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Optimal couples of rearrangement invariant spaces for the Riesz potential on the bounded domain

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Abstract
We prove continuity of the Riesz potential operator in optimal couples of rearrangement invariant function spaces defined in $\mathbf{R}^n$ with the Lebesgue measure.
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1 Introduction

Let $\mathcal{M}$ be the space of all locally integrable functions f on $\Omega \subset \mathbf{R}^n$ with the Lebesgue measure, finite almost everywhere, and let $\mathcal{M}^+$ be the space of all non-negative locally integrable functions on $(0,\infty)$ with respect to the Lebesgue measure, finite almost everywhere. We shall also need the following two subclasses of $\mathcal{M}^+$. The subclass M consists of those elements g of $\mathcal{M}^+$ for which there exists an $m > 0$ such that $t^m g(t)$ is increasing. The subclass M_0 consists of those elements g of $\mathcal{M}^+$ which are decreasing.

The Riesz potential operator R^s_Ω, $0 < s < n$, $n \geq 1$ is defined formally by

$$R^s_\Omega f(x) = \int_\Omega f(y)|x-y|^{s-n}\,dy, \quad f \in \mathcal{M}^+; \qquad |\Omega| = 1. \tag{1.1}$$

We shall consider rearrangement invariant quasi-Banach spaces E, continuously embedded in $L^1(\mathbf{R}^n) + L^\infty(\mathbf{R}^n)$, such that the quasi-norm $\|f\|_E$ in E is generated by a quasi-norm ρ_E, defined on $\mathcal{M}^+$ with values in $[0,\infty]$, in the sense that $\|f\|_E = \rho_E(f^*)$. In this way equivalent quasi-norms ρ_E give the same space E. We suppose that E is nontrivial. Here f^* is the decreasing rearrangement of f, given by

$$f^*(t) = \inf\{\lambda > 0 : \mu_f(\lambda) \leq t\}, \quad t > 0,$$

where μ_f is the distribution function of f, defined by

$$\mu_f(\lambda) = \left|\left\{x \in \mathbf{R}^n : |f(x)| > \lambda\right\}\right|_n,$$

$|\cdot|_n$ denoting the Lebesgue n-measure.

Note that $f^*(t) = 0$, if $t > 1$.

There is an equivalent quasi-norm ρ_p that satisfies the triangle inequality $\rho_p^p(g_1+g_2) \le \rho_p^p(g_1) + \rho_p^p(g_2)$ for some $p \in (0,1)$ that depends only on the space E (see [1]).

We say that the norm ρ_E is K-monotone (*cf.* [2], p.84, and also [3], p.305) if

$$\int_0^t g_1^*(s)\,ds \le \int_0^t g_2^*(s)\,ds \quad \text{implies} \quad \rho_E(g_1^*) \le \rho_E(g_2^*), \quad g_1, g_2 \in \mathcal{M}^+. \tag{1.2}$$

Then ρ_E is monotone, *i.e.*, $g_1 \le g_2$ implies $\rho_E(g_1) \le \rho_E(g_2)$.

We use the notations $a_1 \lesssim a_2$ or $a_2 \gtrsim a_1$ for non-negative functions or functionals to mean that the quotient a_1/a_2 is bounded; also, $a_1 \approx a_2$ means that $a_1 \lesssim a_2$ and $a_1 \gtrsim a_2$. We say that a_1 is equivalent to a_2 if $a_1 \approx a_2$.

We say that the norm ρ_E satisfies the Minkovski inequality if for the equivalent quasi-norm ρ_p,

$$\rho_p^p\Bigl(\sum g_j\Bigr) \lesssim \sum \rho_p^p(g_j), \quad g_j \in \mathcal{M}^+. \tag{1.3}$$

For example, if E is a rearrangement invariant Banach function space as in [3], then by the Luxemburg representation theorem $\|f\|_E = \rho_E(f^*)$ for some norm ρ_E satisfying (1.2) and (1.3). More general example is given by the Riesz-Fischer monotone spaces as in [3], p.305.

Recall the definition of the lower and upper Boyd indices α_E and β_E. Let

$$h_E(u) = \sup\left\{ \frac{\rho_E(g_u^*)}{\rho_E(g^*)} : g \in \mathcal{M}^+ \right\}, \qquad g_u(t) := g(t/u)$$

be the dilation function generated by ρ_E. Then

$$\alpha_E := \sup_{0<t<1} \frac{\log h_E(t)}{\log t} \quad \text{and} \quad \beta_E := \inf_{1<t<\infty} \frac{\log h_E(t)}{\log t}.$$

If ρ_E is monotone, then the function h_E is submultiplicative, increasing, $h_E(1) = 1$, $h_E(u)h_E(1/u) \ge 1$, hence $0 \le \alpha_E \le \beta_E$. If ρ_E is K-monotone, then by interpolation (analogously to [3], p.148), we see that $h_E(s) \le \max(1,s)$. Hence in this case we have also $\beta_E \le 1$.

Using the Minkovski inequality for the equivalent quasi-norm ρ_p and monotonicity of f^*, we see that

$$\rho_E(f^*) \approx \rho_E(f^{**}) \quad \text{if } \beta_E < 1, \tag{1.4}$$

where $f^{**}(t) = \frac{1}{t}\int_0^t f^*(s)\,ds$. The main goal of this paper is to prove continuity of the Riesz potential operator $R_\Omega^s : E \mapsto G$ in optimal couples of rearrangement invariant function spaces E and G, where $\|f\|_G := \rho_G(f^*)$. It is convenient to introduce the following classes of quasi-norms, where the optimality of $R_\Omega^s : E \mapsto G$ is investigated. Let $\mathcal{N}_d$ stand for all domain quasi-norms ρ_E, which are monotone, rearrangement invariant, satisfying Minkowski's inequality, $\rho_E(\chi_{(0,1)}) < \infty$ and

