-variation if and only if*

$$\int_{\mathbb{R}\setminus\{0\}} \big(1 \wedge |x|^p\big) L_X(\mathrm{d}x) < \infty.$$

A less precise but more general result is a characterization of the p-variation index (see (2) for the definition) of a sample function of a Lévy process using the Blumenthal–Gettor index. Let X be a homogeneous Lévy process. The *Blumenthal–Getoor index* of X is defined by

$$\beta(X) := \inf\left\{\alpha > 0: \int_{\mathbb{R}\setminus\{0\}} \big(1 \wedge |x|^\alpha\big) L_X(\mathrm{d}x) < \infty\right\},$$

where L_X is the Lévy measure of X. Note that $0 < \beta(X) \leqslant 2$. If X has no Gaussian part, then, for any $0 < t < \infty$, the p-variation index $\upsilon(X; [0, t]) = \beta(X)$ almost surely. This follows from Theorems 4.1 and 4.2 of Blumenthal and Getoor [2] and from Theorem 2 of Monroe [42].

Hyperbolic Lévy motion. A homogeneous Lévy process X is called a hyperbolic Lévy motion if, in its characteristic exponent (32), $b \equiv 0$ and the Lévy measure $L_X(\mathrm{d}x) = g(x; \alpha, \delta)\,\mathrm{d}x$, where

$$g(x; \alpha, \delta) = \frac{1}{\pi^2 |x|} \int_0^\infty \frac{\exp\{-|x|\sqrt{2y + (\alpha)^2}\}}{y(J_1^2(\delta\sqrt{2y}) + Y_1^2(\delta\sqrt{2y}))}\,\mathrm{d}y + \frac{\exp\{-|x|\}}{|x|}$$

for $\alpha > 0$ and $\delta \geqslant 0$. Here J_1 and Y_1 are the Bessel functions of the first and second kind, respectively. Eberlein and Keller [18] used a hyperbolic Lévy motion to model stock price returns and showed (ibid, p. 295) that $g(x; \alpha, \delta) \sim x^{-2}$ as $x \to 0$. Therefore, by Theorem 18, a hyperbolic Lévy motion X has locally bounded p-variation for each $p > 1$, and so the p-variation index $\upsilon(X; [0, 1]) = 1$ almost surely.

Normal inverse Gaussian Lévy process. A homogeneous Lévy process X is called a normal inverse Gaussian Lévy process if the parameters of the characteristic exponent (32) are given by $b \equiv 0$ and

$$a = (2\alpha\delta/\pi) \int_0^1 \sinh(\beta x) K_1(\alpha x)\,\mathrm{d}x$$

and the Lévy measure $L_X(\mathrm{d}x) = f(x; \alpha, \beta, \delta)\,\mathrm{d}x$, where

$$f(x; \alpha, \beta, \delta) = \frac{\alpha\delta}{\pi} \frac{\mathrm{e}^{-\beta x}}{|x|} K_1(\alpha|x|) = \frac{\delta}{\pi\sqrt{2}} \frac{\mathrm{e}^{\beta x}}{|x|} \int_0^\infty y^{-1/2} \mathrm{e}^{-|x|\sqrt{2y+\alpha^2}}\,\mathrm{d}y$$

for $\alpha > 0$, $0 \leqslant |\beta| < \alpha$, $\mu \in \mathbb{R}$, and $\delta > 0$. Here K_1 denotes the modified Bessel function of third order and index 1. Barndorff-Nielsen [1] showed that $f(x; \alpha, \beta, \delta) \sim (\delta/\pi)x^{-2}$ as $x \to 0$. Therefore, by

Theorem 18, a normal inverse Gaussian Lévy process H has locally bounded p-variation for each $p > 1$, and so the p-variation index $\upsilon(X; [0,1]) = 1$ almost surely.

α-Stable Lévy motion. A homogeneous Lévy process X is called an α-stable Lévy motion of index α if its characteristic exponent (32) is given by $b \equiv 0$ and the Lévy measure $L_X = L_{\alpha,r,q}$, where $L_{\alpha,r,q}(\mathrm{d}x) := rx^{-1-\alpha}\,\mathrm{d}x$ if $x > 0$ and $L_{\alpha,r,q}(\mathrm{d}x) := q(-x)^{-1-\alpha}\,\mathrm{d}x$ if $x < 0$ for $\alpha \in (0,2)$ and $r, q \geqslant 0$ with $r + q > 0$. If $\alpha < 1$, then (33) holds and $\int h\,\mathrm{d}L_X < \infty$. In this case, it is said that an α-stable Lévy motion has no drift if $a + \int h\,\mathrm{d}L_X = 0$. The following result is due to Fristedt and Taylor [20, Theorem 2].

