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Oscillatory behavior of second-order nonlinear neutral differential equations with distributed deviating arguments

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Dedicated to Professor Ivan Kiguradze

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Abstract

We study oscillatory properties of a class of second-order nonlinear neutral functional differential equations with distributed deviating arguments. On the basis of less restrictive assumptions imposed on the neutral coefficient, some new criteria are presented. Three examples are provided to illustrate these results.

MSC: 34C10; 34K11

Keywords: oscillation; neutral differential equation; second-order equation; distributed deviating argument

1 Introduction

This paper is concerned with oscillation of the second-order nonlinear functional differential equation

$$\left(r(t)\left|z'(t)\right|^{\alpha-1}z'(t)\right)' + \int_a^b q(t,\xi)\left|x\left[g(t,\xi)\right]\right|^{\alpha-1}x\left[g(t,\xi)\right]\,\mathrm{d}\sigma(\xi) = 0, \tag{1.1}$$

where $t \geq t_0 > 0$, $\alpha \geq 1$ is a constant, and $z := x + p \cdot x \circ \tau$. Throughout, we assume that the following hypotheses hold:

(H$_1$) $\mathbb{I} := [t_0, \infty)$, $r, p \in \mathrm{C}^1(\mathbb{I}, \mathbb{R})$, $r(t) > 0$, and $p(t) \geq 0$;
(H$_2$) $q \in \mathrm{C}(\mathbb{I} \times [a,b], [0,\infty))$ and $q(t,\xi)$ is not eventually zero on any $[t_\mu, \infty) \times [a,b]$, $t_\mu \in \mathbb{I}$;
(H$_3$) $g \in \mathrm{C}(\mathbb{I} \times [a,b], [0,\infty))$, $\liminf_{t\to\infty} g(t,\xi) = \infty$, and $g(t,a) \leq g(t,\xi)$ for $\xi \in [a,b]$;
(H$_4$) $\tau \in \mathrm{C}^2(\mathbb{I}, \mathbb{R})$, $\tau'(t) > 0$, $\lim_{t\to\infty} \tau(t) = \infty$, and $g(\tau(t),\xi) = \tau[g(t,\xi)]$;
(H$_5$) $\sigma \in \mathrm{C}([a,b], \mathbb{R})$ is nondecreasing and the integral of (1.1) is taken in the sense of Riemann-Stieltjes.

By a solution of (1.1), we mean a function $x \in \mathrm{C}([t_x,\infty), \mathbb{R})$ for some $t_x \geq t_0$, which has the properties that $z \in \mathrm{C}^1([t_x,\infty), \mathbb{R})$, $r|z'|^{\alpha-1}z' \in \mathrm{C}^1([t_x,\infty), \mathbb{R})$, and satisfies (1.1) on $[t_x,\infty)$. We restrict our attention to those solutions x of (1.1) which exist on $[t_x,\infty)$ and satisfy $\sup\{|x(t)| : t \geq T\} > 0$ for any $T \geq t_x$. A solution x of (1.1) is termed oscillatory if it is neither eventually positive nor eventually negative; otherwise, it is called nonoscillatory. Equation (1.1) is said to be oscillatory if all its solutions oscillate.

As is well known, neutral differential equations have a great number of applications in electric networks. For instance, they are frequently used in the study of distributed networks containing lossless transmission lines, which rise in high speed computers, where the lossless transmission lines are used to interconnect switching circuits; see [1]. Hence, there has been much research activity concerning oscillatory and nonoscillatory behavior of solutions to different classes of neutral differential equations, we refer the reader to [2–30] and the references cited therein.

In the following, we present some background details that motivate our research. Recently, Baculíková and Lacková [6], Džurina and Hudáková [12], Li *et al.* [15, 18], and Sun *et al.* [22] established some oscillation criteria for the second-order half-linear neutral differential equation

$$\left(r(t)\left|z'(t)\right|^{\alpha-1}z'(t)\right)' + q(t)\left|x\left(\delta(t)\right)\right|^{\alpha-1}x\left(\delta(t)\right) = 0,$$

where $z := x + p \cdot x \circ \tau$,

$$0 \le p(t) < 1 \quad \text{or} \quad p(t) > 1.$$

Baculíková and Džurina [4, 5] and Li *et al.* [17] investigated oscillatory behavior of a second-order neutral differential equation

$$\left(r(t)\left(x(t) + p(t)x\left[\tau(t)\right]\right)'\right)' + q(t)x\left[\sigma(t)\right] = 0,$$

where

$$0 \le p(t) \le p_0 < \infty \quad \text{and} \quad \tau'(t) \ge \tau_0 > 0. \tag{1.2}$$

Ye and Xu [26] and Yu and Fu [27] considered oscillation of the second-order differential equation

$$\left(x(t) + p(t)x(t-\tau)\right)'' + \int_a^b q(t,\xi)x\left(g(t,\xi)\right)\,\mathrm{d}\sigma(\xi) = 0.$$

Assuming $0 \le p(t) < 1$, Thandapani and Piramanantham [23], Wang [24], Xu and Weng [25], and Zhao and Meng [30] studied oscillation of an equation

$$\left(r(t)\left(x(t) + p(t)x(t-\tau)\right)'\right)' + \int_a^b q(t,\xi)f\left(x\left(g(t,\xi)\right)\right)\,\mathrm{d}\sigma(\xi) = 0.$$

As yet, there are few results regarding the study of oscillatory properties of (1.1) under the conditions $p(t) \ge 1$ or $\lim_{t\to\infty} p(t) = \infty$. Thereinto, Li and Thandapani [19] obtained several oscillation results for (1.1) in the case where (1.2) holds, $\sigma(\xi) = \xi$, and

$$\int_{t_0}^{\infty} \frac{\mathrm{d}t}{r^{1/\alpha}(t)} = \infty. \tag{1.3}$$

In the subsequent sections, we shall utilize the Riccati substitution technique and some inequalities to establish several new oscillation criteria for (1.1) assuming that (1.3) holds or

$$\int_{t_0}^{\infty} \frac{\mathrm{d}t}{r^{1/\alpha}(t)} < \infty. \tag{1.4}$$

All functional inequalities are assumed to hold eventually, that is, they are satisfied for all t large enough.

2 Main results

In what follows, we use the following notation for the convenience of the reader:

$$\begin{aligned}
&Q(t,\xi) := \min\{q(t,\xi), q(\tau(t),\xi)\}, \qquad d_+(t) := \max\{0, d(t)\},\\
&\phi(t) := \frac{\alpha p'[h(t)]h'(t)}{p[h(t)]} - \frac{\tau''(t)}{\tau'(t)}, \qquad \zeta(t) := \frac{\rho_+'(t)}{\rho(t)} + \phi(t),\\
&\varphi(t) := \left(\frac{\rho_+'(t)}{\rho(t)}\right)^{\alpha+1} + \frac{p^\alpha[h(t)](\zeta_+(t))^{\alpha+1}}{\tau'(t)}, \quad \text{and} \quad \delta(t) := \int_{\eta(t)}^{\infty} \frac{\mathrm{d}s}{r^{1/\alpha}(s)},
\end{aligned}$$

where h, ρ, and η will be specified later.