$$E \hookrightarrow L^1(\Omega). \tag{1.5}$$

Let $\mathcal{N}_t$ consist of all target quasi-norms ρ_G that are monotone, satisfy Minkowski's inequality, $\rho_G(\chi_{(0,1)}) < \infty$, $\rho_G(\chi_{(1,\infty)}t^{s/n-1}) < \infty$ and

$$G \hookrightarrow \Lambda^\infty\bigl(t^{1-s/n}\bigr)\bigl(\mathcal{R}^n\bigr), \tag{1.6}$$

where $\chi_{(a,b)}$ is the characteristic function of the interval (a,b), $0 < a < b \leq \infty$. Note that technically it is more convenient not to require that the target quasi-norm ρ_G is rearrangement invariant. Of course, the target space G is rearrangement invariant, since $\|f\|_G = \rho_G(f^*)$. Finally, let $\mathcal{N} := \mathcal{N}_d \times \mathcal{N}_t$.

Definition 1.1 (Admissible couple) We say that the couple $(\rho_E, \rho_G) \in \mathcal{N}$ is admissible for the Riesz potential if the following estimate is valid:

$$\rho_G\bigl(\bigl(R^s_\Omega f\bigr)^{**}\bigr) \lesssim \rho_E\bigl(f^*\bigr). \tag{1.7}$$

Moreover, $\rho_E(E)$ is called domain quasi-norm (domain space), and ρ_G (G) is called a target quasi-norm (target space).

For example, by Theorem 2.2 below (the sufficient part), the couple $E = \Lambda^q(t^{s/n}w)(\Omega)$, $G = \Lambda^q(v)$, $1 \leq q \leq \infty$, is admissible if $\beta_E < 1$ and v is related to w by the Muckenhoupt condition [4]:

$$\left(\int_0^t \bigl[v(s)\bigr]^q \,ds/s\right)^{1/q} \left(\int_t^\infty \bigl[w(s)\bigr]^{-r} \,ds/s\right)^{1/r} \lesssim 1, \qquad 1/q + 1/r = 1. \tag{1.8}$$

Definition 1.2 (Optimal target quasi-norm) Given the domain quasi-norm $\rho_E \in \mathcal{N}_d$, the optimal target quasi-norm, denoted by $\rho_{G(E)}$, is the strongest target quasi-norm, *i.e.*,

$$\rho_G\bigl(g^*\bigr) \lesssim \rho_{G(E)}\bigl(g^*\bigr), \quad g \in \mathcal{M}^+, \tag{1.9}$$

for any target quasi-norm $\rho_G \in \mathcal{N}_t$ such that the couple ρ_E, ρ_G is admissible.

Definition 1.3 (Optimal domain quasi-norm) Given the target quasi-norm $\rho_G \in \mathcal{N}_t$, the optimal domain quasi-norm, denoted by $\rho_{E(G)}$, is the weakest domain quasi-norm, *i.e.*,

$$\rho_{E(G)}\bigl(g^*\bigr) \lesssim \rho_E\bigl(g^*\bigr), \quad g \in \mathcal{M}^+, \tag{1.10}$$

for any domain quasi-norm $\rho_E \in \mathcal{N}_d$ such that the couple ρ_E, ρ_G is admissible.

Definition 1.4 (Optimal couple) The admissible couple ρ_E, ρ_G is said to be optimal if $\rho_E = \rho_{E(G)}$ and $\rho_G = \rho_{G(E)}$.

The optimal quasi-norms are uniquely determined up to equivalence, while the corresponding optimal quasi-Banach spaces are unique.

2 Admissible couples

Here we give a characterization of all admissible couples $(\rho_E, \rho_G) \in \mathcal{N}$. It is convenient to define the case $\beta_E = 1$ as limiting and the case $\beta_E < 1$ as sublimiting.

Theorem 2.1 (General case $\beta_E \le 1$) *The couple* $(\rho_E, \rho_G) \in \mathcal{N}$ *is admissible if and only if*

$$\rho_G(\chi_{(0,1)} S_1 g) \lesssim \rho_E(g), \quad g \in \mathcal{M}^+ \text{ or } g \in M_0, \tag{2.1}$$

where

$$S_1 g(t) := \begin{cases} t^{s/n-1} \int_0^t g(u)\,du + \int_t^1 u^{s/n} g(u)\,du/u, & 0 < t < 1, 0 < s < n, n \ge 1, \\ t^{s/n-1} \int_0^1 g(u)\,du, & t > 1, 0 < s < n, n \ge 1. \end{cases} \tag{2.2}$$

Proof First we prove

$$\left(R^s_\Omega f\right)^{**} \lesssim S_1 f^*. \tag{2.3}$$

We are going to use real interpolation for quasi-Banach spaces. First we recall some basic definitions. Let (A_0, A_1) be a couple of two quasi-Banach spaces (see [2, 5]) and let

$$K(t,f) = K(t,f;A_0,A_1) = \inf_{f=f_0+f_1} \left\{ \|f_0\|_{A_0} + t\|f_1\|_{A_1} \right\}, \quad f \in A_0 + A_1$$

be the K-functional of Peetre (see [2]). By definition, the K-interpolation space $A_\Phi = (A_0, A_1)_\Phi$ has a quasi-norm

$$\|f\|_{A_\Phi} = \left\| K(t,f) \right\|_\Phi,$$

where Φ is a quasi-normed function space with a monotone quasi-norm on $(0,\infty)$ with the Lebesgue measure and such that $\min\{1,t\} \in \Phi$. Then (see [5])

$$A_0 \cap A_1 \hookrightarrow A_\Phi \hookrightarrow A_0 + A_1,$$

where by $X \hookrightarrow Y$ we mean that X is continuously embedded in Y. If $\|g\|_\Phi = (\int_0^\infty t^{-\theta q} \times g^q(t)\,dt/t)^{1/q}$, $0 < \theta < 1$, $0 < q \le \infty$, we write $(A_0, A_1)_{\theta,q}$ instead of $(A_0, A_1)_\Phi$ (see [2]).

