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On approximate homomorphisms: a fixed point approach

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Abstract

Consider the functional equation $\Im_1(f) = \Im_2(f)$ $(\Im)$ in a certain general setting. A function g is an approximate solution of $(\Im)$ if $\Im_1(g)$ and $\Im_2(g)$ are close in some sense. The Ulam stability problem asks whether or not there is a true solution of $(\Im)$ near g. A functional equation is superstable if every approximate solution of the functional equation is an exact solution of it. In this paper, for each $m = 1, 2, 3, 4$, we will find out the general solution of the functional equation

$$f(ax+y)+f(ax-y) = a^{m-2}[f(x+y)+f(x-y)]+2(a^2-1)[a^{m-2}f(x)+\frac{(m-2)(1-(m-2)^2)}{6}f(y)]$$

for any fixed integer a with $a \neq 0, \pm1$.

Using a fixed point method, we prove the generalized Hyers-Ulam stability of homomorphisms in real Banach algebras for this functional equation. Moreover, we establish the superstability of this functional equation by suitable control functions.

Keywords: Banach algebra, Approximate homomorphism, Additive, Quadratic, Cubic and quartic functional equation, Fixed point approach

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Introduction

The problem of stability of functional equations originated from a question of Ulam [1] concerning the stability of group homomorphisms: let $(G_1, *)$ be a group and let $(G_2, \star, d)$ be a metric group with the metric $d(.,.)$. Given $\epsilon > 0$, does there exist a $\delta(\epsilon) > 0$ such that if a mapping $h : G_1 \to G_2$ satisfies the inequality $d(h(x * y), h(x) \star h(y)) < \delta$ for all $x, y \in G_1$, then there exists a homomorphism $H : G_1 \to G_2$ with $d(h(x), H(x)) < \epsilon$ for all $x \in G_1$? If the answer is affirmative, we would say that the equation of homomorphism $H(x * y) = H(x) \star H(y)$ is stable. The concept of stability for a functional equation arises when we replace the functional equation by an inequality which acts as a perturbation of the equation. Thus, the stability question of functional equations is that how do the solutions of the inequality differ from those of the given functional equation? Hyers [2] gave a first affirmative answer to the question of Ulam for Banach spaces. Let X and Y be Banach spaces. Assume that $f : X \to Y$ satisfies $\|f(x+y) - f(x) - f(y)\| \leq \epsilon$ for all $x, y \in X$ and some $\epsilon > 0$. Then, there exists a unique additive mapping $T : X \to Y$ such that $\|f(x) - T(x)\| \leq \epsilon$ for all $x \in X$. A generalized version of the theorem of Hyers for approximately additive mappings was given by Aoki [3] in 1950 (cf. also [4]). In 1978, a generalized solution for approximately linear mappings was given by Th.M. Rassias [5]. He considered a mapping $f : X \to Y$ satisfying the condition $\|f(x+y) - f(x) - f(y)\| \leq \epsilon(\|x\|^p + \|y\|^p)$ for all $x, y \in X$, where $\epsilon \geq 0$ and $0 \leq p < 1$. This result was later extended to all $p \neq 1$ and generalized by Gajda [6], Th.M. Rassias and Semrl [7], Isac and Th.M Rassias [8]. Lee and Jun [9] have improved the stability problem for approximately additive mappings. The problem when $p = 1$ is not true. Counterexamples for the corresponding assertion in the case $p = 1$ were constructed by Gadja [6], Th.M. Rassias and Semrl [7]. Furthermore, a generalization of the Th.M. Rassias' theorem was obtained by

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Găvruta [10], who replaced $\epsilon(\| x \|^p + \| y \|^p)$ by a general control function $\varphi(x, y)$. The functional equation

$$f(x+y) + f(x-y) = 2f(x) + 2f(y) \tag{1.1}$$

is related to a symmetric bi-additive mapping [11,12]. It is natural that this equation is called a *quadratic functional equation*. For more details about various results concerning such problems the reader is referred to [13-25]. Jun and Kim [26] introduced the following cubic functional equation

$$f(2x+y)+f(2x-y) = 2f(x+y)+2f(x-y)+12f(x) \tag{1.2}$$

and they established the general solution and the generalized Hyers-Ulam stability for the functional equation (1.2). Obviously, the function $f(x) = cx^3$ satisfies the functional equation (1.2), which is called a *cubic functional equation*. Lee et. al. [27] considered the following functional equation

$$f(2x+y)+f(2x-y) = 4f(x+y)+4f(x-y)+24f(x)-6f(y) \tag{1.3}$$

It is easy to see that the function $f(x) = dx^4$ is a solution of the functional equation (1.3), which is called a *quartic functional equation*.

Bourgin [4,28] is the first mathematician dealing with stability of (ring) homomorphism $f(xy) = f(x)f(y)$. The topic of approximate homomorphisms was studied by a number of mathematicians, see [29-36].

Let A be a ring. A mapping $f : A \to A$ is called a *quadratic homomorphism* if f is a quadratic mapping satisfying

$$f(xy) = f(x)f(y) \tag{1.4}$$

for all $x, y \in A$. For instance, let A be commutative. Then the mapping $f : A \to A$, defined by $f(x) = x^2$ $(x \in A)$, is a quadratic homomorphism. Eshaghi Gordji and Ghobadipour [37] investigated the generalized Hyers-Ulam stability of quadratic homomorphisms and of quadratic derivations on Banach algebras. In addition, the generalized Hyers-Ulam stability of cubic homomorphisms on Banach algebras has been investigated by Eshaghi Gordji and Bavand Savadkouhi [38].