Theorem 2.1 *Assume* (H_1)-(H_5), (1.3), *and let* $g(t,a) \in C^1(\mathbb{I},\mathbb{R})$, $g'(t,a) > 0$, $g(t,a) \le t$, *and* $g(t,a) \le \tau(t)$ *for* $t \in \mathbb{I}$. *Suppose further that there exists a real-valued function* $h \in C^1(\mathbb{I},\mathbb{R})$ *such that* $p[g(t,\xi)] \le p[h(t)]$ *for* $t \in \mathbb{I}$ *and* $\xi \in [a,b]$. *If there exists a real-valued function* $\rho \in C^1(\mathbb{I},(0,\infty))$ *such that*

$$\limsup_{t\to\infty} \int_{t_0}^{t} \rho(s)\left[\frac{\int_a^b Q(s,\xi)\,\mathrm{d}\sigma(\xi)}{2^{\alpha-1}} - \frac{r[g(s,a)]\varphi(s)}{(\alpha+1)^{\alpha+1}(g'(s,a))^\alpha}\right]\mathrm{d}s = \infty, \tag{2.1}$$

then (1.1) *is oscillatory.*

Proof Let x be a nonoscillatory solution of (1.1). Without loss of generality, we assume that there exists a $t_1 \in \mathbb{I}$ such that $x(t) > 0$, $x[\tau(t)] > 0$, and $x[g(t,\xi)] > 0$ for all $t \ge t_1$ and $\xi \in [a,b]$. Then $z(t) > 0$. Applying (1.1), one has, for all sufficiently large t,

$$\begin{aligned}
&\left(r(t)\left|z'(t)\right|^{\alpha-1}z'(t)\right)' + \int_a^b q(t,\xi)x^\alpha\left[g(t,\xi)\right]\mathrm{d}\sigma(\xi)\\
&\quad + \int_a^b q\left(\tau(t),\xi\right)p^\alpha\left[h(t)\right]x^\alpha\left[g\left(\tau(t),\xi\right)\right]\mathrm{d}\sigma(\xi)\\
&\quad + \frac{p^\alpha[h(t)]}{\tau'(t)}\left(r\left[\tau(t)\right]\left|z'\left[\tau(t)\right]\right|^{\alpha-1}z'\left[\tau(t)\right]\right)' = 0.
\end{aligned}$$

Using the inequality (see [5, Lemma 1])

$$(A+B)^\alpha \le 2^{\alpha-1}\left(A^\alpha + B^\alpha\right), \quad \text{for } A \ge 0, B \ge 0, \text{ and } \alpha \ge 1,$$

the definition of z, $g(\tau(t),\xi) = \tau[g(t,\xi)]$, and $p[g(t,\xi)] \le p[h(t)]$, we conclude that

$$\begin{aligned}&\bigl(r(t)\bigl|z'(t)\bigr|^{\alpha-1}z'(t)\bigr)' + \frac{1}{2^{\alpha-1}}\int_a^b Q(t,\xi)z^{\alpha}\bigl[g(t,\xi)\bigr]\,\mathrm{d}\sigma(\xi)\\&\quad+\frac{p^{\alpha}[h(t)]}{\tau'(t)}\bigl(r\bigl[\tau(t)\bigr]\bigl|z'\bigl[\tau(t)\bigr]\bigr|^{\alpha-1}z'\bigl[\tau(t)\bigr]\bigr)' \le 0.\end{aligned} \tag{2.2}$$

By virtue of (1.1), we get

$$\bigl(r(t)\bigl|z'(t)\bigr|^{\alpha-1}z'(t)\bigr)' \le 0, \quad t \ge t_1. \tag{2.3}$$

Thus, $r|z'|^{\alpha-1}z'$ is nonincreasing. Now we have two possible cases for the sign of z': (i) $z' < 0$ eventually, or (ii) $z' > 0$ eventually.

(i) Assume that $z'(t) < 0$ for $t \ge t_2 \ge t_1$. Then we have by (2.3)

$$r(t)\bigl|z'(t)\bigr|^{\alpha-1}z'(t) \le r(t_2)\bigl|z'(t_2)\bigr|^{\alpha-1}z'(t_2) < 0, \quad t \ge t_2,$$

which yields

$$z(t) \le z(t_2) - r^{1/\alpha}(t_2)\bigl|z'(t_2)\bigr|\int_{t_2}^t r^{-1/\alpha}(s)\,\mathrm{d}s.$$

Then we obtain $\lim_{t\to\infty} z(t) = -\infty$ due to (1.3), which is a contradiction.

(ii) Assume that $z'(t) > 0$ for $t \ge t_2 \ge t_1$. It follows from (2.2) and $g(t,\xi) \ge g(t,a)$ that

$$\begin{aligned}&\bigl(r(t)\bigl(z'(t)\bigr)^{\alpha}\bigr)' + \frac{p^{\alpha}[h(t)]}{\tau'(t)}\bigl(r\bigl[\tau(t)\bigr]\bigl(z'\bigl[\tau(t)\bigr]\bigr)^{\alpha}\bigr)'\\&\quad+\frac{1}{2^{\alpha-1}}z^{\alpha}\bigl[g(t,a)\bigr]\int_a^b Q(t,\xi)\,\mathrm{d}\sigma(\xi) \le 0.\end{aligned} \tag{2.4}$$

We define a Riccati substitution

$$\omega(t) := \rho(t)\frac{r(t)(z'(t))^{\alpha}}{(z[g(t,a)])^{\alpha}}, \quad t \ge t_2. \tag{2.5}$$

Then $\omega(t) > 0$. From (2.3) and $g(t,a) \le t$, we have

$$z'\bigl[g(t,a)\bigr] \ge \bigl(r(t)/r\bigl[g(t,a)\bigr]\bigr)^{1/\alpha}z'(t). \tag{2.6}$$

Differentiating (2.5), we get

$$\begin{aligned}\omega'(t) &= \rho'(t)\frac{r(t)(z'(t))^{\alpha}}{(z[g(t,a)])^{\alpha}} + \rho(t)\frac{(r(t)(z'(t))^{\alpha})'}{(z[g(t,a)])^{\alpha}}\\&\quad-\alpha\rho(t)\frac{r(t)(z'(t))^{\alpha}z^{\alpha-1}[g(t,a)]z'[g(t,a)]g'(t,a)}{(z[g(t,a)])^{2\alpha}}.\end{aligned} \tag{2.7}$$

Therefore, by (2.5), (2.6), and (2.7), we see that

$$\omega'(t) \le \frac{\rho'(t)}{\rho(t)}\omega(t) + \rho(t)\frac{(r(t)(z'(t))^{\alpha})'}{(z[g(t,a)])^{\alpha}} - \frac{\alpha g'(t,a)}{\rho^{1/\alpha}(t)r^{1/\alpha}[g(t,a)]}\omega^{(\alpha+1)/\alpha}(t). \tag{2.8}$$

Similarly, we introduce another Riccati transformation:

$$\upsilon(t) := \rho(t)\frac{r[\tau(t)](z'[\tau(t)])^{\alpha}}{(z[g(t,a)])^{\alpha}}, \quad t \geq t_2. \tag{2.9}$$

Then $\upsilon(t) > 0$. From (2.3) and $g(t,a) \leq \tau(t)$, we obtain

$$z'\big[g(t,a)\big] \geq \big(r\big[\tau(t)\big]/r\big[g(t,a)\big]\big)^{1/\alpha} z'\big[\tau(t)\big]. \tag{2.10}$$