Using the Hardy-Littlewood inequality $\int_{\mathbf{R}^n} |f(x)g(x)|\,dx \le \int_0^\infty f^*(t) g^*(t)\,dt$, we get the well-known mapping property

$$R^s_\Omega : \Lambda^1\left(t^{s/n}\right)(\Omega) \mapsto L^\infty\left(\mathcal{R}^n\right)$$

and by the Minkovski inequality for the norm f^{**} we get

$$R^s_\Omega : L^1(\Omega) \mapsto \Lambda^\infty\left(t^{1-s/n}\right)\left(\mathcal{R}^n\right).$$

Hence

$$t^{1-s/n}\left(R^s_\Omega f\right)^{**}(t) \lesssim K\left(t^{1-s/n}, f; L^1(\Omega), \Lambda^1\left(t^{s/n}\right)(\Omega)\right),$$

therefore (see [2], Section 5.7)

$$t^{1-s/n}\left(R^s_\Omega f\right)^{**}(t) \lesssim \begin{cases} \int_0^t f^*(u)\,du + t^{1-s/n} \int_t^1 u^{s/n} f^*(u)\,du/u, & 0 < t < 1, \\ \int_0^1 f^*(u)\,du, & t > 1, \end{cases}$$

implies

$$\left(R^s_\Omega f\right)^{**}(t) \lesssim S_1 f^*(t).$$

It is clear that (1.7) follows from (2.1) and (2.3).

Now we prove that (1.7) implies (2.1). To this end we choose the test function in the form $f(x) = g(c|x|^n)$, $g \in \mathcal{M}^+$, so that $f^*(t) = g^*(t)$ for some positive constant c (*cf.* [6]). Then

$$R^s_\Omega f(x) = \int_{|y|<|x|} g\left(c|y|^n\right)|x-y|^{s-n}\,dy + \int_{|y|>|x|} g\left(c|y|^n\right)|x-y|^{s-n}\,dy,$$

whence

$$\left|R^s_\Omega f(x)\right| \gtrsim |x|^{s-n}\int_0^{c|x|^n} g(u)\,du + \int_{c|x|^n}^{|\Omega|=1} u^{s/n-1}g(u)\,du \gtrsim \chi_{(0,1)}(S_1 g)\left(c|x|^n\right).$$

Note that $\chi_{(0,1)}S_1 g \approx \chi_{(0,1)}Q_1 T_1' g + \chi_{(0,1)}\int_0^1 g(u)\,du$, where

$$Q_1 g := \int_t^1 g(u)\,du/u, \quad t < 1,$$

and

$$T_1' g(t) := \begin{cases} t^{s/n-1}\int_0^t g(u)\,du, & 0 < t < 1, 0 < s < n, n \geq 1, \\ t^{s/n-1}\int_0^1 g(u)\,du, & t > 1, 0 < s < n, n \geq 1, \end{cases}$$

hence $\chi_{(0,1)}S_1 g$ is decreasing, therefore

$$\left|R^s_\Omega f\right|^*(t) \gtrsim \chi_{(0,1)}S_1 g(t). \tag{2.4}$$

Thus, if (1.7) is given, then (2.4) implies (2.1). □

In the sublimiting case $\beta_E < 1$ we can simplify the condition (2.1), replacing S_1 by T_1. Here

$$T_1 g(t) := \begin{cases} t^{s/n-1}\int_t^1 u^{s/n}g(u)\,du/u, & 0 < t < 1, 0 < s < n, n \geq 1, \\ 0, & t > 1. \end{cases} \tag{2.5}$$

Theorem 2.2 (Sublimiting case $\beta_E < 1$) *The couple $(\rho_E, \rho_G) \in \mathcal{N}$ is admissible if and only if*

$$\rho_G(\chi_{(0,1)}T_1 g) \lesssim \rho_E(g), \quad g \in M, \tag{2.6}$$

where we recall that

$$M := \left\{g \in \mathcal{M}^+ \textit{ and } t^m g(t) \textit{ is increasing for some } m > 0\right\}.$$

Proof Let ρ_E, ρ_G be an admissible couple, then

$$\rho_G(\chi_{(0,1)}S_1 g) \lesssim \rho_E(g).$$

Since $\rho_G(\chi_{(0,1)}T_1 g) \lesssim \rho_G(\chi_{(0,1)}S_1 g)$, it follows that $\rho_G(\chi_{(0,1)}T_1 g) \lesssim \rho_E(g)$, $g \in M$. Now we need to prove sufficiency of (2.6). We have

$$\chi_{(0,1)}S_1 g^* \approx \chi_{(0,1)}T_1 g^{**} + \chi_{(0,1)}g^{**}(1),$$

so

$$\rho_G\bigl(\chi_{(0,1)}S_1 g^*\bigr) \lesssim \rho_G\bigl(\chi_{(0,1)}T_1 g^{**}\bigr) + \rho_G(\chi_{(0,1)})g^{**}(1)$$

implies

$$\rho_G\bigl(\chi_{(0,1)}S_1 g^*\bigr) \lesssim \rho_E\bigl(g^*\bigr). \qquad \square$$

In the subcritical case $\alpha_E > s/n$ we have another simplification of (2.1).

