Research Article
On the Stability of Trigonometric Functional Equations

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The aim of this paper is to study the superstability related to the d'Alembert, the Wilson, the sine functional equations for the trigonometric functional equations as follows: $f(x + y) - f(x - y) = 2f(x)g(y)$, $f(x + y) - f(x - y) = 2g(x)f(y)$.

1. Introduction

Baker et al. [1] and Bourgin [2] introduced that if f satisfies the stability inequality $|E_1(f) - E_2(f)| \le \varepsilon$, then either f is bounded or $E_1(f) = E_2(f)$. This is now frequently referred to as *superstability*.

   The superstability of the cosine functional equation (also called the d'Alembert functional equation)

$$f(x + y) + f(x - y) = 2f(x)f(y) \tag{A}$$

and the sine functional equation

$$f(x)f(y) = f\left(\frac{x + y}{2}\right)^2 - f\left(\frac{x - y}{2}\right)^2 \tag{S}$$

are investigated by Baker [3] and Cholewa [4], respectively.

   The d'Alembert functional equation (A) is generalized to the following functional equations:

$$f(x + y) + f(x - y) = 2f(x)g(y), \tag{A_{fg}}$$

$$f(x + y) + f(x - y) = 2g(x)f(y). \tag{A_{gf}}$$

Equation (A_{fg}), raised by Wilson, is sometimes called the Wilson equation.
We will consider the trigonometric functional equation as follow:

$$f(x+y) - f(x-y) = 2f(x)f(y), \tag{T}$$

$$f(x+y) - f(x-y) = 2f(x)g(y), \tag{T_{fg}}$$

$$f(x+y) - f(x-y) = 2g(x)f(y). \tag{T_{gf}}$$

The cosine-type functional equations (A), (A_{fg}), (A_{gf}) and sine functional equation have been investigated by Badora, Cholewa, Ger, Kannappan, Kim, and so forth [3–9].

Given mappings $f : G \to \mathbb{C}$, we will denote a difference operator $DA : G \times G \to \mathbb{C}$ as

$$DA(x,y) := f(x+y) + f(x-y) - 2f(x)f(y). \tag{1.1}$$

Badora and Ger [6] proved the superstability under the condition $|DA(x,y)| \leq \varphi(x)$ or $\varphi(y)$ for the d'Alembert equation (A).

The aim of this paper is to investigate improved superstability for the trigonometric functional equations (T_{fg}), (T_{gf}) under the following types:

$$\begin{aligned} |DT_{fg}(x,y)| &\leq \varphi(x) \quad \text{or} \quad \varphi(y), \\ |DT_{gf}(x,y)| &\leq \varphi(x) \quad \text{or} \quad \varphi(y). \end{aligned} \tag{1.2}$$

As a consequence, the obtained results imply the superstability for (T) in the same type:

$$|DT(x,y)| \leq \varphi(x) \quad \text{or} \quad \varphi(y), \tag{1.3}$$

and the superstability under the constant bounded for the functional equations (T), (T_{fg}), and (T_{gf}). We have also extended the results obtained on the Abelian group to the Banach algebra.

In this paper, let $(G,+)$ be an Abelian group, $\mathbb{C}$ the field of complex numbers, and $\mathbb{R}$ the field of real numbers. In particular, let $(G,+)$ be a uniquely 2-divisible group whenever the function is related to the sine functional equation (S), it will be denoted by "under 2-divisible" for short. We may assume that f and g are nonzero functions and ε is a nonnegative real constant, a mapping $\varphi : G \to \mathbb{R}$.

2. Stability of the equation (T_{gf})

In this section, we investigate the stability of the trigonometric functional equation (T_{gf}) as related to the cosine-, the sine-, and the mixed-type functional equations (A), (A_{fg}), (A_{gf}), (T_{fg}), (T_{gf}), and (S).

THEOREM 2.1. *Suppose that $f,g : G \to \mathbb{C}$ satisfy the inequality*

$$|f(x+y) - f(x-y) - 2g(x)f(y)| \leq \varphi(x) \tag{2.1}$$

for all $x, y \in G$. Then either f is bounded or g satisfies (A).