Differentiating (2.9), we have

$$\begin{aligned}\upsilon'(t) = {}& \rho'(t)\frac{r[\tau(t)](z'[\tau(t)])^{\alpha}}{(z[g(t,a)])^{\alpha}} + \rho(t)\frac{(r[\tau(t)](z'[\tau(t)])^{\alpha})'}{(z[g(t,a)])^{\alpha}} \\ & - \alpha\rho(t)\frac{r[\tau(t)](z'[\tau(t)])^{\alpha} z^{\alpha-1}[g(t,a)]z'[g(t,a)]g'(t,a)}{(z[g(t,a)])^{2\alpha}}.\end{aligned} \tag{2.11}$$

Therefore, by (2.9), (2.10), and (2.11), we find

$$\upsilon'(t) \leq \frac{\rho'(t)}{\rho(t)}\upsilon(t) + \rho(t)\frac{(r[\tau(t)](z'[\tau(t)])^{\alpha})'}{(z[g(t,a)])^{\alpha}} - \frac{\alpha g'(t,a)}{\rho^{1/\alpha}(t)r^{1/\alpha}[g(t,a)]}\upsilon^{(\alpha+1)/\alpha}(t). \tag{2.12}$$

Combining (2.8) and (2.12), we get

$$\begin{aligned}&\omega'(t) + \frac{p^{\alpha}[h(t)]}{\tau'(t)}\upsilon'(t) \\ &\quad \leq \rho(t)\frac{(r(t)(z'(t))^{\alpha})' + \frac{p^{\alpha}[h(t)]}{\tau'(t)}(r[\tau(t)](z'[\tau(t)])^{\alpha})'}{(z[g(t,a)])^{\alpha}} + \frac{\rho'(t)}{\rho(t)}\omega(t) \\ &\qquad - \frac{\alpha g'(t,a)}{\rho^{1/\alpha}(t)r^{1/\alpha}[g(t,a)]}\omega^{(\alpha+1)/\alpha}(t) + \frac{p^{\alpha}[h(t)]}{\tau'(t)}\frac{\rho'(t)}{\rho(t)}\upsilon(t) \\ &\qquad - \frac{p^{\alpha}[h(t)]}{\tau'(t)}\frac{\alpha g'(t,a)}{\rho^{1/\alpha}(t)r^{1/\alpha}[g(t,a)]}\upsilon^{(\alpha+1)/\alpha}(t).\end{aligned}$$

It follows from (2.4) that

$$\begin{aligned}\omega'(t) + \frac{p^{\alpha}[h(t)]}{\tau'(t)}\upsilon'(t) \leq {}& -\frac{\rho(t)}{2^{\alpha-1}}\int_a^b Q(t,\xi)\,\mathrm{d}\sigma(\xi) + \frac{\rho'_+(t)}{\rho(t)}\omega(t) \\ & - \frac{\alpha g'(t,a)}{\rho^{1/\alpha}(t)r^{1/\alpha}[g(t,a)]}\omega^{(\alpha+1)/\alpha}(t) + \frac{p^{\alpha}[h(t)]}{\tau'(t)}\frac{\rho'_+(t)}{\rho(t)}\upsilon(t) \\ & - \frac{p^{\alpha}[h(t)]}{\tau'(t)}\frac{\alpha g'(t,a)}{\rho^{1/\alpha}(t)r^{1/\alpha}[g(t,a)]}\upsilon^{(\alpha+1)/\alpha}(t).\end{aligned}$$

Integrating the latter inequality from t_2 to t, we obtain

$$\begin{aligned}&\omega(t) - \omega(t_2) + \frac{p^{\alpha}[h(t)]}{\tau'(t)}\upsilon(t) - \frac{p^{\alpha}[h(t_2)]}{\tau'(t_2)}\upsilon(t_2) \\ &\quad \leq -\int_{t_2}^{t}\frac{\rho(s)}{2^{\alpha-1}}\int_a^b Q(s,\xi)\,\mathrm{d}\sigma(\xi)\,\mathrm{d}s\end{aligned}$$

$$+\int_{t_2}^{t}\left[\frac{\rho'_+(s)}{\rho(s)}\omega(s)-\frac{\alpha g'(s,a)}{\rho^{1/\alpha}(s)r^{1/\alpha}[g(s,a)]}\omega^{(\alpha+1)/\alpha}(s)\right]\mathrm{d}s$$
$$+\int_{t_2}^{t}\frac{p^{\alpha}[h(s)]}{\tau'(s)}\left\{\left[\frac{\rho'_+(s)}{\rho(s)}+\phi(s)\right]_+\upsilon(s)-\frac{\alpha g'(s,a)}{\rho^{1/\alpha}(s)r^{1/\alpha}[g(s,a)]}\upsilon^{(\alpha+1)/\alpha}(s)\right\}\mathrm{d}s. \tag{2.13}$$

Define

$$A:=\left[\frac{\alpha g'(t,a)}{\rho^{1/\alpha}(t)r^{1/\alpha}[g(t,a)]}\right]^{\alpha/(\alpha+1)}\omega(t)\quad\text{and}$$
$$B:=\left[\frac{\alpha}{\alpha+1}\frac{\rho'_+(t)}{\rho(t)}\left[\frac{\alpha g'(t,a)}{\rho^{1/\alpha}(t)r^{1/\alpha}[g(t,a)]}\right]^{-\alpha/(\alpha+1)}\right]^{\alpha}.$$

Using the inequality

$$\frac{\alpha+1}{\alpha}AB^{1/\alpha}-A^{(\alpha+1)/\alpha}\le\frac{1}{\alpha}B^{(\alpha+1)/\alpha},\quad\text{for } A\ge 0 \text{ and } B\ge 0, \tag{2.14}$$

we get

$$\frac{\rho'_+(t)}{\rho(t)}\omega(t)-\frac{\alpha g'(t,a)}{\rho^{1/\alpha}(t)r^{1/\alpha}[g(t,a)]}\omega^{(\alpha+1)/\alpha}(t)\le\frac{1}{(\alpha+1)^{\alpha+1}}\frac{r[g(t,a)](\rho'_+(t))^{\alpha+1}}{(\rho(t)g'(t,a))^{\alpha}}.$$

On the other hand, define

$$A:=\left[\frac{\alpha g'(t,a)}{\rho^{1/\alpha}(t)r^{1/\alpha}[g(t,a)]}\right]^{\alpha/(\alpha+1)}\upsilon(t)\quad\text{and}$$
$$B:=\left[\frac{\alpha}{\alpha+1}\zeta_+(t)\left[\frac{\alpha g'(t,a)}{\rho^{1/\alpha}(t)r^{1/\alpha}[g(t,a)]}\right]^{-\alpha/(\alpha+1)}\right]^{\alpha}.$$

Then we have by (2.14)

$$\zeta_+(t)\upsilon(t)-\frac{\alpha g'(t,a)}{\rho^{1/\alpha}(t)r^{1/\alpha}[g(t,a)]}\upsilon^{(\alpha+1)/\alpha}(t)\le\frac{1}{(\alpha+1)^{\alpha+1}}\frac{r[g(t,a)](\zeta_+(t))^{\alpha+1}\rho(t)}{(g'(t,a))^{\alpha}}.$$

Thus, from (2.13), we get

$$\omega(t)-\omega(t_2)+\frac{p^{\alpha}[h(t)]}{\tau'(t)}\upsilon(t)-\frac{p^{\alpha}[h(t_2)]}{\tau'(t_2)}\upsilon(t_2)$$
$$\le-\int_{t_2}^{t}\rho(s)\left\{\frac{\int_a^b Q(s,\xi)\,\mathrm{d}\sigma(\xi)}{2^{\alpha-1}}-\frac{r[g(s,a)]}{(\alpha+1)^{\alpha+1}(g'(s,a))^{\alpha}}\right.$$
$$\left.\times\left[\left(\frac{\rho'_+(s)}{\rho(s)}\right)^{\alpha+1}+\frac{p^{\alpha}[h(s)](\zeta_+(s))^{\alpha+1}}{\tau'(s)}\right]\right\}\mathrm{d}s,$$

which contradicts (2.1). This completes the proof. □

Assuming (1.2), where p_0 and τ_0 are constants, we obtain the following result.