Definition 1.1. Let A,B be two algebras,

(i) A mapping $f : A \to B$ is called an *additive homomorphism* (briefly, 1-homomorphism) if f is an additive mapping satisfying (1.4) for all $x, y \in A$;
(ii) A mapping $f : A \to B$ is called a *quadratic homomorphism* (briefly, 2-homomorphism) if f is a quadratic mapping satisfying (1.4) for all $x, y \in A$;
(iii) A mapping $f : A \to B$ is called a *cubic homomorphism* (briefly, 3-homomorphism) if f is a cubic mapping satisfying (1.4) for all $x, y \in A$;
(iiii) A mapping $f : A \to B$ is called a *quartic homomorphism* (briefly, 4-homomorphism) if f is a quartic mapping satisfying (1.4) for all $x, y \in A$.

Now we will state the following notion of fixed point theory. For the proof, refer to [39]. For an extensive theory of fixed point theorems and other nonlinear methods, the reader is referred to [40]. In 2003, Radu [41] proposed a new method for obtaining the existence of exact solutions and error estimations, based on the fixed point alternative (see also [42-44]).

Let (X, d) be a generalized metric space. An operator $T : X \to X$ satisfies a Lipschitz condition with Lipschitz constant L if there exists a constant $L \geq 0$ such that $d(Tx, Ty) \leq Ld(x, y)$ for all $x, y \in X$. If the Lipschitz constant L is less than 1, then the operator T is called a strictly contractive operator. Note that the distinction between the generalized metric and the usual metric is that the range of the former is permitted to include the infinity. We recall the following theorem by Margolis and Diaz.

Theorem 1.2. *(Cf.* [39,41]*.) Suppose that we are given a complete generalized metric space (Ω, d) and a strictly contractive mapping $T : \Omega \to \Omega$ with Lipschitz constant L. Then for each given $x \in \Omega$, either*

$$d(T^m x, T^{m+1}x) = \infty \text{ for all } m \geq 0,$$

or there exists a natural number m_0 such that

- $d(T^m x, T^{m+1}x) < \infty$ *for all* $m \geq m_0$;
- *the sequence $\{T^m x\}$ is convergent to a fixed point y^* of T;*
- *y^* is the unique fixed point of T in* $\Lambda = \{y \in \Omega : d(T^{m_0}x, y) < \infty\}$;
- $d(y, y^*) \leq \frac{1}{1-L} d(y, Ty)$ *for all* $y \in \Lambda$.

In this paper, we obtain general solution of the functional equation

$$\begin{aligned} f(ax+y) + f(ax-y) = {} & a^{m-2}[f(x+y) + f(x-y)] \\ & + 2(a^2-1)[\,a^{m-2}f(x) \\ & + \frac{(m-2)(1-(m-2)^2)}{6} f(y)] \end{aligned} \tag{1.5}$$

for all $x, y \in X$, $a \neq 0, \pm 1$ and for each $m = 1, 2, 3, 4$. Also, we investigate the generalized Hyers-Ulam stability of homomorphisms in Banach algebras via fixed point method for the functional equation (1.5). Moreover, we establish the superstability of the functional equation (1.5) by suitable control functions.

Solution of Eq. (1.5)

We here present the general solution of (1.5).

Theorem 2.1. *Let X, Y be real vector spaces, and let $f : X \to Y$ be a mapping satisfying (1.5). Then the following assertions hold:*

(a) *Eq. (1.5) with $m = 1$ is equivalent to the additive functional equation. So every solution of Eq. (1.5) with $m = 1$ is also an additive mapping (briefly, 1-function);*
(b) *Eq. (1.5) with $m = 2$ is equivalent to the functional quadratic equation. So every solution of Eq. (1.5) with $m = 2$ is also a quadratic mapping (briefly, 2-function);*
(c) *Eq. (1.5) with $m = 3$ is equivalent to the cubic functional equation. So every solution of Eq. (1.5) with $m = 3$ is also a cubic mapping (briefly, 3-function);*
(d) *Eq. (1.5) with $m = 4$ is equivalent to the quartic equation. So every solution of Eq. (1.5) with $m = 4$ is also a quartic mapping (briefly, 4-function).*

Proof. (a): Let $f : X \to Y$ satisfy the additive functional equation

$$f(x+y) = f(x) + f(y) \tag{2.1}$$

for all $x, y \in X$. Putting $x = y = 0$ in (2.1), we get $f(0) = 0$. Setting $y := -x$ in (2.1), we get $f(-x) = -f(x)$. Letting $y := x$ and $y := 2x$ in (2.1), respectively, we obtain that $f(2x) = 2f(x)$ and $f(3x) = 3f(x)$ for all $x, y \in X$. By induction we lead to $f(kx) = kf(x)$ for all positive integers k. Replacing $x := x + y$ and $y := x - y$ in (2.1), we have

$$f(x+y) + f(x-y) = 2f(x) \tag{2.2}$$

for all $x, y \in X$. Replacing x by ax in (2.2), we get

$$f(ax+y) + f(ax-y) = 2af(x)$$

for all $x, y \in X$. Multiplying the above equation by a, we obtain that

$$af(ax+y) + af(ax-y) = 2a^2 f(x)$$

for all $x, y \in X$. From (2.2) we have

$$f(x+y) + f(x-y) - 2f(x) = 0$$

for all $x, y \in X$. By the last two equations, we infer that

$$af(ax+y)+af(ax-y) = f(x+y)+f(x-y)+2(a^2-1)f(x)$$

for all $x, y \in X$. That is, f satisfy the functional equation (1.5) with $m = 1$.

