THE ANOSOV THEOREM FOR INFRANILMANIFOLDS WITH AN ODD-ORDER ABELIAN HOLONOMY GROUP

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We prove that $N(f) = |L(f)|$ for any continuous map f of a given infranilmanifold with Abelian holonomy group of odd order. This theorem is the analogue of a theorem of Anosov for continuous maps on nilmanifolds. We will also show that although their fundamental groups are solvable, the infranilmanifolds we consider are in general not solvmanifolds, and hence they cannot be treated using the techniques developed for solvmanifolds.

1. Introduction

Let M be a smooth closed manifold and let $f : M \to M$ be a continuous self-map of M. In fixed point theory, two numbers are associated with f to provide information on its fixed points: the Lefschetz number $L(f)$ and the Nielsen number $N(f)$. Inspired by the fact that $N(f)$ gives more information than $L(f)$, but unfortunately $N(f)$ is not readily computable from its definition (while $L(f)$ is much easier to calculate), in literature, a considerable amount of work has been done on investigating the relation between both numbers. In [1] Anosov proved that $N(f) = |L(f)|$ for all continuous maps $f : M \to M$ if M is a nilmanifold, but he also observed that there exists a continuous map $f : K \to K$ of the Klein bottle K such that $N(f) \neq |L(f)|$.

There are two possible ways of trying to generalize this theorem of Anosov. Firstly, one can search classes of maps for which the relation holds for a specific type of manifold. For instance, Kwasik and Lee proved in [10] that the Anosov theorem holds for homotopic periodic maps of infranilmanifolds and in [14] Malfait did the same for virtually unipotent maps of infranilmanifolds. Secondly, one can look for classes of manifolds, other than nilmanifolds, for which the relation holds for all continuous maps of the given manifold, as was established by Keppelmann and McCord for exponential solvmanifolds (see [8]).

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In this article we will work on the class of infranilmanifolds. After the preliminaries we will first describe a class of maps for which the Anosov theorem holds and thereafter we will follow the second approach and work with infranilmanifolds with odd-order Abelian holonomy group. The main result of this paper is that the Anosov theorem always holds for these kinds of infranilmanifolds. This result cannot be extended to the case of even-order Abelian holonomy groups, since Anosov already constructed a counterexample for the Klein bottle, which has $\mathbb{Z}_2$ as holonomy group. A detailed investigation of the case of even-order holonomy is much more delicate and will be dealt with in an other paper.

Throughout the paper we will illustrate all concepts by means of examples. In fact the whole collection of examples together forms one big example. Moreover, by means of this example, we will also show that the manifolds we study are in general not solvmanifolds and therefore cannot be treated by the techniques developed for solvmanifolds.

2. Preliminaries

Let G be a connected, simply connected, nilpotent Lie group. An affine endomorphism of G is an element (g,φ) of the semigroup $G \rtimes \operatorname{Endo}(G)$ with $g \in G$ the translational part and $\varphi \in \operatorname{Endo}(G)$ (= the semigroup of all endomorphisms of G) the linear part. The product of two affine endomorphisms is given by $(g,\varphi)(h,\mu) = (g \cdot \varphi(h), \varphi\mu)$ and (g,φ) maps an element $x \in G$ to $g \cdot \varphi(x)$. If the linear part φ belongs to $\operatorname{Aut}(G)$, then (g,φ) is an invertible affine transformation of G. We write $\operatorname{Aff}(G) = G \rtimes \operatorname{Aut}(G)$ for the group of invertible affine transformations of G.

Example 2.1. One of the best known examples of a connected and simply connected nilpotent Lie group is the Heisenberg group

$$H = \left\{ \begin{pmatrix} 1 & y & z \\ 0 & 1 & x \\ 0 & 0 & 1 \end{pmatrix} \mid x, y, z \in \mathbb{R} \right\}. \tag{2.1}$$

For further use, we will use $h(x,y,z)$ to denote the element $\begin{pmatrix} 1 & y & (1/3)z \\ 0 & 1 & x \\ 0 & 0 & 1 \end{pmatrix}$. (The reason for introducing a 3 in the upper right corner lies in the use of this example later on.) The reader easily computes that

$$h(x_1,y_1,z_1)h(x_2,y_2,z_2) = h(x_1+x_2, y_1+y_2, z_1+z_2+3x_2y_1). \tag{2.2}$$

