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Hodge Theory on Metric Spaces

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Abstract Hodge theory is a beautiful synthesis of geometry, topology, and analysis which has been developed in the setting of Riemannian manifolds. However, spaces of images, which are important in the mathematical foundations of vision and pattern recognition, do not fit this framework. This motivates us to develop a version of Hodge theory on metric spaces with a probability measure. We believe that this constitutes a step toward understanding the geometry of vision.

Appendix B by Anthony Baker discusses a separable, compact metric space with infinite-dimensional α-scale homology.

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1 Introduction

Hodge Theory [22] studies the relationships of topology, functional analysis, and geometry of a manifold. It extends the theory of the Laplacian on domains of Euclidean space or on a manifold.

However, there are a number of spaces, not manifolds, which could benefit from an extension of Hodge theory, and that is the motivation here. In particular, we believe that a deeper analysis in the theory of vision could be led by developments of Hodge type. Spaces of images are important for developing a mathematics of vision (see e.g. Smale, Rosasco, Bouvrie, Caponnetto, and Poggio [35]), but these spaces are far from possessing manifold structures. Other settings include spaces occurring in quantum field theory, such as manifolds with singularities and/or non-uniform measures.

A number of previous papers have given us inspiration and guidance. For example, there are those in combinatorial Hodge theory by Eckmann [16], Dodziuk [13], Friedman [19], and more recently Jiang, Lim, Yao, and Ye [23]. Recent decades have seen extensions of the Laplacian from its classical setting to that of combinatorial graph theory. See e.g. Fan Chung [9]. Robin Forman [18] provides useful extensions from manifolds. Further extensions and relationships to the classical settings are given by Belkin and Niyogi [2], Belkin, De Vito, and Rosasco [3], Coifman and Maggioni [10], and Smale and Zhou [34].

Our approach starts with a metric space X (complete, separable), endowed with a probability measure. For $\ell \geq 0$, an ℓ-form is a function on $(\ell+1)$-tuples of points in X. The coboundary operator δ is defined from ℓ-forms to $(\ell+1)$-forms in the classical way following Čech, Alexander, and Spanier. Using the L^2-adjoint δ^* of δ for a boundary operator, the ℓth order Hodge operator on ℓ-forms is defined by $\Delta_\ell = \delta^*\delta + \delta\delta^*$. The harmonic ℓ-forms on X are solutions of the equation $\Delta_\ell(f) = 0$. The ℓ-harmonic forms reflect the ℓth homology of X, but have geometric features. The harmonic form is a special representative of the homology class, and it may be interpreted as one satisfying an optimality condition. Moreover, the Hodge equation is linear, and by choosing a finite sample from X, one can obtain an approximation of this representative by a linear equation in finite dimension.

There are two avenues for developing this Hodge theory. The first is a kernel version corresponding to a Gaussian or a reproducing kernel Hilbert space. Here the topology is trivial, but the analysis gives a substantial picture. The second version is akin to the adjacency matrix of graph theory and corresponds to a threshold at a given scale α. When X is finite, this picture overlaps with that of the combinatorial Hodge theory referred to above.

For passage to a continuous Hodge theory, one encounters the following problem.

Problem 1 (Poisson Regularity Problem) *If $\Delta_\ell(f) = g$ is continuous, under what conditions is f continuous?*

It is proved that a positive solution of the Poisson Regularity Problem implies a complete Hodge decomposition for continuous ℓ-forms in the "adjacency matrix" setting (at any scale α), provided the L^2-cohomology is finite dimensional. The problem is solved affirmatively for some cases as $\ell = 0$, or X is finite. One special case is the following.

Problem 2 *Under what conditions are harmonic ℓ-forms continuous?*

Here we have a solution for $\ell = 0$ and $\ell = 1$.

The solution of these regularity problems would be progress toward the important cohomology identification problem: To what extent does the L^2-cohomology coincide with the classical cohomology? We have an answer to this question, as well as a full Hodge theory in the special, but important case of Riemannian manifolds. The following theorem is proved in Sect. 9 of this paper.

Theorem 1 *Suppose that M is a compact Riemannian manifold, with strong convexity radius r, and that $k > 0$ is an upper bound on the sectional curvatures. Then, if $0 < \alpha < \max\{r, \sqrt{\pi}/2k\}$, our Hodge theory holds. That is, we have a Hodge decomposition; the kernel of Δ_ℓ is isomorphic to the L^2-cohomology and to the de Rham cohomology of M in degree ℓ.*

More general conditions on a metric space X are given in Sect. 9.