Theorem 2.2 *Suppose* (H_1)-(H_5), (1.2), (1.3), *and let* $g(t,a)\in \mathrm{C}^1(\mathbb{I},\mathbb{R})$, $g'(t,a)>0$, $g(t,a)\le t$, *and* $g(t,a)\le\tau(t)$ *for* $t\in\mathbb{I}$. *If there exists a real-valued function* $\rho\in \mathrm{C}^1(\mathbb{I},(0,\infty))$ *such* *that*

$$\limsup_{t\to\infty}\int_{t_0}^{t}\left[\frac{\rho(s)\int_a^b Q(s,\xi)\,\mathrm{d}\sigma(\xi)}{2^{\alpha-1}}-\frac{1+\frac{p_0{}^\alpha}{\tau_0}}{(\alpha+1)^{\alpha+1}}\frac{r[g(s,a)](\rho_+'(s))^{\alpha+1}}{(\rho(s)g'(s,a))^\alpha}\right]\mathrm{d}s=\infty, \tag{2.15}$$

then (1.1) *is oscillatory.*

Proof As above, let x be an eventually positive solution of (1.1). Proceeding as in the proof of Theorem 2.1, we have $z'(t) > 0$, (2.3), and (2.4) for all sufficiently large t. Using (1.2), (2.3), and (2.4), we obtain

$$\begin{aligned}&\bigl(r(t)\bigl(z'(t)\bigr)^\alpha\bigr)'+\frac{p_0{}^\alpha}{\tau_0}\bigl(r\bigl[\tau(t)\bigr]\bigl(z'\bigl[\tau(t)\bigr]\bigr)^\alpha\bigr)'\\&\quad+\frac{1}{2^{\alpha-1}}z^\alpha\bigl[g(t,a)\bigr]\int_a^b Q(s,\xi)\,\mathrm{d}\sigma(\xi)\le 0.\end{aligned} \tag{2.16}$$

The remainder of the proof is similar to that of Theorem 2.1, and hence it is omitted. □

Theorem 2.3 *Suppose we have* (H_1)-(H_5), (1.3), *and let* $\tau(t)\le t$ *and* $g(t,a)\ge\tau(t)$ *for* $t\in\mathbb{I}$. *Assume also that there exists a real-valued function* $h\in \mathrm{C}^1(\mathbb{I},\mathbb{R})$ *such that* $p[g(t,\xi)]\le p[h(t)]$ *for* $t\in\mathbb{I}$ *and* $\xi\in[a,b]$. *If there exists a real-valued function* $\rho\in\mathrm{C}^1(\mathbb{I},(0,\infty))$ *such that*

$$\limsup_{t\to\infty}\int_{t_0}^{t}\rho(s)\left[\frac{\int_a^b Q(s,\xi)\,\mathrm{d}\sigma(\xi)}{2^{\alpha-1}}-\frac{r[\tau(s)]\varphi(s)}{(\alpha+1)^{\alpha+1}(\tau'(s))^\alpha}\right]\mathrm{d}s=\infty, \tag{2.17}$$

then (1.1) *is oscillatory.*

Proof Let x be a nonoscillatory solution of (1.1). Without loss of generality, we assume that there exists a $t_1\in\mathbb{I}$ such that $x(t)>0$, $x[\tau(t)]>0$, and $x[g(t,\xi)]>0$ for all $t\ge t_1$ and $\xi\in[a,b]$. As in the proof of Theorem 2.1, we obtain (2.3) and (2.4). In view of (2.3), $r|z'|^{\alpha-1}z'$ is nonincreasing. Now we have two possible cases for the sign of z': (i) $z'<0$ eventually, or (ii) $z'>0$ eventually.

(i) Suppose that $z'(t)<0$ for $t\ge t_2\ge t_1$. Then, with a proof similar to the proof of case (i) in Theorem 2.1, we obtain a contradiction.

(ii) Suppose that $z'(t)>0$ for $t\ge t_2\ge t_1$. We define a Riccati substitution

$$\omega(t):=\rho(t)\frac{r(t)(z'(t))^\alpha}{(z[\tau(t)])^\alpha},\quad t\ge t_2. \tag{2.18}$$

Then $\omega(t)>0$. From (2.3) and $\tau(t)\le t$, we have

$$z'\bigl[\tau(t)\bigr]\ge\bigl(r(t)/r\bigl[\tau(t)\bigr]\bigr)^{1/\alpha}z'(t). \tag{2.19}$$

Differentiating (2.18), we obtain

$$\begin{aligned}\omega'(t)&=\rho'(t)\frac{r(t)(z'(t))^\alpha}{(z[\tau(t)])^\alpha}+\rho(t)\frac{(r(t)(z'(t))^\alpha)'}{(z[\tau(t)])^\alpha}\\&\quad-\alpha\rho(t)\frac{r(t)(z'(t))^\alpha z^{\alpha-1}[\tau(t)]z'[\tau(t)]\tau'(t)}{(z[\tau(t)])^{2\alpha}}.\end{aligned} \tag{2.20}$$

Therefore, by (2.18), (2.19), and (2.20), we see that

$$\omega'(t) \le \frac{\rho'(t)}{\rho(t)}\omega(t) + \rho(t)\frac{(r(t)(z'(t))^{\alpha})'}{(z[\tau(t)])^{\alpha}} - \frac{\alpha\tau'(t)}{\rho^{1/\alpha}(t)r^{1/\alpha}[\tau(t)]}\omega^{(\alpha+1)/\alpha}(t). \tag{2.21}$$

Similarly, we introduce another Riccati substitution:

$$\upsilon(t) := \rho(t)\frac{r[\tau(t)](z'[\tau(t)])^{\alpha}}{(z[\tau(t)])^{\alpha}}, \quad t \ge t_2. \tag{2.22}$$

Then $\upsilon(t) > 0$. Differentiating (2.22), we have

$$\begin{aligned}\upsilon'(t) = {} & \rho'(t)\frac{r[\tau(t)](z'[\tau(t)])^{\alpha}}{(z[\tau(t)])^{\alpha}} + \rho(t)\frac{(r[\tau(t)](z'[\tau(t)])^{\alpha})'}{(z[\tau(t)])^{\alpha}} \\ & - \alpha\rho(t)\frac{r[\tau(t)](z'[\tau(t)])^{\alpha}z^{\alpha-1}[\tau(t)]z'[\tau(t)]\tau'(t)}{(z[\tau(t)])^{2\alpha}}.\end{aligned} \tag{2.23}$$

