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Fixed point theorems for w -cone distance contraction mappings in tvs-cone metric spaces

Ljubomir Ćirić^{1*}, Hossein Lakzian² and Vladimir Rakočević³* Correspondence: lciric@rcub.bg.ac.rs¹Faculty of Mechanical Engineering, Kraljice Marije 16, Belgrade, Serbia
Full list of author information is available at the end of the article

Abstract

In this article, we introduce the concept of a w -cone distance on topological vector space (tvs)-cone metric spaces and prove various fixed point theorems for w -cone distance contraction mappings in tvs-cone metric spaces. The techniques of the proof of our theorems are more complex than in the corresponding previously published articles, since a new technique was necessary for the considered class of mappings. Presented fixed point theorems generalize results of Suzuki and Takahashi, Abbas and Rhoades, Pathak and Shahzad, Raja and Veazpour, Hicks and Rhoades and several other results existing in the literature.

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1 Introduction and preliminaries

There exist many generalizations of the concept of metric spaces in the literature. Fixed point theory in abstract metric or K -metric spaces was developed in the middle of 70th years of twentieth century. Huang and Zhang [1] re-introduced and studied the concept of cone metric spaces over a Banach space, and proved several fixed point theorems. Then, there have been a lot of articles in which known fixed point theorems in metric are extended to cone metric spaces. Recently, Du [2] used the scalarization function and investigated the equivalence of vectorial versions of fixed point theorems in K -metric spaces and scalar versions of fixed point theorems in metric spaces. He showed that many of the fixed point theorems for mappings satisfying contractive conditions of a linear type in K -metric spaces can be considered as corollaries of corresponding theorems in metric spaces. Nevertheless, the fixed point theory in K -metric spaces proceeds to be actual, since the method of scalarization function cannot be applied for a wide class of weakly contractive mapping, satisfying nonlinear contractive conditions.

Kada et al. [3] introduced and studied the concept of w -distance on a metric space. They give examples of w -distance and improved Caristi's fixed point theorem, Ekeland's ε -variational's principle and the nonconvex minimization theorem according to Takahashi (see many useful examples and results on w -distance in [4-8] and in references there in).

Definition 1. [3]. Let X be a metric space with metric d . Then a function $p : X \times X \rightarrow [0, \infty)$ is called a w -distance on X if the following are satisfied:

- (1) $p(x, z) \leq p(x, y) + p(y, z)$, for any $x, y, z \in X$,
- (2) for any $x \in X$, $p(x, \cdot): X \rightarrow [0, \infty)$ is lower semicontinuous,
- (3) for any $\varepsilon > 0$, there exist $\delta > 0$ such that $p(z, x) \leq \delta$ and $p(z, y) \leq \delta$ imply $d(x, y) \leq \varepsilon$.

In the following we suppose that E is a real Hausdorff topological vector space (tvs for short) with the zero vector θ . A proper nonempty and closed subset P of E is called a (convex) cone if $P + P \subset P$, $\lambda P \subset P$ for $\lambda \geq 0$ and $P \cap (-P) = \theta$. We shall always assume that the cone P has a nonempty interior $\text{int } P$ (such cones are called *solid*).

Each cone P induces a partial order \preceq on E by $x \preceq y \Leftrightarrow y - x \in P$. $x \prec y$ will stand for $x \preceq y$ and $x \neq y$, while $x \ll y$ will stand for $y - x \in \text{int } P$. The pair (E, P) is an *ordered tvs*.

Let us recall that the algebraic operations in tvs-cone are continuous functions. For the convenience of the reader we give the next result.

Lemma 2. Let E be a tvs over $\mathcal{F} = \mathbb{R}, \mathbb{C}$.

- (1) Suppose that $x_n, y_n, x, y \in E$ and $x_n \rightarrow x$ and $y_n \rightarrow y$. Then $x_n + y_n \rightarrow x + y$.
- (2) Suppose that $x_n, x \in E$, $\lambda_n, \lambda \in \mathcal{F}$, $x_n \rightarrow x$ and $\lambda_n \rightarrow \lambda$. Then $\lambda_n x_n \rightarrow \lambda x$.

