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Dunkl generalization of Szász operators via q-calculus

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Abstract

We construct the linear positive operators generated by the q-Dunkl generalization of the exponential function. We have approximation properties of the operators via a universal Korovkin-type theorem and a weighted Korovkin-type theorem. The rate of convergence of the operators for functions belonging to the Lipschitz class is presented. We obtain the rate of convergence by means of the classical, second order, and weighted modulus of continuity, respectively, as well.

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1 Introduction

In 1912, Bernstein [1] gave the following polynomials for any $f \in C[0,1]$, $x \in [0,1]$:

$$B_n(f;x) := \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} f\left(\frac{k}{n}\right), \quad n \in \mathbb{N}.$$

In 1950, for $x \geq 0$, Szász [2] introduced the operators

$$S_n(f;x) := e^{-nx} \sum_{k=0}^{\infty} \frac{(nx)^k}{k!} f\left(\frac{k}{n}\right),$$

where $f \in C[0,\infty)$.

q-calculus plays an important role in the natural sciences such as mathematics, physics, and chemistry. It has many applications in number theory, orthogonal polynomials, quantum theory, etc. There is a generalization of q-calculus, which is (p,q)-calculus where $0 < q < p \leq 1$. For $p = 1$, (p,q)-integers reduce to q-integers. (p,q)-integers are introduced to unify several forms of q-oscillator algebras in the representation theory of single parameter quantum algebras in physics. There have appeared some papers dealing with (p,q)-calculus in recent years. Details are in [3].

We first mention some notations of q-calculus as found in [4, 5]. Let $n \in \mathbb{N}_0$ and $q \in (0,1)$. The q-integer $[n]_q$ and q-factorial $[n]_q!$ are, respectively, defined by

$$[n]_q = \frac{1-q^n}{1-q} = 1 + q + q^2 + \cdots + q^{n-1},$$

$$[n]_q! = \begin{cases} 1, & \text{if } n = 0, \\ [n]_q [n-1]_q \cdots [1]_q, & \text{if } n = 1, 2, \ldots. \end{cases}$$

For $n \in \mathbb{N}$, we have q-binomial coefficients

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[n-k]_q! [k]_q!}, \quad 1 \le k \le n,$$

with $\begin{bmatrix} n \\ 0 \end{bmatrix}_q = 1$ and $\begin{bmatrix} n \\ k \end{bmatrix}_q = 0$ for $k > n$. Then we give the following known representations:

$$(a;q)_n = \begin{cases} \prod_{k=0}^{n-1}(1 - aq^k), & \text{if } n = 1, 2, \ldots, \\ 1, & \text{if } n = 0 \end{cases} \quad \text{and} \quad (a;q)_\infty = \prod_{k=0}^{\infty}(1 - aq^k).$$

q-Bernstein polynomials were first introduced by Lupaş [6] in 1987. A most useful definition of q-Bernstein polynomials was given by Phillips [7] as follows:

$$B_{n,q}(f;x) := \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q x^k \prod_{s=0}^{n-k-1}(1 - q^s x) f\left(\frac{[k]_q}{[n]_q}\right), \quad n \in \mathbb{N}.$$

Many generalizations of q-Bernstein polynomials were given by authors such as Ostrovska [8], Büyükyazıcı [9, 10], Büyükyazıcı and Sharma [11], Aral [12], Nowak and Gupta [13], Gupta [14], Wang [15, 16], Wang and Wu [17], Phillips [18], Aral *et al.* [19], Acar and Aral [20], Aral and Gupta [21] and Finta and Gupta [22]. On the other hand, some authors dealt with generalizations of Szász-type operators [2, 12, 23–29].

