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Multidimensional common fixed point theorems under probabilistic φ-contractive conditions in multidimensional Menger probabilistic metric spaces

Chuanxi Zhu, Zhe Wei*, Zhaoqi Wu and Wenqing Xu

*Correspondence:
thewayweizhe@163.com
Department of Mathematics, Nanchang University, Nanchang, 330031, P.R. China

Abstract

In this paper, we introduce the new concepts of multidimensional Menger probabilistic metric spaces and related fixed point for a pair of mappings $T: \underbrace{X \times X \times \cdots \times X}_{n} \to X$ and $A: X \to X$. Utilizing the properties of the related triangular norm and the compatibility of A with T, some multidimensional common fixed point problems of hybrid probabilistic contractions with a gauge function φ are studied. The obtained results generalize some coupled and triple common fixed point theorems in the corresponding literature. Finally, an example is given to illustrate our main results.

Keywords: multidimensional Menger probabilistic metric space; fixed point; hybrid probabilistic contractions; compatible

1 Introduction

Coupled fixed points were studied first by Bhaskar and Lakshmikantham [1]. Since then, some new results on the existence and uniqueness of coupled fixed points have been presented in partially ordered metric spaces, cone metric spaces, and fuzzy metric spaces [2–5]. The concept of a probabilistic metric space was initiated and studied by Menger, which is a generalization of the metric space [6]. Many results for the existence of fixed points or solutions of nonlinear equations under various types of conditions in Menger probabilistic spaces (briefly, *PM*-spaces) have been extensively considered by many scholars [7–22]. In 2010, Jachymski established a fixed point theorem for φ-contractions and gave a characterization of a function φ having the property that there exists a probabilistic φ-contraction, which is not a probabilistic k-contraction ($k \in [0,1)$) [23]. In 2011, Xiao *et al.* obtained some common coupled fixed point results for hybrid probabilistic contractions with a gauge function φ in Menger probabilistic metric spaces without assuming any continuity or monotonicity conditions for φ [24]. In 2014, Luo *et al.* introduced the concept of generalized Menger probabilistic metric spaces and obtained some tripled common fixed point results with a gauge function φ with the same properties in generalized Menger probabilistic metric spaces [25].

The purpose of this paper is to introduce the new concepts of multidimensional Menger probabilistic metric spaces and a related fixed point for a pair of mappings T: $\underbrace{X \times X \times \cdots \times X}_{n} \to X$ and A: $X \to X$. Utilizing the properties of the related triangular norm and the compatibility of A with T, some multidimensional common fixed point problems of hybrid probabilistic contractions with a gauge function φ are studied. The obtained results generalize some coupled and triple common fixed point theorems in the corresponding literature. Finally, an example is given to illustrate our main results.

2 Preliminaries

Denote by n any given positive integer which is not smaller than 2, Λ_n the set $\{1, 2, \ldots, n\}$, X^n the product $\underbrace{X \times X \times \cdots \times X}_{n}$, $\mathbb{R}$ the set of the real numbers, $\mathbb{R}^+$ the set of the non-negative real numbers, and $\mathbb{Z}^+$ the set of all positive integers. A mapping $F : \mathbb{R} \to \mathbb{R}^+$ is called a distribution function if it is nondecreasing left-continuous with $\sup_{t\in\mathbb{R}} F(t) = 1$ and $\inf_{t\in\mathbb{R}} F(t) = 0$.

We will denote by $\mathscr{D}$ the set of all distribution functions, by $\mathscr{D}^+ = \{F \in \mathscr{D} : F(t) = 0, \forall t \leq 0\}$, while H will always denote the specific distribution function defined by

$$H(t) = \begin{cases} 0, & t \leq 0, \\ 1, & t > 0. \end{cases}$$

If φ: $\mathbb{R}^+ \to \mathbb{R}^+$ is a function such that $\varphi(0) = 0$, then φ is called a gauge function. If $t \in \mathbb{R}^+$, then $\varphi^n(t)$ denotes the nth iteration of $\varphi(t)$ and $\varphi^{-1}(\{0\}) = \{t \in \mathbb{R}^+ : \varphi(t) = 0\}$.

First, we give *PM*-spaces introduced by Menger with the related triangular norm.

Definition 2.1 [7] A mapping $\Delta : [0,1] \times [0,1] \to [0,1]$ is called a triangular norm (for short, a t-norm) if the following conditions are satisfied for any $a, b, c, d \in [0,1]$:

(1) $\Delta(a, 1) = a$;
(2) $\Delta(a, b) = \Delta(b, a)$;
(3) $\Delta(a, c) \geq \Delta(b, d)$ for $a \geq b$, $c \geq d$;
(4) $\Delta(a, \Delta(b, c)) = \Delta(\Delta(a, b), c)$.

Definition 2.2 [6] A triplet $(X, \mathscr{F}, \Delta)$ is called a Menger probabilistic metric space (for short, a *Menger PM*-space) if X is a nonempty set, Δ is a t-norm, and $\mathscr{F}$ is a mapping from $X \times X$ into $\mathscr{D}^+$ satisfying the following conditions (we denote $\mathscr{F}(x, y)$ by $F_{x,y}$):

(MS-1) $F_{x,y}(t) = H(t)$ for all $t \in R$ if and only if $x = y$;
(MS-2) $F_{x,y}(t) = F_{y,x}(t)$ for all $t \in R$;
(MS-3) $F_{x,y}(t + s) \geq \Delta(F_{x,z}(t), F_{z,y}(s))$ for all $x, y, z \in X$ and $t, s \geq 0$.

Then we give the generalized Menger PM-spaces introduced by Luo *et al.* with the related triangular norm.

Definition 2.3 [8] A mapping $\Delta : [0,1] \times [0,1] \times [0,1] \to [0,1]$ is called a triangular norm (for short, a t-norm) if the following conditions are satisfied for any $a, b, c, d, e, f \in [0,1]$:

(1) $\Delta(a, 1, 1) = a$, $\Delta(0, 0, 0) = 0$;
(2) $\Delta(a, b, c) = \Delta(a, c, b) = \Delta(c, b, a)$;
(3) $\Delta(a, b, c) \geq \Delta(d, e, f)$ for $a \geq d$, $b \geq e$, $c \geq f$;
(4) $\Delta(a, \Delta(b, c, d), e) = \Delta(\Delta(a, b, c), d, e) = \Delta(a, b, \Delta(c, d, e))$.

Definition 2.4 [25] A triplet $(X, \mathscr{F}, \Delta)$ is called a generalized Menger probabilistic metric space (for short, a generalized Menger *PM*-space) if X is a nonempty set, Δ is a t-norm, and $\mathscr{F}$ is a mapping from $X \times X$ into $\mathscr{D}^+$ satisfying the following conditions (we denote $\mathscr{F}(x,y)$ by $F_{x,y}$):

(GPM-1) $F_{x,y}(t) = H(t)$ for all $t \in R$ if and only if $x = y$;

(GPM-2) $F_{x,y}(t) = F_{y,x}(t)$ for all $t \in R$;

(GPM-3) $F_{x,w}(t_1 + t_2 + t_3) \geq \Delta(F_{x,y}(t_1), F_{y,z}(t_2), F_{z,w}(t_3))$ for all $x, y, z, w \in X$ and $t_1, t_2, t_3 \geq 0$.

Now, we introduce the definition of multidimensional Menger probabilistic metric spaces with the related triangular norm.