Proof. Let f be unbounded. Then we can choose a sequence $\{y_n\}$ in G such that

$$0 \neq |f(y_n)| \longrightarrow \infty \quad \text{as } n \longrightarrow \infty. \tag{2.2}$$

Taking $y = y_n$ in (2.1), we obtain

$$\left| \frac{f(x+y_n) - f(x-y_n)}{2f(y_n)} - g(x) \right| \leq \frac{\varphi(x)}{2|f(y_n)|}, \tag{2.3}$$

that is,

$$\lim_{n \to \infty} \frac{f(x+y_n) - f(x-y_n)}{2f(y_n)} = g(x) \tag{2.4}$$

for all $x \in G$. Using (2.1), we have

$$\begin{aligned}
2\varphi(x) &\geq |f(x+(y+y_n)) - f(x-(y+y_n)) - 2g(x)f(y+y_n)| \\
&\quad + |f(x+(y-y_n)) - f(x-(y-y_n)) - 2g(x)f(y-y_n)| \\
&\geq |f(x+(y+y_n)) - f(x-(y+y_n)) - 2g(x)f(y+y_n) \\
&\quad - f(x+(y-y_n)) + f(x-(y-y_n)) + 2g(x)f(y-y_n)|
\end{aligned} \tag{2.5}$$

so that

$$\begin{aligned}
&\left| \frac{f((x+y)+y_n) - f((x+y)-y_n)}{2f(y_n)} \right. \\
&\quad + \left. \frac{f((x-y)+y_n) - f((x-y)-y_n)}{2f(y_n)} - 2g(x)\frac{f(y+y_n) - f(y-y_n)}{2f(y_n)} \right| \\
&\quad \leq \frac{\varphi(x)}{|f(y_n)|}
\end{aligned} \tag{2.6}$$

for all $x, y \in G$. By virtue of (2.2) and (2.4), we have

$$|g(x+y) + g(x-y) - 2g(x)g(y)| \leq 0 \tag{2.7}$$

for all $x, y \in G$. Therefore g satisfies (A). $\qquad\square$

COROLLARY 2.2. *Suppose that $f, g : G \to \mathbb{C}$ satisfy the inequality*

$$|f(x+y) - f(x-y) - 2g(x)f(y)| \leq \varepsilon \tag{2.8}$$

for all $x, y \in G$. Then either f is bounded or g satisfies (A).

COROLLARY 2.3. *Suppose that $f : G \to \mathbb{C}$ satisfy the inequality*

$$|f(x+y) - f(x-y) - 2f(x)f(y)| \leq \varphi(x) \tag{2.9}$$

for all $x, y \in G$. Then either f is bounded or f satisfies (A).

COROLLARY 2.4. *Suppose that* $f : G \to \mathbb{C}$ *satisfy the inequality*

$$| f(x + y) - f(x - y) - 2f(x)f(y) | \le \varepsilon \tag{2.10}$$

for all $x, y \in G$. *Then either* f *is bounded or* f *satisfies* (A).

THEOREM 2.5. *Suppose that* $f, g : G \to \mathbb{C}$ *satisfy the inequality*

$$| f(x + y) - f(x - y) - 2g(x)f(y) | \le \varphi(y) \quad \forall x, y \in G. \tag{2.11}$$

If g *fails to be bounded, then*
   (i) g *satisfies* (A),
  (ii) f *and* g *satisfy* (T_{gf}),
 (iii) f *and* g *satisfy* (A_{fg}).

Proof. (i) If f is bounded, choose $y_0 \in G$ such that $f(y_0) \ne 0$, and then by (2.11) we obtain

$$|g(x)| - \left| \frac{f(x + y_0) - f(x - y_0)}{2f(y_0)} \right| \le \left| \frac{f(x + y_0) - f(x - y_0)}{2f(y_0)} - g(x) \right| \le \frac{\varphi(y_0)}{2|f(y_0)|}, \tag{2.12}$$

from which it follows that g is also bounded on G. Since f is nonzero, the unboundedness of g implies the unboundedness of f. Hence g satisfies (A) by Theorem 2.1.

(ii) For the unbounded g, we can choose a sequence $\{x_n\}$ in G such that $0 \ne |g(x_n)| \to \infty$ as $n \to \infty$.