Proof. (1) Suppose that $W \subset E$ is an open set and $x + y \in W$. Let us define $f: E \times E \rightarrow E$ by $f(u, v) = u + v$, $u, v \in E$. Because f is continuous at (x, y) there is an open set $G \subset E \times E$ such that $(x, y) \in G$ and $f(G) \subset W$. Now there are open sets $U_i, V_i \subset E$, $i \in I$, such that $G = \bigcup_{i \in I} U_i \times V_i$. Hence, there exists $i_0 \in I$ such that $(x, y) \in U_{i_0} \times V_{i_0}$. Because $x \in U_{i_0}$ and $x_n \rightarrow x$, there exists n_1 such that $x_n \in U_{i_0}$ for all $n \geq n_1$. Also, because $y \in V_{i_0}$ and $y_n \rightarrow y$, there exists n_2 such that $y_n \in V_{i_0}$ for all $n \geq n_2$. Hence, $x_n + y_n = f(x_n, y_n) \in f(U_{i_0} \times V_{i_0}) \subset W$ for all $n \geq \max\{n_1, n_2\}$. Thus, $x_n + y_n \rightarrow x + y$.

(2) Suppose that $W \subset E$ is an open set and $\lambda x \in W$. Let us define $g: \mathcal{F} \times E \rightarrow E$ by $g(\mu, v) = \mu v$, $\mu \in \mathcal{F}$, $v \in E$. Because g is continuous at (λ, x) there is an open set $G \subset \mathcal{F} \times E$ such that $(\lambda, x) \in G$ and $g(G) \subset W$. Now there are open sets $U_i \subset \mathcal{F}$, $V_i \subset E$, $i \in I$, such that $G = \bigcup_{i \in I} U_i \times V_i$. Hence, there exists $i_0 \in I$ such that $(\lambda, x) \in U_{i_0} \times V_{i_0}$. Because $\lambda \in U_{i_0}$ and $\lambda_n \rightarrow \lambda$, there exists n_1 such that $\lambda_n \in U_{i_0}$ for all $n \geq n_1$. Also, because $x \in V_{i_0}$ and $x_n \rightarrow x$, there exists n_2 such that $x_n \in V_{i_0}$ for all $n \geq n_2$. Hence, $\lambda_n x_n = g(\lambda_n, x_n) \in g(U_{i_0} \times V_{i_0}) \subset W$ for all $n \geq \max\{n_1, n_2\}$. Thus, $\lambda_n x_n \rightarrow \lambda x$. ■

Following [1,2,9,10] we give the following

Definition 3. Let X be a nonempty set and (E, P) an ordered tvs. A function $d: X \times X \rightarrow E$ is called a tvs-cone metric and (X, d) is called a tvs-cone metric space if the following conditions hold:

- (C1) $\theta \preceq d(x, y)$ for all $x, y \in X$ and $d(x, y) = \theta$ if and only if $x = y$;
- (C2) $d(x, y) = d(y, x)$ for all $x, y \in X$;
- (C3) $d(x, z) \preceq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

Let $x \in X$ and $\{x_n\}$ be a sequence in X . Then, it is said that

(i) $\{x_n\}$ (tvs-cone) converges to x if for every $c \in E$ with $\theta \ll c$ there exists a natural number n_0 such that $d(x_n, x) \ll c$ for all $n > n_0$; we denote it by $\lim_{n \rightarrow \infty} x_n = x$ or $x_n \rightarrow x$ as $n \rightarrow \infty$;

(ii) $\{x_n\}$ is a (tvs-cone) Cauchy sequence if for every $c \in E$ with $0 \ll c$ there exists a natural number n_0 such that $d(x_m, x_n) \ll c$ for all $m, n > n_0$;

(iii) (X, d) is (tvs-cone) complete if every tvs-Cauchy sequence is (tvs) convergent in X .

Let (X, d) be a tvs-cone metric space. Then the following properties are often used (see e.g., [1,2,9-13]).

(p₁) If $u \preceq v$ and $v \ll w$ then $u \ll w$.