Theorem 2.3 (Case $\alpha_E > s/n$) *The couple* $(\rho_E, \rho_G) \in \mathcal{N}$ *is admissible if and only if*

$$\rho_G\bigl(\chi_{(0,1)}T_1' g\bigr) \lesssim \rho_E(g), \qquad g \in M_0 := \bigl\{g \in \mathcal{M}^+, g \textit{ is decreasing}\bigr\}, \tag{2.7}$$

where

$$T_1'g(t) := \begin{cases} t^{s/n-1}\int_0^t g(u)\,du, & 0 < t < 1, 0 < s < n, n \ge 1, \\ t^{s/n-1}\int_0^1 g(u)\,du, & t > 1, 0 < s < n, n \ge 1. \end{cases}$$

Proof Let $(\rho_E, \rho_G) \in \mathcal{N}$ be admissible, then

$$\rho_G(\chi_{(0,1)}S_1 g) \lesssim \rho_E(g), \quad g \in M_0.$$

As

$$\rho_G\bigl(\chi_{(0,1)}T_1' g\bigr) \lesssim \rho_G(\chi_{(0,1)}S_1 g),$$

we have

$$\rho_G\bigl(\chi_{(0,1)}T_1' g\bigr) \lesssim \rho_E(g).$$

For the reverse, it is enough to check that (2.7) implies (2.1) for $g \in M_0$, or

$$\rho_G(\chi_{(0,1)}T_1 g) \lesssim \rho_E(g), \quad g \in M_0.$$

As

$$\chi_{(0,1)}T_1 g \lesssim \chi_{(0,1)}T_1'\bigl(t^{-s/n}\chi_{(0,1)}T_1 g\bigr),$$

so

$$\rho_G(\chi_{(0,1)} T_1 g) \lesssim \rho_E\big(t^{-s/n}\chi_{(0,1)} T_1 g\big) \approx \rho_E\big(t^{-s/n} Q_1\big(t^{s/n} g\big)\big) \lesssim \rho_E(g).$$

Here we use

$$\rho_E\big(Q_1\big(t^{-s/n} g\big)\big) \lesssim \rho_E\big(t^{-s/n} g\big), \quad g \in M_0, \alpha_E > s/n, t < 1. \qquad \square$$

2.1 Optimal quasi-norms

Here we give a characterization of the optimal domain and optimal target quasi-norms. We can define an optimal target quasi-norm by using Theorem 2.1.

Definition 2.4 (Construction of the optimal target quasi-norm) For a given domain quasi-norm $\rho_E \in \mathcal{N}_d$ we set

$$\rho_{G_E}(\chi_{(0,1)} g) := \inf\big\{\rho_E(h) : \chi_{(0,1)} g \le \chi_{(0,1)} S_1 h, h \in \mathcal{M}^+\big\}, \quad g \in \mathcal{M}^+. \tag{2.8}$$

Then

$$\rho_{G(E)}(g) := \rho_{G_E}(\chi_{(0,1)} g) + \sup_{t>1} t^{1-s/n} g.$$

Theorem 2.5 *Let $\rho_E \in \mathcal{N}_d$ be a given domain quasi-norm. Then $\rho_{G(E)} \in \mathcal{N}_t$, the couple ρ_E, $\rho_{G(E)}$ is admissible and the target quasi-norm is optimal. By definition,*

$$G(E) := \Big\{f \in \mathcal{M} : \lim_{t\to\infty} f^*(t) = 0, \rho_{G(E)}\big(f^*\big) < \infty\Big\}. \tag{2.9}$$

Proof To see that $\rho_{G(E)}$ is a quasi-norm, we first prove (1.6), for that we first prove

$$\sup_{0<t<1} t^{1-s/n} g^* \lesssim \rho_{G_E}\big(g^*\big), \quad g \in \mathcal{M}^+. \tag{2.10}$$

Take $g \in \mathcal{M}^+$ and consider an arbitrary $h \in \mathcal{M}^+$ such that, for $t < 1$, $g^* \le S_1 h$. By the Hardy inequality $g^* \lesssim S_1(h^*)$. Then,

$$t^{1-s/n} g^* \le K\big(t^{1-s/n}, h; L^1(\Omega), \Lambda^1\big(t^{s/n}\big)(\Omega)\big).$$

Hence

$$\sup_{0<t<1} t^{1-s/n} g^* \le K\big(1, h; L^1(\Omega), \Lambda^1\big(t^{s/n}\big)(\Omega)\big) \lesssim \rho_E(h).$$

Taking the infimum over all h such that $g^* \le S_1 h$, we get (2.10). Hence $G_E \hookrightarrow \Lambda^\infty(t^{1-s/n})(0, 1)$, also $\rho_G(\chi(1,\infty) g) = \sup_{t>1} t^{1-s/n} g$. And these two together give (1.6). $\rho_{G(E)}$ is indeed a quasi-norm on $\mathcal{M}^+$. Since $\chi_{(0,1)}(R^s_\Omega f)^* \lesssim \chi_{(0,1)} S_1 f^*$, which gives $\rho_{G_E}(\chi_{(0,1)}(R^s_\Omega f)^*) \lesssim \rho_E(f^*)$. Also

$$\sup_{t>1} t^{-s/n}\big(R^s_\Omega f\big)^* \lesssim \sup_{t>1} t^{1-s/n} S_1 f^* = \int_0^1 f^*(u)\, du \lesssim \rho_E\big(f^*\big).$$

Hence ρ_E, $\rho_{G(E)}$ is admissible couple. Now we are going to prove that $\rho_{G(E)}$ is optimal. For this purpose, suppose that the couple $(\rho_E, \rho_{G_1}) \in \mathcal{N}$ is admissible. Then by Theorem 2.1,

$$\rho_{G_1}(\chi_{(0,1)} S_1 g) \lesssim \rho_E(g), \quad g \in \mathcal{M}^+.$$

Therefore if $\chi_{(0,1)} g^* \le \chi_{(0,1)} S_1 h$, $h \in \mathcal{M}^+$, then

$$\rho_{G_1}\big(\chi_{(0,1)} g^*\big) \le \rho_{G_1}(\chi_{(0,1)} S_1 h) \lesssim \rho_E(h),$$

so taking the infimum on the right-hand side, we get

$$\rho_{G_1}\big(\chi_{(0,1)} g^*\big) \lesssim \rho_{G_E}\big(\chi_{(0,1)} g^*\big),$$

hence $\rho_{G_1}(g^*) \lesssim \rho_{G(E)}(g^*)$. □

In the sublimiting case $\beta_E < 1$ we can simplify the optimal target quasi-norm.