An obvious slight change in the steps of the proof applied in Theorem 2.1 with $x = x_n$ in (2.11) gives us

$$\lim_{n \to \infty} \frac{f(x_n + y) - f(x_n - y)}{2g(x_n)} = f(y), \quad y \in G. \tag{2.13}$$

Replacing x by $x_n + x$ and $x_n - x$ in (2.11), dividing both sides by $2g(x_n)$, we have the inequality

$$\begin{aligned}
\left| \frac{f(x_n + (x + y)) - f(x_n - (x + y))}{2g(x_n)} \right. & \\
\left. - \frac{f(x_n + (x - y)) - f(x_n - (x - y))}{2g(x_n)} - 2\frac{g(x_n + x) + g(x_n - x)}{2g(x_n)} \cdot f(y) \right| & \\
\le \frac{\varphi(y)}{|g(x_n)|} &
\end{aligned} \tag{2.14}$$

for all $x, y \in G$ and every $n \in \mathbb{N}$. We take the limit as $n \to \infty$ with the use of (2.13), since g satisfies (A), which states nothing else but (T_{gf}).

(iii) An obvious slight change in the steps of the proof applied after (2.13) in (2.11) gives us the inequality

$$
\begin{aligned}
\bigg| & \frac{f\left(x_n + (x+y)\right) - f\left(x_n - (x+y)\right)}{2g(x_n)} \\
& + \frac{f\left(x_n + (x-y)\right) - f\left(x_n - (x-y)\right)}{2g(x_n)} - 2 \cdot \frac{g(x_n+y) + g(x_n-y)}{2g(x_n)} \cdot f(x) \bigg| \\
& \leq \frac{\varepsilon}{|g(x_n)|}
\end{aligned}
\tag{2.15}
$$

for all $x, y \in G$ and every $n \in \mathbb{N}$. Like last sentence of (ii), the required result (A_{fg}) holds.
$\square$

COROLLARY 2.6. *Suppose that $f, g : G \to \mathbb{C}$ satisfy the inequality*

$$
\left| f(x+y) - f(x-y) - 2g(x)f(y) \right| \leq \varepsilon \quad \forall x, y \in G.
\tag{2.16}
$$

If g fails to be bounded, then
  (i) *g satisfies (A),*
  (ii) *f and g satisfy (T_{gf}),*
  (iii) *f and g satisfy (A_{fg}).*

COROLLARY 2.7. *Let $(G, +)$ be a uniquely 2-divisible group. Suppose that $f, g : G \to \mathbb{C}$ satisfy the inequality*

$$
\left| f(x+y) - f(x-y) - 2g(x)f(y) \right| \leq \min\{\varphi(x), \varphi(y)\} \quad \forall x, y \in G;
\tag{2.17}
$$

  (a) *if f fails to be bounded, then g satisfies (A);*
  (b) *if g fails to be bounded, then*
    (i) *g satisfies (A),*
    (ii) *f and g satisfy (T_{gf}),*
    (iii) *f and g satisfy (A_{fg}).*

3. Stability of the equation (T_{fg})

In this section, we investigate the stability of the trigonometric functional equations (T_{fg}) related to the sine equation (S) and the cosine equation (A).

THEOREM 3.1. *Suppose that $f, g : G \to \mathbb{C}$ satisfy the inequality*

$$
\left| f(x+y) - f(x-y) - 2f(x)g(y) \right| \leq \varphi(y) \quad \forall x, y \in G.
\tag{3.1}
$$

If f fails to be bounded, then
  (i) *g satisfies (S) under 2-divisible,*
  (ii) *in particular, f satisfies (A), f and g are solutions of $g(x+y) - g(x-y) = 2f(x)g(y)$.*

Proof. (i) For the unbounded f, we can choose a sequence $\{x_n\}$ in G such that $0 \neq |f(x_n)| \to \infty$ as $n \to \infty$.

