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Ideal quasi-Cauchy sequences

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available at the end of the article**Abstract**

An ideal I is a family of subsets of positive integers \mathbf{N} which is closed under taking finite unions and subsets of its elements. A sequence (x_n) of real numbers is said to be I -convergent to a real number L if for each $\varepsilon > 0$, the set $\{n : |x_n - L| \geq \varepsilon\}$ belongs to I . We introduce I -ward compactness of a subset of \mathbf{R} , the set of real numbers, and I -ward continuity of a real function in the senses that a subset E of \mathbf{R} is I -ward compact if any sequence (x_n) of points in E has an I -quasi-Cauchy subsequence, and a real function is I -ward continuous if it preserves I -quasi-Cauchy sequences where a sequence (x_n) is called to be I -quasi-Cauchy when (Δx_n) is I -convergent to 0. We obtain results related to I -ward continuity, I -ward compactness, ward continuity, ward compactness, ordinary compactness, ordinary continuity, δ -ward continuity, and slowly oscillating continuity.

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1 Introduction

The concept of continuity and any concept involving continuity play a very important role not only in pure mathematics but also in other branches of sciences involving mathematics, especially in computer science, information theory, and biological science.

A subset E of \mathbf{R} , the set of real numbers, is compact if any open covering of E has a finite subcovering where \mathbf{R} is the set of real numbers. This is equivalent to the statement that any sequence $\mathbf{x} = (x_n)$ of points in E has a convergent subsequence whose limit is in E . A real function f is continuous if and only if $(f(x_n))$ is a convergent sequence whenever (x_n) is. Regardless of limit, this is equivalent to the statement that $(f(x_n))$ is Cauchy whenever (x_n) is. Using the idea of continuity of a real function and the idea of compactness in terms of sequences, Çakallı [1] introduced the concept of ward continuity in the sense that a function f is ward continuous if it preserves quasi-Cauchyness, *i.e.*, $(f(x_n))$ is a quasi-Cauchy sequence whenever (x_n) is, and a subset E of \mathbf{R} is ward compact if any sequence $\mathbf{x} = (x_n)$ of points in E has a quasi-Cauchy subsequence $\mathbf{z} = (z_k) = (x_{n_k})$ of the sequence \mathbf{x} where a sequence (z_k) is called quasi-Cauchy if $\lim_{k \rightarrow \infty} (z_{k+1} - z_k) = 0$ (see also [2], [3] and [4]). In [5] a real-valued function defined on a subset E of \mathbf{R} is called δ -ward continuous if it preserves δ -quasi Cauchy sequences where a sequence $\mathbf{x} = (x_n)$ is defined to be δ -quasi Cauchy if the sequence (Δx_n) is quasi-Cauchy. A subset E of \mathbf{R} is said to be δ -ward compact if any sequence $\mathbf{x} = (x_n)$ of points in E has a δ -quasi-Cauchy subsequence $\mathbf{z} = (z_k) = (x_{n_k})$ of the sequence \mathbf{x} .

A sequence (x_n) of points in \mathbf{R} is slowly oscillating if for any given $\varepsilon > 0$, there exists a $\delta = \delta(\varepsilon) > 0$ and an $N = N(\varepsilon)$ such that $|x_m - x_n| < \varepsilon$ if $n \geq N(\varepsilon)$ and $n \leq m \leq (1 + \delta)n$. A function

defined on a subset E of \mathbf{R} is called slowly oscillating continuous if it preserves slowly oscillating sequences (see [6]). A function defined on a subset E of \mathbf{R} is called quasi-slowly oscillating continuous on E if it preserves quasi-slowly oscillating sequences of points in E where a sequence $\mathbf{x} = (x_n)$ is called quasi-slowly oscillating if $(\Delta x_n) = (x_{n+1} - x_n)$ is a slowly oscillating sequence [7].

The purpose of this paper is to investigate the concept of ideal ward compactness of a subset of \mathbf{R} and the concept of ideal ward continuity of a real function which cannot be given by means of any summability matrix and to prove related theorems.

2 Preliminaries

First of all, some definitions and notation will be given in the following. Throughout this paper, \mathbf{N} and \mathbf{R} will denote the set of all positive integers and the set of all real numbers, respectively. We will use boldface letters $\mathbf{x}, \mathbf{y}, \mathbf{z}, \dots$ for sequences $\mathbf{x} = (x_n), \mathbf{y} = (y_n), \mathbf{z} = (z_n), \dots$ of terms in \mathbf{R} . c and S will denote the set of all convergent sequences and the set of all statistically convergent sequences of points in \mathbf{R} , respectively.