- (p₂) If $a \leq b + c$ for each $c \in \text{int } P$ then $a \leq b$.
- (p₃) If $a \leq \lambda a$, where $a \in P$ and $0 < \lambda < 1$, then $a = \theta$.
- (p₄) If $\varepsilon \in \text{int } P$, $\theta \leq a_n$ and $a_n \rightarrow \theta$, then there exists n_0 such that for all $n > n_0$ we have $a_n \ll \varepsilon$.

2 w-Cone distance in tvs-cone metric spaces

Let (X, d) be a tvs-cone metric space. Then

- (c₁) $T : X \rightarrow X$ is continuous at $x \in X$ if x_n is a sequence in X and $\lim x_n = x$ implies $T(x) = \lim T(x_n)$.
- (c₂) $G : X \rightarrow P$ is lower semicontinuous at $x \in X$ if for any ε in E with $\theta \ll \varepsilon$, there is n_0 in \mathbb{N} such that

$$G(x) \leq G(x_n) + \varepsilon, \quad \text{for all } n \geq n_0, \tag{1}$$

whenever $\{x_n\}$ is a sequence in X and $x_n \rightarrow x$.

- (c₃) For $x \in X$, $T : X \rightarrow X$, $O(x; \infty) = \{x, Tx, T^2x, \dots\}$ is called the orbit of x . $G : X \rightarrow P$ is

T -orbitally lower semicontinuous at x if for any ε in E with $\theta \ll \varepsilon$, there is n_0 in \mathbb{N} such that (1) with $x = u$ holds whenever $x_n \in O(x; \infty)$ and $x_n \rightarrow u$.

Observe that if in definitions (c₁), (c₂) and (c₃) we have $E = \mathbb{R}$, $P = [0, \infty)$, $\|x\| = |x|$, $x \in E$, then we get the well-known definitions of continuity, lower and T -orbitally lower semicontinuity.

Definition 4. Let (X, d) be a tvs-cone metric space. Then, a function $p : X \times X \rightarrow P$ is called a w -cone distance on X if the following are satisfied:

- (w₁) $p(x, z) \leq p(x, y) + p(y, z)$, for any $x, y, z \in X$,
- (w₂) for any $x \in X$, $p(x, \cdot) : X \rightarrow P$ is lower semicontinuous,
- (w₃) for any ε in E with $\theta \ll \varepsilon$, there is δ in E with $\theta \ll \delta$, such that $p(z, x) \ll \delta$ and $p(z, y) \ll \delta$ imply $d(x, y) \ll \varepsilon$.

Example 5. Let (X, d) be a cone metric space. Then, a cone metric d is a w -cone distance p on X .

Proof. Clearly, if $p = d$, a w -cone distance p satisfies (w₁) and (w₃). So we have only to prove (w₂). Suppose that $x, y \in X$, $y_n \in X$, $y_n \rightarrow y$ and ε in E with $\theta \ll \varepsilon$ be arbitrary. Since $y_n \rightarrow y$, then there is n_0 in \mathbb{N} such that $d(y_n, y) \ll \varepsilon$ for all $n \geq n_0$. Define $G(y) = d(x, y)$. Then we have

$$G(y) = d(x, y) \leq d(x, y_n) + d(y_n, y) \leq d(x, y_n) + \varepsilon = G(y_n) + \varepsilon$$

for all $n \geq n_0$. Therefore $p(x, \cdot) = d(x, \cdot)$ is lower semicontinuous.

Remark 6. Wang and Guo [14] defined the concept of c -distance on a cone metric space in the sense of Huang and Zhang [1], which is also a generalization of w -distance of Kada et al. [3]. They proved a common fixed point theorem (Theorem 2.2) by using c -distance in a cone metric space (X, d) , where a cone P is normal with normal constant K . Now we shall present an example (Example 7 below), which shows that there are cone metric spaces where underlying cone is not normed, and so theorems of Wang and Guo [14] cannot be applied. On the other case, our presented fixed point theorems for mappings under contractive conditions expressed in the terms of w -distance can be applied, although the underlying cone is not normed.