THEOREM 19. *Let $X_\alpha = \{X_\alpha(t)\colon\ t \in [0,1]\}$ be an α-stable Lévy motion with $\alpha \in (0,2)$ having no drift if $\alpha < 1$ and with $r = q$ if $\alpha = 1$. For a function $\Phi \in \mathcal{V}$, with probability* 1

$$\lim_{|\kappa|\to 0} s_\Phi(X_\alpha;\kappa) \equiv \lim_{|\kappa|\to 0} \sum_{i=1}^{n} \Phi\big(|X_\alpha(t_i) - X_\alpha(t_{i-1})|\big) = \sum_{(0,1]} \Phi\big(|\Delta^- X_\alpha|\big), \tag{34}$$

where $|\kappa|$ is the mesh of a partition $\kappa = \{t_i\}_{i=0}^n$ of $[0,1]$.

To apply this result it is useful to recall that the right side of (34) is finite almost surely if and only if $\int_{-1}^{1} \Phi(u)u^{-1-\alpha}\,du < \infty$. Xu [60] established necessary and/or sufficient conditions for the boundedness of the p-variation of a sample function of a symmetric α-stable processes with possibly dependent increments. More information about p-variation of stable processes can be found in Fristedt [19].

4. ESTIMATING THE p-VARIATION INDEX

Suppose that we are given a set $G = \{(t_i, x_i)\}_{i=1}^n$ of finitely many points on a plane, where $t_0 := 0 < t_1 < \cdots < t_n = 1$ and $\{x_i\}_{i=1}^n \subset \mathbb{R}$. Assume that the data G is generated by an unknown function f so that $f(t_i) = x_i$ for $i = 1, \ldots, n$. We wish to estimate the p-variation index $\upsilon(f) := \upsilon(f; [0,1])$ defined by (2). Note that this index with values in $[0,\infty]$ exists for any function f. Why should we care about estimating the p-variation index?

The p-variation calculus was used to model stock price changes in [43]. As we have seen in Section 2, solutions of Kolmogorov integral equations with respect to an additive upper-continuous interval function μ exist, provided that μ has bounded p-variation for some $p < 2$. Thus, if we wish to apply the p-variation calculus to model real phenomena, then we should try to find methods allowing one to check our assumptions on the p-variation from real data. In particular, we need to test the hypothesis that the p-variation index is less than 2.

A completely different approach to model financial markets using game-theoretic probability and continuous nonstandard functions of p-variation have been proposed by Shafer and Vovk [53]. One of the conditions for pricing options using games is that the p-variation of such a function should be less than or equal to 2.

As we have seen in Section 3, the p-variation index of sample functions of basic stochastic processes is related to their parameters. For example, relation (30) for a fractional Brownian motion B_H with exponent H and relation $\upsilon(X_\alpha; [0,1]) = \alpha$ for an α-stable Lévy motion X_α. Therefore, an α-stable Lévy process and a fractional Brownian motion with Hurst exponent H, both have the same p-variation index in the case $\alpha = 1/H \in (1,2)$. On the other hand, for each t, the random variable $B_H(t)$ has exponentially small probability tail, while the random variable $X_\alpha(t)$ has no even the second moment. Also the two relations can be used in different ways comparing new and old statistical procedures for estimating different parameters. In this case, an estimate $\hat{\upsilon}$ of the p-variation index can also be interpreted as an estimate of either of the two parameters $1/H$ or α if $\hat{\upsilon} \in (1,2)$ and if the estimation procedure does not assume the continuity. Therefore, estimation of the p-variation index offers a new perspective to analyzing data sets and is closely related to an estimation of fractal dimension (see, e.g., Cutler [5]).

We review two methods of estimating the p-variation index proposed in [44] and [45].

Orey index. First is a method of estimation of the p-variation index of a sample function for a class of Gaussian stochastic processes. Let $X = \{X(t)\colon\ t \geqslant 0\}$ be a mean-zero Gaussian stochastic process with stationary increments and continuous in quadratic mean. Let σ be the incremental variance of X given by

$$\sigma(u) := \sigma_X(t, t+u) = \Big(E\big[X(t+u) - X(t)\big]^2\Big)^{1/2}, \quad t, u \geqslant 0.$$

Following Orey [46], let

$$\gamma_* := \inf\big\{\gamma > 0\colon\ u^\gamma/\sigma(u) \to 0, \text{ as } u \downarrow 0\big\} = \limsup_{u\downarrow 0}\Big(\frac{\log\sigma(u)}{\log u}\Big)$$

and

$$\gamma^* := \sup\big\{\gamma > 0\colon\ u^\gamma/\sigma(u) \to +\infty, \text{ as } u \downarrow 0\big\} = \liminf_{u\downarrow 0}\Big(\frac{\log\sigma(u)}{\log u}\Big).$$

Note that $0 \leqslant \gamma^* \leqslant \gamma_* \leqslant \infty$. If $\gamma_* = \gamma^*$, then we say that X has the *Orey index* $\gamma_X := \gamma_* = \gamma^*$.