Definition 2.5 A mapping $\Delta: \underbrace{[0,1] \times [0,1] \times \cdots \times [0,1]}_{n} \to [0,1]$ is called a triangular norm (for short, a t-norm) if the following conditions are satisfied for any $a_1, a_2, \ldots, a_n, a_{n+1}, \ldots, a_{2n} \in [0,1]$:

(1) $\Delta(a_1, 1, \ldots, 1) = a_1$, $\Delta(0, 0, \ldots, 0) = 0$;

(2) $\Delta(a_1, a_2, \ldots, a_{n-2}, a_{n-1}, a_n) = \Delta(a_1, a_n, \ldots, a_{n-2}, a_{n-1}) = \Delta(a_1, a_n, a_{n-1}, \ldots, a_{n-2}) = \cdots = \Delta(a_1, a_n, a_{n-1}, a_{n-2}, \ldots, a_2) = \Delta(a_n, a_{n-1}, a_{n-2}, \ldots, a_2, a_1)$;

(3) $\Delta(a_1, a_2, \ldots, a_n) \geq \Delta(a_{n+1}, a_{n+2}, \ldots, a_{2n})$ for $a_1 \geq a_{n+1}, a_2 \geq a_{n+2}, \ldots, a_n \geq a_{2n}$;

(4) $\Delta(\Delta(a_1, a_2, \ldots, a_n), a_{n+1}, \ldots, a_{2n-1}) = \Delta(a_1, \Delta(a_2, \ldots, a_{n+1}), a_{n+2} \cdots, a_{2n-1}) = \cdots = \Delta(a_1, \ldots, a_{n-1}, \Delta(a_n, a_{n+1}, \ldots, a_{2n-1}))$.

Two typical examples of t-norm are $\Delta_M(a_1, a_2, \ldots, a_n) = \min\{a_1, a_2, \ldots, a_n\}$ and $\Delta_P(a_1, a_2, \ldots, a_n) = a_1 a_2 \cdots a_n$ for all $a_1, a_2, \ldots, a_n \in [0,1]$.

Definition 2.6 A triplet $(X, \mathscr{F}, \Delta)$ is called a multidimensional Menger probabilistic metric space (for short, a multidimensional Menger *PM*-space) if X is a nonempty set, Δ is a t-norm and $\mathscr{F}$ is a mapping from $X \times X$ into $\mathscr{D}^+$ satisfying the following conditions (we denote $\mathscr{F}(x,y)$ by $F_{x,y}$):

(MPM-1) $F_{x,y}(t) = H(t)$ for all $t \in R$ if and only if $x = y$;

(MPM-2) $F_{x,y}(t) = F_{y,x}(t)$ for all $t \in R$;

(MPM-3) $F_{x_1,x_{n+1}}(t_1 + t_2 + \cdots + t_n) \geq \Delta(F_{x_1,x_2}(t_1), F_{x_2,x_3}(t_2), \ldots, F_{x_n,x_{n+1}}(t_n))$ for all $x_1, x_2, \ldots, x_{n+1} \in X$ and $t_1, t_2, \ldots, t_n \geq 0$.

Remark 2.1 If $n = 2$, the multidimensional Menger *PM*-space is a Menger *PM*-space. While $n = 3$, the multidimensional Menger *PM*-space is a generalized Menger *PM*-space.

Remark 2.2 If $\Delta = \Delta_M$, the multidimensional Menger *PM*-space is a Menger *PM*-space. In fact, let $x_1 = x, x_2 = z, \ldots, x_n = z, x_{n+1} = y$ in (MPM-3), then for any $t, s, \delta \geq 0$, $(n-2)\delta \leq s$, we have

$$F_{x,y}(t+s) \geq \min\bigl\{(F_{x,z}(t), F_{z,z}(\delta), \ldots, F_{z,z}(\delta), F_{z,y}\bigl(s - (n-2)\delta\bigr)\bigr\}.$$

Thus we have

$$F_{x,y}(t+s) \geq \min\bigl\{(F_{x,z}(t), F_{z,y}\bigl(s - (n-2)\delta\bigr)\bigr\}.$$

Taking $\delta \to 0$, we obtain

$$F_{x,y}(t+s) \geq \min\bigl\{(F_{x,z}(t), F_{z,y}(s)\bigr\}.$$

Therefore, if $\Delta = \Delta_M$, the multidimensional Menger *PM*-space is a Menger *PM*-space.

Example 2.1 Suppose that $X = [-1,1]$. Define $\mathscr{F} : X \times X \to \mathscr{D}^+$ by

$$\mathscr{F}_{x,y}(t) = F_{x,y}(t) = \begin{cases} \frac{t}{t+|x-y|}, & t > 0, \\ 0, & t \leq 0, \end{cases}$$

for all $x, y \in X$. It is easy to verify that $(X, \mathscr{F}, \Delta_M)$ satisfies (MPM-1) and (MPM-2). Now we prove it also satisfies (MPM-3). Assume that $t_1, t_2, \ldots, t_n \geq 0$ and $x_1, x_2, \ldots, x_{n+1} \in X$. Then we have

$$\begin{aligned} F_{x_1,x_{n+1}}(t_1 + \cdots + t_n) &= \frac{t_1 + \cdots + t_n}{t_1 + \cdots + t_n + |x_1 - x_{n+1}|} \\ &\geq \frac{t_1 + \cdots + t_n}{t_1 + \cdots + t_n + |x_1 - x_2| + \cdots + |x_n - x_{n+1}|} \\ &\geq \min\left\{\frac{t_1}{t_1 + |x_1 - x_2|}, \ldots, \frac{t_n}{t_n + |x_n - x_{n+1}|}\right\} \\ &= \Delta_M\bigl(F_{x_1,x_2}(t_1), \ldots, F_{x_n,x_{n+1}}(t_n)\bigr). \end{aligned}$$

Hence $(X, \mathscr{F}, \Delta_M)$ a multidimensional Menger *PM*-space.

Proposition 2.1 *Let $(X, \mathscr{F}, \Delta)$ be a multidimensional Menger PM-space and Δ be a continuous t-norm. Then $(X, \mathscr{F}, \Delta)$ is a Hausdorff topological space in the (ϵ, λ)-topology $\mathscr{T}$, i.e., the family of sets*

$$\bigl\{U_x(\epsilon, \lambda) : \epsilon > 0, \lambda \in (0,1], x \in X\bigr\}$$

is a base of neighborhoods of a point x for $\mathscr{F}$, where

$$U_x(\epsilon, \lambda) = \bigl\{y \in X : F_{x,y}(\epsilon) > 1 - \lambda\bigr\}.$$

Proof It suffices to prove that:

(i) for any $x \in X$, there exists an $U = U_x(\epsilon, \lambda)$ such that $x \in U$;

(ii) for any given $U_x(\epsilon_1, \lambda_1)$ and $U_x(\epsilon_2, \lambda_2)$, there exist $\epsilon > 0$ and $\lambda > 0$, such that $U_x(\epsilon, \lambda) \subset U_x(\epsilon_1, \lambda_1) \cap U_x(\epsilon_2, \lambda_2)$;

(iii) for any $y \in U_x(\epsilon, \lambda)$, there exist $\epsilon' > 0$ and $\lambda' > 0$, such that $U_y(\epsilon', \lambda') \subset U_x(\epsilon, \lambda)$;

(iv) for any $x, y \in X$, $x \neq y$, there exist $U_x(\epsilon_1, \lambda_1)$ and $U_y(\epsilon_2, \lambda_2)$, such that $U_x(\epsilon_1, \lambda_1) \cap U_y(\epsilon_2, \lambda_2) = \emptyset$.