Therefore, by (2.22) and (2.23), we get

$$\upsilon'(t) = \frac{\rho'(t)}{\rho(t)}\upsilon(t) + \rho(t)\frac{(r[\tau(t)](z'[\tau(t)])^{\alpha})'}{(z[\tau(t)])^{\alpha}} - \frac{\alpha\tau'(t)}{\rho^{1/\alpha}(t)r^{1/\alpha}[\tau(t)]}\upsilon^{(\alpha+1)/\alpha}(t). \tag{2.24}$$

Combining (2.21) and (2.24), we have

$$\begin{aligned}\omega'(t) + \frac{p^{\alpha}[h(t)]}{\tau'(t)}\upsilon'(t) \le {} & \rho(t)\frac{(r(t)(z'(t))^{\alpha})' + \frac{p^{\alpha}[h(t)]}{\tau'(t)}(r[\tau(t)](z'[\tau(t)])^{\alpha})'}{(z[\tau(t)])^{\alpha}} + \frac{\rho'(t)}{\rho(t)}\omega(t) \\ & - \frac{\alpha\tau'(t)}{\rho^{1/\alpha}(t)r^{1/\alpha}[\tau(t)]}\omega^{(\alpha+1)/\alpha}(t) + \frac{p^{\alpha}[h(t)]}{\tau'(t)}\frac{\rho'(t)}{\rho(t)}\upsilon(t) \\ & - \frac{p^{\alpha}[h(t)]}{\tau'(t)}\frac{\alpha\tau'(t)}{\rho^{1/\alpha}(t)r^{1/\alpha}[\tau(t)]}\upsilon^{(\alpha+1)/\alpha}(t).\end{aligned}$$

It follows from (2.4) and $g(t,a) \ge \tau(t)$ that

$$\begin{aligned}\omega'(t) + \frac{p^{\alpha}[h(t)]}{\tau'(t)}\upsilon'(t) \le {} & -\frac{\rho(t)}{2^{\alpha-1}}\int_a^b Q(t,\xi)\,\mathrm{d}\sigma(\xi) + \frac{\rho'_+(t)}{\rho(t)}\omega(t) \\ & - \frac{\alpha\tau'(t)}{\rho^{1/\alpha}(t)r^{1/\alpha}[\tau(t)]}\omega^{(\alpha+1)/\alpha}(t) + \frac{p^{\alpha}[h(t)]}{\tau'(t)}\frac{\rho'_+(t)}{\rho(t)}\upsilon(t) \\ & - \frac{p^{\alpha}[h(t)]}{\tau'(t)}\frac{\alpha\tau'(t)}{\rho^{1/\alpha}(t)r^{1/\alpha}[\tau(t)]}\upsilon^{(\alpha+1)/\alpha}(t).\end{aligned}$$

Integrating the latter inequality from t_2 to t, we obtain

$$\begin{aligned}& \omega(t) - \omega(t_2) + \frac{p^{\alpha}[h(t)]}{\tau'(t)}\upsilon(t) - \frac{p^{\alpha}[h(t_2)]}{\tau'(t_2)}\upsilon(t_2) \\ & \quad \le -\int_{t_2}^t \frac{\rho(s)}{2^{\alpha-1}}\int_a^b Q(s,\xi)\,\mathrm{d}\sigma(\xi)\,\mathrm{d}s + \int_{t_2}^t \left[\frac{\rho'_+(s)}{\rho(s)}\omega(s) - \frac{\alpha\tau'(s)}{\rho^{1/\alpha}(s)r^{1/\alpha}[\tau(s)]}\omega^{(\alpha+1)/\alpha}(s)\right]\mathrm{d}s \\ & \qquad + \int_{t_2}^t \frac{p^{\alpha}[h(s)]}{\tau'(s)}\left\{\left[\frac{\rho'_+(s)}{\rho(s)} + \phi(s)\right]_+\upsilon(s) - \frac{\alpha\tau'(s)}{\rho^{1/\alpha}(s)r^{1/\alpha}[\tau(s)]}\upsilon^{(\alpha+1)/\alpha}(s)\right\}\mathrm{d}s.\end{aligned} \tag{2.25}$$

Define

$$A := \left[\frac{\alpha\tau'(t)}{\rho^{1/\alpha}(t)r^{1/\alpha}[\tau(t)]}\right]^{\alpha/(\alpha+1)}\omega(t) \quad \text{and}$$

$$B := \left[\frac{\alpha}{\alpha+1}\frac{\rho'_+(t)}{\rho(t)}\left[\frac{\alpha\tau'(t)}{\rho^{1/\alpha}(t)r^{1/\alpha}[\tau(t)]}\right]^{-\alpha/(\alpha+1)}\right]^{\alpha}.$$

Using inequality (2.14), we have

$$\frac{\rho'_+(t)}{\rho(t)}\omega(t) - \frac{\alpha\tau'(t)}{\rho^{1/\alpha}(t)r^{1/\alpha}[\tau(t)]}\omega^{(\alpha+1)/\alpha}(t) \le \frac{1}{(\alpha+1)^{\alpha+1}}\frac{r[\tau(t)](\rho'_+(t))^{\alpha+1}}{(\rho(t)\tau'(t))^{\alpha}}.$$

On the other hand, define

$$A := \left[\frac{\alpha\tau'(t)}{\rho^{1/\alpha}(t)r^{1/\alpha}[\tau(t)]}\right]^{\alpha/(\alpha+1)}\upsilon(t) \quad \text{and}$$

$$B := \left[\frac{\alpha}{\alpha+1}\zeta_+(t)\left[\frac{\alpha\tau'(t)}{\rho^{1/\alpha}(t)r^{1/\alpha}[\tau(t)]}\right]^{-\alpha/(\alpha+1)}\right]^{\alpha}.$$

Then, by (2.14), we obtain

$$\zeta_+(t)\upsilon(t) - \frac{\alpha\tau'(t)}{\rho^{1/\alpha}(t)r^{1/\alpha}[\tau(t)]}\upsilon^{(\alpha+1)/\alpha}(t) \le \frac{1}{(\alpha+1)^{\alpha+1}}\frac{r[\tau(t)](\zeta_+(t))^{\alpha+1}\rho(t)}{(\tau'(t))^{\alpha}}.$$

Thus, from (2.25), we get

$$\begin{aligned}
&\omega(t) - \omega(t_2) + \frac{p^{\alpha}[h(t)]}{\tau'(t)}\upsilon(t) - \frac{p^{\alpha}[h(t_2)]}{\tau'(t_2)}\upsilon(t_2) \\
&\quad\le -\int_{t_2}^{t}\rho(s)\left\{\frac{\int_a^b Q(s,\xi)\,\mathrm{d}\sigma(\xi)}{2^{\alpha-1}} - \frac{r[\tau(s)]}{(\alpha+1)^{\alpha+1}(\tau'(s))^{\alpha}}\right. \\
&\qquad\left.\times\left[\left(\frac{\rho'_+(s)}{\rho(s)}\right)^{\alpha+1} + \frac{p^{\alpha}[h(s)](\zeta_+(s))^{\alpha+1}}{\tau'(s)}\right]\right\}\mathrm{d}s,
\end{aligned}$$

which contradicts (2.17). This completes the proof. □

Assuming we have (1.2), where p_0 and τ_0 are constants, we get the following result.