On the other hand, let f satisfy (1.5) with $m = 1$. Letting $x = y = 0$ in (1.5), we get $f(0) = 0$. Putting $x = 0$, we see that f is odd. Setting $y = 0$ in (1.5), we get

$$f(ax) = af(x) \tag{2.3}$$

for all $x \in X$. Putting $y := x + ay$ in (1.5), we get

$$\begin{aligned} af(a(x+y)+x) + af(a(x-y)-x) &= f(2x+ay) + f(-ay) \\ &\quad + 2(a^2-1)f(x) \end{aligned} \tag{2.4}$$

for all $x, y \in X$. Letting $y := -y$ in (2.4), we obtain that

$$\begin{aligned} af(a(x-y)+x) + af(a(x+y)-x) &= f(2x-ay) + f(ay) \\ &\quad + 2(a^2-1)f(x) \end{aligned} \tag{2.5}$$

for all $x, y \in X$. Adding Eq. (2.4) to (2.5) and using the oddness of f, we see that

$$\begin{aligned} af(a(x+y)+x) &+ af(a(x+y)-x) + af(a(x-y)+x) \\ &+ af(a(x-y)-x) = f(2x+ay) + f(2x-ay) \\ &+ 4(a^2-1)f(x) \end{aligned} \tag{2.6}$$

for all $x, y \in X$. Replacing x and y by $x + y$ and x in (1.5), respectively, we obtain

$$\begin{aligned} af(a(x+y)+x) + af(a(x+y)-x) &= f(2x+y) + f(y) \\ &\quad + 2(a^2-1)f(x+y) \end{aligned} \tag{2.7}$$

for all $x, y \in X$. Replacing y by $-y$ in (2.7), we get

$$\begin{aligned} af(a(x-y)+x) + af(a(x-y)-x) &= f(2x-y) + f(-y) \\ &\quad + 2(a^2-1)f(x-y) \end{aligned} \tag{2.8}$$

for all $x, y \in X$. Adding Eq. (2.7) to (2.8), and using the oddness of f, we obtain that

$$\begin{aligned} af(a(x+y)+x) &+ af(a(x+y)-x) + af(a(x-y)+x) \\ &+ af(a(x-y)-x) = f(2x+y) + f(2x-y) \\ &+ 2(a^2-1)[f(x+y) + f(x-y)] \end{aligned} \tag{2.9}$$

for all $x, y \in X$. By (2.6) and (2.9), we have

$$\begin{aligned} f(2x+ay) + f(2x-ay) + 4(a^2-1)f(x) &= f(2x+y) \\ + f(2x-y) + 2(a^2-1)[f(x+y) + f(x-y)] \end{aligned} \tag{2.10}$$

for all $x, y \in X$. Replacing x and y by $2x$ and ay in (1.5), respectively, and using (2.3), we see that

$$\begin{aligned} f(2x+ay) + f(2x-ay) &= a^2 f(2x+y) + a^2 f(2x-y) \\ &\quad + 2(1-a^2)f(x) \end{aligned} \tag{2.11}$$

for all $x, y \in X$. By (2.10) and (2.11), we have

$$f(2x+y)+f(2x-y) = 2f(x+y)+2f(x-y)+2f(2x)-4f(x) \tag{2.12}$$

for all $x, y \in X$. Replacing x and y by $x+y$ and $x-y$ in (2.12), respectively, we obtain that

$$f(3x+y)+f(x+3y) = 2f(2x)+2f(2y)+2f(2x+2y) - 4f(x+y) \tag{2.13}$$

for all $x, y \in X$. Replacing y by $x+y$ in (2.12) and using the oddness of f, we obtain that

$$f(3x+y)+f(x-y) = 2f(2x+y)+2f(2x)-2f(y)-4f(x) \tag{2.14}$$

for all $x, y \in X$. By (2.14), we get the relation

$$f(x+3y)-f(x-y) = 2f(x+2y)-2f(x)+2f(2y)-4f(y) \tag{2.15}$$

for all $x, y \in X$. Combining (2.14) with (2.15) and using (2.13), one gets

$$f(2x+2y) = f(2x+y)+f(x+2y)+2f(x+y)-3f(x)-3f(y) \tag{2.16}$$

for all $x, y \in X$. Replacing y by $-y$ in (2.16) and then adding the result to (2.16), we obtain

$$f(2x+2y)+f(2x-2y) = f(2x+y)+f(2x-y)+f(x+2y) + f(x-2y)+2f(x+y)+2f(x-y)-6f(x) \tag{2.17}$$

for all $x, y \in X$. In turn, substituting $2y$ for y in (2.12), we obtain

$$f(2x+2y)+f(2x-2y) = 2f(x+2y)+2f(x-2y)+2f(x)-4f(x) \tag{2.18}$$

for all $x, y \in X$. It follows from (2.12), (2.17) and (2.18) that

$$f(x+2y)+f(x-2y) = 4f(x+y)+4f(x-y)-6f(x) \tag{2.19}$$

for all $x, y \in X$. Letting $y = x$ in (2.12) and using $f(0) = 0$, we get $f(3x) = 4f(2x) - 5f(x)$ for all $x \in X$. Setting $y = 2x$ in (2.12) and using the oddness of f, we get $f(4x) = 10f(2x) - 16f(x)$ for all $x \in X$. By induction, we get the relation