Let us fix the following elements for use throughout the paper: $a = h(1,0,0)$, $b = h(0,1,0)$ and $c = h(0,0,1)$. The group N generated by the elements a, b, c has a presentation of the form

$$N = \langle a,b,c \mid [b,a] = c^3,\ [c,a] = [c,b] = 1 \rangle. \tag{2.3}$$

(We use the convention that $[b,a] = b^{-1}a^{-1}ba$.) Obviously the group N consists exactly of all elements $h(x,y,z)$, for which $x,y,z \in \mathbb{Z}$.

For any connected, simply connected nilpotent Lie group G with Lie algebra $\mathfrak{g}$, it is known that the exponential map $\exp : \mathfrak{g} \to G$ is bijective and we denote by log the inverse of exp.

Example 2.2. The Lie algebra of H, is the Lie algebra of matrices of the form

$$\mathfrak{h} = \left\{ \begin{pmatrix} 0 & y & z \\ 0 & 0 & x \\ 0 & 0 & 0 \end{pmatrix} \mid x, y, z \in \mathbb{R} \right\}. \tag{2.4}$$

The exponential map is given by

$$\exp : \mathfrak{h} \longrightarrow H : \begin{pmatrix} 0 & y & z \\ 0 & 0 & x \\ 0 & 0 & 0 \end{pmatrix} \longmapsto \begin{pmatrix} 1 & y & z + \dfrac{xy}{2} \\ 0 & 1 & x \\ 0 & 0 & 1 \end{pmatrix}. \tag{2.5}$$

Hence

$$\log : H \longrightarrow \mathfrak{h} : \begin{pmatrix} 1 & y & z \\ 0 & 1 & x \\ 0 & 0 & 1 \end{pmatrix} \longmapsto \begin{pmatrix} 0 & y & z - \dfrac{xy}{2} \\ 0 & 0 & x \\ 0 & 0 & 0 \end{pmatrix}. \tag{2.6}$$

For later use, we fix the following basis of $\mathfrak{h}$:

$$\begin{gathered} C = \begin{pmatrix} 0 & 0 & \dfrac{1}{3} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \log(c), \qquad B = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \log(b), \\ A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} = \log(a). \end{gathered} \tag{2.7}$$

For any endomorphism φ of the Lie group G to itself there exists a unique endomorphism φ_* of the Lie algebra $\mathfrak{g}$ (namely the differential of φ), making the following diagram commutative:

$$\begin{tikzcd} G \arrow[r, "\varphi"] \arrow[d, shift left, "\log"] & G \arrow[d, shift left, "\log"] \\ \mathfrak{g} \arrow[r, "\varphi_*"'] \arrow[u, shift left, "\exp"] & \mathfrak{g} \arrow[u, shift left, "\exp"] \end{tikzcd} \tag{2.8}$$

Conversely, every endomorphism φ_* of $\mathfrak{g}$ appears as the differential of an endomorphism of G.

Example 2.3. Let H and $\mathfrak{h}$ be as before. With respect to the basis C, B and A (in this order!), any endomorphism φ_* is given by a matrix of the form

$$\begin{pmatrix} k_1l_2 - k_2l_1 & l_3 & k_3 \\ 0 & l_2 & k_2 \\ 0 & l_1 & k_1 \end{pmatrix}. \tag{2.9}$$

This follows from the fact that $3C = [B,A]$ and hence $3\varphi_*(C) = [\varphi_*(B),\varphi_*(A)]$. Conversely, any such a matrix represents an endomorphism of $\mathfrak{g}$. The corresponding endomorphism φ of H satisfies

$$\begin{aligned} \varphi(h(x,y,z)) &= \exp\left(\varphi_*\left(\log\left(h(x,y,z)\right)\right)\right) \\ &= h\Bigg(k_1x + l_1y, k_2x + l_2y, 3k_3x + 3l_3y + \frac{3(k_1x + l_1y)(k_2x + l_2y)}{2} \\ &\qquad + (k_1l_2 - k_2l_1)\left(z - \frac{3xy}{2}\right)\Bigg). \end{aligned} \tag{2.10}$$