Theorem 2.6 *If $\rho_E \in \mathcal{N}_d$ be a given domain quasi-norm. Then for $g \in \mathcal{M}^+$,*

$$\begin{aligned} &\rho_{G_E}\big(\chi_{(0,1)} g^*\big) \approx \rho\big(\chi_{(0,1)} g^*\big), \\ &\rho(\chi_{(0,1)} g) := \inf\big\{\rho_E(h) : \chi_{(0,1)} g \le \chi_{(0,1)} T_1 h, h \in M\big\}, \end{aligned} \tag{2.11}$$

i.e.,

$$\rho_{G(E)}(g) \approx \rho(\chi_{(0,1)} g) + \sup_{t>1} t^{1-s/n} g.$$

Proof If $\chi_{(0,1)} g^* \le \chi_{(0,1)} T_1 h$, $h \in M$, then $\chi_{(0,1)} g^* \le \chi_{(0,1)} S_1 h$, therefore

$$\rho_{G_E}\big(\chi_{(0,1)} g^*\big) \le \rho_E(h)$$

and taking the infimum, we get

$$\rho_{G_E}\big(\chi_{(0,1)} g^*\big) \le \rho\big(\chi_{(0,1)} g^*\big).$$

Now for the reverse, let $\chi_{(0,1)} g^* \le \chi_{(0,1)} S_1 h$, $h \in \mathcal{M}^+$.
Then

$$\chi_{(0,1)} g^* \lesssim \chi_{(0,1)} S_1\big(h^*\big) \approx \chi_{(0,1)} T_1\big(h^{**}\big) + \chi_{(0,1)} f^{**}(1),$$

so

$$\chi_{(0,1)} g^* - \chi_{(0,1)} f^{**}(1) \lesssim \chi_{(0,1)} T_1\big(h^{**}\big),$$

which gives, since $h^{**} \in M$,

$$\rho\big(\chi_{(0,1)} g^* - \chi_{(0,1)} f^{**}(1)\big) \lesssim \rho_E\big(h^{**}\big) \approx \rho_E\big(h^*\big) \approx \rho_E(h),$$

and this implies

$$\rho\big(\chi_{(0,1)}g^*\big) \lesssim \rho_E(h) + f^{**}(1),$$

which gives

$$\rho\big(\chi_{(0,1)}g^*\big) \lesssim \rho_E(h).$$

Taking the infimum, we get $\rho(\chi_{(0,1)}g^*) \lesssim \rho_{G_E}(\chi_{(0,1)}g^*)$, hence $\rho(\chi_{(0,1)}g^*) \approx \rho_{G_E}(\chi_{(0,1)}g^*)$. □

A simplification of the optimal target quasi-norm is possible also in the subcritical case $\alpha_E > s/n$.

Theorem 2.7 *Let $\rho_E \in \mathcal{N}_d$ be a given domain quasi-norm. Then for $g \in \mathcal{M}^+$,*

$$\begin{aligned} &\rho_{G_E}\big(\chi_{(0,1)}g^*\big) \approx \rho_1\big(\chi_{(0,1)}g^*\big), \\ &\rho_1(\chi_{(0,1)}g) := \inf\big\{\rho_E(h) : \chi_{(0,1)}g \le T_1'h, h \in M_0\big\}, \end{aligned} \tag{2.12}$$

i.e.,

$$\rho_{G(E)}(g) \approx \rho_1(\chi_{(0,1)}g) + \sup_{t>1} t^{1-s/n}g.$$

Proof If $\chi_{(0,1)}g^* \le \chi_{(0,1)}T_1'h$, $h \in M_0$, then

$$\chi_{(0,1)}g^* \le \chi_{(0,1)}S_1h.$$

Therefore

$$\rho_{G_E}\big(\chi_{(0,1)}g^*\big) \le \rho_E(h),$$

and taking the infimum, we get

$$\rho_{G_E}\big(\chi_{(0,1)}g^*\big) \le \rho_1\big(\chi_{(0,1)}g^*\big).$$

For the reverse, let $\chi_{(0,1)}g^* \le \chi_{(0,1)}S_1h$. Then $\chi_{(0,1)}g^* \lesssim \chi_{(0,1)}T_1(h^*) + \chi_{(0,1)}T_1'(h^*)$. As

$$\chi_{(0,1)}T_1g \lesssim \chi_{(0,1)}T_1'\big(t^{-s/n}\chi_{(0,1)}T_1g\big),$$

we get

$$\chi_{(0,1)}g^* \lesssim \chi_{(0,1)}T_1'\big(h^* + t^{-s/n}\chi_{(0,1)}T_1\big(h^*\big)\big),$$

whence

$$\begin{aligned} \rho_1\big(\chi_{(0,1)}g^*\big) &\lesssim \rho_E\big(t^{-s/n}\chi_{(0,1)}T_1\big(h^*\big)\big) + \rho_E(h) \\ &\approx \rho_E\big(t^{-s/n}Q_1\big(t^{s/n}h^*\big)\big) + \rho_E(h) \\ &\lesssim \rho_E(h), \end{aligned}$$

where we use

$$\rho_E\big(Q_1\big(t^{-s/n}g\big)\big) \lesssim \rho_E\big(t^{-s/n}g\big), \quad g \in M_0, \alpha_E > s/n, t < 1.$$

Therefore, taking the infimum we arrive at

$$\rho_1\big(g^*\big) \lesssim \rho_{G_E}\big(g^*\big). \qquad \square$$

We can construct an optimal domain quasi-norm $\rho_{E(G)}$ by Theorem 2.1 as follows.