Example 7. Let $E = C[0, 1]$ be the Banach space of real-valued continuous functions with the usual norm

$$\|f(t) - g(t)\| = \max_{0 \leq t \leq 1} |f(t) - g(t)|$$

and let a cone P be defined by $P = \{f \in E : f(t) \geq 0 \text{ for } t \in [0, 1]\}$. This cone is normal in the Banach-space topology on E . Let τ^* be the strongest locally convex topology on the linear vector space E . Then, $\text{int}P \neq \emptyset$, but the cone P is not normal in the topology τ^* . Indeed, if we suppose, to the contrary, that P is normal cone, then the topology τ^* is normed (see, e.g., [15]). But if the cone of an ordered tvs is solid and normal, then such tvs must be an ordered normed space, which is impossible because an infinite dimensional space with the strongest locally convex topology cannot be metrizable (see, e.g., [13]). Let now $X = [0, +\infty)$ and $d : X \times X \rightarrow (E, \tau^*)$ be defined by $d(x, y)(t) = |x - y| \cdot e^t$. Then (X, d) is a tvs-cone metric space over the non normedzable linear tvs (E, τ^*) .

The following lemma is crucial and is an extension of Lemma 1 in [3] for a w -metric distance to a w -cone distance.

Lemma 8. Let (X, d) be a tvs-cone metric space and let p be a w -cone distance on X . Let $\{x_n\}$ and $\{y_n\}$ be sequences in X , let $\{\alpha_n\}$ with $\theta \leq \alpha_n$, and $\{\beta_n\}$ with $\theta \leq \beta_n$, be sequences in E converging to θ , and let $x, y, z \in X$. Then the following hold:

(i) If $p(x_n, y) \leq \alpha_n$ and $p(x_n, z) \leq \beta_n$ for any $n \in \mathbb{N}$, then $y = z$. In particular, if $p(x, y) = \theta$

and $p(x, z) = \theta$, then $y = z$.

(ii) If $p(x_n, y_n) \leq \alpha_n$ and $p(x_n, z) \leq \beta_n$ for any $n \in \mathbb{N}$, then $\{y_n\}$ converges to z .

(iii) If $p(x_n, x_m) \leq \alpha_n$ for any $n, m \in \mathbb{N}$ with $m > n$, then $\{x_n\}$ is a Cauchy sequence.

(iv) If $p(y, x_n) \leq \alpha_n$ for any $n \in \mathbb{N}$, then $\{x_n\}$ is a Cauchy sequence.

Proof. We shall prove (ii). Let ε in E with $\theta \ll \varepsilon$ be arbitrary. From (w_3) of Definition 4 there is δ in E with $\theta \ll \delta$, such that $p(x_n, y_n) \ll \delta$ and $p(y_n, z) \ll \delta$ imply $d(y_n, z) \ll \varepsilon$. Since $\alpha_n \rightarrow \theta$ and $\beta_n \rightarrow \theta$, there is $n_0 \in \mathbb{N}$ such that $\alpha_n \leq \delta$ and $\beta_n \leq \delta$ for all $n \geq n_0$. Then for all $n \geq n_0$ we have

$$p(x_n, y_n) \leq \alpha_n \ll \delta \quad \text{and} \quad p(x_n, z) \leq \beta_n \ll \delta.$$

Thus from (w_3) , $d(y_n, z) \ll \varepsilon$. Hence $y_n \rightarrow z$ as $n \rightarrow \infty$. Similarly, following lines of the proof of Lemma 1 in [3], one can prove (i), (iii) and (iv). ■

3 Fixed point theorems for w -cone distance contraction mappings in K -metric spaces

We note that the method of Du [2] for cone contraction mappings cannot be applied for a w -cone distance contraction mappings.

In the following theorem, which extends and improves Theorem 2 of [3] and Theorem 1 of [5], we give an estimate for a w -cone distance $p(x_n, z)$ of an approximate value x_n and a fixed point z .