Sucu [24] defined a Dunkl analog of Szász operators via a generalization of the exponential function given by [30] as

$$S_n^*(f;x) := \frac{1}{e_\mu(nx)} \sum_{k=0}^{\infty} \frac{(nx)^k}{\gamma_\mu(k)} f\left(\frac{k + 2\mu\theta_k}{n}\right), \tag{1.1}$$

where $\mu \ge 0$, $n \in \mathbb{N}$, $x \ge 0$, $f \in C[0,\infty)$, and $e_\mu(x) = \sum_{n=0}^{\infty} \frac{x^n}{\gamma_\mu(n)}$. Here

$$\gamma_\mu(2k) = \frac{2^{2k} k! \Gamma(k + \mu + 1/2)}{\Gamma(\mu + 1/2)}$$

and

$$\gamma_\mu(2k+1) = \frac{2^{2k+1} k! \Gamma(k + \mu + 3/2)}{\Gamma(\mu + 1/2)}.$$

There is a recursion relation for γ_μ,

$$\gamma_\mu(k+1) = (k + 1 + 2\mu\theta_{k+1})\gamma_\mu(k), \quad k \in \mathbb{N}_0,$$

where

$$\theta_k = \begin{cases} 0, & \text{if } k \in 2\mathbb{N}, \\ 1, & \text{if } k \in 2\mathbb{N} + 1 \end{cases}$$

(details are in [24]). İçöz and Çekim [25] investigated a Stancu-type generalization of a Kantorovich-type integral modification of the Dunkl analog of Szász operators by

$$T_n^*(f;x) := \frac{n}{e_\mu(nx)} \sum_{k=0}^{\infty} \frac{(nx)^k}{\gamma_\mu(k)} \int_{\frac{k+2\mu\theta_k}{n}}^{\frac{k+1+2\mu\theta_k}{n}} f\left(\frac{nt+\alpha}{n+\beta}\right) dt,$$

where $\mu \geq 0$, $n \in \mathbb{N}$, $x \geq 0$, $\alpha, \beta \in \mathbb{R}$ $(0 \leq \alpha \leq \beta)$, and $f \in C[0,\infty)$.

Ben Cheikh *et al.* [31] stated the q-Dunkl classical q-Hermite-type polynomials. They gave definitions of q-Dunkl analogs of exponential functions, recursion relations, and notations for $\mu > -\frac{1}{2}$ and $0 < q < 1$, respectively:

$$e_{\mu,q}(x) = \sum_{n=0}^{\infty} \frac{x^n}{\gamma_{\mu,q}(n)}, \qquad E_{\mu,q}(x) = \sum_{n=0}^{\infty} \frac{q^{\frac{n(n-1)}{2}} x^n}{\gamma_{\mu,q}(n)} \quad (x \in \mathbb{R}), \tag{1.2}$$

$$\gamma_{\mu,q}(n+1) = \left(\frac{1-q^{2\mu\theta_{n+1}+n+1}}{1-q}\right)\gamma_{\mu,q}(n), \quad n \in \mathbb{N}, \tag{1.3}$$

$$\theta_n = \begin{cases} 0, & n \in 2\mathbb{N}, \\ 1, & n \in 2\mathbb{N}+1. \end{cases} \tag{1.4}$$

An explicit formula of $\gamma_{\mu,q}(n)$ is

$$\gamma_{\mu,q}(n) = \frac{(q^{2\mu+1}, q^2)_{[\frac{n+1}{2}]}(q^2, q^2)_{[\frac{n}{2}]}}{(1-q)^n}. \tag{1.5}$$

One can find some of the special cases $\gamma_{\mu,q}(n)$ below:

$$\begin{aligned}
&\gamma_{\mu,q}(0) = 1, \qquad \gamma_{\mu,q}(1) = \frac{1-q^{2\mu+1}}{1-q}, \\
&\gamma_{\mu,q}(2) = \left(\frac{1-q^{2\mu+1}}{1-q}\right)\left(\frac{1-q^2}{1-q}\right), \\
&\gamma_{\mu,q}(3) = \left(\frac{1-q^{2\mu+1}}{1-q}\right)\left(\frac{1-q^2}{1-q}\right)\left(\frac{1-q^{2\mu+3}}{1-q}\right), \\
&\gamma_{\mu,q}(4) = \left(\frac{1-q^{2\mu+1}}{1-q}\right)\left(\frac{1-q^2}{1-q}\right)\left(\frac{1-q^{2\mu+3}}{1-q}\right)\left(\frac{1-q^4}{1-q}\right).
\end{aligned}$$