Following the idea given in a 1946 American Mathematical Monthly problem [8], a number of authors such as Posner [9], Iwinski [10], Srinivasan [11], Antoni [12], Antoni and Salat [13], Spigel and Krupnik [14] have studied A -continuity defined by a regular summability matrix A . Some authors, Öztürk [15], Savaş and Das [16], Borsik and Salat [17], have studied A -continuity for methods of almost convergence or for related methods.

The idea of statistical convergence was formerly given under the name 'almost convergence' by Zygmund in the first edition of his celebrated monograph published in Warsaw in 1935 [18]. The concept was formally introduced by Fast [19] and later was reintroduced by Schoenberg [20], and also independently by Buck [21]. Although statistical convergence was introduced over nearly the last seventy years, it has been an active area of research for twenty years. This concept has been applied in various areas such as number theory [22], measure theory [23], trigonometric series [18], summability theory [24], locally convex spaces [25], in the study of strong integral summability [26], turnpike theory [27–29], Banach spaces [30], metrizable topological groups [31], and topological spaces [32, 33]. It should be also mentioned that the notion of statistical convergence has been considered, in other contexts, by several people like R.A. Bernstein and Z. Frolik *et al.* The concept of statistical convergence is a generalization of the usual notion of convergence that, for real-valued sequences, parallels the usual theory of convergence. For a subset M of \mathbf{N} , the asymptotic density of M denoted by $\delta(M)$ is given by

$$\delta(M) = \lim_{n \rightarrow \infty} \frac{1}{n} |\{k \leq n : k \in M\}|,$$

if this limit exists where $|\{k \leq n : k \in M\}|$ denotes the cardinality of the set $\{k \leq n : k \in M\}$. A sequence (x_n) is statistically convergent to ℓ if

$$\delta(\{n : |x_n - \ell| \geq \epsilon\}) = 0$$

for every $\epsilon > 0$. In this case, ℓ is called the statistical limit of \mathbf{x} . Schoenberg [20] studied some basic properties of statistical convergence and also studied the statistical convergence as a summability method. Fridy [34] gave characterizations of statistical convergence. Caserta, Maio, and Kočinac [35] studied statistical convergence in function spaces, while Caserta and Kočinac [36] investigated statistical exhaustiveness.

By a lacunary sequence $\theta = (k_r)$, we mean an increasing sequence $\theta = (k_r)$ of positive integers such that $k_0 = 0$ and $h_r : k_r - k_{r-1} \rightarrow \infty$. The intervals determined by θ will be denoted by $I_r = (k_{r-1}, k_r]$, and the ratio $\frac{k_r}{k_{r-1}}$ will be abbreviated as q_r . In this paper, we assume that $\liminf_r q_r > 1$. The notion of lacunary statistical convergence was introduced and studied by Fridy and Orhan in [37] and [38] (see also [39] and [40]). A sequence (α_k) of points in \mathbf{R} is called lacunary statistically convergent to an element ℓ of \mathbf{R} if

$$\lim_{r \rightarrow \infty} \frac{1}{h_r} \left| \left\{ k \in I_r : |\alpha_k - \ell| \geq \varepsilon \right\} \right| = 0,$$

for every positive real number ε . The condition assumed a few lines above ensures the regularity of the lacunary statistical sequential method.

The concept of I -convergence, which is a generalization of statistical convergence, was introduced by Kostyrko, Šalát, and Wilczyński [41] by using the ideal I of subsets of \mathbf{N} and further studied in [42–46], and [47]. The concept was also studied for double sequences in [48–51], and [52]. Although an ideal is defined as a hereditary and additive family of subsets of a non-empty arbitrary set X , here in our study, it suffices to take I as a family of subsets of \mathbf{N} such that $A \cup B \in I$ for each $A, B \in I$, and each subset of an element of I is an element of I . A non-trivial ideal I is called *admissible* if and only if $\{\{n\} : n \in \mathbf{N}\} \subset I$. A non-trivial ideal I is maximal if there cannot exist any non-trivial ideal $J \neq I$ containing I as a subset. Further details on ideals can be found in Kostyrko *et al.* (see [41]). Throughout this paper, we assume I is a non-trivial admissible ideal in \mathbf{N} . Recall that a sequence $\mathbf{x} = (x_n)$ of points in \mathbf{R} is said to be I -convergent to a real number ℓ if $\{n \in \mathbf{N} : |x_n - \ell| \geq \varepsilon\} \in I$ for every $\varepsilon > 0$. In this case, we write $I\text{-}\lim x_n = \ell$. We say that a sequence (x_n) of points in \mathbf{R} is I -quasi-Cauchy if $I\text{-}\lim(x_{n+1} - x_n) = 0$. We see that I -convergence of a sequence (x_n) implies I -quasi-Cauchyness of (x_n) . Throughout the paper, $I(\mathbf{R})$ and ΔI will denote the set of all I -convergent sequences and the set of all I -quasi-Cauchy sequences of points in \mathbf{R} , respectively.