Let X has the Orey index γ_X. By the definitions of γ_* and γ^* it follows that

$$\inf\big\{p > 0\colon\ v_p(\sigma_X; [0,t]) < \infty\big\} = 1/\gamma_X$$

for each $0 < t < \infty$. Thus, by Corollary 15, for each $0 < t < \infty$, with probability 1, the p-variation index $\upsilon(X; [0,t]) = 1/\gamma_X$. Therefore, an estimation of the p-variation index of X means an estimation of the Orey index.

The idea of estimating the Orey index of X is based on a result of Gladyshev [9] concerning a rescaled quadratic-variation property of a Gaussian stochastic process X with stationary increments. To state his result in the case where X is a fractional Brownian motion B_H, let $\lambda_m = \{i2^{-m}\colon\ i = 0, \ldots, 2^m\}$, $m = 1, 2, \ldots$, be a nested sequence of dyadic partitions of $[0,1]$. Gladyshev [9, Section 3] proved that the relation

$$\lim_{m\to\infty}\frac{\log\sqrt{s_2(B_H; \lambda_m)/2^m}}{\log(1/2^m)} = H$$

holds for almost all sample functions of B_H. Since the incremental variance $\sigma(u) \equiv u^H$ when $X = B_H$, it follows that the Orey index γ_{B_H} of B_H exists and equals the Hurst exponent H, and so the right side of the preceding formula is equal to the Orey index of B_H.

More generally, one can formulate Gladyshev's idea as follows. Again, let X be a mean-zero Gaussian stochastic process with stationary increments, continuous in quadratic mean, and let σ be its incremental variance. We say that the *strong limit theorem* on $[0,1]$ holds for X if there is a sequence $\eta = \{N_m\}_{m\geqslant 1}$ of integers strictly increasing to infinity and a finite constant C such that the limit

$$\lim_{m\to\infty}\frac{s_2(X; \lambda_m)}{N_m\sigma(1/N_m)^2} = C \tag{35}$$

exists almost surely, where $\{\lambda_m\}_{m\geqslant 1}$ is a sequence of partitions $\lambda_m = \{i/N_m\}_{i=0}^{N_m}$ of $[0,1]$ associated to η. Note that if X is a Brownian motion B, then the incremental variance $\sigma(u) \equiv \sqrt{u}$ and the strong limit theorem for B means the existence of the quadratic variation for B. Gladyshev [9] and others proved the strong limit theorem (35) for a large class of stochastic processes (see [44] for a review of results).

Suppose that the strong limit theorem (35) holds for X. For each $m \geqslant 1$, we have the identity

$$\log\sqrt{\frac{s_2(X; \lambda_m)}{N_m}} = \log\sigma_X(1/N_m) + \frac{1}{2}\log C + \frac{1}{2}\log\Big(1 + \frac{s_2(X; \lambda_m)}{CN_m\sigma(1/N_m)^2} - 1\Big).$$

Then, also assuming that X has the Orey index γ_X, it follows that the limit

$$\lim_{m\to\infty} \frac{\log\sqrt{s_2(X;\lambda_m)/N_m}}{\log(1/N_m)} = \gamma_X \tag{36}$$

exists for almost all sample functions of X. This relation was used in [44] to estimate the Orey index from a sample function of a stochastic process X given its values at finitely many points.

Oscillation summing index. Let f be a real-valued function defined on an interval $[0,1]$. Let $\eta = \{N_m\}_{m\geqslant 1}$ be a sequence of strictly increasing positive integers. Again with η we associate a sequence $\{\lambda_m\}_{m\geqslant 1}$ of partitions $\lambda_m = \{i/N_m\}_{i=0}^{N_m}$ of $[0,1]$. For $i = 1,\ldots,N_m$, the subintervals $\Delta_{i,m} := [(i-1)/N_m, i/N_m]$ have the same length $1/N_m$. For each $m \geqslant 1$, let

$$Q(f;\lambda_m) = \sum_{i=1}^{N_m} \mathrm{Osc}(f;\Delta_{i,m}), \tag{37}$$

where, for a subset $A \subset [0,1]$,

$$\mathrm{Osc}(f;A) := \sup\big\{|f(t)-f(s)|:\ \ s,t\in A\big\} = \sup_{t\in A} f(t) - \inf_{s\in A} f(s).$$