It is easy to check that (i)-(iii) are true. Now we prove that (iv) is also true. In fact, suppose that $x, y \in X$ and $x \neq y$. Then there exist $t_0 > 0$ and $0 < a < 1$, such that $F_{x,y}(t_0) = a$. Let

$$U_x = \left\{r : F_{x,r}\left(\frac{t_0}{n}\right) > b\right\}, \qquad U_y = \left\{r : F_{y,r}\left(\frac{t_0}{n}\right) > b\right\},$$

where $0 < b < 1$ and $\Delta(b,\underbrace{1,\ldots,1}_{n-2},b) > a$ (since Δ is continuous and $\Delta(1,\ldots,1) = 1$, such b exists). Now suppose that there exists a point $v \in U_x \cap U_y$, which implies that $F_{x,v}(\frac{t_0}{n}) > b$ and $F_{y,v}(\frac{t_0}{n}) > b$. Then we have

$$a = F_{x,y}(t_0) \geq \Delta\left(F_{x,v}\left(\frac{t_0}{n}\right),\underbrace{F_{v,v}\left(\frac{t_0}{n}\right),\ldots,F_{v,v}\left(\frac{t_0}{n}\right)}_{n-2},F_{v,y}\left(\frac{t_0}{n}\right)\right) \geq \Delta(b,\underbrace{1,\ldots,1}_{n-2},b) > a,$$

which is a contradiction. Thus the conclusion (iv) is proved. This completes the proof. □

Definition 2.7 Let $(X,\mathscr{F},\Delta)$ be a multidimensional Menger *PM*-space, Δ be a continuous t-norm.

(i) A sequence $\{x_m\}$ in X is said to be $\mathscr{T}$-convergent to $x \in X$ if $\lim_{m\to\infty} F_{x_m,x} = 1$ for all $t > 0$;

(ii) a sequence $\{x_m\}$ in X is said to be a $\mathscr{T}$-Cauchy sequence, if for any given $\epsilon > 0$ and $\lambda \in (0,1]$, there exists a positive integer $N = N(\epsilon,\lambda)$, such that $F_{x_m,x_k}(\epsilon) > 1-\lambda$, whenever $m,k \geq N$;

(iii) $(X,\mathscr{F},\Delta)$ is said to be $\mathscr{T}$-complete, if each $\mathscr{T}$-Cauchy sequence in X is $\mathscr{T}$-convergent to some point in X.

Definition 2.8 A t-norm Δ is said to be H-type if the family of functions $\{\Delta^m(t)\}_{m=1}^{\infty}$ is equi-continuous at $t = 1$, where

$$\Delta^1(t) = \Delta(t,\ldots,t), \qquad \Delta^{m+1}(t) = \Delta\bigl(\underbrace{t,\ldots,t}_{n-1},\Delta^m(t)\bigr), \quad m = 1,2,\ldots, t \in [0,1].$$

Definition 2.9 Let X be a nonempty set, $T : X^n \to X$ and $A : X \to X$ be two mappings. A is said to be commutative with T, if $AT(x_1,\ldots,x_n) = T(Ax_1,\ldots,Ax_n)$ for all $x_1,\ldots x_n \in X$. A point $u \in X$ is called a multidimensional common fixed point of T and A, if $u = Au = T(u,\ldots,u)$.

Definition 2.10 Let X be a nonempty set, $T : X^n \to X$ and $A : X \to X$ be two mappings. Let $\{x_m^1\},\ldots,\{x_m^n\}$ be n sequences in X and $\sigma_1,\ldots,\sigma_n$ be n permutations of Λ_n. A and T are said to be compatible in $(X,\mathscr{F},\Delta)$ if

$$\lim_{m\to\infty} F_{AT(x_m^{\sigma_i(1)},\ldots,x_m^{\sigma_i(n)}),T(Ax_m^{\sigma_i(1)},\ldots,Ax_m^{\sigma_i(n)})}(t) = 1$$

for all $i = 1,\ldots,n$ and $t > 0$, whenever

$$\lim_{m\to\infty} T\bigl(x_m^{\sigma_i(1)},\ldots,x_m^{\sigma_i(n)}\bigr) = \lim_{m\to\infty} Ax_m^i \in X$$

for all $i = 1,\ldots,n$;

A and T are said to be compatible in (X,d) where (X,d) is a usual metric space if

$$\lim_{m\to\infty} d\bigl(AT\bigl(x_m^{\sigma_i(1)},\ldots,x_m^{\sigma_i(n)}\bigr),T\bigl(Ax_m^{\sigma_i(1)},\ldots,Ax_m^{\sigma_i(n)}\bigr)\bigr) = 0$$

for all $i = 1, \dots, n$ and $t > 0$, whenever

$$\lim_{m\to\infty} T\bigl(x_m^{\sigma_i(1)}, \dots, x_m^{\sigma_i(n)}\bigr) = \lim_{m\to\infty} Ax_m^i \in X$$

for all $i = 1, \dots, n$.

Obviously, if T and A are commutative, then they are compatible, but the converse does not hold.

The following lemmas play an important role in proving our main results in Section 3.

Lemma 2.1 [23] *Suppose that $F \in \mathscr{D}^+$. For every $m \in Z^+$, let $F_m : R \to [0,1]$ be nondecreasing and $g_m : (0,+\infty) \to (0,+\infty)$ satisfy $\lim_{m\to\infty} g_m(t) = 0$ for any $t > 0$. If $F_m(g_m(t)) \geq F(t)$ for any $t > 0$, then $\lim_{m\to\infty} F_m(t) = 1$ for any $t > 0$.*

Lemma 2.2 *Let X be a nonempty set, and $T : X^n \to X$ and $A : X \to X$ be two mappings. If $T(X^n) \subset A(X)$, then there exist n sequences $\{x_m^1\}_{m=0}^\infty, \dots, \{x_m^n\}_{m=0}^\infty$ in X, such that $Ax_{m+1}^1 = T(x_m^1, x_m^2, \dots, x_m^n), Ax_{m+1}^2 = T(x_m^2, x_m^3, \dots, x_m^n, x_m^1), \dots, Ax_{m+1}^n = T(x_m^n, x_m^1, \dots, x_m^{n-1})$.*

Proof Let $x_0^1, x_0^2, \dots, x_0^n$ be any given points in X. Since $T(X^n) \subset A(X)$, we can choose $x_1^1, x_1^2, \dots, x_1^n \in X$ such that $Ax_1^1 = T(x_0^1, x_0^2, \dots, x_0^n), Ax_1^2 = T(x_0^2, x_0^3, \dots, x_0^n, x_0^1), \dots, Ax_1^n = T(x_0^n, x_0^1, \dots, x_0^{n-1})$. Continuing this process, we can construct n sequences $\{x_m^1\}_{m=0}^\infty, \dots, \{x_m^n\}_{m=0}^\infty$ in X, such that

$$\begin{aligned} &Ax_{m+1}^1 = T\bigl(x_m^1, x_m^2, \dots, x_m^n\bigr), \qquad Ax_{m+1}^2 = T\bigl(x_m^2, x_m^3, \dots, x_m^n, x_m^1\bigr), \qquad \dots, \\ &Ax_{m+1}^n = T\bigl(x_m^n, x_m^1, \dots, x_m^{n-1}\bigr). \end{aligned}$$

□

Lemma 2.3 [13] *Let (X, d) is a usual metric space. Define $\mathscr{F} : X \times X \to \mathscr{D}^+$ by*

$$F_{x,y} = H\bigl(t - d(x,y)\bigr), \quad \textit{for } x, y \in X \textit{ and } t > 0.$$

Then $(X, \mathscr{F}, \Delta_M)$ is a Menger PM-space and is called the induced Menger PM-space by (X, d). It is complete if (X, d) is complete.