If we take $I = I_{\text{fin}} = \{A \subseteq \mathbf{N} : A \text{ is a finite subset of } \mathbf{N}\}$, then I_{fin} will be a non-trivial admissible ideal in \mathbf{N} , and the corresponding convergence will coincide with the usual convergence. If we take $I = I_\delta = \{A \subseteq \mathbf{N} : \delta(A) = 0\}$, then I_δ will be a non-trivial admissible ideal of \mathbf{N} , and the corresponding convergence will coincide with the statistical convergence.

Connor and Grosse-Erdman [53] gave sequential definitions of continuity for real functions calling G -continuity instead of A -continuity. Their results cover the earlier works related to A -continuity where a method of sequential convergence, or briefly a method, is a linear function G defined on a linear subspace of s , denoted by c_G , into \mathbf{R} . A sequence $\mathbf{x} = (x_n)$ is said to be G -convergent to ℓ if $\mathbf{x} \in c_G$ and $G(\mathbf{x}) = \ell$. In particular, \lim denotes the limit function $\lim \mathbf{x} = \lim_n x_n$ on the linear space c , and $st\text{-}\lim$ denotes the statistical limit function $st\text{-}\lim \mathbf{x} = st\text{-}\lim_n x_n$ on the linear space S . Also $I\text{-}\lim$ denotes the I -limit function $I\text{-}\lim \mathbf{x} = I\text{-}\lim_n x_n$ on the linear space $I(\mathbf{R})$. A method G is called regular if every convergent sequence $\mathbf{x} = (x_n)$ is G -convergent with $G(\mathbf{x}) = \lim \mathbf{x}$. A method is called subsequential if whenever \mathbf{x} is G -convergent with $G(\mathbf{x}) = \ell$, then there is a subsequence (x_{n_k}) of \mathbf{x} with $\lim_k x_{n_k} = \ell$. Since the ordinary convergence implies ideal convergence, I is a regular sequential method [54]. Recently, Cakalli studied new sequential definitions of compactness in [55, 56] and slowly oscillating compactness in [6].

3 Ideal sequential compactness

First, we recall the definition of G -sequentially compactness of a subset E of \mathbf{R} . A subset E of \mathbf{R} is called G -sequentially compact if whenever (x_n) is a sequence of points in E , there is subsequence $\mathbf{y} = (y_k) = (x_{n_k})$ of (x_n) such that $G(\mathbf{y}) = \lim \mathbf{y}$ in E (see [57]). For regular methods, any sequentially compact subset E of \mathbf{R} is also G -sequentially compact and the converse is not always true. For any regular subsequential method G , a subset E of \mathbf{R} is G -sequentially compact if and only if it is sequentially compact in the ordinary sense.

Although I -sequential compactness is a special case of G -sequential compactness when $G = \lim$, we state the definition of I -sequential compactness of a subset E of \mathbf{R} as follows.

Definition 1 A subset E of \mathbf{R} is called I -sequentially compact if whenever (x_n) is a sequence of points in E , there is an I -convergent subsequence $\mathbf{y} = (y_k) = (x_{n_k})$ of (x_n) such that $I\text{-}\lim \mathbf{y}$ is in E .

Lemma 1 [41] *Sequential method I is regular, i.e., if $\lim x_n = \ell$, then $I\text{-}\lim x_n = \ell$.*

Lemma 2 ([58], Proposition 3.2) *Any I -convergent sequence of points in \mathbf{R} with an I -limit ℓ has a convergent subsequence with the same limit ℓ in the ordinary sense.*

We have the following result which states that any non-trivial admissible ideal I is a regular subsequential sequential method.