Definition 2.8 (Construction of an optimal domain quasi-norm) For a given target quasi-norm $\rho_G \in \mathcal{N}_t$, we construct an optimal domain quasi-norm $\rho_{E(G)}$ by

$$\rho_{E(G)}(g) := \rho_G\big(\chi_{(0,1)}S_1 g^*\big), \quad g \in \mathcal{M}^+. \tag{2.13}$$

Theorem 2.9 *If $\rho_G \in \mathcal{N}_t$ is a given target quasi-norm, then the domain quasi-norm $\rho_{E(G)}$ is optimal. Moreover, if $\beta_G < 1 - s/n$, then the couple $\rho_{E(G)}$, ρ_G is optimal.*

Proof Since $\chi_{(0,1)}S_1 g^* \approx \chi_{(0,1)} T_1 g^{**} + \chi_{(0,1)} g^{**}(1)$, so

$$\rho_{E(G)}(g) \approx \rho_G\big(\chi_{(0,1)} T_1 g^{**} + \chi_{(0,1)} g^{**}(1)\big),$$

it follows that $\rho_{E(G)}$ is a quasi-norm. To prove the property (1.5), we notice that

$$\begin{aligned}\rho_{E(G)}\big(f^*\big) &= \rho_G\big(\chi_{(0,1)}S_1 f^*\big) \geq \rho_G(\chi_{(0,1)})\big(Sf^*\big)(1) \\ &\gtrsim \int_0^1 f^*(t)\,dt \approx \|f\|_{L^1(\Omega)}.\end{aligned}$$

The couple $\rho_{E(G)}$, ρ_G is admissible since $\rho_{E(G)}(g) = \rho_G(\chi_{(0,1)}S_1 g^*) \geq \rho_G(\chi_{(0,1)}S_1 g)$. Moreover, $\rho_{E(G)}$ is optimal, since for any admissible couple $(\rho_{E_1}, \rho_G) \in \mathcal{N}$ we have $\rho_G(\chi_{(0,1)}S_1 h) \lesssim \rho_{E_1}(h)$, $h \in \mathcal{M}^+$. Therefore,

$$\rho_{E(G)}\big(g^*\big) \leq \rho_{E_1}\big(g^*\big).$$

To check that if $\beta_G < 1 - s/n$, the couple $\rho_{E(G)}$, ρ_G is optimal, we need only to prove that ρ_G is an optimal target quasi-norm, *i.e.*, $\rho(g^*) \lesssim \rho_G(g^*)$, where $\rho = \rho_{G(E(G))}$ is defined by (2.11), since $\beta_{E(G)} < 1$. We have $\chi_{(0,1)}g^{**}(t) - \chi_{(0,1)}g^{**}(1) = \chi_{(0,1)} T_1 h$, where $h(t) = t^{-s/n}[g^{**}(t) - g^*(t)] \in M$, $t < 1$, therefore,

$$\rho_{G_{E(G)}}\big(\chi_{(0,1)}g^{**}(t) - \chi_{(0,1)}g^{**}(1)\big) \leq \rho_{E(G)}(h) = \rho_G\big(\chi_{(0,1)}S_1 h^*\big)$$

implies

$$\rho_{G_{E(G)}}\big(\chi_{(0,1)}g^{**}(t)\big) \lesssim \rho_G\big(\chi_{(0,1)}S_1 h^*\big) + g^{**}(1),$$

since

$$\chi_{(0,1)}S_1 h^* = \chi_{(0,1)}t^{s/n}h^{**} + \chi_{(0,1)} T_1 h^* \lesssim \chi_{(0,1)}t^{s/n}h^{**} + \chi_{(0,1)} T_1 h^{**},$$

so

$$\rho_{G_{E(G)}}\big(\chi_{(0,1)}g^{**}(t)\big) \lesssim \rho_G\big(\chi_{(0,1)}t^{s/n}h^{**}\big) + \rho_G\big(\chi_{(0,1)}T_1h^{**}\big) + g^{**}(1).$$

Now we define

$$P_1g(t) := \frac{1}{t}\int_0^t g(u)\,du, \quad t < 1.$$

For $t < 1$, since $h^* \lesssim Q_1h$, we have $h^{**} = P_1h^* \lesssim Q_1P_1h$, therefore $T_1h^{**} \lesssim T_1Q_1(P_1h) \lesssim T_1(P_1h)$. Also $T_1(P_1h) \approx T_1h + t^{s/n}P_1h$ and $P_1h \le h^{**}$. Therefore,

$$\begin{aligned}\rho_{G_{E(G)}}\big(\chi_{(0,1)}g^*\big) &\lesssim \rho_G(\chi_{(0,1)}T_1h) + \rho_G\big(\chi_{(0,1)}t^{s/n}h^{**}\big) + g^{**}(1)\\ &\lesssim \rho_G\big(\chi_{(0,1)}g^{**}\big) + \rho_G\big(\chi_{(0,1)}t^{s/n}h^{**}\big) + g^{**}(1).\end{aligned}$$

For $t < 1$, since $h(t) \le t^{-s/n}g^{**}(t)$ we have $h^*(t) \le t^{-s/n}g^{**}$, therefore using $\beta_G < 1 - s/n$, Minkowski's inequality, and monotonicity of ρ_G, we have

$$\rho_G\big(\chi_{(0,1)}t^{s/n}h^{**}\big) \lesssim \rho_G\big(\chi_{(0,1)}g^{**}\big).$$

Thus

$$\rho_{G_{E(G)}}\big(\chi_{(0,1)}g^*\big) \lesssim \rho_G\big(\chi_{(0,1)}g^{**}\big) \approx \rho_G\big(\chi_{(0,1)}g^*\big),$$

hence $\rho(g^*) \lesssim \rho_G(g^*)$. □

Example 2.10 If $G = C_0$ consists of all bounded continuous functions such that $f^*(\infty) = 0$ and $\rho_G(g) = g^*(0) = g^{**}(0)$, then $\alpha_G = \beta_G = 0$ and $\rho_{E(G)}(g) \approx \int_0^1 t^{s/n}g^{**}\,dt/t$, *i.e.*, $E = \Gamma^1(t^{s/n})(\Omega)$ and the couple E, G is optimal.