Now, in this paper, we define a q-Dunkl analog of Szász operators as follows:

$$D_{n,q}(f;x) = \frac{1}{e_{\mu,q}([n]_q x)} \sum_{k=0}^{\infty} \frac{([n]_q x)^k}{\gamma_{\mu,q}(k)} f\left(\frac{1-q^{2\mu\theta_k+k}}{1-q^n}\right), \tag{1.6}$$

where $\mu > \frac{1}{2}$, $n \in \mathbb{N}$, $x \geq 0$, $0 < q < 1$ and $f \in C[0,\infty)$. Here $e_{\mu,q}$ and $\gamma_{\mu,q}$ are in (1.2), (1.5), respectively. Note that, when we take $q \to 1$, then we have (1.1).

2 Approximation properties

In this section, the convergence of the operators $D_{n,q}$ is examined via a universal Korovkin-type theorem and a weighted approximation theorem given by [32].

Lemma 1 *The operators $D_{n,q}$ given by* (1.6) *satisfy the following*:

$$D_{n,q}(1;x) = 1, \tag{2.1}$$

$$D_{n,q}(t;x) = x, \tag{2.2}$$

$$\begin{aligned}[1-2\mu]_q q^{2\mu}\frac{e_{\mu,q}(q[n]_q x)}{e_{\mu,q}([n]_q x)}\frac{x}{[n]_q} &\leq D_{n,q}\bigl(t^2;x\bigr)-x^2 \\ &\leq [1+2\mu]_q\frac{x}{[n]_q},\end{aligned} \tag{2.3}$$

$$\begin{aligned}D_{n,q}\bigl(t^3;x\bigr)-x^3 &\geq (2q+1)[1-2\mu]_q\frac{e_{\mu,q}(q[n]_q x)}{e_{\mu,q}([n]_q x)}\frac{x^2}{[n]_q} \\ &\quad + q^{4\mu}[1-2\mu]_q^2\frac{e_{\mu,q}(q^2[n]_q x)}{e_{\mu,q}([n]_q x)}\frac{x}{[n]_q^2},\end{aligned} \tag{2.4}$$

$$\begin{aligned}D_{n,q}\bigl(t^4;x\bigr) &\leq x^4 + 6[1+2\mu]_q\frac{x^3}{[n]_q} + 7[1+2\mu]_q^2\frac{x^2}{[n]_q^2} \\ &\quad + [1+2\mu]_q^3\frac{x}{[n]_q^3}.\end{aligned} \tag{2.5}$$

Proof For $f(t) = 1$, we have

$$D_{n,q}(1;x) = \frac{1}{e_{\mu,q}([n]_q x)}\sum_{k=0}^{\infty}\frac{([n]_q x)^k}{\gamma_{\mu,q}(k)} = 1.$$

From (1.3) and (1.6), we get

$$\begin{aligned}D_{n,q}(t;x) &= \frac{1}{e_{\mu,q}([n]_q x)}\sum_{k=0}^{\infty}\frac{([n]_q x)^k}{\gamma_{\mu,q}(k)}\left(\frac{1-q^{2\mu\theta_k+k}}{1-q^n}\right) \\ &= \frac{1}{[n]_q}\frac{1}{e_{\mu,q}([n]_q x)}\sum_{k=1}^{\infty}\frac{([n]_q x)^k}{\gamma_{\mu,q}(k-1)} \\ &= x.\end{aligned}$$

By (1.3) and (1.6), we obtain

$$\begin{aligned}D_{n,q}\bigl(t^2;x\bigr) &= \frac{1}{e_{\mu,q}([n]_q x)}\sum_{k=0}^{\infty}\frac{([n]_q x)^k}{\gamma_{\mu,q}(k)}\left(\frac{1-q^{2\mu\theta_k+k}}{1-q^n}\right)^2 \\ &= \frac{1}{[n]_q^2}\frac{1}{e_{\mu,q}([n]_q x)}\sum_{k=1}^{\infty}\frac{([n]_q x)^k}{\gamma_{\mu,q}(k-1)}\left(\frac{1-q^{2\mu\theta_k+k}}{1-q}\right) \\ &= \frac{1}{[n]_q^2}\frac{1}{e_{\mu,q}([n]_q x)}\sum_{k=0}^{\infty}\frac{([n]_q x)^{k+1}}{\gamma_{\mu,q}(k)}\left(\frac{1-q^{2\mu\theta_{k+1}+k+1}}{1-q}\right).\end{aligned}$$