Theorem 2.4 *Suppose we have* (H_1)-(H_5), (1.2), (1.3), *and let* $\tau(t) \le t$ *and* $g(t,a) \ge \tau(t)$ *for* $t \in \mathbb{I}$. *If there exists a real-valued function* $\rho \in C^1(\mathbb{I},(0,\infty))$ *such that*

$$\limsup_{t\to\infty}\int_{t_0}^{t}\left[\frac{\rho(s)\int_a^b Q(s,\xi)\,\mathrm{d}\sigma(\xi)}{2^{\alpha-1}} - \frac{1}{(\alpha+1)^{\alpha+1}}\left(1+\frac{{p_0}^{\alpha}}{\tau_0}\right)\frac{r[\tau(s)](\rho'_+(s))^{\alpha+1}}{(\tau_0\rho(s))^{\alpha}}\right]\mathrm{d}s = \infty, \tag{2.26}$$

then (1.1) *is oscillatory.*

Proof Assume again that x is an eventually positive solution of (1.1). As in the proof of Theorem 2.1, we have $z'(t) > 0$, (2.3), and (2.4) for all sufficiently large t. By virtue of (1.2), (2.3), and (2.4), we have (2.16) for all sufficiently large t. The rest of the proof is similar to that of Theorem 2.3, and so it is omitted. □

In the following, we present some oscillation criteria for (1.1) in the case where (1.4) holds.

Theorem 2.5 *Suppose we have* (H_1)-(H_5), (1.2), (1.4), *and let* $g(t,a) \in C^1(\mathbb{I},\mathbb{R})$, $g'(t,a) > 0$, $g(t,a) \leq \tau(t) \leq t$ *for* $t \in \mathbb{I}$, *and* $g(t,\xi) \leq g(t,b)$ *for* $\xi \in [a,b]$. *Assume further that there exists a real-valued function* $\rho \in C^1(\mathbb{I},(0,\infty))$ *such that* (2.15) *is satisfied. If there exists a real-valued function* $\eta \in C^1(\mathbb{I},\mathbb{R})$ *such that* $\eta(t) \geq t$, $\eta(t) \geq g(t,b)$, $\eta'(t) > 0$ *for* $t \in \mathbb{I}$, *and*

$$\limsup_{t\to\infty} \int_{t_0}^{t} \left[\frac{\int_a^b Q(s,\xi)\,\mathrm{d}\sigma(\xi)}{2^{\alpha-1}} \delta^\alpha(s) - \left(1 + \frac{p_0{}^\alpha}{\tau_0}\right)\left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1} \frac{\eta'(s)}{\delta(s) r^{1/\alpha}[\eta(s)]} \right] \mathrm{d}s = \infty, \tag{2.27}$$

then (1.1) *is oscillatory.*

Proof Let x be a nonoscillatory solution of (1.1). Without loss of generality, we assume that there exists a $t_1 \in \mathbb{I}$ such that $x(t) > 0$, $x[\tau(t)] > 0$, and $x[g(t,\xi)] > 0$ for all $t \geq t_1$ and $\xi \in [a,b]$. Then $z(t) > 0$. As in the proof of Theorem 2.1, we get (2.2). By virtue of (1.1), we have (2.3). Thus, $r|z'|^{\alpha-1}z'$ is nonincreasing. Now we have two possible cases for the sign of z': (i) $z' < 0$ eventually, or (ii) $z' > 0$ eventually.

(i) Suppose that $z'(t) > 0$ for $t \geq t_2 \geq t_1$. Then, by the proof of Theorem 2.2, we obtain a contradiction to (2.15).

(ii) Suppose that $z'(t) < 0$ for $t \geq t_2 \geq t_1$. It follows from (2.2), (2.3), and $g(t,\xi) \leq g(t,b)$ that

$$\begin{aligned}&\left(-r(t)\left(-z'(t)\right)^\alpha\right)' + \frac{p_0{}^\alpha}{\tau_0}\left(-r\left[\tau(t)\right]\left(-z'\left[\tau(t)\right]\right)^\alpha\right)' \\ &\quad + \frac{1}{2^{\alpha-1}} z^\alpha\left[g(t,b)\right] \int_a^b Q(t,\xi)\,\mathrm{d}\sigma(\xi) \leq 0.\end{aligned} \tag{2.28}$$

We define the function u by

$$u(t) := -\frac{r(t)(-z'(t))^\alpha}{z^\alpha[\eta(t)]}, \quad t \geq t_2. \tag{2.29}$$

Then $u(t) < 0$. Noting that $r(-z')^\alpha$ is nondecreasing, we get

$$z'(s) \leq \frac{r^{1/\alpha}(t)}{r^{1/\alpha}(s)} z'(t), \quad s \geq t \geq t_2.$$

Integrating this inequality from $\eta(t)$ to l, we obtain

$$z(l) \leq z\left[\eta(t)\right] + r^{1/\alpha}(t) z'(t) \int_{\eta(t)}^{l} \frac{\mathrm{d}s}{r^{1/\alpha}(s)}.$$

Letting $l \to \infty$, we have

$$0 \leq z\left[\eta(t)\right] + r^{1/\alpha}(t) z'(t)\delta(t).$$

That is,

$$-\delta(t)\frac{r^{1/\alpha}(t)z'(t)}{z[\eta(t)]} \le 1.$$

Thus, we get by (2.29)

$$-\delta^{\alpha}(t)u(t) \le 1. \tag{2.30}$$

Similarly, we define another function v by

$$v(t) := -\frac{r[\tau(t)](-z'[\tau(t)])^{\alpha}}{z^{\alpha}[\eta(t)]}, \quad t \ge t_2. \tag{2.31}$$

Then $v(t) < 0$. Noting that $r(-z')^{\alpha}$ is nondecreasing and $\tau(t) \le t$, we get

$$r(t)\big(-z'(t)\big)^{\alpha} \ge r\big[\tau(t)\big]\big(-z'\big[\tau(t)\big]\big)^{\alpha}.$$

Thus, $0 < -v(t) \le -u(t)$. Hence, by (2.30), we see that

$$-\delta^{\alpha}(t)v(t) \le 1. \tag{2.32}$$

Differentiating (2.29), we obtain

$$u'(t) = \frac{(-r(t)(-z'(t))^{\alpha})'z^{\alpha}[\eta(t)] + \alpha r(t)(-z'(t))^{\alpha}z^{\alpha-1}[\eta(t)]z'[\eta(t)]\eta'(t)}{z^{2\alpha}[\eta(t)]}.$$

By (2.3) and $\eta(t) \ge t$, we have $z'[\eta(t)] \le (r(t)/r[\eta(t)])^{1/\alpha}z'(t)$, and so

$$u'(t) \le \frac{(-r(t)(-z'(t))^{\alpha})'}{z^{\alpha}[\eta(t)]} - \alpha\frac{\eta'(t)}{r^{1/\alpha}[\eta(t)]}\big(-u(t)\big)^{(\alpha+1)/\alpha}. \tag{2.33}$$

Similarly, we see that

$$v'(t) \le \frac{(-r[\tau(t)](-z'[\tau(t)])^{\alpha})'}{z^{\alpha}[\eta(t)]} - \alpha\frac{\eta'(t)}{r^{1/\alpha}[\eta(t)]}\big(-v(t)\big)^{(\alpha+1)/\alpha}. \tag{2.34}$$