As one sees, although the map φ_* is linear and thus easy to describe, the corresponding φ is much more complicated. In order to be able to continue presenting examples, we will use a matrix representation of the semigroup $H \rtimes \mathrm{Endo}(H)$. Given an endomorphism φ of H, let us denote by M_φ the 4×4-matrix

$$M_\varphi = \begin{pmatrix} P & 0 \\ 0 & 1 \end{pmatrix}, \tag{2.11}$$

where P denotes the 3×3-matrix, representing φ_* with respect to the basis C, B, A (again in this fixed order). Define the map

$$\psi : H \rtimes \mathrm{Endo}(H) \longrightarrow M_4(\mathbb{R}) : (h(x,y,z),\varphi) \longmapsto \begin{pmatrix} 1 & -\dfrac{3x}{2} & \dfrac{3y}{2} & -\dfrac{3xy}{2} + z \\ 0 & 1 & 0 & y \\ 0 & 0 & 1 & x \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot M_\varphi. \tag{2.12}$$

We leave it to the reader to verify that ψ defines a faithful representation of the semigroup $H \rtimes \mathrm{Endo}(H)$ into the semigroup $M_4(\mathbb{R})$ (respectively of the group $\mathrm{Aff}(H)$ into the group $\mathrm{Gl}(4,\mathbb{R})$).

Remark 2.4. An analogous matrix representation can be obtained for any $G \rtimes \mathrm{Endo}(G)$ in case G is two-step nilpotent. (Recall that a group G is said to be k-step nilpotent if the $k+1$'th term of the lower central series $\gamma_{k+1}(G) = 1$, where $\gamma_1(G) = G$ and $\gamma_{i+1}(G) = [G,\gamma_i(G)]$. For example, the Heisenberg group is 2-step nilpotent.) This is proved in [3] for the group $\mathrm{Aff}(G)$, but the details in that paper can easily be adjusted to the case of the semigroup $G \rtimes \mathrm{Endo}(G)$.

2.1. Infranilmanifolds and continuous maps. In this section we quickly recall the notion of almost-crystallographic groups and infranilmanifolds. We refer the reader to [4] for more details.

An almost-crystallographic group is a subgroup E of $\mathrm{Aff}(G)$, such that its subgroup of pure translations $N = E \cap G$, is a uniform lattice (by which we mean a discrete and cocompact subgroup) of G and moreover, N is of finite index in E. Therefore the quotient group $F = E/N$ is finite and is called the holonomy group of E. Note that the group F is isomorphic to the image of E under the natural projection $\mathrm{Aff}(G) \to \mathrm{Aut}(G)$, and hence F can be viewed as a subgroup of $\mathrm{Aut}(G)$ and of $\mathrm{Aff}(G)$.

Any almost-crystallographic group acts properly discontinuously on (the corresponding) G and the orbit space $E\backslash G$ is compact. Recall that an action of a group E on a locally compact space X is said to be properly discontinuous, if for every compact subset C of X, the set $\{\gamma \in E \mid \gamma C \cap C \neq \varnothing\}$ is finite. When E is a torsion free almost-crystallographic group, it is referred to as an almost-Bieberbach group and the orbit space $M = E\backslash G$ is called an infranilmanifold. In this case E equals the fundamental group $\pi_1(M)$ of the infranilmanifold, and we will also talk about F as being the holonomy group of M.

Any almost-crystallographic group determines a faithful representation $T : F \to \mathrm{Aut}(G)$, which is induced by the natural projection $p : \mathrm{Aff}(G) = G \rtimes \mathrm{Aut}(G) \to \mathrm{Aut}(G)$, and which is referred to as the holonomy representation.

Remark 2.5. As isomorphic crystallographic subgroups are conjugated inside $\mathrm{Aff}(G)$ (see Theorem 2.7 below or [13]), it follows that the holonmy representation of an almost-crystallographic group is completely determined from the algebraic structure of E up to conjugation by an element of $\mathrm{Aff}(G)$.