Lemma 2.4 [14] *Let $\varphi(t) : \mathbb{R}^+ \to \mathbb{R}^+$ be a function. Let $a, b, t \in \mathbb{R}^+$. Then we have*

$$H(t - a) \geq H\bigl(\varphi(t) - b\bigr) \quad \textit{if and only if} \quad \varphi(b) \leq a.$$

3 Main results

In this section, we shall give the main results of this paper.

Theorem 3.1 *Let $(X, \mathscr{F}, \Delta)$ be a complete multidimensional Menger PM-space with Δ a continuous related t-norm of H-type, φ: $\mathbb{R}^+ \to \mathbb{R}^+$ be a gauge function such that $\varphi^{-1}(\{0\}) = \{0\}$, $\varphi(t) < t$, and $\lim_{m\to+\infty} \varphi^m(t) = 0$ for any $t > 0$. Let T: $X^n \to X$ and A: $X \to X$ be two mappings satisfying the following conditions*:

$$F_{T(x_1,x_2,\dots,x_n),T(y_1,y_2,\dots,y_n)}\bigl(\varphi(t)\bigr) \geq \bigl[F_{Ax_1,Ay_1}(t)F_{Ax_2,Ay_2}(t)\cdots F_{Ax_n,Ay_n}(t)\bigr]^{\frac{1}{n}} \tag{3.1}$$

for all $x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n \in X$, *and* $t > 0$, *where* $T(X^n) \subset A(X)$, A *is continuous and compatible with* T. *Then* T *and* A *have a unique multidimensional common fixed point in* X.

Proof By Lemma 2.2, we can construct n sequences $\{x_m^1\}_{m=0}^{\infty}, \ldots, \{x_m^n\}_{m=0}^{\infty}$ in X, such that $Ax_{m+1}^1 = T(x_m^1, x_m^2, \ldots, x_m^n), Ax_{m+1}^2 = T(x_m^2, x_m^3, \ldots, x_m^n, x_m^1), \ldots, Ax_{m+1}^n = T(x_m^n, x_m^1, \ldots, x_m^{n-1})$.

From (3.1), for all $t > 0$, we have

$$\begin{aligned}
F_{Ax_m^1, Ax_{m+1}^1}\big(\varphi(t)\big) &= F_{T(x_{m-1}^1, x_{m-1}^2, \ldots, x_{m-1}^n), T(x_m^1, x_m^2, \ldots, x_m^n)}\big(\varphi(t)\big) \\
&\geq \big[F_{Ax_{m-1}^1, Ax_m^1}(t) F_{Ax_{m-1}^2, Ax_m^2}(t) \cdots F_{Ax_{m-1}^n, Ax_m^n}(t)\big]^{\frac{1}{n}}, \\
F_{Ax_m^2, Ax_{m+1}^2}\big(\varphi(t)\big) &= F_{T(x_{m-1}^2, x_{m-1}^3, \ldots, x_{m-1}^1), T(x_m^2, x_m^3, \ldots, x_m^1)}\big(\varphi(t)\big) \\
&\geq \big[F_{Ax_{m-1}^2, Ax_m^2}(t) F_{Ax_{m-1}^3, Ax_m^3}(t) \cdots F_{Ax_{m-1}^1, Ax_m^1}(t)\big]^{\frac{1}{n}}, \\
&\vdots \\
F_{Ax_m^n, Ax_{m+1}^n}\big(\varphi(t)\big) &= F_{T(x_{m-1}^n, x_{m-1}^1, \ldots, x_{m-1}^{n-1}), T(x_m^n, x_m^1, \ldots, x_m^{n-1})}\big(\varphi(t)\big) \\
&\geq \big[F_{Ax_{m-1}^n, Ax_m^n}(t) F_{Ax_{m-1}^1, Ax_m^1}(t) \cdots F_{Ax_{m-1}^{n-1}, Ax_m^{n-1}}(t)\big]^{\frac{1}{n}}.
\end{aligned} \tag{3.2}$$

Denote $P_m(t) = [F_{Ax_{m-1}^1, Ax_m^1}(t) F_{Ax_{m-1}^2, Ax_m^2}(t) \cdots F_{Ax_{m-1}^n, Ax_m^n}(t)]^{\frac{1}{n}}$. From (3.2), we have

$$\begin{aligned}
P_{m+1}\big(\varphi(t)\big) &= \big[F_{Ax_m^1, Ax_{m+1}^1}\big(\varphi(t)\big) F_{Ax_m^2, Ax_{m+1}^2}\big(\varphi(t)\big) \cdots F_{Ax_m^n, Ax_{m+1}^n}\big(\varphi(t)\big)\big]^{\frac{1}{n}} \\
&\geq \big[\underbrace{P_m(t) P_m(t) \cdots P_m(t)}_{n}\big]^{\frac{1}{n}} = P_m(t),
\end{aligned}$$

which implies that

$$\begin{aligned}
&F_{Ax_m^1, Ax_{m+1}^1}\big(\varphi^m(t)\big) \geq P_m\big(\varphi^{m-1}(t)\big) \geq \cdots P_1(t), \\
&F_{Ax_m^2, Ax_{m+1}^2}\big(\varphi^m(t)\big) \geq P_m\big(\varphi^{m-1}(t)\big) \geq \cdots P_1(t), \\
&\vdots \\
&F_{Ax_m^2, Ax_{m+1}^2}\big(\varphi^m(t)\big) \geq P_m\big(\varphi^{m-1}(t)\big) \geq \cdots P_1(t).
\end{aligned} \tag{3.3}$$

Since $P_1(t) = [F_{Ax_0^1, Ax_1^1}(t) F_{Ax_0^2, Ax_1^2}(t) \cdots F_{Ax_0^n, Ax_1^n}(t)]^{\frac{1}{n}} \in \mathscr{D}^+$ and $\lim_{m \to \infty} \varphi^m(t) = 0$ for each $t > 0$, using Lemma 2.1, we have

$$\lim_{m \to \infty} F_{Ax_m^1, Ax_{m+1}^1}(t) = 1, \qquad F_{Ax_m^2, Ax_{m+1}^2}(t) = 1, \qquad \ldots, \qquad F_{Ax_m^n, Ax_{m+1}^n}(t) = 1. \tag{3.4}$$

Thus

$$\lim_{m \to \infty} P_m(t) = 1, \quad \forall t > 0. \tag{3.5}$$

We claim that, for any $k \in \mathbb{Z}^+$ and $t > 0$,

$$\begin{aligned} &F_{Ax_m^1, Ax_{m+k}^1}(t) \geq \Delta^k\left(P_m\left(\frac{t-\varphi(t)}{n-1}\right)\right), \\ &F_{Ax_m^2, Ax_{m+k}^2}(t) \geq \Delta^k\left(P_m\left(\frac{t-\varphi(t)}{n-1}\right)\right), \\ &\vdots \\ &F_{Ax_m^n, Ax_{m+k}^n}(t) \geq \Delta^k\left(P_m\left(\frac{t-\varphi(t)}{n-1}\right)\right). \end{aligned} \tag{3.6}$$

In fact, by (3.2) and $\varphi(t) < t$, we can conclude that (3.6) holds for $k = 1$ since $F_{Ax_m^1, Ax_{m+1}^1}(t) \geq F_{Ax_m^1, Ax_{m+1}^1}(\varphi(t)) \geq P_m(t) \geq P_m(\frac{t-\varphi(t)}{n-1}) \geq \Delta^1(P_m(\frac{t-\varphi(t)}{n-1}))$. Assume that (3.6) holds for some k. Since $\varphi(t) < t$, by the first inequality of (3.2), we have $F_{Ax_m^1, Ax_{m+1}^1}(t) \geq F_{Ax_m^1, Ax_{m+1}^1}(\varphi(t)) \geq P_m(t)$. By (3.1) and (3.6), we have

$$\begin{aligned} F_{Ax_{m+1}^1, Ax_{m+k+1}^1}\bigl(\varphi(t)\bigr) &\geq \bigl[F_{Ax_m^1, Ax_{m+k}^1}(t) F_{Ax_m^2, Ax_{m+k}^2}(t) \cdots F_{Ax_m^n, Ax_{m+k}^n}(t)\bigr]^{\frac{1}{n}} \\ &\geq \Delta^k\left(P_m\left(\frac{t-\varphi(t)}{n-1}\right)\right). \end{aligned}$$

