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Research Article

T-Stability of Picard Iteration in Metric Spaces

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We establish a general result for the stability of Picard's iteration. Several theorems in the literature are obtained as special cases.

Let (X,d) be a complete metric space and T a self-map of X. Let $x_{n+1}=f(T,x_n)$ be some iteration procedure. Suppose that $F(T)$, the fixed point set of T, is nonempty and that x_n converges to a point $q\in F(T)$. Let $\{y_n\}\subset X$ and define $\epsilon_n=d(y_{n+1},f(T,y_n))$. If $\lim\epsilon_n=0$ implies that $\lim y_n=q$, then the iteration procedure $x_{n+1}=f(T,x_n)$ is said to be T-stable. Without loss of generality, we may assume that $\{y_n\}$ is bounded, for if $\{y_n\}$ is not bounded, then it cannot possibly converge. If these conditions hold for $x_{n+1}=Tx_n$, that is, Picard's iteration, then we will say that Picard's iteration is T-stable.

We will obtain sufficient conditions that Picard's iteration is T-stable for an arbitrary self-map, and then demonstrate that a number of contractive conditions are Picard T-stable.

We will need the following lemma from [1].

Lemma 1. *Let* $\{x_n\}$, $\{\epsilon_n\}$ *be nonnegative sequences satisfying* $x_{n+1}\le hx_n+\epsilon_n$, *for all* $n\in\mathbb{N}$, $0\le h<1$, $\lim\epsilon_n=0$. *Then,* $\lim x_n=0$.

Theorem 1. *Let* (X,d) *be a nonempty complete metric space and* T *a self-map of* X *with* $F(T)\neq\varnothing$. *If there exist numbers* $L\ge 0$, $0\le h<1$, *such that*

$$d(Tx,q)\le Ld(x,Tx)+hd(x,q) \tag{1}$$

for each $x \in X$, $q \in F(T)$, and, in addition,

$$\lim d(y_n, Ty_n) = 0, \tag{2}$$

then Picard's iteration is T-stable.

Proof. First, we show that the fixed point q of T is unique. Suppose p is another fixed point of T, then

$$d(p,q) = d(Tp,q) \leq Ld(p,Tp) + hd(p,q) = hd(p,q). \tag{3}$$

Since $0 \leq h < 1$, so $d(p,q) = 0$, that is, $p = q$.

Let $\{y_n\} \subset X$, $\epsilon_n = d(y_{n+1}, Ty_n)$, and $\lim \epsilon_n = 0$. We need to show that $\lim y_n = q$.

Using (1), (2), and Lemma 1,

$$d(y_{n+1}, q) \leq d(y_{n+1}, Ty_n) + d(Ty_n, q) \leq \epsilon_n + Ld(y_n, Ty_n) + hd(y_n, q), \tag{4}$$

and $\lim y_n = q$. □

Corollary 1. *Let (X,d) be a nonempty complete metric space and T a self-map of X satisfying the following: there exists $0 \leq h < 1$, such that, for each $x, y \in X$,*

$$d(Tx,Ty) \leq h \max\{d(x,y), d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)\}. \tag{5}$$

Then, Picard's iteration is T-stable.

Proof. From [2, Theorem 11], T has a unique fixed point q. Also, T satisfies (1). It remains to show that (2) is satisfied.

Define p_n to be the diameter of the orbit of y_n; that is, $p_n = \delta(O(y_n, Ty_n, \ldots))$. First, we show that p_n is bounded:

$$\begin{aligned} d(Ty_n, q) &\leq h \max\{d(y_n,q), d(y_n,Ty_n), d(y_n,Tq), d(q,Ty_n), d(q,Tq)\} \\ &\leq h \max\{d(y_n,q), d(y_n,Ty_n), d(y_n,q), d(q,Ty_n), 0\} \\ &= h \max\{d(y_n,q), d(y_n,Ty_n), d(y_n,q), d(q,Ty_n)\}. \end{aligned} \tag{6}$$

Hence, $d(Ty_n,q) \leq hd(y_n,q)$ or $d(Ty_n,q) \leq hd(y_n,Ty_n)$ or $d(Ty_n,q) \leq hd(q,Ty_n)$.

If $d(Ty_n,q) \leq hd(y_n,q)$, it is clear that

$$d(Ty_n,q) \leq hd(y_n,q) \leq \frac{h}{1-h} d(y_n,q). \tag{7}$$

If $d(Ty_n,q) \leq hd(q,Ty_n)$, then

$$d(Ty_n,q) = 0 \leq \frac{h}{1-h} d(y_n,q). \tag{8}$$

If $d(Ty_n, q) \le hd(y_n, Ty_n)$, then

$$d(y_n, Ty_n) \le d(Ty_n, q) + d(y_n, q) \le hd(y_n, Ty_n) + d(y_n, q). \tag{9}$$

Hence, $d(Ty_n, q) \le (h/(1-h))d(y_n, q)$. Now it is easy to see that $\{Ty_n\}$ is bounded and so is $\{p_n\}$, since $\{y_n\}$ is bounded.

For any $i, j \ge n$, using (5),

$$d(Ty_i, Ty_j) \le h \max\{d(y_i, y_j), d(y_i, Ty_i), d(y_j, Ty_j), d(y_i, Ty_j), d(y_j, Ty_i)\} \le hp_n. \tag{10}$$

Thus,

$$d(y_i, Ty_j) \le d(y_i, Ty_{i-1}) + d(Ty_{i-1}, Ty_j) \le \epsilon_{i-1} + hp_{n-1}. \tag{11}$$

But

$$d(y_i, y_j) \le d(y_i, Ty_{i-1}) + d(Ty_{i-1}, Ty_{j-1}) + d(Ty_{j-1}, y_j) \le \epsilon_{i-1} + hp_{n-1} + \epsilon_{i-1}, \tag{12}$$

which implies that

$$p_n \le 2\epsilon_{i-1} + hp_{n-1}, \tag{13}$$

and $\lim p_n = 0$ by Lemma 1. Since $d(y_n, Ty_n) \le p_n$, $\lim d(y_n, Ty_n) = 0$.

The conclusion now follows from Theorem 1. □

Corollary 2 (see [3, Theorem 1]). *Let (X, d) be a nonempty complete metric space and T a self-map of X satisfying*

$$d(Tx, Ty) \le Ld(x, Tx) + ad(x, y) \tag{14}$$

for all $x, y \in X$, where $L \ge 0$, $0 \le a < 1$. Suppose that T has a fixed point p. Then, T is Picard T-stable.

Proof. Since T satisfies (14) for all $x, y \in X$, then T satisfies inequality (1) of our paper. Let $\{y_n\} \subset X$ and define $\epsilon_n = d(y_{n+1}, y_n)$. From the proof of Theorem 1 of [3], $\lim d(y_n, Ty_n) = 0$. Therefore, by our theorem (Theorem 1), T is Picard T-stable. □

Definition (5) of this paper is actually Definition (24) of [2]. Therefore, many contractive conditions are special cases of (5), and, for each of these, Picard's iteration is T-stable. For example, Theorems 1 and 2 of [4] and Theorem 1 of [5] are special cases of Corollary 1.

We will not examine the analogues of Theorem 1 for Mann, Ishikawa, Kirk, or any other iteration scheme since, if one obtains convergence to a fixed point for a map using Picard's iteration, there is no point in considering any other more complicated iteration procedure.

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