$$f(kx) = \frac{k(k^2-1)}{6}f(2x) + \frac{k(4-k^2)}{3}f(x) \tag{2.20}$$

for all $x \in X$ and each positive integer k. By using (2.20) for $k = a$ and (2.3), we obtain $f(2x) = 2f(x)$ for all $x \in X$. Replacing x by $2x$ in (2.19) and using $f(2x) = 2f(x)$, we get

$$2f(2x+y)+2f(2x-y) = f(x+y)+f(x-y)+6f(x) \tag{2.21}$$

for all $x, y \in X$.

The rest of the proof is similar to Theorem 2.1 of [45].

For $m = 2, 4$, Lee and Chung [46,47] showed that Eq. (1.5) is equivalent to the quadratic functional equation and the quartic functional equation, respectively. Moreover, Najati [48] solved the solution of (1.5) for $m = 3$. □

Approximation of homomorphisms in Banach algebras

In this section, we prove the generalized Hyers-Ulam stability of homomorphisms in real Banach algebras for the functional equation (1.5).

Throughout this section we suppose that X is a normed algebra, and Y is a Banach algebra. For convenience, we use the following abbreviation for a given mapping $f : X \to Y$:

$$\Delta_m f(x,y) = f(ax+y)+f(ax-y)-a^{m-2}[f(x+y)+f(x-y)] - 2(a^2-1)[a^{m-2}f(x)+\frac{(m-2)(1-(m-2)^2)}{6}f(y)]$$

for all $x, y \in X$ and any fixed integer $a \neq 0, \pm 1$.

From now on, let m be a positive integer less than 5.

Theorem 3.1. *Let $f : X \to Y$ be a mapping for which there exist functions $\varphi_m, \psi_m : X \times X \to [0, \infty)$ such that*

$$\|\Delta_m f(x,y)\| \leq \varphi_m(x,y), \tag{3.1}$$

$$\|f(xy) - f(x)f(y)\| \leq \psi_m(x,y) \tag{3.2}$$

for all $x, y \in X$. If there exists a constant $0 < L < 1$ such that

$$\varphi_m\left(\frac{x}{a}, \frac{y}{a}\right) \leq \frac{L}{a^m}\varphi_m(x,y), \tag{3.3}$$

$$\psi_m\left(\frac{x}{a}, \frac{y}{a}\right) \leq \frac{L}{a^{2m}}\psi_m(x,y) \tag{3.4}$$

for all $x, y \in X$, then there exists a unique m-homomorphism $H : X \to Y$ such that

$$\|f(x) - H(x)\| \leq \frac{L}{2a^m(1-L)}\varphi_m(x,0), \tag{3.5}$$

$$H(x)[H(y) - f(y)] = [H(x) - f(x)]H(y) = 0 \tag{3.6}$$

for all $x, y \in X$.

Proof. It follows from (3.3) and (3.4) that

$$\lim_{n\to\infty} a^{mn}\varphi_m\left(\frac{x}{a^n}, \frac{y}{a^n}\right) = 0, \tag{3.7}$$

$$\lim_{n\to\infty} a^{2mn}\psi_m\left(\frac{x}{a^n}, \frac{y}{a^n}\right) = 0 \tag{3.8}$$

for all $x, y \in X$. By (3.7), $\lim_{n\to\infty} a^{mn}\varphi_m(0,0) = 0$. Hence $\varphi_m(0,0) = 0$. Letting $x = y = 0$ in (3.1), we get $f(0) \leq \varphi_m(0,0) = 0$. So $f(0) = 0$.

Let $\Omega = \{g |\ g : X \to Y,\ g(0) = 0\}$. We introduce a generalized metric on Ω as follows:

$$d(g,h) = d_{\varphi_m}(g,h) = \inf\{K \in (0,\infty) :\ \|g(x) - h(x)\| \leq K\varphi_m(x,0),\ x \in X\}$$

It is easy to show that (Ω, d) is a complete generalized metric space [43].

Now we consider the mapping $T : \Omega \to \Omega$ defined by $Tg(x) = a^m\, g(\frac{x}{a})$ for all $x \in X$ and all $g \in \Omega$. Note that for all $g, h \in \Omega$,

$$\begin{aligned} d(g,h) < K \;\Rightarrow\; & \|g(x) - h(x)\| \\ & \le K\varphi_m(x,0), \qquad \textit{for all } x \in X, \\ \Rightarrow\; & \|a^m g\left(\frac{x}{a}\right) - a^m h\left(\frac{x}{a}\right)\| \\ & \le a^m\, K\, \varphi_m\left(\frac{x}{a}, 0\right), \qquad \textit{for all } x \in X, \\ \Rightarrow\; & \|a^m g\left(\frac{x}{a}\right) - a^m h\left(\frac{x}{a}\right)\| \\ & \le L\, K\, \varphi_m(x,0), \qquad \textit{for all } x \in X, \\ \Rightarrow\; & d(Tg, Th) \le L\, K. \end{aligned}$$

Hence we see that

$$d(Tg, Th) \le L\, d(g,h)$$

for all $g, h \in \Omega$, that is, T is a strictly self-function of Ω with the Lipschitz constant L.