Hence, by the monotonicity of Δ, we have

$$\begin{aligned} F_{Ax_m^1, Ax_{m+k+1}^1}(t) &= F_{Ax_m^1, Ax_{m+k+1}^1}\bigl(t - \varphi(t) + \varphi(t)\bigr) \\ &\geq \Delta\left(F_{Ax_m^1, Ax_{m+1}^1}\left(\frac{t-\varphi(t)}{n-1}\right), \ldots, F_{Ax_m^1, Ax_{m+1}^1}\left(\frac{t-\varphi(t)}{n-1}\right),\right. \\ &\qquad \left. F_{Ax_{m+1}^1, Ax_{m+k+1}^1}\bigl(\varphi(t)\bigr)\right) \\ &\geq \Delta\left(P_m\left(\frac{t-\varphi(t)}{n-1}\right), \ldots, P_m\left(\frac{t-\varphi(t)}{n-1}\right), \Delta^k\left(P_m\left(\frac{t-\varphi(t)}{n-1}\right)\right)\right) \\ &= \Delta^{k+1}\left(P_m\left(\frac{t-\varphi(t)}{n-1}\right)\right). \end{aligned}$$

Similarly, we have $F_{Ax_m^2, Ax_{m+k+1}^2}(t) \geq \Delta^{k+1}(P_m(\frac{t-\varphi(t)}{n-1})), \ldots, F_{Ax_m^n, Ax_{m+k+1}^n}(t) \geq \Delta^{k+1}(P_m(\frac{t-\varphi(t)}{n-1}))$. Therefore, by induction, (3.6) holds for all $k \in \mathbb{Z}^+$ and $t > 0$.

Suppose that $\lambda \in (0,1]$ is given. Since Δ is a t-norm of H-type, there exists $\delta > 0$ such that

$$\Delta^k(s) > 1 - \lambda, \quad s \in (1-\delta, 1], k \in \mathbb{Z}^+. \tag{3.7}$$

By (3.5), there exists $M \in \mathbb{Z}^+$, such that $P_m(\frac{t-\varphi(t)}{n-1}) > 1 - \delta$ for all $m \geq M$. Hence, from (3.6) and (3.7), we get $F_{Ax_m^1, Ax_{m+k}^1}(t) > 1 - \lambda, F_{Ax_m^2, Ax_{m+k}^2}(t) > 1 - \lambda, \ldots, F_{Ax_m^n, Ax_{m+k}^n}(t) > 1 - \lambda$ for all $m \geq M$, $k \in \mathbb{Z}^+$. Therefore $\{Ax_m^1\}, \{Ax_m^2\}, \ldots, \{Ax_m^n\}$ are n Cauchy sequences.

Since $(X, \mathscr{F}, \Delta)$ is complete, there exist $u^1, u^2, \ldots, u^n \in X$, such that

$$\lim_{m\to\infty} Ax_m^1 = u^1, \qquad \lim_{m\to\infty} Ax_m^2 = u^2, \qquad \ldots, \qquad \lim_{m\to\infty} Ax_m^n = u^n.$$

By the continuity of A, we have

$$\lim_{m\to\infty} AAx_m^1 = Au^1, \qquad \lim_{m\to\infty} AAx_m^2 = Au^2, \qquad \ldots, \qquad \lim_{m\to\infty} AAx_m^n = Au^n.$$

The compatibility of A with T implies that

$$\lim_{m\to\infty} F_{AT(x_m^1,x_m^2,\ldots,x_m^n),T(Ax_m^1,Ax_m^2,\ldots,Ax_m^n)}(t) = 1, \qquad \ldots,$$
$$\lim_{m\to\infty} F_{AT(x_m^n,x_m^1,\ldots,x_m^{n-1}),T(Ax_m^n,Ax_m^1,\ldots,Ax_m^{n-1})}(t) = 1,$$

where $\sigma_1 = (1,2,\ldots,n), \sigma_2 = (2,3,\ldots,1), \ldots, \sigma_n = (n,1,\ldots,n-1)$.

From (3.1) and $\varphi(t) < t$, we obtain

$$\begin{aligned} F_{AAx_{m+1}^1,T(u^1,u^2,\ldots,u^n)}(t) &= F_{AAx_{m+1}^1,T(u^1,u^2,\ldots,u^n)}\bigl(t-\varphi(t)+\varphi(t)\bigr) \\ &\geq \Delta\biggl(F_{AAx_{m+1}^1,T(Ax_m^1,Ax_m^2,\ldots,Ax_m^n)}\biggl(\frac{t-\varphi(t)}{n-1}\biggr), \\ &\quad F_{T(Ax_m^1,Ax_m^2,\ldots,Ax_m^n),T(Ax_m^1,Ax_m^2,\ldots,Ax_m^n)}\biggl(\frac{t-\varphi(t)}{n-1}\biggr),\ldots, \\ &\quad F_{T(Ax_m^1,Ax_m^2,\ldots,Ax_m^n),T(Ax_m^1,Ax_m^2,\ldots,Ax_m^n)}\biggl(\frac{t-\varphi(t)}{n-1}\biggr), \\ &\quad F_{T(Ax_m^1,Ax_m^2,\ldots,Ax_m^n),T(u^1,u^2,\ldots,u^n)}\bigl(\varphi(t)\bigr)\biggr) \\ &= \Delta\biggl(F_{AAx_{m+1}^1,T(Ax_m^1,Ax_m^2,\ldots,Ax_m^n)}\biggl(\frac{t-\varphi(t)}{n-1}\biggr),1,\ldots,1, \\ &\quad F_{T(Ax_m^1,Ax_m^2,\ldots,Ax_m^n),T(u^1,u^2,\ldots,u^n)}\bigl(\varphi(t)\bigr)\biggr). \end{aligned} \tag{3.8}$$

From (3.1), we have

$$F_{T(Ax_m^1,Ax_m^2,\ldots,Ax_m^n),T(u^1,u^2,\ldots,u^n)}\bigl(\varphi(t)\bigr) \geq \bigl[F_{AAx_m^1,Au^1}(t)F_{AAx_m^2,Au^2}(t)\cdots F_{AAx_m^n,Au^n}(t)\bigr]^{\frac{1}{n}}. \tag{3.9}$$

Combining (3.8) with (3.9) and letting $m \to \infty$, we obtain $\lim_{m\to\infty} AAx_m^1 = T(u^1,u^2,\ldots,u^n)$. Hence $T(u^1,u^2,\ldots,u^n) = Au^1$. Similarly, we can show that $T(u^2,u^3,\ldots,u^1) = Au^2, T(u^3,u^4,\ldots,u^2) = Au^3, \ldots, T(u^n,u^1,\ldots,u^{n-1}) = Au^n$.