Let $\mathfrak{g}$ denote the Lie algebra of G. By taking differentials, the holonomy representation also induces a faithful representation

$$T_* : F \longrightarrow \mathrm{Aut}(\mathfrak{g}) : x \longmapsto T_*(x) := d(T(x)). \tag{2.13}$$

Example 2.6. Let φ be the automorphism of H, whose differential φ_* is given by the matrix $\left(\begin{smallmatrix} 1 & -3/2 & 0 \\ 0 & -1 & 1 \\ 0 & -1 & 0 \end{smallmatrix}\right)$. Let $\alpha = (h(0,0,1/3), \varphi) \in \mathrm{Aff}(H)$. Then the group E generated by a, b, c and α has a presentation of the form

$$E = \left\langle \begin{array}{llll} a,b,c,\alpha \mid [b,a] = c^3 & [c,a] = 1 & [c,b] = 1 & \\ \alpha a = b\alpha & \alpha b = a^{-1}b^{-1}\alpha & \alpha c = c\alpha & \alpha^3 = c \end{array} \right\rangle. \tag{2.14}$$

(This is easily checked using the matrix representation (2.12).) E is an almost-crystallographic group with translation subgroup $N = H \cap E = \langle a,b,c\rangle$ and a holonomy group $F = E/N \cong \mathbb{Z}_3$ of order three. (See also [4, page 164, type 13].) We have that

$$T_*(F) = \left\{ I_3, \begin{pmatrix} 1 & -\frac{3}{2} & 0 \\ 0 & -1 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & -\frac{3}{2} & 0 \\ 0 & -1 & 1 \\ 0 & -1 & 0 \end{pmatrix}^2 = \begin{pmatrix} 1 & 0 & -\frac{3}{2} \\ 0 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix} \right\}. \tag{2.15}$$

(Of course I_n will denote the $n \times n$-identity matrix.)

As E is torsion-free, it is an almost-Bieberbach group and it determines an infranilmanifold $M = E\backslash H$.

Essential for our purposes is the following result due to K. B. Lee (see [11]).

THEOREM 2.7. *Let $E, E' \subset \mathrm{Aff}(G)$ be two almost-crystallographic groups. Then for any homomorphism $\theta : E \to E'$, there exists a $g = (d, D) \in G \rtimes \mathrm{Endo}(G)$ such that $\theta(\alpha) \cdot g = g \cdot \alpha$ for all $\alpha \in E$.*

Important for us is the following corollary of this theorem (we refer to [11] for a detailed proof).

COROLLARY 2.8. *Let $M = E\backslash G$ be an infranilmanifold and $f : M \to M$ a continuous map of M. Then f is homotopic to a map $h : M \to M$ induced by an affine endomorphism $(d, D) : G \to G$.*

We say that (d, D) is a homotopy lift of f. Note that one can find the homotopy lift of a given f, by using Theorem 2.7 for the homomorphism $f_* : \pi_1(M) \to \pi_1(M)$ induced by f. In fact, using this method one can characterize all continuous maps, up to homotopy, of a given infranilmanifold M.

Example 2.9. Let E be the almost Bieberbach group of the previous example, then there is a homomorphism $\theta_1 : E \to E$, which is determined by the images of the generators as follows:

$$\theta_1(a) = b^2c^3, \qquad \theta_1(b) = a^2c^3, \qquad \theta_1(c) = c^{-4}, \qquad \theta_1(\alpha) = c^{-2}\alpha^2. \tag{2.16}$$

Using the matrix representation (2.12) it is easy to check that θ_1 really determines an endomorphism of E and that this endomorphism is induced by the affine endomorphism $(h(0,0,0), D_1)$, where

$$D_{1,*} = \begin{pmatrix} -4 & 3 & 3 \\ 0 & 0 & 2 \\ 0 & 2 & 0 \end{pmatrix}. \tag{2.17}$$

Another example is given by the morphism θ_2 determined by

$$\theta_2(a) = a^4b^4c^{20}, \qquad \theta_2(b) = a^{-4}c^{-10}, \qquad \theta_2(c) = c^{16}, \qquad \theta_2(\alpha) = c^5\alpha, \tag{2.18}$$

and induced by $(h(0,0,0), D_2)$, where

$$D_{2,*} = \begin{pmatrix} 16 & -10 & -4 \\ 0 & 0 & 4 \\ 0 & -4 & 4 \end{pmatrix}. \tag{2.19}$$