One can easily see that

$$[2\mu\theta_{k+1}+k+1]_q = [2\mu\theta_{k+1}+k]_q + q^{2\mu\theta_k+k}\bigl[2\mu(-1)^k+1\bigr]_q. \tag{2.6}$$

Using (2.6) and writing odd and even terms separately, we have

$$\begin{aligned} D_{n,q}\bigl(t^2;x\bigr) &= \frac{1}{[n]_q^2}\frac{1}{e_{\mu,q}([n]_q x)}\sum_{k=0}^{\infty}\frac{([n]_q x)^{k+1}}{\gamma_{\mu,q}(k)}\left(\frac{1-q^{2\mu\theta_k+k}}{1-q}\right) \\ &\quad + \frac{1}{[n]_q^2}\frac{[1+2\mu]_q}{e_{\mu,q}([n]_q x)}\sum_{k=0}^{\infty}\frac{([n]_q x)^{2k+1}}{\gamma_{\mu,q}(2k)}q^{2\mu\theta_{2k}+2k} \\ &\quad + \frac{1}{[n]_q^2}\frac{[1-2\mu]_q}{e_{\mu,q}([n]_q x)}\sum_{k=0}^{\infty}\frac{([n]_q x)^{2k+2}}{\gamma_{\mu,q}(2k+1)}q^{2\mu\theta_{2k+1}+2k+1}. \end{aligned}$$

Using the inequality

$$[1-2\mu]_q \le [1+2\mu]_q, \tag{2.7}$$

it follows that

$$\begin{aligned} D_{n,q}\bigl(t^2;x\bigr) &\ge x^2 + \frac{1}{q[n]_q^2}\frac{[1-2\mu]_q}{e_{\mu,q}([n]_q x)}\sum_{k=0}^{\infty}\frac{([n]_q xq)^{2k+1}}{\gamma_{\mu,q}(2k)} \\ &\quad + \frac{q^{2\mu-1}}{[n]_q^2}\frac{[1-2\mu]_q}{e_{\mu,q}([n]_q x)}\sum_{k=0}^{\infty}\frac{([n]_q xq)^{2k+2}}{\gamma_{\mu,q}(2k+1)} \\ &\ge x^2 + [1-2\mu]_q q^{2\mu}\frac{e_{\mu,q}(q[n]_q x)}{e_{\mu,q}([n]_q x)}\frac{x}{[n]_q}. \end{aligned}$$

On the other hand, from (2.7), we have

$$\begin{aligned} D_{n,q}\bigl(t^2;x\bigr) &\le x^2 + \frac{1}{[n]_q^2}\frac{[1+2\mu]_q}{e_{\mu,q}([n]_q x)}\sum_{k=0}^{\infty}\frac{([n]_q x)^{k+1}}{\gamma_{\mu,q}(k)} \\ &= x^2 + [1+2\mu]_q\frac{x}{[n]_q}. \end{aligned}$$

So we have (2.3).

By the same computations, one gets (2.4) and (2.5). □

Lemma 2 *The first, second, and fourth moments of the operators $D_{n,q}$ are*

$$D_{n,q}(t-x;x) = 0, \tag{2.8}$$

$$D_{n,q}\bigl((t-x)^2;x\bigr) \le [1+2\mu]_q\frac{x}{[n]_q}, \tag{2.9}$$

$$\begin{aligned} D_{n,q}\bigl((t-x)^4;x\bigr) &\le \left(12\frac{[1+2\mu]_q}{[n]_q} - 4\frac{(2q+1)[1-2\mu]_q}{[n]_q}q^{2\mu+1}\frac{e_{\mu,q}(q[n]_q x)}{e_{\mu,q}([n]_q x)}\right)x^3 \\ &\quad + \frac{[1+2\mu]_q^3}{[n]_q^3}x \\ &\quad + \left(\frac{7[1+2\mu]_q^2}{[n]_q^2} - 4\frac{[1-2\mu]_q^2}{[n]_q^2}q^{4\mu}\frac{e_{\mu,q}(q^2[n]_q x)}{e_{\mu,q}([n]_q x)}\right)x^2. \end{aligned} \tag{2.10}$$