Theorem 1 *The sequential method I is regular and subsequential.*

Proof Regularity of I follows from Lemma 1, and subsequentiality of I follows from Lemma 2. \square

Theorem 2 ([55], Corollary 3) *A subset of \mathbf{R} is sequentially compact if and only if it is I -sequentially compact.*

Although I -sequential continuity is a special case of G -sequential continuity when $G = \lim$, we state the definition of I -sequential continuity of a function defined on a subset E of \mathbf{R} as follows.

Definition 2 ([54], Definition 2) A function $f : E \rightarrow \mathbf{R}$ is I -sequentially continuous at a point x_0 if, given a sequence (x_n) of points in E , $I\text{-}\lim \mathbf{x} = x_0$ implies that $I\text{-}\lim f(\mathbf{x}) = f(x_0)$.

Theorem 3 *Any I -sequentially continuous function at a point x_0 is continuous at x_0 in the ordinary sense.*

Proof Although there is a proof in [54], we give a different proof for completeness. Let f be any I -sequentially continuous function at a point x_0 . Since any proper admissible ideal is a regular subsequential method by Theorem 1, it follows from [57], Theorem 13, that f is continuous in the ordinary sense. \square

Theorem 4 ([54], Theorem 2.2) *Any continuous function at a point x_0 is I -sequentially continuous at x_0 .*

Combining Theorem 3 and Theorem 4 we have the following corollary.

Corollary 1 *A function is I -sequentially continuous at a point x_0 if and only if it is continuous at x_0 .*

As statistical limit is an I -sequential method, we get ([56], Theorem 2).

Corollary 2 *A function is statistically continuous at a point x_0 if and only if it is continuous at x_0 in the ordinary sense.*

As lacunary statistical limit is an I -sequential method, we get ([56], Theorem 5).

Corollary 3 *A function is lacunarily statistically continuous at a point x_0 if and only if it is continuous at x_0 in the ordinary sense.*

Corollary 4 *For any regular subsequential method G , a function is G -sequentially continuous at a point x_0 , then it is I -sequentially continuous at x_0 .*

Proof The proof follows from [57], Theorem 13. □

Corollary 5 *Any ward continuous function on a subset E of \mathbf{R} is I -sequentially continuous on E .*

Theorem 5 *If a function is slowly oscillating continuous on a subset E of \mathbf{R} , then it is I -sequentially continuous on E .*

Proof Let f be any slowly oscillating continuous function on E . It follows from [6], Theorem 2.1, that f is continuous. By Theorem 4, we see that f is I -sequentially continuous on E . This completes the proof. □

Theorem 6 *If a function is δ -ward continuous on a subset E of \mathbf{R} , then it is I -sequentially continuous on E .*

Proof Let f be any δ -ward continuous function on E . It follows from [5], Corollary 2, that f is continuous. By Theorem 4, we obtain that f is I -sequentially continuous on E . This completes the proof. □

Corollary 6 *If a function is quasi-slowly oscillating continuous on a subset E of \mathbf{R} , then it is I -sequentially continuous on E .*

Proof Let f be any quasi-slowly oscillating continuous function on E . It follows from [7], Theorem 3.2, that f is continuous. By Theorem 4, we deduce that f is I -sequentially continuous on E . This completes the proof. □

4 Ideal ward continuity

We say that a sequence $\mathbf{x} = (x_n)$ is I -ward convergent to a number ℓ if $I\text{-}\lim_{n \rightarrow \infty} \Delta x_n = \ell$. For the special case $\ell = 0$, we say that \mathbf{x} is ideal quasi-Cauchy or I -quasi-Cauchy.

Now, we give the definition of I -ward compactness of a subset of \mathbf{R} .

Definition 3 A subset E of \mathbf{R} is called I -ward compact if whenever $\mathbf{x} = (x_n)$ is a sequence of points in E , there is an I -quasi-Cauchy subsequence of \mathbf{x} .

We note that this definition of I -ward compactness cannot be obtained by any G -sequential compactness, *i.e.*, by any summability matrix A , even by the summability matrix $A = (a_{nk})$ defined by $a_{nk} = -1$ if $k = n$ and $a_{kn} = 1$ if $k = n + 1$ and

$$G(x) = I\text{-}\lim A\mathbf{x} = I\text{-}\lim_{k \rightarrow \infty} \sum_{n=1}^{\infty} a_{kn}x_n = I\text{-}\lim_{k \rightarrow \infty} \Delta x_k. \tag{1}$$

Despite the fact that G -sequential compact subsets of \mathbf{R} should include the singleton set $\{0\}$, I -ward compact subsets of \mathbf{R} do not have to include the singleton $\{0\}$.