2.2. Lefschetz and Nielsen numbers on infranilmanifolds. Let M be a compact manifold and assume $f : M \to M$ is a continuous map. The Lefschetz number $L(f)$ is defined by

$$L(f) = \sum_i (-1)^i \operatorname{Trace}\left(f_* : H_i(M, \mathbb{Q}) \longrightarrow H_i(M, \mathbb{Q})\right). \tag{2.20}$$

The set $\mathrm{Fix}(f)$ of fixed points of f is partitioned into equivalence classes, referred to as fixed point classes, by the relation: $x, y \in \mathrm{Fix}(f)$ are f-equivalent if and only if there is a path w from x to y such that w and fw are (rel. endpoints) homotopic. To each class one assigns an integer index. A fixed point class is said to be essential if its index is nonzero. The Nielsen number of f is the number of essential fixed point classes of f. The relation between $L(f)$ and $N(f)$ is given by the property that $L(f)$ is exactly the sum of the indices of all fixed point classes. For more details we refer to [2, 7] or [9].

In this paper, we examine the relation $N(f) = |L(f)|$ for continuous maps $f : M \to M$ on an infranilmanifold M. Since $L(f)$ and $N(f)$ are homotopy invariants, one can restrict to those maps which are induced by an affine endomorphism of the covering Lie group G.

In fact, this is exploited completely in the following theorem of K. B. Lee (see [11]), which will play a crucial role throughout the rest of this paper.

Theorem 2.10. *Let $f : M \to M$ be a continuous map of an infranilmanifold M and let $T : F \to \mathrm{Aut}(G)$ be the associated holonomy representation. Let $(d,D) \in G \rtimes \mathrm{Endo}(G)$ be a homotopy lift of f. Then*

$$\begin{aligned} N(f) = L(f) &\Longleftrightarrow \det\left(I_n - T_*(x)D_*\right) \geq 0, \quad \forall x \in F, \text{ and respectively,} \\ N(f) = -L(f) &\Longleftrightarrow \det\left(I_n - T_*(x)D_*\right) \leq 0, \quad \forall x \in F. \end{aligned} \tag{2.21}$$

Remark 2.11. Recently J. B. Lee and K. B. Lee generalized (see [12]) this theorem by proving that the following formulas for $L(f)$ and $N(f)$ hold on infranilmanifolds. Using the notations from above:

$$\begin{aligned} L(f) &= \frac{1}{|F|} \sum_{x \in F} \det\left(I_n - T_*(x)D_*\right), \\ N(f) &= \frac{1}{|F|} \sum_{x \in F} \left|\det\left(I_n - T_*(x)D_*\right)\right|. \end{aligned} \tag{2.22}$$

3. A class of maps for which the Anosov theorem holds

With Theorem 2.10 in mind, we can describe a class of maps on infranilmanifolds, for which the Anosov theorem always holds. Note that we do not claim that such maps exist on all infranilmanifolds.

Proposition 3.1. Let M be an infranilmanifold with holonomy group F and associated holonomy representation $T : F \to \mathrm{Aut}(G)$. Let $f : M \to M$ be a continuous map and (d,D) a homotopy lift of f.

Suppose that for all $x \in F$, $x \neq 1 : T_*(x)D_* \neq D_*T_*(x)$. Then

$$\forall x \in F : \det\left(I_n - D_*\right) = \det\left(I_n - T_*(x)D_*\right), \tag{3.1}$$

and hence $N(f) = |L(f)|$.

Proof. Let $1 \neq x \in F$. Since (d,D) is obtained from Theorem 2.7, we know that there exists an $y \in F$ such that $T(y)_*D_* = D_*T(x)_*$. Indeed, if $\tilde{x}$ is a pre-image of $x \in E = \pi_1(M)$, then y is the natural projection of $f_*(\tilde{x})$, where f_* denote the morphism induced by f on $\pi_1(M)$.