Theorem 1 *Let $D_{n,q}$ be the operators given by* (1.6). *Then for any $f \in C[0,\infty) \cap E$, the following relation*:

$$\lim_{n\to\infty} D_{n,q}(f;x) = f(x)$$

holds uniformly on each compact subset of $[0,\infty)$, *where*

$$E := \left\{ f : x \in [0,\infty), \frac{f(x)}{1+x^2} \text{ is convergent as } x \to \infty \right\}.$$

Proof The proof is based on the well-known universal Korovkin-type theorem (see details in [33, 34]). □

We recall the weighted spaces of the functions which are defined on the positive semi-axis $\mathbb{R}^+ = [0,\infty)$ as follows:

$$B_\rho(\mathbb{R}^+) = \{ f : |f(x)| \le M_f \rho(x) \},$$
$$C_\rho(\mathbb{R}^+) = \{ f : f \in B_\rho(\mathbb{R}^+) \cap C[0,\infty) \},$$
$$C_\rho^k(\mathbb{R}^+) = \left\{ f : f \in C_\rho(\mathbb{R}^+) \text{ and } \lim_{x\to\infty} \frac{f(x)}{\rho(x)} = k \ (k \text{ is a constant}) \right\},$$

where $\rho(x) = 1 + x^2$ is a weight function and M_f is a constant depending only on f. $C_\rho(\mathbb{R}^+)$ is a normed space with the norm $\|f\|_\rho := \sup_{x\ge 0} \frac{|f(x)|}{\rho(x)}$.

Theorem 2 *Let $D_{n,q}$ be the operators given by* (1.6). *Then for any $f \in C_\rho^k(\mathbb{R}^+)$, we have*

$$\lim_{n\to\infty} \left\| D_{n,q}(f;x) - f(x) \right\|_\rho = 0.$$

Proof Using Lemma 1, one can easily prove the theorem. □

3 Rate of convergence

In this section, we compute the rate of convergence of the operators $D_{n,q}$ with the help of Lipschitz class functions, and the classical, second order, and weighted modulus of continuity. For the sake of simplicity, we just give the theorems and lemmas without proofs in this section.

Lemma 3 *Let $f \in \mathrm{Lip}_M(\alpha)$ $(0 < \alpha \le 1, M > 0)$, i.e.*

$$\mathrm{Lip}_M(\alpha) := \left\{ f : |f(\xi) - f(\eta)| \le M|\xi - \eta|^\alpha, \xi, \eta \in [0,\infty) \right\}.$$

Then

$$\left| D_{n,q}(f;x) - f(x) \right| \le M\left(\vartheta_n(x)\right)^{\frac{\alpha}{2}}$$

holds where $\vartheta_n(x) = D_{n,q}((t-x)^2;x)$.

Theorem 3 *Let $f \in \widetilde{C}[0,\infty)$. Then the operators $D_{n,q}$ verify*

$$\left|D_{n,q}(f;x)-f(x)\right| \leq \left\{1+\sqrt{x[2\mu+1]_q}\right\}\omega\left(f;\frac{1}{\sqrt{[n]_q}}\right),$$

where $\widetilde{C}[0,\infty)$ is the space of uniformly continuous functions on $[0,\infty)$, i.e. $\omega(f;\delta)$ is the modulus of continuity of the function $f \in \widetilde{C}[0,\infty)$ defined by

$$\omega(f;\delta) = \sup_{x,y\in[0,\infty)}\left\{\left|f(x)-f(y)\right| : |x-y| \leq \delta\right\}.$$

Lemma 4 *Let $g \in C_B^2[0,\infty)$. Then we get*

$$\left|D_{n,q}(g;x)-g(x)\right| \leq \frac{\vartheta_n(x)}{2}\|g\|_{C_B^2[0,\infty)},$$

where $\vartheta_n(x)$ is given in Lemma 3 and $C_B[0,\infty)$ is the space of all bounded and continuous functions on $[0,\infty)$ and

$$C_B^2[0,\infty) = \left\{g \in C_B[0,\infty) : g', g'' \in C_B[0,\infty)\right\}$$

with the norm

$$\|g\|_{C_B^2[0,\infty)} = \|g\|_{C_B[0,\infty)} + \left\|g'\right\|_{C_B[0,\infty)} + \left\|g''\right\|_{C_B[0,\infty)}.$$