Theorem 9. Let (X, d) be a complete tvs-cone metric space with w -cone distance p on X . Suppose that for some $0 \leq k < 1$ a mapping $T : X \rightarrow X$ satisfies the following condition:

$$p(Tx, T^2x) \leq kp(x, Tx), \quad \text{for all } x \in X. \tag{2}$$

Assume that either of the following holds:

(i) If $y \neq Ty$, there exists $c \in \text{int}(P)$, $c \neq \theta$, such that

$$c \ll p(x, y) + p(x, Tx), \quad \text{for all } x \in X;$$

(ii) T is continuous.

Then, there exists $z \in X$, such that $z = Tz$ and

$$p(T^n x, z) \leq \frac{k^n}{1-k} \cdot p(x, Tx) \quad \text{for } n \geq 1. \tag{3}$$

Moreover, if $v = Tv$ for some $v \in X$, then $p(v, v) = \theta$.

Proof. Let $x \in X$ and define a sequence $\{x_n\}$ by $x_0 = x$; $x_n = T^n x$ for any $n \in \mathbb{N}$. Then from (2) we have, for any $n \in \mathbb{N}$,

$$p(x_n, x_{n+1}) = p(Tx_{n-1}, Tx_n) \leq kp(x_{n-1}, x_n) \leq \dots \leq k^n p(x, Tx). \tag{4}$$

Thus, if $m > n$, then from (w_1) of Definition 4 and (4),

$$\begin{aligned} p(x_n, x_m) &\leq p(x_n, x_{n+1}) + \dots + p(x_{m-1}, x_m) \\ &\leq k^n p(x, Tx) + \dots + k^{m-1} p(x, Tx) \\ &\leq \frac{k^n}{1-k} \cdot p(x, Tx). \end{aligned} \tag{5}$$

Hence, by (iii) of Lemma 8 with $\alpha_n = [k^n/(1-k)] \cdot p(x, Tx)$, $\{x_n\}$ is a Cauchy sequence in X . Since X is complete, $\{x_n\}$ converges to some point $z \in X$.

Now we shall prove the estimate (3). Define a function $G : X \rightarrow P$ by $G(x) = p(x_n, x)$, where n is any fixed positive integer. Since $x_n \rightarrow z$ as $n \rightarrow \infty$, from (w_2) of Definition 4 and (2) we have that for any ε in E with $\theta \ll \varepsilon$, there is m_0 in \mathbb{N} such that

$$p(x_n, z) \leq p(x_n, x_m) + \varepsilon \quad \text{for } m > m_0.$$

Thus by (5) we get

$$p(x_n, z) \leq \frac{k^n}{1-k} \cdot p(x, Tx) + \varepsilon. \tag{6}$$

Hence, as $x_n = T^n x$,

$$p(T^n x, z) \leq \frac{k^n}{1-k} \cdot p(x, Tx) + \varepsilon \quad \text{for any } \varepsilon \text{ in } E \text{ with } \theta \ll \varepsilon.$$

Thus, taking $\varepsilon = \varepsilon/i$ we have

$$p(T^n x, z) \leq \frac{k^n}{1-k} \cdot p(x, Tx) + \frac{\varepsilon}{i} \quad \text{for each } i \geq 1. \tag{7}$$

From (7) and by definition of the partial order \leq on E , we obtain

$$\frac{\varepsilon}{i} - p(T^n x, z) + \frac{k^n}{1-k} \cdot p(x, Tx) \in P.$$

By Lemma 2, it is easy to show that

$$\lim_{i \rightarrow \infty} \left[\frac{\varepsilon}{i} - p(T^n x, z) + \frac{k^n}{1-k} \cdot p(x, Tx) \right] = -p(T^n x, z) + \frac{k^n}{1-k} \cdot p(x, Tx).$$

Therefore, as P is closed,

$$-p(T^n x, z) + \frac{k^n}{1-k} \cdot p(x, Tx) \in P. \tag{8}$$

From the definition of partial order \leq , (8) is equivalent to (3). Thus we proved (3).