An obvious slight change in the steps applied at the start of Theorem 2.5 gives us the existence of a limit function:

$$h_1(x) := \lim_{n \to \infty} \frac{f(x_n + x) + f(x_n - x)}{f(x_n)}, \tag{3.2}$$

where the function $h_1 : G \to \mathbb{C}$ satisfies the equation

$$g(x + y) - g(x - y) = h_1(x)g(y), \quad x, y \in G. \tag{3.3}$$

From the definition of h_1, we get the equality $h_1(0) = 2$, which, jointly with (3.3), implies that g is an odd function. Keeping this in mind, by means of (3.3), we infer the equality

$$\begin{aligned}
g(x + y)^2 - g(x - y)^2 &= [g(x + y) + g(x - y)][g(x + y) - g(x - y)] \\
&= [g(x + y) + g(x - y)]h_1(x)g(y) \\
&= [g(2x + y) + g(2x - y)]g(y) \\
&= [g(y + 2x) - g(y - 2x)]g(y) \\
&= h_1(y)g(2x)g(y).
\end{aligned} \tag{3.4}$$

Since the oddness of g forces it to vanish at 0, putting $x = y$ in (3.3) we get the equation

$$g(2y) = h_1(y)g(y), \quad \forall y \in G. \tag{3.5}$$

This, in return, leads to the equation

$$g(x + y)^2 - g(x - y)^2 = g(2x)g(2y), \tag{3.6}$$

valid for all $x, y \in G$ which, in the light of the unique 2-divisibility of G, states nothing else but (S).

(ii) In particular case f satisfies (A), (3.2) means that $h_1 = 2f$. Hence, from (3.3), f and g are solutions of $g(x + y) - g(x - y) = 2f(x)g(y)$.   □

COROLLARY 3.2. *Suppose that* $f, g : G \to \mathbb{C}$ *satisfy the inequality*

$$|f(x + y) - f(x - y) - 2f(x)g(y)| \le \varepsilon \quad \forall x, y \in G. \tag{3.7}$$

If f fails to be bounded, then
   (i) *g satisfies (S) under 2-divisible,*
   (ii) *in particular, f satisfies (A), f and g are solutions of $g(x + y) - g(x - y) = 2f(x)g(y)$.*

COROLLARY 3.3. *Suppose that* $f : G \to \mathbb{C}$ *satisfy the inequality*

$$|f(x + y) - f(x - y) - 2f(x)f(y)| \le \varphi(y) \quad \forall x, y \in G. \tag{3.8}$$

Then either f is bounded or f satisfies (S) under 2-divisible.

COROLLARY 3.4. *Suppose that $f : G \to \mathbb{C}$ satisfy the inequality*

$$|f(x+y) - f(x-y) - 2f(x)f(y)| \leq \varepsilon \quad \forall\, x, y \in G. \tag{3.9}$$

Then either f is bounded or f satisfies (S) under 2-divisible.

THEOREM 3.5. *Suppose that $f, g : G \to \mathbb{C}$ satisfy the inequality*

$$|f(x+y) - f(x-y) - 2f(x)g(y)| \leq \varphi(x) \quad \forall\, x, y \in G. \tag{3.10}$$

If g fails to be bounded, then
   (i) *f and g are solutions of (T_{fg}),*
   (ii) *f satisfies (S) under 2-divisible and one of the cases $f(0) = 0$, $f(x) = f(-x)$,*
   (iii) *in particular, g satisfies (A) or (T), and f and g are solutions of (A_{fg}).*

Proof. (i) As with the earlier theorems, consider a sequence $\{y_n\}$ in G such that $0 \neq |g(y_n)| \to \infty$ as $n \to \infty$, then we have

$$f(x) = \lim_{n \to \infty} \frac{f(x+y_n) - f(x-y_n)}{2g(y_n)} \quad \forall\, x \in G. \tag{3.11}$$

Replacing x by $x + y_n$ and $x - y_n$ in (3.10), we have

$$\left| \frac{f((x+y)+y_n) - f((x+y)-y_n)}{2g(y_n)} \right.$$
$$\left. - \frac{f((x-y)+y_n) - f((x-y)-y_n)}{2g(y_n)} - 2 \cdot \frac{f(x+y_n) - f(x-y_n)}{2g(y_n)} \cdot g(y) \right| \tag{3.12}$$
$$\leq \frac{\varphi(x+y_n) + \varphi(x-y_n)}{2|g(y_n)|},$$

which gives, with an application of (3.11), the required result (T_{fg}).