Certain previous studies show how topology questions can give insight into the study of images. Lee, Pedersen, and Mumford [25] have investigated 3×3 pixel images from real-world data bases to find evidence for the occurrence of homology classes of degree 1. Moreover, Carlsson, Ishkhanov, de Silva, and Zomorodian [6] have found evidence for homology of surfaces in the same data base. Here we attempt to give some foundations to these studies. Moreover, this general Hodge theory could yield optimal representatives of the homology classes and provide systematic algorithms. Note that the theory we describe here is defined over the real numbers; its homology and cohomology groups are real vector spaces.

The problem of recognizing a surface is quite complex; in particular, the cohomology of a non-oriented surface has torsion, and it may seem naive to attempt to recover such information from computations over $\mathbb{R}$. Nevertheless, we shall argue that Hodge theory provides a rich set of tools for object recognition, going strictly beyond ordinary real cohomology.

Related in spirit to our L^2-cohomology, but in a quite different setting, is the L^2-cohomology as introduced by Atiyah [1]. This is defined either via L^2-differential forms [1] or combinatorially [14], but again with an L^2 condition. Questions like the Hodge decomposition problem also arise in this setting, and its failure gives rise to additional invariants, the Novikov–Shubin invariants. This theory has been extensively studied; compare e.g. [8, 26, 27, 32] for important properties and geometric as well as algebraic applications. In [15, 28, 33] approximation of the L^2-Betti numbers for infinite simplicial complexes in terms of associated finite simplicial complexes is discussed in increasing generality. Complete calculations of the spectrum of the associated Laplacian are rarely possible, but compare [11] for one of these cases. The monograph [29] provides rather complete information about this theory. Of particular relevance for the present paper is Pansu's [31], where in Sect. 4 he introduces an L^2-Alexander–Spanier complex similar to ours. He uses it to prove homotopy invariance of L^2-cohomology—that way identifying its cohomology with L^2-de Rham cohomology and L^2-simplicial cohomology (under suitable assumptions).

Here is some background to the writing of this paper. Essentially Sects. 2 through 8 were in a finished paper by Nat Smale and Steve Smale, February 20, 2009. That version stated that the coboundary operator of Theorem 4, Sect. 4 must have a closed image. Thomas Schick pointed out that this assertion was wrong, and in fact produced a counterexample, now Appendix A of this paper. Moreover, Schick and Laurent Bartholdi set in motion the proofs that give the sufficient conditions for the finite dimensionality of the L^2-cohomology groups in Sect. 9 of this paper, and hence the property that the image of the coboundary is closed. In particular, Theorems 7 and 8 were proved by Schick and Laurent Bartholdi.

Some conversations with Shmuel Weinberger were helpful.

2 An L^2-Hodge Theory

In this section we construct a general Hodge theory for certain L^2-spaces over X, making use only of a probability measure on a set X.

As expected, our main result (Theorem 2) shows that homology is trivial under these general assumptions. This is a backbone for our subsequent elaborations, in which a metric will be taken into account to obtain non-trivial homology.

This is akin to the construction of Alexander-Spanier cohomology in topology, in which a chain complex with trivial homology (which does not see the space's topology) is used to manufacture the standard Alexander-Spanier complex.

The amount of structure needed for our theory is minimal. First, let us introduce some notation used throughout the section. X will denote a set endowed with a probability measure μ ($\mu(X) = 1$). The ℓ-fold Cartesian product of X will be denoted as X^ℓ and μ_ℓ will denote the product measure on X^ℓ. A useful example to keep in mind is: X a compact domain in Euclidean space, μ the normalized Lebesgue measure. More generally, one may take μ to be a Borel measure, which need not be the Euclidean measure.

Furthermore, we will assume the existence of a kernel function $K\colon X^2 \to \mathbb{R}$, a non-negative, measurable, symmetric function which we will assume is in $L^\infty(X \times X)$, and for certain results, we will impose additional assumptions on K.

We may consider, for simplicity, the constant kernel $K \equiv 1$; but most proofs, in this section, cover with no difficulty the general case, so we do not yet impose any restriction to K. However, later sections will concentrate on $K \equiv 1$, which already provides a very rich theory.