Putting $y = 0$ in (3.1), we have

$$\|2f(ax) - 2a^m f(x)\| \le \varphi_m(x,0) \tag{3.9}$$

for all $x \in X$. So

$$\|f(x) - a^m f\left(\frac{x}{a}\right)\| \le \frac{1}{2}\varphi_m\left(\frac{x}{a}, 0\right) \le \frac{L}{2a^m}\, \varphi_m(x,0)$$

for all $x \in X$, that is, $d(f, Tf) \le \frac{L}{2a^m} < \infty$.

Now, from the fixed point alternative, it follows that there exists a fixed point H of T in Ω such that

$$H(x) = \lim_{n\to\infty} a^{mn} f\left(\frac{x}{a^n}\right) \tag{3.10}$$

for all $x \in X$, since $\lim_{n\to\infty} d(T^n f, H) = 0$.

On the other hand it follows from (3.1), (3.7) and (3.10) that

$$\begin{aligned} \|\Delta_m H(x,y)\| &= \lim_{n\to\infty} a^{mn} \|\Delta_m f\left(\frac{x}{a^n}, \frac{y}{a^n}\right)\| \\ &\le \lim_{n\to\infty} a^{mn} \varphi_m\left(\frac{x}{a^n}, \frac{y}{a^n}\right) = 0 \end{aligned}$$

for all $x, y \in X$. So $\Delta_m H(x,y) = 0$. By Theorem 2.1, H is an m-function. So it follows from the definition of H, (3.2) and (3.8) that

$$\begin{aligned} \|H(xy) - H(x)H(y)\| &= \lim_{n\to\infty} a^{2mn} \|f\left(\frac{xy}{a^{2n}}\right) - f\left(\frac{x}{a^n}\right) f\left(\frac{y}{a^n}\right)\| \\ &\le \lim_{n\to\infty} a^{2mn} \psi_m\left(\frac{x}{a^n}, \frac{y}{a^n}\right) = 0 \end{aligned}$$

for all $x, y \in X$. So $H(xy) = H(x)H(y)$. Similarly, we have from (3.2) and (3.8) that

$$H(xy) = H(x) f(y), \quad H(xy) = f(x) H(y) \tag{3.11}$$

for all $x, y \in X$. Since $H(xy) = H(x)H(y)$, we get (3.6) from (3.11).

According to the fixed point alterative, since H is the unique fixed point of T in the set $\Lambda = \{g \in \Omega : d(f,g) < \infty\}$, H is the unique function such that

$$\|f(x) - H(x)\| \le K\, \varphi_m(x,0)$$

for all $x \in X$ and $K > 0$. Again using the fixed point alterative, gives

$$d(f,H) \le \frac{L}{1-L} d(f, Tf) \le \frac{L}{2a^m(1-L)}$$

so we conclude that

$$\|f(x) - H(x)\| \le \frac{L}{2a^m(1-L)}\, \varphi_m(x,0)$$

for all $x \in X$. This completes the proof. □

Corollary 3.2. *Let θ, r, s be non-negative real numbers with $r > m$ and $s > 2m$. Suppose that $f : X \to Y$ is a mapping such that*

$$\|\Delta_m f(x,y)\| \le \theta(\|x\|^r + \|y\|^r),$$

$$\|f(xy) - f(x)f(y)\| \le \theta(\|x\|^s + \|y\|^s)$$

for all $x, y \in X$. Then there exists a unique m-homomorphism $H : X \to Y$ satisfying

$$\|f(x) - H(x)\| \le \frac{\theta}{2(a^r - a^m)} \|x\|^r,$$

$$H(x)[H(y) - f(y)] = [H(x) - f(x)]\, H(y) = 0$$

for all $x, y \in X$.

Proof. The proof follows from Theorem 3.1 by taking

$$\varphi_m(x,y) := \theta(\|x\|^r + \|y\|^r),\; \psi_m(x,y) := \theta(\|x\|^s + \|y\|^s)$$

for all $x \in X$. Then we can choose $L = a^{m-r}$ and we get the desired results. □

Remark 3.3. Let $f : X \to Y$ be a mapping with $f(0) = 0$ for which there exist functions $\varphi_m, \psi_m : X \times X \to [0, \infty)$ satisfying (3.1) and (3.2). Let $0 < L < 1$ be a constant such that $\varphi_m(ax, ay) \le a^m L \varphi_m(x,y)$ and $\psi_m(ax, ay) \le a^{2m} L \psi_m(x,y)$ for all $x, y \in X$. By a similar method to the proof of Theorem 3.1, one can show that there exists a unique m-homomorphism $H : X \to Y$ satisfying (3.6) and

$$\|f(x) - H(x)\| \le \frac{1}{2a^m(1-L)} \varphi_m(x,0)$$

for all $x \in X$.

For the case $\varphi_m(x,y) := \delta + \theta(\|x\|^r + \|y\|^r)$ and $\psi_m(x,y) := \delta + \theta(\|x\|^s + \|y\|^s)$ (where θ, δ are non-negative real numbers and $0 < r, s < m$), there exists a unique m-homomorphism $H : X \to Y$ satisfying

$$\|f(x) - H(x)\| \le \frac{\delta}{2(a^m - a^r)} + \frac{\theta}{2(a^m - a^r)} \|x\|^r$$

for all $x \in X$.