   (ii) Using the same method as in Theorem 3.1, that is, replacing y by $y + y_n$ and $-y + y_n$ in (3.10), and taking the limit as $n \to \infty$ with the use of (3.11), we conclude that, for every $y \in G$, there exists

$$h_2(y) := \lim_{n \to \infty} \frac{g(y_n + y) + g(y_n - y)}{g(y_n)}, \tag{3.13}$$

where the function $h_2 : G \to \mathbb{C}$ satisfies the equation

$$f(x+y) + f(x-y) = f(x)h_2(y), \quad x, y \in G. \tag{3.14}$$

Applying the case $f(0) = 0$ in (3.14), we see that f is an odd function.

The similar method applied after (3.3) of Theorem 3.1 in (3.14) shows us that f satisfies (S).

Next, for the case $f(x) = f(-x)$, it is enough to show that $f(0) = 0$. Suppose that this is not the case.

Putting $x = 0$ in (3.10), from the above assumption and a given condition, we obtain the inequality

$$|g(y)| \leq \frac{\varphi(0)}{2|f(0)|}, \quad y \in G. \tag{3.15}$$

This inequality means that g is globally bounded—a contradiction. Thus the claim $f(0) = 0$ holds.

(iii) In the case g satisfies (A), we know that the limit function h_2 is $2g$. So (3.14) becomes (A_{fg}).

Finally, let g satisfy (T). Replacing y by $y + y_n$ and $y - y_n$ in (3.10), we have

$$\left| \frac{f((x+y)+y_n) - f((x+y)-y_n)}{2g(y_n)} \right.$$
$$+ \frac{f((x-y)+y_n) - f((x-y)-y_n)}{2g(y_n)} - 2f(x) \cdot \left. \frac{g(y+y_n) - g(y-y_n)}{2g(y_n)} \right| \tag{3.16}$$
$$\leq \frac{\varphi(x)}{|g(y_n)|}$$

for all $x, y \in G$. Taking the limit as $n \to \infty$ with the use of (3.11), we conclude that f and g are solutions of (A_{fg}). □

COROLLARY 3.6. *Suppose that $f, g : G \to \mathbb{C}$ satisfy the inequality*

$$|f(x+y) - f(x-y) - 2f(x)g(y)| \leq \varepsilon \quad \forall x, y \in G. \tag{3.17}$$

If g fails to be bounded, then
  (i) *f and g are solutions of (T_{fg}),*
  (ii) *f satisfies (S) under 2-divisible and one of the cases $f(0) = 0$, $f(x) = f(-x)$,*
  (iii) *in particular, g satisfies (A) or (T), and f and g are solutions of (A_{fg}),*
  (iv) *g satisfies (S) under 2-divisible.*

Proof. As proof (i) of Theorem 2.5, we know that g is also bounded whenever f is bounded. Hence, by contraposition, g satisfies (S) from (i) of Theorem 2.1. The other cases are trivial by Theorem 3.5. □

COROLLARY 3.7. *Suppose that $f : G \to \mathbb{C}$ satisfy the inequality*

$$|f(x+y) - f(x-y) - 2f(x)f(y)| \leq \varphi(x) \quad \forall x, y \in G, \tag{3.18}$$

If f fails to be bounded, then
  (i) *f is solution of (T),*
  (ii) *f satisfies (S) under 2-divisible and one of the cases $f(0) = 0$, $f(x) = f(-x)$.*

COROLLARY 3.8. *Let $(G, +)$ be a uniquely 2-divisible group. Suppose that $f, g : G \to \mathbb{C}$ satisfy the inequality*

$$|f(x+y) - f(x-y) - 2f(x)g(y)| \leq \min\{\varphi(x), \varphi(y)\} \quad \forall x, y \in G; \tag{3.19}$$

(a) *if f fails to be bounded, then*
  (i) *g satisfies (S) under 2-divisible,*
  (ii) *in particular, f satisfies (A), f and g are solutions of $g(x + y) - g(x - y) = 2f(x)g(y)$;*
(b) *if g fails to be bounded, then*
  (i) *f and g are solutions of (T_{fg}),*
  (ii) *f satisfies (S) under 2-divisible and one of the cases $f(0) = 0$, $f(x) = f(-x)$*
  (iii) *in particular, g satisfies (A) or (T), and f and g are solutions of (A_{fg}),*
  (iv) *g satisfies (S) under 2-divisible.*

Proof. Above results except for (iv) are trivial by Theorems 3.1 and 3.5. It is sufficient by Theorem 3.1 to show that g is also bounded whenever f is bounded for (iv) of (b). The proof of it runs along the same line as (i) of Theorem 2.5.   □

4. Extension to Banach algebra

All obtained results can be extended to the stability on the Banach algebra. To simplify, we combine four theorems in one, and we will prove one of them.