The kernel K may be used to conveniently encode the notion of locality in our probability space X, for instance by defining it as the Gaussian kernel $K(x, y) = \mathrm{e}^{-\frac{\|x-y\|^2}{\sigma}}$, for some $\sigma > 0$.

Recall that a chain complex of vector spaces is a sequence of vector spaces V_j and linear maps $d_j\colon V_j \to V_{j-1}$ such that the composition $d_{j-1}\circ d_j = 0$. A cochain complex is the same, except that $d_j\colon V_j \to V_{j+1}$. The basic spaces in this section are $L^2(X^\ell)$, from which we will construct chain and cochain complexes:

$$\cdots \xrightarrow{\partial_{\ell+1}} L^2\bigl(X^{\ell+1}\bigr) \xrightarrow{\partial_\ell} L^2\bigl(X^\ell\bigr) \xrightarrow{\partial_{\ell-1}} \cdots \xrightarrow{\partial_1} L^2(X) \xrightarrow{\partial_0} 0 \tag{1}$$

and

$$0 \longrightarrow L^2(X) \xrightarrow{\delta_0} L^2\bigl(X^2\bigr) \xrightarrow{\delta_1} \cdots \xrightarrow{\delta_{\ell-1}} L^2\bigl(X^{\ell+1}\bigr) \xrightarrow{\delta_\ell} \cdots. \tag{2}$$

Here, both ∂_ℓ and δ_ℓ will be bounded linear maps, satisfying $\partial_{\ell-1}\circ\partial_\ell = 0$ and $\delta_\ell\circ\delta_{\ell-1} = 0$. When there is no confusion, we will omit the subscripts of these operators.

We first define $\delta = \delta_{\ell-1}\colon L^2(X^\ell) \to L^2(X^{\ell+1})$ by

$$\delta f(x_0,\dots,x_\ell) = \sum_{i=0}^{\ell}(-1)^i \prod_{j\neq i}\sqrt{K(x_i,x_j)}\, f\bigl(x_0,\dots,\hat{x}_i,\dots,x_\ell\bigr), \tag{3}$$

where $\hat{x}_i$ means that x_i is deleted. This is similar to the coboundary operator of Alexander–Spanier cohomology (see Spanier [36]). The square root in the formula is unimportant for most of the sequel, and is there so that when we define the Laplacian on $L^2(X)$, we recover the operator as defined in Gilboa and Osher [20]. We also note that in the case where X is a finite set, δ_0 is essentially the same as the gradient operator developed by Zhou and Schölkopf [39] in the context of learning theory.

Proposition 1 *For all $\ell \geq 0$, $\delta\colon L^2(X^\ell) \to L^2(X^{\ell+1})$ is a bounded linear map.*

Proof Clearly δf is measurable, as K is measurable. Since $\|K\|_\infty < \infty$, it follows from the Schwarz inequality in $\mathbb{R}^\ell$ that

$$\begin{aligned}\bigl|\delta f(x_0,\dots,x_\ell)\bigr|^2 &\leq \|K\|_\infty^\ell \Biggl(\sum_{i=0}^{\ell}\bigl|f\bigl(x_0,\dots,\hat{x}_i,\dots,x_\ell\bigr)\bigr|\Biggr)^2 \\ &\leq \|K\|_\infty^\ell(\ell+1)\sum_{i=0}^{\ell}\bigl|f\bigl(x_0,\dots,\hat{x}_i,\dots,x_\ell\bigr)\bigr|^2.\end{aligned}$$

Now, integrating both sides of the inequality with respect to $d\mu_{\ell+1}$, using Fubini's theorem on the right side and the fact that $\mu(X) = 1$ gives us

$$\|\delta f\|_{L^2(X^{\ell+1})} \leq \sqrt{\|K\|_\infty^\ell(\ell+1)}\,\|f\|_{L^2(X^\ell)},$$

completing the proof. □

Essentially the same proof shows that δ is a bounded linear map on L^p, $p \geq 1$.

Proposition 2 *For all* $\ell \geq 1$, $\delta_\ell \circ \delta_{\ell-1} = 0$.