The sequence

$$Q_\eta(f) := \big\{Q(f;\lambda_m)\big\}_{m\geqslant 1}$$

is called the *oscillation η-summing* sequence. For a bounded nonconstant function f on $[0,1]$ and any sequence η as above, let

$$\delta_\eta^-(f) := \liminf_{m\to\infty} \frac{\log(Q(f;\lambda_m)/N_m)}{\log(1/N_m)} \quad \text{and} \quad \delta_\eta^+(f) := \limsup_{m\to\infty} \frac{\log(Q(f;\lambda_m)/N_m)}{\log(1/N_m)}.$$

Then we have

$$0 \leqslant \delta_\eta^-(f) \leqslant \delta_\eta^+(f) \leqslant 1. \tag{38}$$

The leftmost inequality follows from the bound

$$\frac{Q(f;\lambda_m)}{N_m} \leqslant \mathrm{Osc}\big(f;[0,1]\big) < \infty,$$

which is valid for each $m \geqslant 1$. The rightmost inequality in (38) holds, since $Q(f;\lambda_m) \geqslant C/2 > 0$ for all sufficiently large $m \geqslant 1$. Indeed, if f is continuous, then $C = v_1(f;[0,1])$. Otherwise, f has a jump at some point $t \in [0,1]$, so that C can be taken to be a saltus at t if $t \notin \lambda_m$ for all sufficiently large m, or C can be taken to be a one-sided nonzero saltus at t if $t \in \lambda_m$ for infinitely many m.

Let f be any nonconstant real-valued function on $[0,1]$, and let $\eta = \{N_m\}_{m\geqslant 1}$ be a sequence of strictly increasing positive integers as before. If $\delta_\eta^-(f) = \delta_\eta^+(f)$, then we say that f has the *oscillation η-summing index* $\delta_\eta(f)$, which is then defined by

$$\delta_\eta(f) := \delta_\eta^-(f) = \delta_\eta^+(f) = \lim_{m\to\infty} \frac{\log(Q(f;\lambda_m)/N_m)}{\log(1/N_m)}. \tag{39}$$

As explained below in more detail, one can show that whenever the metric entropy index or the Hausdorff–Besicovitch dimension of the graph of f has a suitable value, then the oscillation η-summing index $\delta_\eta(f)$ exists and

$$\delta_\eta(f) = \frac{1}{1 \vee \upsilon(f)}. \tag{40}$$

The relations (40) and (39) were used in [45] to estimate the p-variation index $\upsilon(f;[0,1])$ of a sample function $f=X(\cdot,\omega)$ of a stochastic process X given its values at finitely many points.

To formulate sufficient conditions of existence of the oscillation η-summing index, we recall some notation. Let E be a nonempty bounded subset of the plane $\mathbb{R}^2$, and let $N(E;\varepsilon)$, $\varepsilon>0$, be the minimum number of closed balls of diameter ε required to cover E. The *lower* and *upper metric entropy indices* of the set E are defined respectively by

$$\Delta^-(E):=\liminf_{\varepsilon\downarrow 0}\frac{\log N(E;\varepsilon)}{\log(1/\varepsilon)}\quad\text{and}\quad \Delta^+(E):=\limsup_{\varepsilon\downarrow 0}\frac{\log N(E;\varepsilon)}{\log(1/\varepsilon)}.$$

If $\Delta^-(E)=\Delta^+(E)$, then the common value, denoted by $\Delta(E)$, is called the *metric entropy index* of the set E. To recall the Hausdorff–Besicovitch dimension of E, let $\operatorname{diam}(A)$ denote the diameter of a set $A\subset\mathbb{R}^2$. An ε-covering of a bounded set $E\subset\mathbb{R}^2$ is a countable collection $\{E_k\}_{k\geqslant 1}$ of sets such that $E\subset\cup_k E_k$ and $\sup_k\operatorname{diam}(E_k)\leqslant\varepsilon$. For $s>0$, the Hausdorff s-measure of E is defined by

$$\mathcal{H}^s(E):=\liminf_{\varepsilon\downarrow 0}\left\{\sum_{k\geqslant 1}\big(\operatorname{diam}(E_k)\big)^s:\ \{E_k\}_{k\geqslant 1}\text{ is an }\varepsilon\text{-covering of }E\right\}.$$

Clearly, $0\leqslant\mathcal{H}^s(E)\leqslant\infty$, and the function $s\mapsto\mathcal{H}^s(E)$ is nonincreasing. In fact, given any subset $E\subset\mathbb{R}^2$, there is a critical value s_c such that $\mathcal{H}^s(E)=\infty$ for $s<s_c$ and $\mathcal{H}^s(E)=0$ for $s>s_c$. This critical value s_c is called the *Hausdorff–Besicovitch dimension* of E and is denoted by $\dim_{HB}(E)$.