Suppose that the case (i) is satisfied. We have to prove that $Tz = z$. Suppose, to the contrary, that $z \neq Tz$. Then from (i) there exists $c \in \text{int}(P)$ such that

$$c \ll p(x, z) + p(x, Tx), \quad \text{for all } x \in X. \tag{9}$$

From (6) and (3) we conclude that there exists $n_0 \in \mathbb{N}$ such that

$$p(x_n, z) \ll \frac{c}{4} \quad \text{and} \quad p(x_n, x_{n+1}) \ll \frac{c}{4} \quad \text{for all } n > n_0. \tag{10}$$

Since $x_n \rightarrow z$ as $n \rightarrow \infty$, by (ii) of Definition 4 with $x = x_{n_0}$, there exists $m_0 > n_0$, such that Then from (9) with $x = x_{m_0}$ and from (10), we have

$$c \ll p(x_{m_0}, z) + p(x_{m_0}, Tx_{m_0}) = p(x_{m_0}, z) + p(x_{m_0}, x_{m_0+1}) \leq \frac{c}{4} + \frac{c}{4} = \frac{c}{2},$$

a contradiction, as $c \in \text{int}(P)$ Therefore, our assumption $z \neq Tz$ was wrong and so $z = Tz$.

If $v = Tv$ then we have,

$$p(v, v) = p(Tv, T^2v) \leq kp(v, Tv) = k^n p(v, v), \\ k^n p(v, v) - p(v, v) \in P.$$

and by (\mathbf{p}_3) we have $p(v, v) = 0$.

If $v = Tv$ then we have, by using (4),

$$p(v, v) = p(T^n v, T^{n+1} v) = p(v_n, v_{n+1}) \leq k^n p(v, Tv) = k^n p(v, v). \tag{11}$$

Hence

$$k^n p(v, v) - p(v, v) \in P.$$

Since P is closed, by Lemma 2, we get

$$\lim_{n \rightarrow \infty} [k^n p(v, v) - p(v, v)] = -p(v, v),$$

and P is closed, we get $-p(v, v) \in P$. Since also $p(v, v) \in P$, then $p(v, v) = 0$.

To complete the proof, we have to prove (ii). Suppose that T is continuous. Then from (\mathbf{c}_1) , as $\{x_n\}$ converge to z , we have

$$T(z) = \lim_{i \rightarrow \infty} T(x_n) = \lim_{i \rightarrow \infty} x_{n+1} = z.$$

Thus we proved that $T(z) = z$ and so the proof is complete. \blacksquare

Now we shall present an example where our Theorem 9 can be applied, but the main Theorem 2.2 of Wang and Guo [14] cannot.

Example 10. Let $X = [0, +\infty)$ and $d : X \times X \rightarrow (E, \tau^*)$ be defined by $d(x, y)(t) = |x - y| \cdot e^t$, as in Example 7 above. Then (X, d) is a tvs-cone metric space over the non-normed linear tvs (E, τ^*) . Define a mapping $T : X \rightarrow X$ by $Tx = x/2$. Then T satisfies the following condition:

$$p(Tx, T^2x) \leq \frac{1}{2}p(x, Tx), \quad \text{for all } x \in X,$$

and if $y \neq Ty$, there exists $c \in \text{int}(P)$, $c \neq \theta$, such that

$$c \ll p(x, y) + p(x, Tx), \quad \text{for all } x \in X.$$

Thus all hypotheses of our Theorem 9 are satisfied and $z = 0$ is a fixed point of T . Note that the mapping $T : X \rightarrow X$ satisfies the condition (2.1) in the main Theorem 2.2 of Wang and Guo [14] with $g(x) = x$ and $a_1 = 1/2$, $a_2 = a_3 = a_4 = 0$, but Theorem 2.2 cannot be applied since a cone P is not normed.

The following corollary implies the recent result Theorem 3.5. of [4].

Corollary 11. Let (X, d) be a complete tvs-cone metric space with w -cone distance p on X and $0 \leq r < 1/2$. Suppose $T : X \rightarrow X$ and

$$p(Tx, T^2x) \leq rp(x, T^2x), \quad \text{for all } x \in X. \tag{12}$$

Assume that either (i) or (ii) of Theorem 9 holds. Then, there exists $z \in X$, such that $z = Tz$. Moreover, if $v = Tv$, then $p(v, v) = \theta$.