Also

$$\|g\|_{C_B[0,\infty)} = \sup_{x\in[0,\infty)}\left|g(x)\right|.$$

Theorem 4 *For $f \in C_B[0,\infty)$ and $x \in [0,\infty)$, we get*

$$\left|D_{n,q}(f;x)-f(x)\right| \leq 2M\left\{\omega_2\left(f;\sqrt{\frac{\vartheta_n(x)}{4}}\right) + \min\left(1,\frac{\vartheta_n(x)}{4}\right)\|f\|_{C_B[0,\infty)}\right\}, \tag{3.1}$$

where M is a positive constant and $\omega_2(f;\delta)$ is the second order modulus of continuity of the function $f \in C_B[0,\infty)$ defined as

$$\omega_2(f;\delta) := \sup_{0<t\leq\delta}\left\|f(\cdot+2t)-2f(\cdot+t)+f(\cdot)\right\|_{C_B[0,\infty)}$$

and $K_2(f;\delta)$ is the Peetre K-functional defined by

$$K_2(f;\delta) = \inf_{g\in C_B^2[0,\infty)}\left\{\|f-g\|_{C_B[0,\infty)} + \delta\|g\|_{C_B^2[0,\infty)}\right\}.$$

Theorem 5 *Let $f \in C_\rho^k(\mathbb{R}^+)$. Then*

$$\sup_{x\in[0,\infty)}\frac{|D_{n,q}(f;x)-f(x)|}{(1+x^2)^2} \leq S_\mu\left(1+\frac{1}{[n]_q}\right)\Omega\left(f;\frac{1}{\sqrt{[n]_q}}\right)$$

holds. Here S_μ is a constant independent of n.

4 Auxiliary results

In the section, we prove the theorems and lemmas given in the previous section.

Proof of Lemma 3 Since $f \in \mathrm{Lip}_M(\alpha)$ and by linearity of the function f, we get

$$\begin{aligned}\left|D_{n,q}(f;x)-f(x)\right| &\le \left|D_{n,q}\big(f(t)-f(x);x\big)\right| \le D_{n,q}\big(\left|f(t)-f(x)\right|;x\big)\\ &\le MD_{n,q}\big(|t-x|^{\alpha};x\big).\end{aligned}$$

By using Lemma 1 and the Hölder inequality, one gets

$$\begin{aligned}\left|D_{n,q}(f;x)-f(x)\right| &\le M\frac{1}{e_{\mu,q}([n]_qx)}\sum_{k=0}^{\infty}\left(\frac{([n]_qx)^k}{\gamma_{\mu,q}(k)}\right)^{\frac{2-\alpha}{2}}\left(\frac{([n]_qx)^k}{\gamma_{\mu,q}(k)}\right)^{\frac{\alpha}{2}}\left|\frac{1-q^{2\mu\theta_k+k}}{1-q^n}-x\right|^{\alpha}\\ &\le M\frac{1}{e_{\mu,q}([n]_qx)}\left(\sum_{k=0}^{\infty}\frac{([n]_qx)^k}{\gamma_{\mu,q}(k)}\right)^{\frac{2-\alpha}{2}}\\ &\quad\times\left(\sum_{k=0}^{\infty}\frac{([n]_qx)^k}{\gamma_{\mu,q}(k)}\left|\frac{1-q^{2\mu\theta_k+k}}{1-q^n}-x\right|^2\right)^{\frac{\alpha}{2}}\\ &= M\big(D_{n,q}\big((t-x)^2;x\big)\big)^{\frac{\alpha}{2}}.\end{aligned}$$