Firstly, we note that any finite subset of \mathbf{R} is I -ward compact, the union of two I -ward compact subsets of \mathbf{R} is I -ward compact, and the intersection of any I -ward compact subsets of \mathbf{R} is I -ward compact. Furthermore, any subset of an I -ward compact set is I -ward compact, and any bounded subset of \mathbf{R} is I -ward compact. Any compact subset of \mathbf{R} is also I -ward compact, and the set \mathbf{N} is not I -ward compact. We note that any slowly oscillating compact subset of \mathbf{R} is I -ward compact (see [6] for the definition of slowly oscillating compactness). These observations suggest that we have the following result.

Theorem 7 A subset E of \mathbf{R} is ward compact if and only if it is I -ward compact.

Proof Let us suppose first that E is ward compact. It follows from [2], Lemma 2, that E is bounded. Then for any sequence (x_n) , there exists a convergent subsequence (x_{n_k}) of (x_n) whose limit may be in E or not. Then the sequence (Δx_{n_k}) is a null sequence. Since I is a regular method, (Δx_{n_k}) is I -convergent to 0, so it is I -quasi-Cauchy. Thus, E is I -ward compact.

Now, to prove the converse, suppose that E is I -ward compact. Take any sequence (x_n) of points in E . Then there exists an I -quasi-Cauchy subsequence (x_{n_k}) of (x_n) . Since I is sub-sequential, there exists a convergent subsequence $(x_{n_{k_m}})$ of (x_{n_k}) . The sequence $(x_{n_{k_m}})$ is a quasi-Cauchy subsequence of the sequence (x_n) . Thus, E is ward compact. This completes the proof of the theorem. □

Theorem 8 A subset E of \mathbf{R} is bounded if and only if it is I -ward compact.

Proof Using an idea in the proof of [2], Lemma 2, and the preceding theorem, the proof can be obtained easily, so it is omitted. □

Now, we give the definition of I -ward continuity of a real function.

Definition 4 A function f is called I -ward continuous on a subset E of \mathbf{R} if $I\text{-}\lim_{n \rightarrow \infty} \Delta f(x_n) = 0$ whenever $I\text{-}\lim_{n \rightarrow \infty} \Delta x_n = 0$ for a sequence $\mathbf{x} = (x_n)$ of terms in E .

We note that this definition of continuity cannot be obtained by any A -continuity, *i.e.*, by any summability matrix A , even by the summability matrix $A = (a_{nk})$ defined by (1). However, for this special summability matrix A , if A -continuity of f at the point 0 implies I -ward continuity of f , then $f(0) = 0$; and if I -ward continuity of f implies A -continuity of f at the point 0, then $f(0) = 0$.

We note that the sum of two I -ward continuous functions is I -ward continuous, but the product of two I -ward continuous functions need not be I -ward continuous as it can be seen by considering a product of the I -ward continuous function $f(x) = x$ with itself.

In connection with I -quasi-Cauchy sequences and I -convergent sequences, the problem arises to investigate the following types of ‘continuity’ of functions on \mathbf{R} :

$$(\delta i): (x_n) \in \Delta I \Rightarrow (f(x_n)) \in \Delta I,$$

$$(\delta ic): (x_n) \in \Delta I \Rightarrow (f(x_n)) \in c,$$

$$(c): (x_n) \in c \Rightarrow (f(x_n)) \in c,$$

$$(c\delta i): (x_n) \in c \Rightarrow (f(x_n)) \in \Delta I,$$

$$(i): (x_n) \in I(\mathbf{R}) \Rightarrow (f(x_n)) \in I(\mathbf{R}).$$

We see that (δi) is I -ward continuity of f , (i) is I -continuity of f and (c) states the ordinary continuity of f . It is easy to see that (δic) implies (δi) and (δi) does not imply (δic) ; (δi) implies $(c\delta i)$ and $(c\delta i)$ does not imply (δi) ; (δic) implies (c) and (c) does not imply (δic) ; and (c) is equivalent to $(c\delta i)$.

Now, we give the implication (δi) implies (i) , i.e., any I -ward continuous function is I -sequentially continuous.