Proof. Let $x \in X$. From (12) we have $p(Tx, T^2x) \leq rp(x, T^2x) \leq r[p(x, Tx) + p(Tx, T^2x)]$. Hence we get

$$p(Tx, T^2x) \leq kp(x, Tx),$$

where $0 \leq k = r/(1 - r) < 1$. Now, Corollary 11 follows from Theorem 9. ■

If $f : X \rightarrow X$ and $F(f)$ is a set of all fixed points of f , then in a general case $F(f) \neq F(f^n)$. Abbas and Rhoades [11] studied cases when $F(f) = F(f^n)$ for each $n \in \mathbb{N}$, that is, when f has a property P . The following theorem extends and improves Theorem 2 of [11].

Theorem 12. Let (X, d) be a complete tvs-cone metric space with w -cone distance p on X . Suppose $T : X \rightarrow X$ satisfies the following condition:

$$p(Tx, T^2x) \leq kp(x, Tx), \quad \text{for all } x \in X, \tag{13}$$

where $0 \leq k < 1$, or T satisfies strict the inequality (13) with $k = 1$, for all $x \in X$ with $x \neq Tx$. If $F(T) \neq \emptyset$, then T has property P .

Proof. Let $u \in F(T^n)$ for some $n > 1$. Suppose that T satisfies (13). Then

$$p(u, Tu) = p(T^n u, TT^n u) \leq kp(T^{n-1}u, TT^{n-1}u) \leq \dots \leq k^n p(u, Tu). \tag{14}$$

Similarly as from (11) we get $p(v, v) = \theta$, from (14) we obtain $p(u, Tu) = \theta$. Then from (14), $p(T^i u, T^{i+1}u) \leq k^i p(u, Tu) = \theta$ for all $i \in \mathbb{N}$. Now, from (w_1) of Definition 4 and $T^n u = u$ we get

$$p(u, u) \leq p(u, Tu) + p(Tu, T^2u) + \dots + p(T^{n-1}u, T^n u) = \theta.$$

Thus $p(u, u) = \theta$. Hence, and by (i) of Lemma 8 with $x = u$, $y = Tu$ and $z = u$, we have $Tu = u$. Now suppose that T satisfies strict the inequality (13) with $k = 1$ and let $u \in F(T^n)$. If we suppose that $Tu \neq u$, then we have $p(Tu, T^2u) < p(u, Tu)$. If we suppose that $Tu = T^2u$, then $T^i u = TT^i u$ for all $i \in \mathbb{N}$. Thus we have

$$p(Tu, T^2u) = p(T^2u, TT^2u) = \dots = p(T^n u, T^{n+1}u) = p(u, Tu),$$

a contradiction with $p(Tu, T^2u) < p(u, Tu)$. Therefore, $Tu \neq T^2u$. Then we have $p(T^2u, T^3u) < p(Tu, T^2u) < p(u, Tu)$. Continuing this process we obtain

$$p(u, Tu) = p(T^n u, TT^n u) < p(T^{n-1} u, TT^{n-1} u) < \dots < p(u, Tu),$$

a contradiction. Therefore, our supposition $Tu \neq u$ was wrong. Thus we proved that $F(T^n) = F(T)$. ■

The following theorem extends Theorem 2.1 of [16] and implies Theorem 3.7 of [17].

Theorem 13. *Let (X, d) be a complete tvs-cone metric space with w -cone distance p on X and $0 \leq k < 1$. Suppose $T : X \rightarrow X$ and there exists an $x \in X$ such that*

$$p(Ty, T^2y) \leq kp(y, Ty), \quad \text{for all } y \in O(x, \infty).$$

Then,

(i) $\lim T^n x = z$ exists and

$$p(T^n x, z) \leq \frac{k^n}{1-k} \cdot p(x, Tx) \quad \text{for } n \geq 1.$$

(ii) $p(z, Tz) = \theta$ if and only if $G(x) = p(x, Tx)$ is T -orbitally lower semicontinuous at z .