Next we show that $Au^1 = u^1, Au^2 = u^2, \ldots, Au^n = u^n$. In fact, from (3.1), for all $t > 0$, we have

$$\begin{aligned} F_{Au^1,Ax_m^1}\bigl(\varphi(t)\bigr) &= F_{T(u^1,u^2,\ldots,u^n),T(x_{m-1}^1,x_{m-1}^2,\ldots,x_{m-1}^n)}\bigl(\varphi(t)\bigr) \\ &\geq \bigl[F_{Au^1,Ax_{m-1}^1}(t),F_{Au^2,Ax_{m-1}^2}(t),\ldots,F_{Au^n,Ax_{m-1}^n}(t)\bigr]^{\frac{1}{n}}, \\ F_{Au^2,Ax_m^2}\bigl(\varphi(t)\bigr) &= F_{T(u^2,u^3,\ldots,u^1),T(x_{m-1}^2,x_{m-1}^3,\ldots,x_{m-1}^1)}\bigl(\varphi(t)\bigr) \\ &\geq \bigl[F_{Au^2,Ax_{m-1}^2}(t),F_{Au^3,Ax_{m-1}^3}(t),\ldots,F_{Au^1,Ax_{m-1}^1}(t)\bigr]^{\frac{1}{n}}, \\ &\vdots \end{aligned} \tag{3.10}$$

$$\begin{aligned} F_{Au^n,Ax_m^n}\big(\varphi(t)\big) &= F_{T(u^n,u^1,\ldots,u^{n-1}),T(x_{m-1}^n,x_{m-1}^1,\ldots,x_{m-1}^{n-1})}\big(\varphi(t)\big) \\ &\geq \big[F_{Au^n,Ax_{m-1}^n}(t),F_{Au^1,Ax_{m-1}^1}(t),\ldots,F_{Au^{n-1},Ax_{m-1}^{n-1}}(t)\big]^{\frac{1}{n}}. \end{aligned}$$

Denote $Q_m(t) = [F_{Au^1,Ax_m^1}(t),F_{Au^2,Ax_m^2}(t),\ldots,F_{Au^n,Ax_m^n}(t)]^{\frac{1}{n}}$. By (3.10), we have $Q_m(\varphi(t)) \geq Q_{m-1}(t)$, and hence for all $t > 0$

$$Q_m\big(\varphi^m(t)\big) \geq Q_{m-1}\big(\varphi^{m-1}(t)\big) \geq \cdots \geq Q_0(t).$$

Thus, for all $t > 0$, we have

$$\begin{aligned} &F_{Au^1,A_m^1}\big(\varphi^m(t)\big) \geq Q_0(t), \qquad F_{Au^2,A_m^2}\big(\varphi^m(t)\big) \geq Q_0(t), \qquad \ldots, \\ &F_{Au^n,A_m^n}\big(\varphi^m(t)\big) \geq Q_0(t). \end{aligned}$$

Since $Q_0(t) \in \mathscr{D}^+$ and $\lim_{m\to\infty}(\varphi^m(t)) = 0$ for all $t > 0$, by Lemma 2.1, we conclude that

$$\lim_{m\to\infty} Ax_m^1 = Au^1, \qquad \lim_{m\to\infty} Ax_m^1 = Au^1, \qquad \ldots, \qquad \lim_{m\to\infty} Ax_m^n = Au^n. \tag{3.11}$$

This shows that $Au^1 = u^1, Au^2 = u^2, \ldots, Au^n = u^n$. Hence $u^1 = T(u^1,u^2,\ldots,u^n), u^2 = T(u^2,u^3,\ldots,u^1),\ldots,u^n = T(u^n,u^1,\ldots,u^{n-1})$. Finally, we prove that $u^1 = u^2 = \cdots = u^n$.

$$\begin{aligned} F_{u^1,u^2}\big(\varphi(t)\big) &= F_{T(u^1,u^2,\ldots,u^{n-1},u^n),T(u^2,u^3,\ldots,u^n,u^1)}\big(\varphi(t)\big) \\ &\geq \big[F_{Au^1,Au^2}(t),F_{Au^2,Au^3}(t),\ldots,F_{Au^{n-1},Au^n}(t),F_{Au^n,Au^1}(t)\big]^{\frac{1}{n}} \\ &= \big[F_{u^1,u^2}(t),F_{u^2,u^3}(t),\ldots,F_{u^{n-1},u^n}(t),F_{u^n,u^1}(t)\big]^{\frac{1}{n}}, \\ F_{u^2,u^3}\big(\varphi(t)\big) &= F_{T(u^2,u^3,\ldots,u^n,u^1),T(u^3,u^4,\ldots,u^1,u^2)}\big(\varphi(t)\big) \\ &\geq \big[F_{Au^2,Au^3}(t),F_{Au^3,Au^3}(4),\ldots,F_{Au^n,Au^1}(t),F_{Au^1,Au^2}(t)\big]^{\frac{1}{n}} \\ &= \big[F_{u^1,u^2}(t),F_{u^2,u^3}(t),\ldots,F_{u^{n-1},u^n}(t),F_{u^n,u^1}(t)\big]^{\frac{1}{n}}, \\ &\vdots \\ F_{u^n,u^1}\big(\varphi(t)\big) &= F_{T(u^n,u^1,\ldots,u^{n-2},u^{n-1}),T(u^1,u^2,\ldots,u^{n-1},u^n)}\big(\varphi(t)\big) \\ &\geq \big[F_{Au^n,Au^1}(t),F_{Au^1,Au^2}(t),\ldots,F_{Au^{n-2},Au^{n-1}}(t),F_{Au^{n-1},Au^n}(t)\big]^{\frac{1}{n}} \\ &= \big[F_{u^1,u^2}(t),F_{u^2,u^3}(t),\ldots,F_{u^{n-1},u^n}(t),F_{u^n,u^1}(t)\big]^{\frac{1}{n}}. \end{aligned} \tag{3.12}$$

Denote $R(t) = [F_{u^1,u^2}(t),F_{u^2,u^3}(t),\ldots,F_{u^{n-1},u^n}(t),F_{u^n,u^1}(t)]^{\frac{1}{n}}$. From (3.12), we have

$$R\big(\varphi^m(t)\big) \geq R\big(\varphi^{m-1}(t)\big) \geq \cdots \geq R(t).$$

Since $R(t) \in \mathscr{D}^+$, by Lemma 2.1, we get $u^1 = u^2 = \cdots = u^n$. Hence, there exists $u \in X$, such that $u = Au = T(u,\ldots,u)$.

Finally, we show the uniqueness of the multidimensional common fixed point of T and A. Suppose that v is another the multidimensional common fixed point of T and A, *i.e.*, $v = Av = T(v,\dots,v)$. By (3.1), for all $t > 0$, we have

$$\begin{aligned} F_{u,v}\bigl(\varphi(t)\bigr) &= F_{T(u,u,\dots,u),T(v,v,\dots,v)}\bigl(\varphi(t)\bigr) \\ &\geq \bigl[F_{Au,Av}(t)F_{Au,Av}(t)\cdots F_{Au,Av}(t)\bigr]^{\frac{1}{n}} \\ &= F_{Au,Av}(t) = F_{u,v}(t), \end{aligned} \tag{3.13}$$

which implies that $F_{u,v}(\varphi^m(t)) \geq F_{u,v}(t)$ for all $t > 0$. Using Lemma 2.1, we have $F_{u,v}(t) = 1$ for all $t > 0$, *i.e.*, $u = v$. This completes the proof. □

Remark 3.1 If $n = 2$, Theorem 3.1 generalizes Theorem 2.2 in [24]. While $n = 3$, Theorem 3.1 generalizes Theorem 3.1 in [25].

From Theorem 3.1, we can obtain the following corollaries.