Next, we formulate and prove a theorem in superstability of m-homomorphisms for the functional equation (1.5).

Theorem 3.4. *Suppose there exist functions $\varphi_m, \psi_m : X \times X \to [0, \infty)$ such that*

$$\lim_{n\to\infty} a^{mn}\varphi_m\left(0, \frac{y}{a^n}\right) = 0, \tag{3.12}$$

$$\lim_{n\to\infty} a^{2mn}\psi_m\left(\frac{x}{a^n}, \frac{y}{a^n}\right) = 0 \tag{3.13}$$

for all $x, y \in X$. Moreover, assume that $f : X \to Y$ is a mapping such that

$$\|\Delta_m f(x,y)\| \le \varphi_m(0,y), \tag{3.14}$$

$$\|f(xy) - f(x)f(y)\| \le \psi_m(x,y) \tag{3.15}$$

for all $x, y \in X$. Then f is an m-homomorphism.

Proof. $f(0) = 0$, since $\varphi_m(0,0) = 0$. Letting $y = 0$ in (3.14), we get $f(ax) = a^m f(x)$ for all $x \in X$. By using induction we obtain that

$$f(a^n x) = a^{mn} f(x)$$

for all $x \in X$ and $n \in \mathbb{N}$. So

$$f(x) = a^{mn} f\left(\frac{x}{a^n}\right) \tag{3.16}$$

for all $x \in X$ and $n \in \mathbb{N}$. It follows from (3.15) and (3.16) that

$$\begin{aligned}\|f(xy) - f(x)f(y)\| &= a^{2mn}\|f\left(\frac{xy}{a^{2n}}\right) - f\left(\frac{x}{a^n}\right)f\left(\frac{y}{a^n}\right)\| \\ &\le a^{2mn}\psi_m\left(\frac{x}{a^n}, \frac{y}{a^n}\right)\end{aligned} \tag{3.17}$$

for all $x, y \in X$ and $n \in \mathbb{N}$. Hence, by $n \to \infty$ in (3.17) and using (3.13), we have $f(xy) = f(x)f(y)$ for all $x, y \in X$. On the other hand, we have

$$\|\Delta_m f(x,y)\| = a^{mn}\|\Delta_m f\left(\frac{x}{a^n}, \frac{y}{a^n}\right)\| \le a^{mn}\varphi_m\left(0, \frac{y}{a^n}\right) \tag{3.18}$$

for all $x, y \in X$ and $n \in \mathbb{N}$. So, by $n \to \infty$ in (3.18) and using (3.12), we have $\Delta_m f(x,y) = 0$ for all $x, y \in X$.

Therefore, f is an m-homomorphism. □

Corollary 3.5. *Let θ, r, s be non-negative real numbers with $r > m$ and $s > 2m$. Suppose that $f : X \to Y$ is a mapping such that*

$$\|\Delta_m f(x,y)\| \le \theta\|y\|^r, \quad \|f(xy) - f(x)f(y)\| \le \theta(\|x\|^s + \|y\|^s)$$

for all $x, y \in X$. Then f is an m-homomorphism.

Remark 3.6. Let θ, r be non-negative real numbers with $r < m$. Suppose there exists a function $\psi_m : X \times X \to [0, \infty)$ such that

$$\lim_{n\to\infty} \frac{1}{a^{2mn}}\psi_m(a^n x, a^n y) = 0$$

for all $x, y \in X$. Moreover, assume that $f : X \to Y$ is a mapping such that

$$\|\Delta_m f(x,y)\| \le \theta\|y\|^r, \quad \|f(xy) - f(x)f(y)\| \le \psi_m(x,y)$$

for all $x, y \in X$. Then f is an m-homomorphism.

For the case $\psi_m(x,y) := \theta(\|x\|^s + \|y\|^s)$ (where θ is a non-negative real number and $0 < s < 2m$), f is an m-homomorphism.

Example 3.7. *Let $X = \mathbb{R}^{10}$. We define*

$$\begin{aligned}(a_1, a_2, \ldots, a_{10})(b_1, b_2, \ldots, b_{10}) &:= (0, a_1b_5, a_1b_6 + a_2b_8, a_1b_7 \\ &+ a_2b_9 + a_3b_{10}, 0, a_5b_8, a_5b_9 + a_6b_{10}, 0, a_8b_{10}, 0)\end{aligned}$$

for all $a_1, \ldots, a_{10}, b_1, \ldots, b_{10} \in \mathbb{R}$ and

$$\|(a_1, a_2, \ldots, a_{10})\| := \sum_{i=1}^{10} |a_i| \quad (a_i \in \mathbb{R}).$$

Then X is a Banach algebra. Let

$$b := (0, 1, 1, 1, 0, 1, 1, 0, 0, 0)$$

be fixed, and we define $f : X \to X$ by $f(x) = x^4 + b$, and

$$\varphi_4(x,y) := \|\Delta_4 f(x,y)\| = 2a^2(a^2-1)\|b\| = 10a^2(a^2-1),$$

$$\psi_4(x,y) := \|f(xy) - f(x)f(y)\| = \|b\| = 5.$$

Then we have

$$\sum_{i=0}^{\infty} \frac{1}{a^{4i}}\varphi_4(a^i x, a^i y) = \sum_{i=0}^{\infty} \frac{10a^2(a^2-1)}{a^{4i}} = \frac{10a^6}{a^2+1},$$

$$\lim_{n\to\infty} \frac{1}{a^{8n}}\psi_4(a^n x, a^n y) = 0.$$

Also,

$$H(x) = \lim_{n\to\infty} \frac{1}{a^{4n}} f(a^n x) = \lim_{n\to\infty}\left(x^4 + \frac{b}{a^{4n}}\right) = x^4.$$

So

$$H(xy) = (xy)^4 = x^4y^4 = H(x)H(y)$$

for all $x, y \in X$. Furthermore $\Delta_4 H(x,y) = 0$ for all $x, y \in X$. Thus, H is a 4-homomorphism.