Combining (2.33) and (2.34), we get

$$\begin{aligned} u'(t) + \frac{{p_0}^{\alpha}}{\tau_0}v'(t) \le{}& \frac{(-r(t)(-z'(t))^{\alpha})'}{z^{\alpha}[\eta(t)]} + \frac{{p_0}^{\alpha}}{\tau_0}\frac{(-r[\tau(t)](-z'[\tau(t)])^{\alpha})'}{z^{\alpha}[\eta(t)]} \\ &- \alpha\frac{\eta'(t)}{r^{1/\alpha}[\eta(t)]}\big(-u(t)\big)^{(\alpha+1)/\alpha} - \frac{\alpha {p_0}^{\alpha}}{\tau_0}\frac{\eta'(t)}{r^{1/\alpha}[\eta(t)]}\big(-v(t)\big)^{(\alpha+1)/\alpha}. \end{aligned} \tag{2.35}$$

Using (2.28), (2.35), and $g(t,b) \le \eta(t)$, we obtain

$$\begin{aligned} u'(t) + \frac{{p_0}^{\alpha}}{\tau_0}v'(t) \le{}& -\frac{\int_a^b Q(t,\xi)\,\mathrm{d}\sigma(\xi)}{2^{\alpha-1}} - \alpha\frac{\eta'(t)}{r^{1/\alpha}[\eta(t)]}\big(-u(t)\big)^{(\alpha+1)/\alpha} \\ &- \frac{\alpha {p_0}^{\alpha}}{\tau_0}\frac{\eta'(t)}{r^{1/\alpha}[\eta(t)]}\big(-v(t)\big)^{(\alpha+1)/\alpha}. \end{aligned} \tag{2.36}$$

Multiplying (2.36) by $\delta^\alpha(t)$ and integrating the resulting inequality from t_2 to t, we have

$$\begin{aligned}
&u(t)\delta^\alpha(t) - u(t_2)\delta^\alpha(t_2) + \alpha\int_{t_2}^t \frac{\delta^{\alpha-1}(s)\eta'(s)u(s)}{r^{1/\alpha}[\eta(s)]}\,\mathrm{d}s + \alpha\int_{t_2}^t \frac{\eta'(s)\delta^\alpha(s)}{r^{1/\alpha}[\eta(s)]}\bigl(-u(s)\bigr)^{(\alpha+1)/\alpha}\,\mathrm{d}s \\
&\quad + \frac{p_0{}^\alpha}{\tau_0}v(t)\delta^\alpha(t) - \frac{p_0{}^\alpha}{\tau_0}v(t_2)\delta^\alpha(t_2) + \frac{\alpha p_0{}^\alpha}{\tau_0}\int_{t_2}^t \frac{\delta^{\alpha-1}(s)\eta'(s)v(s)}{r^{1/\alpha}[\eta(s)]}\,\mathrm{d}s \\
&\quad + \frac{\alpha p_0{}^\alpha}{\tau_0}\int_{t_2}^t \frac{\eta'(s)\delta^\alpha(s)}{r^{1/\alpha}[\eta(s)]}\bigl(-v(s)\bigr)^{(\alpha+1)/\alpha}\,\mathrm{d}s + \int_{t_2}^t \frac{\int_a^b Q(s,\xi)\,\mathrm{d}\sigma(\xi)}{2^{\alpha-1}}\delta^\alpha(s)\,\mathrm{d}s \le 0.
\end{aligned}$$

Set

$$\begin{aligned}
&A := -\left[\frac{\eta'(t)\delta^\alpha(t)}{r^{1/\alpha}[\eta(t)]}\right]^{(\alpha+1)/\alpha} u(t) \quad \text{and} \\
&B := \left[\frac{\alpha}{\alpha+1}\frac{\delta^{\alpha-1}(t)\eta'(t)}{r^{1/\alpha}[\eta(t)]}\left[\frac{\eta'(t)\delta^\alpha(t)}{r^{1/\alpha}[\eta(t)]}\right]^{-\alpha/(\alpha+1)}\right]^\alpha.
\end{aligned}$$

Using inequality (2.14), we get

$$\frac{\delta^{\alpha-1}(t)\eta'(t)u(t)}{r^{1/\alpha}[\eta(t)]} + \frac{\eta'(t)\delta^\alpha(t)}{r^{1/\alpha}[\eta(t)]}\bigl(-u(t)\bigr)^{(\alpha+1)/\alpha} \ge -\frac{1}{\alpha}\left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1}\frac{\eta'(t)}{\delta(t)r^{1/\alpha}[\eta(t)]}.$$

Similarly, we set

$$\begin{aligned}
&A := -\left[\frac{\eta'(t)\delta^\alpha(t)}{r^{1/\alpha}[\eta(t)]}\right]^{(\alpha+1)/\alpha} v(t) \quad \text{and} \\
&B := \left[\frac{\alpha}{\alpha+1}\frac{\delta^{\alpha-1}(t)\eta'(t)}{r^{1/\alpha}[\eta(t)]}\left[\frac{\eta'(t)\delta^\alpha(t)}{r^{1/\alpha}[\eta(t)]}\right]^{-\alpha/(\alpha+1)}\right]^\alpha.
\end{aligned}$$

Then we have by (2.14)

$$\frac{\delta^{\alpha-1}(t)\eta'(t)v(t)}{r^{1/\alpha}[\eta(t)]} + \frac{\eta'(t)\delta^\alpha(t)}{r^{1/\alpha}[\eta(t)]}\bigl(-v(t)\bigr)^{(\alpha+1)/\alpha} \ge -\frac{1}{\alpha}\left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1}\frac{\eta'(t)}{\delta(t)r^{1/\alpha}[\eta(t)]}.$$

Thus, from (2.30) and (2.32), we find

$$\begin{aligned}
&\int_{t_2}^t \left[\frac{\int_a^b Q(s,\xi)\,\mathrm{d}\sigma(\xi)}{2^{\alpha-1}}\delta^\alpha(s) - \left(1+\frac{p_0{}^\alpha}{\tau_0}\right)\left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1}\frac{\eta'(s)}{\delta(s)r^{1/\alpha}[\eta(s)]}\right]\mathrm{d}s \\
&\quad \le u(t_2)\delta^\alpha(t_2) + \frac{p_0{}^\alpha}{\tau_0}v(t_2)\delta^\alpha(t_2) + 1 + \frac{p_0{}^\alpha}{\tau_0},
\end{aligned}$$

which contradicts (2.27). This completes the proof. □

With a proof similar to the proof of Theorems 2.4 and 2.5, we obtain the following result.

Theorem 2.6 *Suppose we have* (H_1)-(H_5), (1.2), (1.4), *and let* $\tau(t) \le t$, $g(t,a) \ge \tau(t)$ *for* $t \in \mathbb{I}$, *and* $g(t,\xi) \le g(t,b)$ *for* $\xi \in [a,b]$. *Assume also that there exists a real-valued function* $\rho \in \mathrm{C}^1(\mathbb{I},(0,\infty))$ *such that* (2.26) *is satisfied. If there exists a real-valued function* $\eta \in \mathrm{C}^1(\mathbb{I},\mathbb{R})$ *such that* $\eta(t) \ge t$, $\eta(t) \ge g(t,b)$, $\eta'(t) > 0$ *for* $t \in \mathbb{I}$, *and* (2.27) *holds, then* (1.1) *is oscillatory.*

3 Applications and discussion

In this section, we provide three examples to illustrate the main results.