Let f be a regulated function on $[0,1]$, let $\operatorname{gr}(f)$ be the graph of f in $\mathbb{R}^2$, and let $\eta=\{N_m\}_{m\geqslant 1}$ be a sequence of strictly increasing positive integers as before. In Appendix A of [45], it is proved that

$$\big\{2-\delta_\eta^+(f)=\big\}\quad\limsup_{m\to\infty}\frac{\log(N_m Q(f;\lambda_m))}{\log N_m}\leqslant 2-\frac{1}{1\vee\upsilon(f)}$$

and

$$\Delta^-\big(\operatorname{gr}(f)\big)\leqslant\liminf_{m\to\infty}\frac{\log(N_m Q(f;\lambda_m))}{\log N_m}\quad\big\{=2-\delta_\eta^-(f)\big\}.$$

Now, these two bounds give us Theorem 1 of [45]:

THEOREM 20. *For any sequence η, a regulated nonconstant function f on* $[0,1]$ *has the oscillation η-summing index $\delta_\eta(f)$ and* (40) *holds provided the metric entropy index $\Delta(gr(f))$ of the graph $gr(f)$ of f is defined and*

$$\Delta(gr(f))=2-\frac{1}{1\vee\upsilon(f)}.$$

Alternatively, a similar sufficient condition can be formulated using the Hausdorff–Besicovitch dimension of the graph. It is easy to show that, for any bounded subset $E\subset\mathbb{R}^2$, $\dim_{HB}(E)\leqslant\Delta^-(E)$. Therefore, we have

COROLLARY 21. *For any sequence η, a regulated nonconstant function f on* $[0,1]$ *has the oscillation η-summing index $\delta_\eta(f)$ and* (40) *holds, provided that*

$$\dim_{HB}(gr(f))=2-\frac{1}{1\vee\upsilon(f)}.\tag{41}$$

The Hausdorff–Besicovitch dimension seems to be an excellent tool in pure mathematics, and there are lots of results concerning a theoretical computation of this dimension for various sets. According to Tricot [56, p. 258] it has no practical utility for estimating fractal dimensions. Nevertherless, it seems to us that a knowledge of the Hausdorff–Besicovitch dimension of some sets can be used in justifying more practical estimation methods as shown above. In support of this claim we give next few examples.

Let X_α be a symmetric α-stable process for some $0<\alpha\leqslant 2$. Then, by Theorem B of Blumenthal and Getoor [3], for almost all sample functions of X_α, we have

$$\dim_{HB}\big(\mathrm{gr}(X_\alpha)\big)=2-\frac{1}{1\vee\alpha}=2-\frac{1}{1\vee\upsilon(X_\alpha)}.$$

For another example, let X be a mean-zero Gaussian stochastic process with stationary increments, continuous in quadratic mean, and having the Orey index $\gamma_X\in(0,1)$. Then, by Theorem 1 of Orey [46], for almost all sample functions of X, we have

$$\dim_{HB}\big(\mathrm{gr}(X)\big)=2-\gamma_X=2-\frac{1}{\upsilon(X)}.$$

For these stochastic processes, by Corollary 21, we have the following result.

COROLLARY 22. *Let $X=\{X(t)\colon\ 0\leqslant t\leqslant 1\}$ be a stochastic process, and let $\eta=\{N_m\}_{m\geqslant 1}$ be a sequence of strictly increasing positive integers. The relation*

$$\delta_\eta(X)\equiv\lim_{m\to\infty}\frac{\log(Q(X;\lambda_m)/N_m)}{\log(1/N_m)}=\frac{1}{1\vee\upsilon(X)}$$

holds for almost all sample functions of X, provided that either (a) or (b) holds, where

(a) *X is a symmetric α-stable process for some $\alpha\in(0,2]$, that is, X is a cadlag stochastic process with independent stationary increments having a symmetric α-stable distribution if $\alpha\in(0,2)$ and X is a standard Brownian motion if $\alpha=2$;*

(b) *X is a mean-zero Gaussian stochastic process with stationary increments continuous in quadratic mean and such that the Orey index γ_X exists.*

Similar results for more general processes than (a) and (b) of the preceding corollary can be respectively found in Pruitt and Taylor [49, Section 8] and in Kôno [27].

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