Example 3.8. *Let $X = \mathbb{R}^6$. We define*

$$(a_1, a_2, \ldots, a_6)(b_1, b_2, \ldots, b_6) := (0, a_1b_4, a_1b_5 + a_2b_6, 0, a_4b_6, 0)$$

for all $a_1, \ldots, a_6, b_1, \ldots, b_6 \in \mathbb{R}$ and

$$\|(a_1, a_2, \ldots, a_6)\| := \sum_{i=1}^{6} |a_i| \ (a_i \in \mathbb{R}).$$

Then X is a Banach algebra. Let

$$b := (0, 1, 2, 0, 1, 0)$$

be fixed, and we define $f : X \to X$ *by* $f(x) = x^3 + b$, *and*

$$\varphi_3(x, y) := \|\Delta_3 f(x, y)\| = 2|a^3 - 1|\|b\| = 8|a^3 - 1|,$$

$$\psi_3(x, y) := \|f(xy) - f(x)f(y)\| = \|b\| = 4.$$

Then we have

$$\sum_{i=0}^{\infty} \frac{1}{a^{3i}} \varphi_3(a^i x, a^i y) = \sum_{i=0}^{\infty} \frac{8|a^3 - 1|}{a^{3i}} = 8|a|^3,$$

$$\lim_{n\to\infty} \frac{1}{a^{6n}} \psi_3(a^n x, a^n y) = 0.$$

Also,

$$H(x) = \lim_{n\to\infty} \frac{1}{a^{3n}} f(a^n x) = \lim_{n\to\infty} \left(x^3 + \frac{b}{a^{3n}} \right) = x^3.$$

So

$$H(xy) = (xy)^3 = x^3 y^3 = H(x)H(y)$$

for all $x, y \in X$. *Furthermore* $\Delta_3 H(x, y) = 0$ *for all* $x, y \in X$. *Thus, H is a 3-homomorphism.*

One can obtain two similar examples to Examples 3.7 and 3.8 for 2-homomorphism and 1-homomorphism. Also from these examples, it is clear that the superstability of the system of functional equations

$$f(ax+y)+f(ax-y) = a^{m-2}[f(x+y)+f(x-y)] + 2(a^2-1) \times [a^{m-2} f(x) + \frac{(m-2)(1-(m-2)^2)}{6} f(y)],$$

$$f(xy) = f(x)f(y),$$

with the control functions in Remark 3.6 does not hold.