This ends the proof. □

Proof of Theorem 3 From Lemma 1, the property of the modulus of continuity, and the Cauchy-Schwarz inequality, we have

$$\begin{aligned}\left|D_{n,q}(f;x)-f(x)\right| &\le \frac{1}{e_{\mu,q}([n]_qx)}\sum_{k=0}^{\infty}\frac{([n]_qx)^k}{\gamma_{\mu,q}(k)}\left|f\left(\frac{1-q^{2\mu\theta_k+k}}{1-q^n}\right)-f(x)\right|\\ &\le \left(1+\frac{1}{\delta}\frac{1}{e_{\mu,q}([n]_qx)}\sum_{k=0}^{\infty}\frac{([n]_qx)^k}{\gamma_{\mu,q}(k)}\left|\frac{1-q^{2\mu\theta_k+k}}{1-q^n}-x\right|\right)\omega(f;\delta)\\ &\le \left\{1+\frac{1}{\delta}\left(\frac{1}{e_{\mu,q}([n]_qx)}\sum_{k=0}^{\infty}\frac{([n]_qx)^k}{\gamma_{\mu,q}(k)}\left(\frac{1-q^{2\mu\theta_k+k}}{1-q^n}-x\right)^2\right)^{\frac{1}{2}}\right\}\omega(f;\delta)\\ &\le \left\{1+\frac{1}{\delta}\sqrt{\frac{x}{[n]_q}[2\mu+1]_q}\right\}\omega(f;\delta).\end{aligned}$$

If we choose $\delta=\delta_n=\sqrt{\frac{1}{[n]_q}}$, then we have desired result. □

Proof of Lemma 4 Using the generalized mean value theorem in the Taylor series expansion for $g \in C_B^2[0,\infty)$, we have

$$g(t)=g(x)+g'(x)(t-x)+g''(\xi)\frac{(t-x)^2}{2},\quad \xi\in(x,t).$$

By the linearity property of the operator $D_{n,q}$, we obtain

$$D_{n,q}(g;x)-g(x)=g'(x)D_{n,q}\big((t-x);x\big)+\frac{g''(\xi)}{2}D_{n,q}\big((t-x)^2;x\big).$$

From the above equality and Lemma 2, we conclude that

$$\begin{aligned}\left|D_{n,q}(g;x)-g(x)\right| &\leq \frac{x}{2[n]_q}[2\mu+1]_q\left\|g''\right\|_{C_B[0,\infty)}\\ &\leq \frac{x}{2[n]_q}[2\mu+1]_q\|g\|_{C_B^2[0,\infty)}.\end{aligned}$$

This ends the proof. □

Proof of Theorem 4 Let $g \in C_B^2[0,\infty)$. From Lemma 4, we have

$$\begin{aligned}\left|D_{n,q}(f;x)-f(x)\right| &\leq \left|D_{n,q}(f-g;x)\right| + \left|D_{n,q}(g;x)-g(x)\right| + \left|f(x)-g(x)\right|\\ &\leq 2\|f-g\|_{C_B[0,\infty)} + \frac{\vartheta_n(x)}{2}\|g\|_{C_B^2[0,\infty)}.\end{aligned}$$

When we take the infimum over all $g \in C_B^2[0,\infty)$, then we obtain

$$\left|D_{n,q}(f;x)-f(x)\right| \leq 2K_2\left(f;\frac{\vartheta_n(x)}{4}\right).$$

Now we recall the relation

$$K_2(f;\delta) \leq C\left\{\omega_2(f;\sqrt{\delta}) + \min(1,\delta)\|f\|\right\},$$

where we have an absolute constant $C > 0$ [35], and we get (3.1). □

For arbitrary $f \in C_\rho^k(\mathbb{R}^+)$,the weighted modulus of continuity is defined by

$$\Omega(f;\delta) = \sup_{x\in[0,\infty),|h|\leq\delta}\frac{|f(x+h)-f(x)|}{(1+h^2)(1+x^2)}$$

and was introduced by Atakut and İspir in [23]. There are two main properties of this modulus of continuity, which are $\lim_{\delta\to 0}\Omega(f;\delta) = 0$ and

$$\left|f(t)-f(x)\right| \leq 2\left(1+\frac{|t-x|}{\delta}\right)\left(1+\delta^2\right)\left(1+x^2\right)\left(1+(t-x)^2\right)\Omega(f;\delta), \tag{4.1}$$

where $f \in C_\rho^k(\mathbb{R}^+)$ and $t,x \in [0,\infty)$. One can find many properties of the weighted modulus of continuity in [23].