Theorem 9 *If f is I -ward continuous on a subset E of \mathbf{R} , then it is I -sequentially continuous on E .*

Proof Although the following proof is similar to that of [59], Theorem 1, and that of [7], Theorem 3.2, we give it for completeness. Suppose that f is an I -ward continuous function on a subset E of \mathbf{R} . Let (x_n) be an I -quasi-Cauchy sequence of points in E . Then the sequence

$$(x_1, x_0, x_2, x_0, x_3, x_0, \dots, x_{n-1}, x_0, x_n, x_0, \dots)$$

is an I -quasi-Cauchy sequence. Since f is I -ward continuous, the sequence

$$(y_n) = (f(x_1), f(x_0), f(x_2), f(x_0), \dots, f(x_n), f(x_0), \dots)$$

is an I -quasi-Cauchy sequence. Therefore, $I\text{-}\lim_{n \rightarrow \infty} \Delta y_n = 0$. Hence, $I\text{-}\lim_{n \rightarrow \infty} [f(x_n) - f(x_0)] = 0$. It follows that the sequence $(f(x_n))$ I -converges to $f(x_0)$. This completes the proof of the theorem.

The converse is not always true for the function $f(x) = x^2$ is an example since $I\text{-}\lim_{n \rightarrow \infty} \Delta x_n = 0$ for the sequence $(x_n) = (\sqrt{n})$. But $I\text{-}\lim_{n \rightarrow \infty} \Delta f(x_n) \neq 0$, because $f(\sqrt{n}) = (n)$. □

Theorem 10 *If f is I -ward continuous on a subset E of \mathbf{R} , then it is continuous on E in the ordinary sense.*

Proof Let f be an I -ward continuous function on E . By Theorem 9, f is I -sequentially continuous on E . It follows from Theorem 3 that f is continuous on E in the ordinary sense. Thus, the proof is completed. □

Theorem 11 *An I-ward continuous image of any I-ward compact subset of \mathbf{R} is I-ward compact.*

Proof Suppose that f is an I-ward continuous function on a subset E of \mathbf{R} , and E is an I-ward compact subset of \mathbf{R} . Let $\mathbf{y} = (y_n)$ be a sequence of points in $f(E)$. Write $y_n = f(x_n)$, where $x_n \in E$ for each $n \in \mathbf{N}$. I-ward compactness of E implies that there is an I-quasi-Cauchy subsequence $\mathbf{z} = (z_k) = (x_{n_k})$ of (x_n) . As f is I-ward continuous, $(f(z_k))$ is an I-quasi-Cauchy subsequence of \mathbf{y} . Thus, $f(E)$ is I-ward compact. This completes the proof of the theorem. \square

Corollary 7 *An I-ward continuous image of any compact subset of \mathbf{R} is compact.*

Proof The proof of this theorem follows from Theorem 3. \square

Corollary 8 *An I-ward continuous image of any bounded subset of \mathbf{R} is bounded.*

Proof The proof follows from Theorem 8 and Theorem 10. \square

Corollary 9 *An I-ward continuous image of a G-sequentially compact subset of \mathbf{R} is G-sequentially compact for any regular subsequential method G.*

It is a well-known result that a uniform limit of a sequence of continuous functions is continuous. It is true for slowly oscillating continuous functions ([56], Theorem 12), and quasi-slowly oscillating continuous functions ([7], Theorem 3.5), (see also [59], Theorem 5). This is also true in the case of I-ward continuity, i.e., a uniform limit of a sequence of I-ward continuous functions is I-ward continuous.

Theorem 12 *If (f_n) is a sequence of I-ward continuous functions defined on a subset E of \mathbf{R} , and (f_n) is uniformly convergent to a function f , then f is I-ward continuous on E .*

Proof Let $\varepsilon > 0$ and (x_n) be a sequence of points in E such that $I\text{-}\lim_{n \rightarrow \infty} \Delta x_n = 0$. By the uniform convergence of (f_n) , there exists a positive integer N such that $|f_n(x) - f(x)| < \frac{\varepsilon}{3}$ for all $x \in E$ whenever $n \geq N$. By the definition of ideal convergence, for all $x \in E$, we have

$$\left\{ n \in \mathbf{N} : |f_n(x) - f(x)| \geq \frac{\varepsilon}{3} \right\} \in I.$$

As f_N is I-ward continuous on E , we have

$$\left\{ n \in \mathbf{N} : |f_N(x_{n+1}) - f_N(x_n)| \geq \frac{\varepsilon}{3} \right\} \in I.$$