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References

1. Ulam, SM: Problems in Modern Mathematics, Chapter VI , science Editions. Wiley, New York (1964)
2. Hyers, DH: On the stability of the linear functional equation. Proc. Nat. Acad. Sci. USA. **27**, 222–224 (1941)
3. Aoki, T: On the stability of the linear transformation in Banach spaces. J. Math. Soc. Japan. **2**, 64–66 (1950)
4. Bourgin, DG: Classes of transformations and bordering transformations. Bull. Amer. Math. Soc. **57**, 223–237 (1951)
5. Rassias, ThM: On the stability of the linear mapping in Banach spaces. Proc. Amer. Math. Soc. **72**, 297–300 (1978)
6. Gajda, Z: On stability of additive mappings. Int. J. Math. Math. Sci. **14**, 431–434 (1991)
7. Rassias, ThM, Semrl, P: On the behavior of mappings which do not satisfy Hyers-Ulam stability. Proc. Amer. Math. Soc. **114**, 989–993 (1992)
8. Isac, G, Rassias, ThM, On the Hyers-Ulam stability of ψ-additive mappings. J. Approx. Theory. **72**, 131–137 (1993)
9. Lee, Y, Jun, K: On the stability of approximately additive mappings. Proc. Amer. Math. Soc. **128**, 1361–1369 (2000)
10. Gávruta, P: A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings. J. Math. Anal. Appl. **184**, 431–436 (1994)
11. Aczel, J, Dhombres, J: Functional Equations in Several Variables. Cambridge University Press, Cambridge (1989)
12. Kannappan, P: Quadratic functional equation and inner product spaces. Results Math. **27**, 368–372 (1995)
13. Czerwik, S: On the stability of the quadratic mapping in normed spaces. Abh. Math. Sem. Univ. Hamburg. **62**, 59–64 (1992)
14. Eshaghi, Gordji, M, Khodaei, H: Solution and stability of generalized mixed type cubic, quadratic and additive functional equation in quasi-Banach spaces. Nonlinear Analysis.–TMA. **71**, 5629–5643 (2009)
15. Eshaghi Gordji, M, Khodaei, H: On the generalized Hyers-Ulam-Rassias stability of quadratic functional equations. Abstract and Appl. Anal. **2009**, 11 (2009). Article ID 923476
16. Forti, GL: An existence and stability theorem for a class of functional equations. Stochastica. **4**, 23–30 (1980)
17. Forti, GL: Elementary remarks on Ulam-Hyers stability of linear functional equations. J. Math. Anal. Appl. **328**, 109–118 (2007)
18. Jung, S: Hyers-Ulam-Rassias stability of Jensen's equation and its application. Proc. Amer. Math. Soc. **126**, 3137–3143 (1998)
19. Jung, S: Hyers-Ulam-Rassias Stability of Functional Equations in Mathematical Analysis. Hadronic Press Inc., Palm Harbor, Florida (2001)
20. Khodaei, H, Rassias, ThM: Approximately generalized additive functions in several variables. Int. J. Nonlinear Anal. Appl. **1**, 22–41 (2010)
21. Park, C: On an approximate automorphism on a C^*-algebra. Proc. Amer. Math. Soc. **132**, 1739–1745 (2004)
22. Rassias, JM: On Approximation of approximately linear mappings by linear mappings. J. Funct. Anal. **46**, 126–130 (1982)
23. Rassias, JM: Solution of a problem of Ulam. J. Approx. Theory. **57**, 268–273 (1989)
24. Rassias, ThM: New characterization of inner product spaces. Bull. Sci. Math. **108**, 95–99 (1984)
25. Rassias, ThM: On the stability of functional equations and a problem of Ulam. Acta Appl. Math. **62**, 23–130 (2000)
26. Jun, K, Kim, H: The generalized Hyers-Ulam-Rassias stability of a cubic functional equation. J. Math. Anal. Appl. **274**, 867–878 (2002)
27. Lee, S, Im, S, Hawng, I: Quartic functional equation. J. Math. Anal. Appl. **307**, 387–394 (2005)
28. Bourgin, DG: Approximately isometric and multiplicative transformations on continuous function rings. Duke Math. J. **16**, 385–397 (1949)
29. Badora, R: On approximate ring homomorphisms. J. Math. Anal. Appl. **276**, 589–597 (2002)
30. Baker, J, Lawrence, J, Zorzitto, F: The stability of the equation $f(x+y) = f(x)f(y)$. Proc. Amer. Math. Soc. **74**, 242–246 (1979)
31. Eshaghi Gordji, M, Karimi, T, Kaboli Gharetapeh, S: Approximately n-Jordan homomorphisms on Banach algebras. J. Ineq. Appl. Appl. **8** (2009). Article ID 870843
32. Eshaghi Gordji, M, Najati, A: Approximately J^*-homomorphisms: A fixed point approach. J. Geometry and Phys. **60**, 809–814 (2010)
33. Gordji, ME, Savadkouhi, MB: Approximation of generalized homomorphisms in quasi–Banach algebras. Analele Univ. Ovidius Constata, Math Ser. **17**(2), 203–214 (2009)
34. Park, C, Jang, S-Y: Cauchy–Rassias stability of sesquilinear n-quadratic mappings in Banach modules. Rocky Mountain J. Math. **39**(6), 2015–2027 (2009)
35. Park, C: Homomorphisms between Lie JC^*-algebras and Cauchy-Rassias stability of Lie JC^*-algebra derivations. J. Lie Theory. **15**, 393–414 (2005)
36. Park, C: Lie $*$-homomorphisms between Lie C^*-algebras and Lie $*$-derivations on Lie C^*-algebras. J. Math. Anal. Appl. **293**, 419–434 (2004)

37. Gordgi, ME, Ghobadipour, N: Hyers-Ulam-Aoki-Rassias Stability and Ulam-Găvruta-Rassias Stability of the Quadratic Homomorphisms and Quadratic Derivations on Banach Algebras. Nova Science Publishers Inc (2010)
38. Gordgi, ME, Bavand Savadkouhi, M: On approximate cubic homomorphisms. Adv. Difference Equations. **2009**, 11 (2009). Article ID 618463
39. Margolis, B, Diaz, JB: A fixed point theorem of the alternative for contractions on the generalized complete metric space. Bull. Amer. Math. Soc. **126**, 305–309 (1968)
40. Hyers, DH, Isac, G, Rassias, ThM: Stability of Functional Equations in Several Variables (1998)
41. Radu, V: The fixed point alternative and the stability of functional equations. Fixed Point Theory. **4**, 91–96 (2003)
42. Cădariu, L, Radu, V: Fixed points and the stability of Jensen functional equation. J. Ineq. Pure Appl. Math. **4**, 7 (2003). Article 4
43. Cădariu, L, Radu, V: On the stability of the Cauchy functional equation: a fixed point approach. Grazer Math. Ber. **346**, 43–52 (2004)
44. Că, dariu, L, Radu, V: Fixed point methods for the generalized stability of functional equations in a single variable. Fixed Point Theory and App. **2008**, 15 (2008). Article ID 749392
45. Bae, J, Park, W: A functional equation having monomials as solutions. Appl. Mathematics and Comput. **216**, 87–94 (2010)
46. Lee, Y, Chung, S: Stability for quadratic functional equation in the spaces of generalized functions. J. Math. Anal. Appl. **336**, 101–110 (2007)
47. Lee, Y, Chung, S: Stability of quartic functional equation in the spaces of generalized functions. Adv. Difference Equations. **2009**, 16 (2009). Article ID 838347
48. Najati, A: The generalized Hyers-Ulam-Rassias stability of a cubic functional equation. Turk. J. Math. **31**, 395–408 (2007)

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