Because of the condition on T_* and D_* we know that $x \neq y$. Then

$$\begin{aligned}\det(I_n - D_*) &= \det(T_*(x) - D_* T_*(x)) \det(T_*(x^{-1})) \\ &= \det(T_*(x) - T_*(y)D_*) \det(T_*(x^{-1})) \\ &= \det(I_n - T_*(x^{-1}y)D_*).\end{aligned} \tag{3.2}$$

Since $x \neq y$ and T_* is faithful, we have that $T_*(x^{-1}y) \neq I_n$. Moreover, for any other $1 \neq x' \in F$, with $x \neq x'$ and $T_*(y')D_* = D_* T_*(x')$, we have that $x^{-1}y \neq x'^{-1}y'$. Indeed, suppose that there exists an $x' \in F$, $x \neq x'$, such that $x^{-1}y = x'^{-1}y'$. Then

$$\begin{aligned}T_*(x^{-1}y)D_* = T_*(x'^{-1}y')D_* &\Longleftrightarrow T_*(x^{-1})D_* T_*(x) = T_*(x'^{-1})D_* T_*(x') \\ &\Longleftrightarrow D_* T_*(xx'^{-1}) = T_*(xx'^{-1})D_*.\end{aligned} \tag{3.3}$$

This last equality is only satisfied when $xx'^{-1} = 1$. This proves the proposition because any $x \in F$ determines an unique element $x^{-1}y \in F$, and thus all elements of F are obtained. The last conclusion easily follows from Theorem 2.10. □

Example 3.2. Let $M = E\backslash H$ be the infra-nilmanifold from before and suppose that $f_1 : M \to M$ is a continuous map inducing the endomorphism θ_1 on $E = \pi_1(M)$. We know already that $f_* = \theta_1$ is induced by $(1, D_1)$ and it is easy to check that

$$\varphi_* D_{1,*} = \begin{pmatrix} 1 & -\frac{3}{2} & 0 \\ 0 & -1 & 1 \\ 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} -4 & 3 & 3 \\ 0 & 0 & 2 \\ 0 & 2 & 0 \end{pmatrix} = \begin{pmatrix} -4 & 3 & 3 \\ 0 & 0 & 2 \\ 0 & 2 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & -\frac{3}{2} \\ 0 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix} = D_{1,*}\varphi_*^2 \tag{3.4}$$

which implies that the map f (or $D_{1,*}$) satisfies the criteria of the theorem, and indeed we have that

$$\det(I_3 - D_{1,*}) = \det(I_3 - \varphi_* D_{1,*}) = \det(I_3 - \varphi_*^2 D_{1,*}) = -15. \tag{3.5}$$

4. Infranilmanifolds with Abelian holonomy group of odd order

In this section, we concentrate on the infranilmanifolds with an odd-order Abelian holonomy group F and show that the Anosov theorem can be generalized to this class of manifolds.

Let $T : F \to \mathrm{Aut}(G)$ denote the holonomy representation as before, then, for any $x \in F$, we have that $T_*(x)$ is of finite order, since F is finite, and so the eigenvalues $T_*(x)$ are roots of unity. Moreover, since the order of $T_*(x)$ has to be odd, we know that the only eigenvalues of $T_*(x)$ are 1 or not real. The usefulness of this observation follows from the next lemma concerning commuting matrices.

Lemma 4.1. *Let $B,C \in \mathrm{M_n}(\mathbb{R})$ be two real matrices such that $BC = CB$ and suppose that B has only nonreal eigenvalues. Then the (algebraic) multiplicity of any real eigenvalue of C must be even which implies that* $\det(I_n - C) \geq 0$.

Proof. We prove this lemma by induction on n. Note that n is even because B only has non real eigenvalues.

Suppose $n = 2$ and λ is a real eigenvalue of C with eigenvector v such that $Cv = \lambda v$. Then Bv is also an eigenvector of C, since $CBv = BCv = \lambda Bv$. Moreover, v and Bv are linearly independent over $\mathbb{R}$. Otherwise there would exist a $\mu \in \mathbb{R}$ such that $Bv = \mu v$ contradicting the fact that B has no real eigenvalues. So the dimension of the eigenspace of λ is 2 and therefore the multiplicity of λ must be 2.