On the other hand, we have

$$\begin{aligned} & \left\{ n \in \mathbf{N} : |f(x_{n+1}) - f(x_n)| \geq \varepsilon \right\} \\ & \subseteq \left\{ \left\{ n \in \mathbf{N} : |f(x_{n+1}) - f_N(x_{n+1})| \geq \frac{\varepsilon}{3} \right\} \right\} \\ & \cup \left\{ \left\{ n \in \mathbf{N} : |f_N(x_{n+1}) - f_N(x_n)| \geq \frac{\varepsilon}{3} \right\} \cup \left\{ n \in \mathbf{N} : |f_N(x_n) - f(x_n)| \geq \frac{\varepsilon}{3} \right\} \right\}. \end{aligned} \tag{3}$$

Since I is an admissible ideal, the right-hand side of the relation (3) belongs to I , we have

$$\{n \in \mathbf{N} : |f(x_{n+1}) - f(x_n)| \geq \varepsilon\} \in I.$$

This completes the proof of the theorem. \square

Theorem 13 *The set of all I -ward continuous functions on a subset E of \mathbf{R} is a closed subset of the set of all continuous functions on E , i.e., $\overline{\Delta \text{iwc}(E)} = \Delta \text{iwc}(E)$, where $\Delta \text{iwc}(E)$ is the set of all I -ward continuous functions on E , $\overline{\Delta \text{iwc}(E)}$ denotes the set of all cluster points of $\Delta \text{iwc}(E)$.*

Proof Let f be an element in $\overline{\Delta \text{iwc}(E)}$. Then there exists a sequence (f_n) of points in $\Delta \text{iwc}(E)$ such that $\lim_{n \rightarrow \infty} f_n = f$. To show that f is I -ward continuous, consider an I -quasi-sequence (x_n) of points in E . Since (f_n) converges to f , there exists a positive integer N such that for all $x \in E$ and for all $n \geq N$, $|f_n(x) - f(x)| < \frac{\varepsilon}{3}$. By the definition of an ideal for all $x \in E$, we have

$$\left\{n \in \mathbf{N} : |f_n(x) - f(x)| \geq \frac{\varepsilon}{3}\right\} \in I.$$

As f_N is I -ward continuous on E , we have

$$\left\{n \in \mathbf{N} : |f_N(x_{n+1}) - f_N(x_n)| \geq \frac{\varepsilon}{3}\right\} \in I.$$

On the other hand, we have

$$\begin{aligned} & \{n \in \mathbf{N} : |f(x_{n+1}) - f(x_n)| \geq \varepsilon\} \\ & \subseteq \left\{ \left\{n \in \mathbf{N} : |f(x_{n+1}) - f_N(x_{n+1})| \geq \frac{\varepsilon}{3}\right\} \right. \\ & \quad \left. \cup \left\{n \in \mathbf{N} : |f_N(x_{n+1}) - f_N(x_n)| \geq \frac{\varepsilon}{3}\right\} \cup \left\{n \in \mathbf{N} : |f_N(x_n) - f(x_n)| \geq \frac{\varepsilon}{3}\right\} \right\}. \end{aligned} \quad (4)$$

Since I is an admissible ideal, the right-hand side of the relation (3) belongs to I , we have

$$\left\{n \in \mathbf{N} : |f(x_{n+1}) - f(x_n)| \geq \frac{\varepsilon}{3}\right\} \in I.$$

This completes the proof of the theorem. \square

We note that [59], Theorem 6, [56], Theorem 10, and [7], Theorem 2.2, Theorem 3.9, supply some other closed subsets of the set of all continuous functions.

Corollary 10 *The set of all I -ward continuous functions on a subset E of \mathbf{R} is a complete subspace of the space of all continuous functions on E .*

Proof The proof follows from the preceding theorem. \square

In this paper, two new concepts, namely the concept of I -ward continuity of a real function and the concept of I -ward compactness of a subset of \mathbf{R} , were introduced and investigated. In this investigation, we have obtained theorems related to I -ward continuity, I -ward compactness, compactness, sequential continuity, and uniform continuity. We also introduced and studied some other continuities involving I -quasi-Cauchy sequences, statistical sequences, and convergent sequences of points in \mathbf{R} . The present work also contains a generalization of results of the paper [1], and some results in [2] and [4].

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

BH drafted the manuscript. HC checked and organized the manuscript to be its final form and make the revision as the corresponding author.

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