Example 3.1 Consider the second-order neutral functional differential equation

$$\left[x(t)+x(t-2\pi)\right]''+\int_{-\frac{5\pi}{2}}^{\frac{\pi}{2}} x[t+\xi]\,\mathrm{d}\xi=0,\quad t\geq 10. \tag{3.1}$$

Let $\alpha=1$, $a=-5\pi/2$, $b=\pi/2$, $r(t)=1$, $p(t)=1$, $\tau(t)=t-2\pi$, $q(t,\xi)=1$, $g(t,\xi)=t+\xi$, $\sigma(\xi)=\xi$, and $\rho(t)=1$. Then $Q(t,\xi)=\min\{q(t,\xi),q(\tau(t),\xi)\}=1$, $g'(t,a)=1$, $g(t,a)=t-5\pi/2\leq t+\xi$ for $\xi\in[-5\pi/2,\pi/2]$, and $g(t,a)\leq\tau(t)\leq t$. Moreover, letting $\tau_0=1$, then

$$\begin{aligned}&\limsup_{t\to\infty}\int_{t_0}^{t}\left[\frac{\rho(s)\int_a^b Q(s,\xi)\,\mathrm{d}\sigma(\xi)}{2^{\alpha-1}}-\frac{1}{(\alpha+1)^{\alpha+1}}\left(1+\frac{{p_0}^{\alpha}}{\tau_0}\right)\frac{r[g(s,a)](\rho'_+(s))^{\alpha+1}}{(\rho(s)g'(s,a))^{\alpha}}\right]\mathrm{d}s\\&\quad=3\pi\limsup_{t\to\infty}\int_{10}^{t}\mathrm{d}s=\infty.\end{aligned}$$

Hence, by Theorem 2.2, (3.1) is oscillatory. As a matter of fact, one such solution is $x(t)=\sin t$.

Example 3.2 Consider the second-order neutral functional differential equation

$$\left[x(t)+tx(t-\beta)\right]''+\int_0^1\frac{\xi+1}{t}x[t+\xi]\,\mathrm{d}\xi=0,\quad t\geq 1, \tag{3.2}$$

where $\beta\geq 0$ is a constant. Let $\alpha=1$, $a=0$, $b=1$, $r(t)=1$, $p(t)=t$, $\tau(t)=t-\beta$, $q(t,\xi)=(\xi+1)/t$, $g(t,\xi)=t+\xi$, $\sigma(\xi)=\xi$, and $\rho(t)=1$. Then $Q(t,\xi)=\min\{q(t,\xi),q(\tau(t),\xi)\}=(\xi+1)/t$, $g(t,a)=g(t,0)=t\leq t+\xi$ for $\xi\in[0,1]$, $\tau(t)=t-\beta\leq t$, and $g(t,a)\geq\tau(t)$ for $t\geq 1$. Further, setting $h(t)=t+1$,

$$\begin{aligned}\phi(t)&=\frac{\alpha p'[h(t)]h'(t)}{p[h(t)]}-\frac{\tau''(t)}{\tau'(t)}=\frac{1}{t+1},\\ \zeta(t)&=\frac{\rho'_+(t)}{\rho(t)}+\phi(t)=\frac{1}{t+1},\end{aligned}$$

and

$$\varphi(t)=\left(\frac{\rho'_+(t)}{\rho(t)}\right)^{\alpha+1}+\frac{p^{\alpha}[h(t)](\zeta_+(t))^{\alpha+1}}{\tau'(t)}=\frac{1}{t+1}.$$

Therefore, we have

$$\begin{aligned}&\limsup_{t\to\infty}\int_{t_0}^{t}\rho(s)\left[\frac{\int_a^b Q(s,\xi)\,\mathrm{d}\sigma(\xi)}{2^{\alpha-1}}-\frac{r[\tau(s)]\varphi(s)}{(\alpha+1)^{\alpha+1}(\tau'(s))^{\alpha}}\right]\mathrm{d}s\\&\quad=\limsup_{t\to\infty}\int_1^t\left[\int_0^1\frac{\xi+1}{s}\,\mathrm{d}\xi-\frac{1}{4(s+1)}\right]\mathrm{d}s=\limsup_{t\to\infty}\int_1^t\left[\frac{3}{2s}-\frac{1}{4(s+1)}\right]\mathrm{d}s=\infty.\end{aligned}$$

Hence, (3.2) is oscillatory due to Theorem 2.3.

Example 3.3 Consider the second-order neutral functional differential equation

$$\left[t^2\bigl(x(t)+p(t)x(t-\beta)\bigr)'\right]' + \int_0^1 (\xi+1)x[t+\xi]\,\mathrm{d}\xi = 0, \quad t \geq 1, \tag{3.3}$$

where $0 \leq p(t) \leq p_0$, p_0 and β are positive constants. Let $\alpha = 1$, $a = 0$, $b = 1$, $r(t) = t^2$, $\tau(t) = t-\beta$, $q(t,\xi) = \xi+1$, $g(t,\xi) = t+\xi$, $\sigma(\xi) = \xi$, $\rho(t) = 1$, and $\eta(t) = t+1$. Then $Q(t,\xi) = \min\{q(t,\xi), q(\tau(t),\xi)\} = \xi+1$, $\tau_0 = 1$, $g(t,a) = g(t,0) = t \leq t+\xi$ for $\xi \in [0,1]$, $\tau(t) = t-\beta \leq t$, $g(t,a) \geq \tau(t)$ for $t \geq 1$, and $\delta(t) = 1/t$. Further,

$$\begin{aligned}&\limsup_{t\to\infty}\int_{t_0}^t \left[\frac{\rho(s)\int_a^b Q(s,\xi)\,\mathrm{d}\sigma(\xi)}{2^{\alpha-1}} - \frac{1}{(\alpha+1)^{\alpha+1}}\left(1+\frac{p_0{}^\alpha}{\tau_0}\right)\frac{r[\tau(s)](\rho'_+(s))^{\alpha+1}}{(\tau_0\rho(s))^\alpha}\right]\mathrm{d}s \\ &\quad = \frac{3}{2}\limsup_{t\to\infty}\int_1^t \mathrm{d}s = \infty\end{aligned}$$

and

$$\begin{aligned}&\limsup_{t\to\infty}\int_{t_0}^t \left[\frac{\int_a^b Q(s,\xi)\,\mathrm{d}\sigma(\xi)}{2^{\alpha-1}}\delta^\alpha(s) - \left(1+\frac{p_0{}^\alpha}{\tau_0}\right)\left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1}\frac{\eta'(s)}{\delta(s)r^{1/\alpha}[\eta(s)]}\right]\mathrm{d}s \\ &\quad = \left(\frac{3}{2} - \frac{1+p_0}{4}\right)\limsup_{t\to\infty}\int_1^t \frac{\mathrm{d}s}{s+1} = \infty, \quad \text{if } p_0 < 5.\end{aligned}$$

Hence, by Theorem 2.6, (3.3) is oscillatory when $0 \leq p(t) \leq p_0 < 5$.

Remark 3.1 In this paper, we establish some new oscillation theorems for (1.1) in the case where p is finite or infinite on $\mathbb{I}$. The criteria obtained extend the results in [22] and improve those reported in [19]. Similar results can be presented under the assumption that $0 < \alpha \leq 1$. In this case, using [5, Lemma 2], one has to replace $Q(t,\xi) := \min\{q(t,\xi), q(\tau(t),\xi)\}$ with $Q(t,\xi) := 2^{\alpha-1}\min\{q(t,\xi), q(\tau(t),\xi)\}$ and proceed as above. It would be interesting to find another method to investigate (1.1) in the case where $g(\tau(t),\xi) \not\equiv \tau[g(t,\xi)]$.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to this work. They all read and approved the final version of the manuscript.

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