Corollary 3.1 *Let $(X, \mathscr{F}, \Delta)$ be a complete multidimensional Menger PM-space with Δ a continuous related t-norm of H-type, φ: $\mathbb{R}^+ \to \mathbb{R}^+$ be a gauge function such that $\varphi^{-1}(\{0\}) = \{0\}$, $\varphi(t) < t$, and $\lim_{m\to\infty} \varphi^m(t) = 0$ for any $t > 0$. Let T: $X^n \to X$ and A: $X \to X$ be two mappings satisfying the following conditions*:

$$F_{T(x_1,x_2,\dots,x_n),T(y_1,y_2,\dots,y_n)}\bigl(\varphi(t)\bigr) \geq \bigl[F_{Ax_1,Ay_1}(t)F_{Ax_2,Ay_2}(t)\cdots F_{Ax_n,Ay_n}(t)\bigr]^{\frac{1}{n}} \tag{3.14}$$

for all $x_1, x_2, \dots, x_n, y_1, y_2, \dots, y_n \in X$, and $t > 0$, where $T(X^n) \subset A(X)$, A is continuous and commutative with T. Then T and A have a unique multidimensional common fixed point in X.

If $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ be a gauge function such that $\lim_{m\to\infty} \sum_{m=1}^{\infty} \varphi^m(t) < \infty$ for any $t > 0$, we can obtain $\lim_{m\to\infty} \varphi^m(t) = 0$. Hence we have Corollary 3.2 as follows.

Corollary 3.2 *Let $(X, \mathscr{F}, \Delta)$ be a complete multidimensional Menger PM-space with Δ a continuous related t-norm of H-type, and $\Delta \geq \Delta_P$, φ: $\mathbb{R}^+ \to \mathbb{R}^+$ be a gauge function such that $\varphi^{-1}(\{0\}) = \{0\}$, $\varphi(t) < t$, and $\lim_{m\to\infty} \sum_{m=1}^{\infty} \varphi^m(t) < \infty$ for any $t > 0$. Let T: $X^n \to X$ and A: $X \to X$ be two mappings satisfying the following conditions*:

$$F_{T(x_1,x_2,\dots,x_n),T(y_1,y_2,\dots,y_n)}\bigl(\varphi(t)\bigr) \geq \bigl[\Delta\bigl(F_{Ax_1,Ay_1}(t), F_{Ax_2,Ay_2}(t),\dots, F_{Ax_n,Ay_n}(t)\bigr)\bigr]^{\frac{1}{n}} \tag{3.15}$$

for all $x_1, x_2, \dots, x_n, y_1, y_2, \dots, y_n \in X$, and $t > 0$, where $T(X^n) \subset A(X)$, A is continuous and commutative with T. Then T and A have a unique multidimensional common fixed point in X.

Let $A = I$ (I is the identity mapping) in Corollary 3.2, we can obtain the following corollary.

Corollary 3.3 *Let $(X, \mathscr{F}, \Delta)$ be a complete multidimensional Menger PM-space with Δ a continuous related t-norm of H-type, and $\Delta \geq \Delta_P$, φ: $\mathbb{R}^+ \to \mathbb{R}^+$ be a gauge function such* *that* $\varphi^{-1}(\{0\}) = \{0\}$, $\varphi(t) < t$, *and* $\lim_{m\to\infty}\sum_{m=1}^{\infty}\varphi^m(t) < \infty$ *for any* $t > 0$. *Let* $T\colon X^n \to X$ *be a mapping satisfying the following conditions*:

$$F_{T(x_1,x_2,\ldots,x_n),T(y_1,y_2,\ldots,y_n)}\bigl(\varphi(t)\bigr) \geq \bigl[\Delta\bigl(F_{x_1,y_1}(t), F_{x_2,y_2}(t), \ldots, F_{x_n,y_n}(t)\bigr)\bigr]^{\frac{1}{n}} \tag{3.16}$$

for all $x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n \in X$, *and* $t > 0$. *Then* T *has a unique multidimensional fixed point in* X.

Letting $\varphi(t) = \alpha t$ $(0 < \alpha < 1)$ in Corollary 3.2, we can obtain the following corollary.

Corollary 3.4 *Let* $(X, \mathscr{F}, \Delta)$ *be a complete multidimensional Menger PM-space with* Δ *a continuous related t-norm of H-type, and* $\Delta \geq \Delta_P$. *Let* $T: X^n \to X$ *and* $A: X \to X$ *be two mappings satisfying the following conditions*:

$$F_{T(x_1,x_2,\ldots,x_n),T(y_1,y_2,\ldots,y_n)}(\alpha t) \geq \bigl[\Delta\bigl(F_{Ax_1,Ay_1}(t), F_{Ax_2,Ay_2}(t), \ldots, F_{Ax_n,Ay_n}(t)\bigr)\bigr]^{\frac{1}{n}} \tag{3.17}$$

for all $x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n \in X$, *and* $t > 0$, *where* $T(X^n) \subset A(X)$, A *is continuous and commutative with* T. *Then* T *and* A *have a unique multidimensional common fixed point in* X.

From the proof of Theorem 3.1, we can similarly prove the following result.

Theorem 3.2 *Let* $(X, \mathscr{F}, \Delta)$ *be a complete multidimensional Menger PM-space with* Δ *a continuous related t-norm of H-type*, $\varphi\colon \mathbb{R}^+ \to \mathbb{R}^+$ *be a gauge function such that* $\varphi^{-1}(\{0\}) = \{0\}$, $\varphi(t) > t$, *and* $\lim_{m\to\infty}\varphi^m(t) = +\infty$ *for any* $t > 0$. *Let* $T: X^n \to X$ *and* $A: X \to X$ *be two mappings satisfying the following conditions*:

$$F_{T(x_1,x_2,\ldots,x_n),T(y_1,y_2,\ldots,y_n)}(t) \geq \min\bigl\{F_{Ax_1,Ay_1}\bigl(\varphi(t)\bigr), F_{Ax_2,Ay_2}\bigl(\varphi(t)\bigr), \ldots, F_{Ax_n,Ay_n}\bigl(\varphi(t)\bigr)\bigr\} \tag{3.18}$$

for all $x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n \in X$, *and* $t > 0$, *where* $T(X^n) \subset A(X)$ *and* A *is continuous and compatible with* T. *Then* T *and* A *have a unique multidimensional common fixed point in* X.

Remark 3.2 If $n = 2$, Theorem 3.2 generalizes Theorem 2.3 in [24]. While $n = 3$, Theorem 3.2 generalizes Theorem 3.2 in [25].

Letting $A = I$ (I is the identity mapping) in Theorem 3.2, we can obtain the following corollary.

Corollary 3.5 *Let* $(X, \mathscr{F}, \Delta)$ *be a complete multidimensional Menger PM-space with* Δ *a continuous related t-norm of H-type*, $\varphi\colon \mathbb{R}^+ \to \mathbb{R}^+$ *be a gauge function such that* $\varphi^{-1}(\{0\}) = \{0\}$, $\varphi(t) > t$, *and* $\lim_{m\to\infty}\varphi^m(t) = \infty$ *for any* $t > 0$. *Let* $T\colon X^n \to X$ *be a mapping satisfying the following conditions*:

$$F_{T(x_1,x_2,\ldots,x_n),T(y_1,y_2,\ldots,y_n)}(t) \geq \min\bigl\{F_{x_1,y_1}\bigl(\varphi(t)\bigr), F_{x_2,y_2}\bigl(\varphi(t)\bigr), \ldots, F_{x_n,y_n}\bigl(\varphi(t)\bigr)\bigr\} \tag{3.19}$$

for all $x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n \in X$, *and* $t > 0$. *Then* T *and* A *have a unique multidimensional common fixed point in* X.