Example 2.11 Let $G = \Lambda^\infty(v)$ with $\beta_G < 1 - s/n$ and let

$$\rho_E(g) = \sup v(t)\int_t^1 u^{s/n}g^{**}(u)\,du/u.$$

Then, the couple E, G is optimal and $\beta_E < 1$. In particular, this is true if v is slowly varying since then $\alpha_G = \beta_G = 0$ and $\alpha_E = \beta_E = s/n < 1$.

2.2 Subcritical case

Here we suppose that $s/n < \alpha_E$.

Theorem 2.12 (Sublimiting case $\beta_E < 1$) *For a given domain quasi-norm $\rho_E \in \mathcal{N}_d$ with $\rho_E(\chi_{(0,1)}(t)t^{-s/n}) < \infty$, we have*

$$\rho_{G_E}\big(\chi_{(0,1)}g^*\big) \approx \rho_E\big(t^{-s/n}g^*\big) \approx \rho_E\big(t^{-s/n}g^{**}\big), \tag{2.14}$$

i.e.,

$$\rho_{G(E)}\big(g^*\big) \approx \rho_{G_E}\big(\chi_{(0,1)}g^*\big) + \sup_{t>1} t^{1-s/n}g.$$

Moreover, the couple ρ_E, $\rho_{G(E)}$ is optimal.

Proof If $\chi_{(0,1)}g^* \le \chi_{(0,1)} T_1' h$, $h \in M_0$, then for $t < 1$, $t^{-s/n} g^* \le h^{**}$, whence

$$\rho_E(t^{-s/n} g^*) \lesssim \rho_E(h^{**}) \approx \rho_E(h^*) \approx \rho_E(h).$$

Taking the infimum, we get

$$\rho_E(t^{-s/n} g^*) \lesssim \rho_{G_E}(\chi_{(0,1)} g^*).$$

For the reverse, we notice that $\chi_{(0,1)} T_1'(t^{-s/n} g^*) \gtrsim \chi_{(0,1)} g^* = g^*$, hence $\rho_{G_E}(\chi_{(0,1)} g^*) \lesssim \rho_E(t^{-s/n} g^*)$.

It remains to prove that the domain quasi-norm ρ_E is also optimal. Let ρ_{E_1}, $\rho_{G(E)}$ be an admissible couple in $\mathcal{N}$. Then

$$\begin{aligned}
\rho_{E_1}(g^*) &\gtrsim \rho_{G(E)}(\chi_{(0,1)} S_1 g^*) \\
&= \rho_{G_E}(\chi_{(0,1)} S_1 g^*) + \sup_{t>1} t^{1-s/n} \chi_{(0,1)} S_1 g^* \\
&\approx \rho_E(t^{-s/n} \chi_{(0,1)} S_1 g^*) + 0 \\
&\gtrsim \rho_E(t^{-s/n} \chi_{(0,1)} T_1' g^*) \\
&\gtrsim \rho_E(\chi_{(0,1)} g^{**}) \\
&\approx \rho_E(\chi_{(0,1)} g^*) \\
&\approx \rho_E(g^*).
\end{aligned}$$

□

Now we give an example.

Example 2.13 Let

$$E = \Lambda^q(t^\alpha w_1)(\Omega) \cap \Lambda^r(t^\beta w_2)(\Omega), \quad s/n < \alpha < \beta < 1, 0 < q, r \le \infty,$$

where w_1 and w_2 are slowly varying. Then we have $\alpha_E = \alpha$, $\beta_E = \beta$. Now by applying the previous theorem, we get

$$G(E) = \Lambda_0^q(t^{\alpha - s/n} w_1) \cap \Lambda_0^r(t^{\beta - s/n} w_2),$$

and the couple $(E, G(E))$ is optimal.

In the limiting case $\beta_E = 1$, the formula for the optimal target quasi-norm is more complicated.

Theorem 2.14 (Limiting case) *Let*

$$\rho_E(g) := \rho_H(\chi_{(0,1)} g^{**}), \qquad \rho_{G_1}(g) := \rho_H\Big(t^{-1} \sup_{0<u<t} u^{1-s/n} g(u)\Big),$$

where ρ_H is a monotone quasi-norm with $\alpha_H = \beta_H = 1$, $\rho_H(\chi_{(0,1)}) < \infty$, $\rho_H(\chi_{(1,\infty)} t^{-1}) < \infty$ and let

$$\begin{aligned}
E &:= \{f \in \mathcal{M} : tf^{**}(t) \to 0 \text{ as } t \to 0 \text{ and } \rho_E(f^*) < \infty\}, \\
G_1 &:= \Big\{f \in \mathcal{M} : \sup_{0<u<t} u^{1-s/n} f^{**}(u) \to 0 \text{ as } t \to 0 \text{ and } \rho_G(f^*) < \infty\Big\}.
\end{aligned}$$

Define

$$\rho_G(g) := \rho_{G_1}(\chi_{(0,1)}g) + \sup_{t>1} t^{1-s/n}g.$$

Then the couple ρ_E, ρ_G *is optimal.*

Proof Note that

$$E \hookrightarrow L^1(\Omega).$$

Indeed, $\rho_E(f^*) = \rho_H(\chi_{(0,1)}g^{**}) \gtrsim f^{**}(1) = \int_0^1 f^*(u)\,du$. Hence the above embedding follows. Consequently, $\rho_E \in \mathcal{N}_d$. On the other hand,

$$\begin{aligned}\rho_G(f^*) &\geq \rho_H\Bigl(\chi_{(1,\infty)}t^{-1}\sup_{0<u<t} u^{1-s/n}f^*(u)\Bigr)\\ &\geq \sup_{0<u<1} u^{1-s/n}f^*(u)\rho_H\bigl(\chi_{(1,\infty)}t^{-1}\bigr).\end{aligned}$$