Proof of Theorem 5 From Lemma 2 and (4.1), we have

$$\begin{aligned}\left|D_{n,q}(f;x)-f(x)\right| &\leq \frac{1}{e_{\mu,q}([n]_q x)}\sum_{k=0}^{\infty}\frac{([n]_q x)^k}{\gamma_{\mu,q}(k)}\left|f\left(\frac{1-q^{2\mu\theta_k+k}}{1-q^n}\right)-f(x)\right|\\ &\leq 2\left(1+\delta^2\right)\left(1+x^2\right)\Omega(f;\delta)\frac{1}{e_{\mu,q}([n]_q x)}\\ &\quad\times\sum_{k=0}^{\infty}\frac{([n]_q x)^k}{\gamma_{\mu,q}(k)}\left(1+\frac{\left|\frac{1-q^{2\mu\theta_k+k}}{1-q^n}-x\right|}{\delta}\right)\left(1+\left(\frac{1-q^{2\mu\theta_k+k}}{1-q^n}-x\right)^2\right)\end{aligned}$$

$$
\begin{aligned}
&= 2\left(1+\delta^2\right)\left(1+x^2\right)\Omega(f;\delta)\frac{1}{e_{\mu,q}([n]_q x)} \\
&\quad \times \left\{\sum_{k=0}^{\infty}\frac{([n]_q x)^k}{\gamma_{\mu,q}(k)} + \sum_{k=0}^{\infty}\frac{([n]_q x)^k}{\gamma_{\mu,q}(k)}\left(\frac{1-q^{2\mu\theta_k+k}}{1-q^n}-x\right)^2\right. \\
&\quad + \frac{1}{\delta}\sum_{k=0}^{\infty}\frac{([n]_q x)^k}{\gamma_{\mu,q}(k)}\left|\frac{1-q^{2\mu\theta_k+k}}{1-q^n}-x\right| \\
&\quad \left. + \frac{1}{\delta}\sum_{k=0}^{\infty}\frac{([n]_q x)^k}{\gamma_{\mu,q}(k)}\left|\frac{1-q^{2\mu\theta_k+k}}{1-q^n}-x\right|\left(\frac{1-q^{2\mu\theta_k+k}}{1-q^n}-x\right)^2\right\}.
\end{aligned}
$$

Applying the Cauchy-Schwarz inequality for the above series, we obtain

$$
\begin{aligned}
&\left|D_{n,q}(f;x)-f(x)\right| \\
&\quad \le 2\left(1+\delta^2\right)\left(1+x^2\right)\Omega(f;\delta)\left(1+D_{n,q}\left((t-x)^2;x\right)\right. \\
&\quad\quad \left. + \frac{1}{\delta}\sqrt{D_{n,q}\left((t-x)^2;x\right)} + \frac{1}{\delta}\sqrt{D_{n,q}\left((t-x)^2;x\right)D_{n,q}\left((t-x)^4;x\right)}\right).
\end{aligned}
$$

From (2.9) and (2.10), we find

$$
\begin{aligned}
&\left|D_{n,q}(f;x)-f(x)\right| \\
&\quad \le 2\left(1+\delta^2\right)\left(1+x^2\right)\Omega(f;\delta)\left\{1+[1+2\mu]_q\frac{x}{[n]_q}+\frac{1}{\delta}\sqrt{[1+2\mu]_q\frac{x}{[n]_q}}\right. \\
&\quad\quad \left. + \frac{1}{\delta}\sqrt{\left([1+2\mu]_q\frac{x}{[n]_q}\right)\left(12\frac{[1+2\mu]_q}{[n]_q}x^3+\frac{7[1+2\mu]_q^2}{[n]_q^2}x^2+\frac{[1+2\mu]_q^3}{[n]_q^3}x\right)}\right\}.
\end{aligned}
$$

Choosing $\delta = \frac{1}{\sqrt{[n]_q}}$ then the proof is completed. □

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors contributed equally in writing this article and collaborated in its design in coordination. All authors read and approved the final paper.

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