Suppose the lemma holds for $r \times r$ matrices with r even and $r < n$. We then have to show that the lemma holds for $n \times n$ matrices. Again, let λ be a real eigenvalue of C and v an eigenvector of C such that $Cv = \lambda v$. Then, for any $m \in \mathbb{N}$, we have that $B^m v$ is an eigenvector of C. Indeed, $CB^m v = B^m Cv = \lambda B^m v$. Let S be the subspace of $\mathbb{R}^n$ generated by all vectors $B^m v$ with $m \in \mathbb{N}$. Then, for any $s \in S$, we have that $Cs = \lambda s$, so S is part of the eigenspace of λ and secondly $Bs \in S$, which implies that S is a B-invariant subspace of $\mathbb{R}^n$. Let $\{v_1, \ldots, v_k\}$ be a basis for S, then we can complete this basis with $v_{k+1}, \ldots, v_n$ to obtain a basis for $\mathbb{R}^n$. Writing (the matrices of the linear transformations determined by) B and C with respect to this new basis, implies the existence of a matrix $P \in \mathrm{Gl}(n, \mathbb{R})$ such that

$$PCP^{-1} = \begin{pmatrix} \lambda I_k & C_2 \\ 0 & C_3 \end{pmatrix}, \qquad PBP^{-1} = \begin{pmatrix} B_1 & B_2 \\ 0 & B_3 \end{pmatrix} \tag{4.1}$$

with B_1 a real $k \times k$ matrix; B_2, C_2 real $k \times (n-k)$ matrices; and B_3, C_3 real $(n-k) \times (n-k)$ matrices. Of course, the eigenvalues of B_1 and B_3 are also not real and $B_3C_3 = C_3B_3$. Therefore, k has to be even and we can proceed by induction on B_3 and C_3 to conclude that the real eigenvalues of C indeed have even multiplicities.

To prove the second claim of the lemma, we suppose that $\lambda_1, \ldots, \lambda_r$ are the real eigenvalues of C with even multiplicities $m_1, \ldots, m_r$ and that $\mu_1, \overline{\mu_1}, \ldots, \mu_t, \overline{\mu_t}$ are the complex eigenvalues of C with multiplicities $n_1, \ldots, n_t$. Then

$$\begin{aligned} &\det\left(I_n - C\right) \\ &\quad = \left(1-\lambda_1\right)^{m_1} \cdots \left(1-\lambda_r\right)^{m_r} \left(1-\mu_1\right)^{n_1} \left(1-\overline{\mu_1}\right)^{n_1} \cdots \left(1-\mu_t\right)^{n_t} \left(1-\overline{\mu_t}\right)^{n_t} \\ &\quad = \left(1-\lambda_1\right)^{m_1} \cdots \left(1-\lambda_r\right)^{m_r} \left(\left(1-\mu_1\right)\left(\overline{1-\mu_1}\right)\right)^{n_1} \cdots \left(\left(1-\mu_t\right)\left(\overline{1-\mu_t}\right)\right)^{n_t} \\ &\quad = \left(1-\lambda_1\right)^{m_1} \cdots \left(1-\lambda_r\right)^{m_r} \left|1-\mu_1\right|^{2n_1} \cdots \left|1-\mu_t\right|^{2n_t}. \end{aligned} \tag{4.2}$$

This last expression is clearly nonnegative since the m_i are even. □

We are now ready to prove the main theorem of this paper.

Theorem 4.2. *Let M be an n-dimensional infranilmanifold with Abelian holonomy group F of odd order. Then, for any continuous map $f : M \to M$, $N(f) = |L(f)|$.*

Proof. Let $T : F \to \mathrm{Aut}(G)$ be the associated holonomy representation and suppose that (d, D) is a homotopy lift of f. To apply Theorem 2.10, we have to calculate the determinants $\det(I_n - T_*(x)D_*)$ for any $x \in F$. If D_* does not commute with $T(x)_*$ for all $1 \neq x \in F$, we can use Proposition 3.1 to obtain that $N(f) = |L(f)|$.