Hence $G_1 \hookrightarrow \Lambda^\infty(t^{1-s/n})(0,1)$. This together with $\rho_G(\chi_{(1,\infty)}) = \sup_{t>1} t^{1-s/n}g$ gives $G \hookrightarrow \Lambda^\infty(t^{1-s/n})$. Then from the conditions on G_1 it follows that $\rho_G \in \mathcal{N}_t$. Also, $\alpha_E = \beta_E = 1$ and $\alpha_G = \beta_G = 1 - s/n$. On the other hand, if $0 < u < 1$, then

$$u^{1-s/n}\bigl(R^s_\Omega f\bigr)^{**}(u) \lesssim \int_0^u f^*(v)\,dv + u^{1-s/n}\int_u^1 v^{s/n-1}f^*(v)\,dv.$$

For every $\varepsilon > 0$, we can find a $\delta > 0$, such that $vf^{**}(v) < \varepsilon$ for all $0 < v < \delta$. Then for $0 < t < 1$,

$$\sup_{0<u<t} u^{1-s/n}\bigl(R^s_\Omega f\bigr)^{**}(u) \lesssim \int_0^t f^*(v)\,dv + \varepsilon + t^{1-s/n}\int_\delta^1 v^{s/n-1}f^*(v)\,dv. \tag{2.15}$$

Now it is easy to check that $\lim_{t\to 0}\sup_{0<u<t} u^{1-s/n}(R^s_\Omega f)^{**} = 0$ if $f \in E$.

To prove that $R^s : E \to G$ we need to check that the couple ρ_E, ρ_G is admissible. We write for $t < 1$,

$$T_1'g(t) = T_1'g^*(t) = t^{s/n}g^{**}(t), \quad g \in M_0.$$

Then

$$\begin{aligned}\rho_G\bigl(\chi_{(0,1)}T_1'g\bigr) &= \rho_{G_1}\bigl(\chi_{(0,1)}T_1'g\bigr) + \sup_{t>1} t^{1-s/n}\chi_{(0,1)}T_1'g\\ &= \rho_H\Bigl(\chi_{(0,1)}t^{-1}\sup_{0<u<t} u^{1-s/n}T_1'g(u)\Bigr) + \sup_{t>1} t^{1-s/n}\chi_{(0,1)}T_1'g\\ &= \rho_H\bigl(\chi_{(0,1)}g^{**}\bigr)\\ &= \rho_E(g).\end{aligned}$$

To prove that the target space is optimal, notice first that

$$\sup_{0<u<t} u^{1-s/n}f^{**}(u) \approx K\bigl(t^{1-s/n}, f; \Lambda^\infty\bigl(t^{1-s/n}\bigr), L^\infty\bigr).$$

If $f \in G$, then by [2]

$$\begin{aligned}\sup_{0<u<t} u^{1-s/n} f^{**}(u) &\approx \int_0^{t^{1-s/n}} h_1(u)\,du \quad \text{(where } h_1\text{, is decreasing)}\\ &\approx \int_0^t h_1\bigl(v^{1-s/n}\bigr) v^{-s/n}\,dv \quad \text{(by a change of variables)}.\end{aligned}$$

If $h(v) = h_1(v^{1-s/n})v^{-s/n}$ then obviously $h \in M_0$ and

$$\sup_{0<u<t} u^{1-s/n} f^{**}(u) \approx \int_0^t h(v)\,dv = th^{**}(t),$$

whence

$$\rho_E(h) \approx \rho_H\bigl(\chi_{(0,1)} h^{**}\bigr) \approx \rho_H\Bigl(\chi_{(0,1)} t^{-1} \sup_{0<u<t} u^{1-s/n} f^{**}(u)\Bigr) \approx \rho_{G_1}\bigl(\chi_{(0,1)} f^*\bigr).$$

On the other hand,

$$\sup_{0<u<t} u^{1-s/n} f^{**}(u) \approx th^{**}(t)$$

implies $t^{1-s/n} f^*(t) \lesssim th^{**}(t)$, which gives $f^* \lesssim t^{s/n} h^{**}$, which implies $\chi_{(0,1)} f^* \lesssim \chi_{(0,1)} T_1' h$, and therefore

$$\rho_{G_E}\bigl(\chi_{(0,1)} f^*\bigr) \lesssim \rho_E(h) \lesssim \rho_{G_1}\bigl(\chi_{(0,1)} f^*\bigr),$$

proving optimality of G. To check optimality of E, we notice that

$$\begin{aligned}\rho_{E(G)}(h) = \rho_G\bigl(\chi_{(0,1)} S_1 h^*\bigr) &\gtrsim \rho_G\bigl(\chi_{(0,1)} T_1 h^{**}\bigr)\\ &\approx \rho_H\Bigl(t^{-1} \sup_{0<u<t} u^{1-s/n} \chi_{(0,1)} T_1 h^{**}(u)\Bigr)\\ &\gtrsim \rho_H\bigl(\chi_{(0,1)} h^{**}\bigr).\end{aligned}$$

Hence

$$\rho_{E(G)}(h) \gtrsim \rho_E(h). \qquad \square$$

Example 2.15 Let $E = \Gamma_0^\infty(tw)(\Omega)$, consisting of all $f \in \Gamma^\infty(tw)(\Omega)$ such that $tf^{**}(t) \to 0$ as $t \to 0$, w is slowly varying. Then $\beta_E = 1$. If $G = \Gamma_1^\infty(t^{1-s/n}v) \cap \Gamma^\infty(tw)$, where $v(t) = \sup_{u>t} w(u)$ and

$$\Gamma_1^\infty(v) := \Bigl\{f \in \Gamma^\infty(v) : \sup_{0<u<t} u^{1-s/n} f^*(u) \to 0 \text{ as } t \to 0\Bigr\},$$

then this couple is optimal. In particular, if $w = 1$, then $E = L^1(\Omega)$ and $G = \Gamma_1^\infty(t^{1-s/n})$.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors read and approved the final manuscript.

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