THEOREM 4.1. *Let $(E, \|\cdot\|)$ be a semisimple commutative Banach algebra. Assume that $f, g : G \rightarrow E$ and $\varphi : G \rightarrow \mathbb{R}$ satisfy one of the following inequalities:*

$$\|f(x + y) - f(x - y) - 2g(x)f(y)\| \leq \begin{cases} \text{(i) } \varphi(x) \\ \text{(ii) } \varphi(y) \end{cases} \quad \forall\, x, y \in G \qquad (4.1)$$

or

$$\|f(x + y) - f(x - y) - 2f(x)g(y)\| \leq \begin{cases} \text{(i) } \varphi(y) \\ \text{(ii) } \varphi(x) \end{cases} \quad \forall\, x, y \in G. \qquad (4.2)$$

For an arbitrary linear multiplicative functional $x^* \in E^*$,
  (a) *if the superposition $x^* \circ f$ fails to be bounded, then*
    (i) *g satisfies (A) in the case (i) of (4.1),*
    (ii) *g satisfies (S) under 2-divisible in the case (i) of (4.2),*
    (iii) *in particular, f satisfies (A), f and g are solutions of $g(x + y) - g(x - y) = 2f(x)g(y)$ in the case (i) of (4.2);*
  (b) *if the superposition $x^* \circ g$ fails to be bounded, then*
    (i) *g satisfies (A) in the case (ii) of (4.1),*
    (ii) *f and g satisfy (T_{gf}) in the case (ii) of (4.1),*
    (iii) *f and g satisfy (A_{fg}) in the case (ii) of (4.1),*
    (iv) *f and g are solutions of (T_{fg}) in the case (ii) of (4.2),*
    (v) *f satisfies (S) under 2-divisible and one of the cases $(x^* \circ f)(0) = 0$, $(x^* \circ f)(x) = (x^* \circ f)(-x)$ in the case (ii) of (4.2),*
    (vi) *in particular, g satisfies (A) or (T), and f and g are solutions of (A_{fg}) in the case (ii) of (4.2).*

Proof. Take the case (i) of (a). Assume that (i) of (4.1) holds, and fix arbitrarily a linear multiplicative functional $x^* \in E$. As well known, we have $\|x^*\| = 1$ whence, for every

$x, y \in G$, we have

$$
\begin{aligned}
\varphi(x) &\geq \left\| f(x+y) - f(x-y) - 2g(x)f(y) \right\| \\
&= \sup_{\|y^*\|=1} \left| y^* \left(f(x+y) - f(x-y) - 2g(x)f(y) \right) \right| \\
&\geq \left| x^* \left(f(x+y) \right) - x^* \left(f(x-y) \right) - 2x^* \left(g(x) \right) x^* \left(f(y) \right) \right|,
\end{aligned}
\tag{4.3}
$$

which states that the superpositions $x^* \circ f$ and $x^* \circ g$ yield solutions of inequality (2.1). Since, by assumption, the superposition $x^* \circ f$ is unbounded, an appeal to Theorem 2.1 shows that the function $x^* \circ g$ solves (A). In other words, bearing the linear multiplicativity of x^* in mind, for all $x, y \in G$, the difference $DA(x, y)$ falls into the kernel of x^*. Therefore, in view of the unrestricted choice of x^*, we infer that

$$
DA(x, y) \in \cap \left\{ \ker x^* : x^* \text{ is a multiplicative member of } E^* \right\}
\tag{4.4}
$$

for all $x, y \in G$. Since the algebra E has been assumed to be semisimple, the last term of the above formula coincides with the singleton $\{0\}$, that is,

$$
f(x+y) - f(x-y) - 2g(x)f(y) = 0 \quad \forall x, y \in G,
\tag{4.5}
$$

as claimed. The other cases are similar, so their proofs will be omitted.   □

Remark 4.2. By applying $g = f$ or $\varphi(y) = \varphi(x) = \varepsilon$ in Theorem 4.1, we can obtain a number of corollaries.

References

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