Theorem 3.3 *Let (X,d) be a complete metric space, $\varphi\colon \mathbb{R}^+ \to \mathbb{R}^+$ be a gauge function such that $\varphi^{-1}(\{0\}) = \{0\}$, $\varphi(t) > t$, and $\lim_{m\to\infty} \varphi^m(t) = +\infty$ for any $t > 0$. Let $T\colon X^n \to X$ and $A\colon X \to X$ be two mappings satisfying the following conditions*:

$$\begin{aligned}&\varphi\bigl(d\bigl(T(x_1,x_2,\ldots,x_n),T(y_1,y_2,\ldots,y_n)\bigr)\bigr)\\&\quad\leq \max\bigl\{d(Ax_1,Ay_1),d(Ax_2,Ay_2),\ldots,d(Ax_n,Ay_n)\bigr\}\end{aligned}\tag{3.20}$$

for all $x_1,x_2,\ldots,x_n,y_1,y_2,\ldots,y_n \in X$, and $t > 0$, where $T(X^n) \subset A(X)$, A is continuous and compatible with T. Then T and A have a unique multidimensional common fixed point in X.

Proof Take $\Delta = \Delta_M$ and $F_{x,y}(t) = H(t - d(x,y))$. Then by Lemma 2.3 and Remark 2.2, $(X,\mathscr{F},\Delta_M)$ is a complete multidimensional Menger *PM*-space (or a Menger *PM*-space). From Lemma 2.4 and (3.20), we have

$$\begin{aligned}F_{T(x_1,x_2,\ldots,x_n),T(y_1,y_2,\ldots,y_n)}(t) &= H\bigl(t - d\bigl(T(x_1,x_2,\ldots,x_n),T(y_1,y_2,\ldots,y_n)\bigr)\bigr)\\&\geq H\bigl(\varphi(t) - \max\bigl\{d(Ax_1,Ay_1),d(Ax_2,Ay_2),\ldots,d(Ax_n,Ay_n)\bigr\}\bigr)\\&= \min\bigl\{H\bigl(\varphi(t) - d(Ax_1,Ay_1)\bigr),\ldots,H\bigl(\varphi(t) - d(Ax_n,Ay_n)\bigr)\bigr\}\\&= \min\bigl\{F_{Ax_1,Ay_1}\bigl(\varphi(t)\bigr),\ldots,F_{Ax_n,Ay_n}\bigl(\varphi(t)\bigr)\bigr\}.\end{aligned}\tag{3.21}$$

Hence the conclusion follows from Theorem 3.2. □

4 An application

In this section, we will provide an example to exemplify the validity of the main result of this paper.

Example 4.1 Suppose that $X \in [-1,1] \subset R$, $\Delta = \Delta_M$. Then Δ_M is a t-norm of H-type and $\Delta_M \geq \Delta_P$. Define $\mathscr{F}\colon X \times X \to \mathscr{D}$ by

$$\mathscr{F}_{x,y}(t) = F_{x,y}(t) = \begin{cases} e^{-\frac{|x-y|}{t}}, & t > 0, x,y \in X,\\ 0, & t \leq 0, x,y \in X.\end{cases}$$

We claim that $(X,\mathscr{F},\Delta_M)$ is a multidimensional Menger *PM*-space. In fact, it is easy to verify (MPM-1) and (MPM-2). Assume that for any $t_1,t_2,\ldots,t_n > 0$, and $x_1,x_2,\ldots,x_{n+1} \in X$,

$$\Delta_M\bigl(F_{x_1,x_2}(t_1),F_{x_2,x_3}(t_2),\ldots,F_{x_n,x_{n+1}}(t_n)\bigr) = \min\bigl\{e^{-\frac{|x_1-x_2|}{t_1}},e^{-\frac{|x_2-x_3|}{t_2}},e^{-\frac{|x_n-x_{n+1}|}{t_n}}\bigr\} = e^{-\frac{|x_1-x_2|}{t_1}}.$$

Then we have $t_1|x_2 - x_3| \leq t_2|x_1 - x_2|, t_1|x_3 - x_4| \leq t_3|x_1 - x_2|,\ldots,t_1|x_n - x_{n+1}| \leq t_n|x_1 - x_2|$, and so $\frac{t_1+t_2+\cdots+t_n}{t_1}|x_1 - x_2| \geq |x_1 - x_2| + |x_2 - x_3| + \cdots + |x_n - x_{n+1}| \geq |x_1 - x_{n+1}|$. It follows that

$$\begin{aligned}F_{x_1,x_{n+1}}(t_1 + t_2 + \cdots + t_n) &= e^{-\frac{|x_1-x_{n+1}|}{t_1+t_2+\cdots+t_n}} \geq e^{-\frac{|x_1-x_2|}{t_1}}\\&= \Delta_M\bigl(F_{x_1,x_2}(t_1),F_{x_2,x_3}(t_2),\ldots,F_{x_n,x_{n+1}}(t_n)\bigr).\end{aligned}$$

Hence (MPM-3) holds. It is obvious that $(X, \mathscr{F}, \Delta_M)$ is complete. Suppose that $\varphi(t) = \frac{t}{n}$, then it is easy to verify that $\varphi^{-1}(\{0\}) = \{0\}$, $\varphi(t) < t$, and $\lim_{m\to\infty} \sum_{m=1}^{\infty} \varphi^m(t) < \infty$ for any $t > 0$. For $x_1, x_2, \ldots, x_n \in X$, define $T: X^n \to X$ as follows:

$$T(x_1, x_2, \ldots, x_n) = \frac{1}{n^4} - \frac{x_1^2}{n^4} - \frac{x_2^2}{n^4} - \cdots - \frac{x_{n-1}^2}{n^4} - \frac{|x_n|}{n^3}.$$

Then, for each $t > 0$ and $x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n \in X$, we have

$$\begin{aligned} &\bigl|\bigl(x_1^2 - y_1^2\bigr) + \cdots + \bigl(x_{n-1}^2 - y_{n-1}^2\bigr) + n\bigl(|x_n| - |y_n|\bigr)\bigr| \\ &\quad \le |x_1 - y_1|\bigl(|x_1| + |y_1|\bigr) + \cdots + |x_{n-1} - y_{n-1}|\bigl(|x_{n-1}| + |y_{n-1}|\bigr) + n\bigl(|x_n| - |y_n|\bigr) \\ &\quad \le n^2 \max\bigl\{|x_1 - y_1|, \ldots, |x_n - y_n|\bigr\}, \end{aligned}$$

and so

$$\begin{aligned} F_{T(x_1,x_2,\ldots,x_{n-1},x_n),T(y_1,y_2,\ldots,y_{n-1},y_n)}\bigl(\varphi(t)\bigr) &= F_{T(x_1,x_2,\ldots,x_{n-1},x_n),T(y_1,y_2,\ldots,y_{n-1},y_n)}\left(\frac{t}{n}\right) \\ &= e^{-\frac{|(x_1^2-y_1^2)+\cdots+(x_{n-1}^2-y_{n-1}^2)+n(|x_n|-y_n)|}{n^3 t}} \\ &\ge \min\bigl\{e^{-\frac{|x_1-y_1|}{nt}}, e^{-\frac{|x_2-y_2|}{nt}}, \ldots, e^{-\frac{|x_n-y_n|}{nt}}\bigr\} \\ &= \bigl[\Delta_M\bigl(F_{x_1,y_1}(t), F_{x_2,y_2}(t), \ldots, F_{x_n,y_n}(t)\bigr)\bigr]^{\frac{1}{n}}. \end{aligned}$$

Thus, all conditions of Corollary 3.3 are satisfied. Therefore, T has a unique fixed point in X.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally. All authors read and approved the final manuscript.

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