Proof. Observe that (i) follows from the proof of Theorem 9. Now we prove (ii). It is clear that $p(z, Tz) = \theta$ implies $G(z) = p(z, Tz) = \theta$ and hence $G(z) \leq G(x_n) + \varepsilon$ for any ε in E with $\theta \ll \varepsilon$ and all $x_n \in O(x, \infty)$. Suppose now that G is T -orbitally lower semicontinuous at z . Then from (c_3) , as $x_n = T^n x \rightarrow z$, for any ε in E with $\theta \ll \varepsilon$ there is n_0 in \mathbb{N} such that

$$\theta \leq p(z, Tz) = G(z) \leq G(x_n) + \varepsilon = p(x_n, x_{n+1}) + \varepsilon \leq k^n p(x, Tx) + \frac{1}{i} \varepsilon,$$

for all $n \geq n_0$ and $i \geq 1$. Hence $k^n p(x, Tx) + \frac{1}{i} \varepsilon - p(z, Tz) \in P$. Letting $n \rightarrow \infty$, By Lemma 2, we get $\theta \leq p(z, Tz) \leq \frac{1}{i} \varepsilon$. Hence, we get $p(z, Tz) = \theta$. ■

The following theorem extends and unifies Theorem 2 of [6] and results of ([1,18]).

Theorem 14. *Let (X, d) be a complete tvs-cone metric space with w -cone distance p on X and $0 \leq k < 1$. Suppose that $T : X \rightarrow X$ is a p -contractive mapping i.e.,*

$$p(Tx, Ty) \leq kp(x, y), \quad \text{for all } x, y \in X. \tag{15}$$

Then, T has a unique fixed point $z \in X$, and $p(z, z) = \theta$.

Proof. Let $x \in X$ and define $x_{n+1} = T^n x$ for any $n \in \mathbb{N}$. Then, from the proof of Theorem 9, $\lim T^n x = z \in X$ and (3) holds for all $n \geq 1$. From (3) we have

$$p(T^n x, z) \leq \frac{k^n}{1-k} \cdot p(x, Tx) \quad \text{for } n \geq 1,$$

and from (15) and (3) we get

$$p(T^n x, Tz) \leq kp(T^{n-1} x, z) \leq k \frac{k^{n-1}}{1-k} \cdot p(x, Tx) = \frac{k^n}{1-k} \cdot p(x, Tx) \quad \text{for } n \geq 1.$$

Thus from Lemma 2 and (i) of Lemma 8, with $\alpha_n = \beta_n = [k^n/(1-k)] \cdot p(x, Tx)$ we obtain $Tz = z$. Then $p(z, z) = p(Tz, T^2z) \leq kp(z, Tz) = kp(z, z)$. Hence, $p(z, z) = \theta$. Suppose that $u \in X$ is also a fixed point of T . Then $p(z, u) = p(Tz, Tu) \leq kp(z, u)$ and hence $p(z, u) = \theta$. From $p(z, u) = \theta$, $p(z, z) = \theta$ and by (i) of Lemma 8, we have $u = z$. ■

In 1998, Ume [8] proved the w -distance version of Ćirić's [19] results for quasi-contraction on metric space. Recently, cone metric version of Ćirić's results has been proved [12]. Now, a natural question arises:

Question. Let (X, d) be a complete tvs-cone metric space with w -cone distance p on X . Suppose $f: X \rightarrow X$ such that for some constant $\lambda \in (0, 1)$ and for every $x, y \in X$, there exists $v \in \{p(x, y), p(x, fx), p(y, fy), p(x, fy), p(y, fx)\}$, such that $p(fx, fy) \leq \lambda \cdot v$. Does there exist a unique fixed point $z \in X$ of f , and $p(z, z) = \theta$?

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Author details

¹Faculty of Mechanical Engineering, Kraljice Marije 16, Belgrade, Serbia ²Department of Mathematics, Payame Noor University, 19395-4697 Tehran, Islamic Republic of Iran ³Faculty of Sciences and Mathematics, University in Niš, Visegradska 33, Niš, Serbia

Authors' contributions

All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

Competing interests

The authors declare that they have no competing interests.

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