Now assume that there exists an $x_0 \in F$, $x_0 \neq 1$, such that $T_*(x_0)D_* = D_*T_*(x_0)$. Since $T_*(x_0)$ is of finite odd order, the eigenvalues of $T_*(x_0)$ are 1 or non real and $T_*(x_0)$ is diagonalizable (over $\mathbb{C}$). This implies that there exists a $P \in \mathrm{Gl}(n, \mathbb{R})$ such that

$$PT_*(x_0)P^{-1} = \begin{pmatrix} I_{n_1} & 0 \\ 0 & A_2 \end{pmatrix}, \tag{4.3}$$

with n_1 the multiplicity of the eigenvalue 1 and A_2 an $(n - n_1) \times (n - n_1)$-matrix having non real eigenvalues. Note that we do not exclude the case where $n_1 = 0$ (i.e., the case where 1 is not an eigenvalue of $T_*(x_0)$). Since PD_*P^{-1} now commutes with $PT_*(x_0)P^{-1}$, we must have that

$$PD_*P^{-1} = \begin{pmatrix} D_1 & 0 \\ 0 & D_2 \end{pmatrix}, \tag{4.4}$$

with D_1 an $n_1 \times n_1$-matrix and D_2 an $(n - n_1) \times (n - n_1)$-matrix commuting with A_2. Moreover, since F is Abelian, all $T_*(x)$ commute with $T_*(x_0)$, and hence

$$\forall x \in F : PT_*(x)P^{-1} = \begin{pmatrix} T_1'(x) & 0 \\ 0 & T_2'(x) \end{pmatrix}, \tag{4.5}$$

with $T_1' : F \to \mathrm{Gl}(n_1, \mathbb{R})$ and $T_2' : F \to \mathrm{Gl}(n - n_1, \mathbb{R})$. So we obtain for any $x \in F$

$$\begin{aligned} \det(I_n - T_*(x)D_*) &= \det\left(I_n - PT_*(x)P^{-1}PD_*P^{-1}\right) \\ &= \det\left(I_{n_1} - T_1'(x)D_1\right)\det\left(I_{n-n_1} - T_2'(x)D_2\right). \end{aligned} \tag{4.6}$$

On the second factor of the above expression we can apply Lemma 4.1 since A_2 commutes with $T_2'(x)D_2$, for any $x \in F$, and A_2 only has non real eigenvalues. So the second factor is always positive or zero. (In case $n_1 = 0$, there is no "first factor" and the proof finishes here.)

To calculate the first factor, we define $F_1 = F/\ker T_1'$ and consider the faithful representation $T_{1*} : F_1 \to \mathrm{Gl}(n_1, \mathbb{R}) : \overline{x} \mapsto T_1'(x)$. One can easily verify that T_{1*} is well defined. Note that $|F_1| < |F|$ since $x_0 \in \ker(T_1')$ and so we can proceed by induction on the order. This induction process ends when $F_1 = 1$ or when for any $x_1 \in F_1 : T_{1*}(x_1)D_1 \neq D_1T_{1*}(x_1)$. □

Example 4.3. Let $M = E\backslash H$ as before and let $f_2 : M \to M$ be a continuous map inducing the endomorphism θ_2 on $E = \pi_1(M)$. Then we have that

$$\det\left(I_3 - D_{2,*}\right) = \det\left(I_3 - \varphi_*^2 D_{2,*}\right) = -195, \qquad \det\left(I_3 - \varphi_* D_{2,*}\right) = -375. \tag{4.7}$$

Although these determinants are no longer all equal, they still have the same sign, implying $N(f) = |L(f)|$. (In fact here $N(f) = -L(f)$.)

Finally, we would like to remark that although the fundamental group of an infranil-manifold with an Abelian holonomy group is always solvable (in fact polycyclic), these manifolds do not need to be solvmanifolds in general and so the Nielsen theory on these manifolds cannot be treated by the techniques developed for solvmanifolds (as in e.g. [5, 6, 8]).

Example 4.4. The almost-Bieberbach group $E = \langle a, b, c, \alpha \rangle$ is not the fundamental group of a solvmanifold. Indeed, suppose that E is the fundamental group of a solvmanifold, then it is known that the manifold admits a fibering over a torus with a nilmanifold as fibre. On the level of the fundamental group, this implies that there exists a short exact sequence

$$1 \longrightarrow \Gamma \longrightarrow E \longrightarrow A \longrightarrow 1, \tag{4.8}$$

where Γ is a finitely generated torsion free nilpotent group and A is a free Abelian group of finite rank. However, it is easy to see that $[E, E]$ is of finite index in E, and therefore, the only free Abelian quotient of E is the trivial group. Therefore, there does not exist a normal nilpotent group $\Gamma \subseteq E$, with E/Γ free Abelian. This shows that E is not the fundamental group of a solvmanifold.

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