Proof The proof is standard when $K \equiv 1$. For $f \in L^2(X^\ell)$ we have

$$\begin{aligned}
&\delta_\ell(\delta_{\ell-1} f)(x_0, \dots, x_{\ell+1}) \\
&\quad = \sum_{i=0}^{\ell+1} (-1)^i \prod_{j \neq i} \sqrt{K(x_i, x_j)} (\delta_{\ell-1} f)\big(x_0, \dots, \hat{x}_i, \dots, x_{\ell+1}\big) \\
&\quad = \sum_{i=0}^{\ell+1} (-1)^i \prod_{j \neq i} \sqrt{K(x_i, x_j)} \sum_{k=0}^{i-1} (-1)^k \\
&\qquad \times \prod_{n \neq k,i} \sqrt{K(x_k, x_n)} f\big(x_0, \dots, \hat{x}_k, \dots, \hat{x}_i, \dots, x_{\ell+1}\big) \\
&\qquad + \sum_{i=0}^{\ell+1} (-1)^i \prod_{j \neq i} \sqrt{K(x_i, x_j)} \sum_{k=i+1}^{\ell+1} (-1)^{k-1} \\
&\qquad \times \prod_{n \neq k,i} \sqrt{K(x_k, x_n)} f\big(x_0, \dots, \hat{x}_i, \dots, \hat{x}_k, \dots, x_{\ell+1}\big).
\end{aligned}$$

Now we note that on the right side of the second equality for given i, k with $k < i$, the corresponding term in the first sum

$$(-1)^{i+k} \prod_{j \neq i} \sqrt{K(x_i, x_j)} \prod_{n \neq k,i} \sqrt{K(x_k, x_n)} f\big(x_0, \dots, \hat{x}_k, \dots, \hat{x}_i, \dots, x_{\ell+1}\big)$$

cancels the term in the second sum where i and k are reversed

$$(-1)^{k+i-1} \prod_{j \neq k} \sqrt{K(x_k, x_j)} \prod_{n \neq k,i} \sqrt{K(x_k, x_n)} f\big(x_0, \dots, \hat{x}_k, \dots, \hat{x}_i, \dots, x_{\ell+1}\big)$$

because, using the symmetry of K,

$$\prod_{j \neq i} \sqrt{K(x_i, x_j)} \prod_{n \neq k,i} \sqrt{K(x_k, x_n)} = \prod_{j \neq k} \sqrt{K(x_k, x_j)} \prod_{n \neq k,i} \sqrt{K(x_i, x_n)}.$$

□

It follows that (2) and (3) define a cochain complex. We now define, for $\ell > 0$, $\partial_\ell \colon L^2(X^{\ell+1}) \to L^2(X^\ell)$ by

$$\partial_\ell g(x) = \sum_{i=0}^{\ell} (-1)^i \int_X \left(\prod_{j=0}^{\ell-1} \sqrt{K(t, x_j)} \right) g(x_0, \dots, x_{i-1}, t, x_i, \dots, x_{\ell-1}) \, \mathrm{d}\mu(t), \quad (4)$$

where $x = (x_0, \dots, x_{\ell-1})$ and for $\ell = 0$ we define $\partial_0 \colon L^2(X) \to 0$.

Proposition 3 *For all* $\ell \geq 0$, $\partial_\ell \colon L^2(X^{\ell+1}) \to L^2(X^\ell)$ *is a bounded linear map.*

Proof For $g \in L^2(X^{\ell+1})$, we have

$$\begin{aligned}
&\left|\partial_\ell g(x_0, \ldots, x_{\ell-1})\right| \\
&\quad \le \|K\|_\infty^{(\ell-1)/2} \sum_{i=0}^{\ell} \int_X \left|g(x_0, \ldots, x_{i-1}, t, \ldots, x_{\ell-1})\right| \mathrm{d}\mu(t) \\
&\quad \le \|K\|_\infty^{(\ell-1)/2} \sum_{i=0}^{\ell} \left(\int_X \left|g(x_0, \ldots, x_{i-1}, t, \ldots, x_{\ell-1})\right|^2 \mathrm{d}\mu(t)\right)^{\frac{1}{2}} \\
&\quad \le \|K\|_\infty^{(\ell-1)/2} \sqrt{\ell+1} \left(\sum_{i=0}^{\ell} \int_X \left|g(x_0, \ldots, x_{i-1}, t, \ldots, x_{\ell-1})\right|^2 \mathrm{d}\mu(t)\right)^{\frac{1}{2}},
\end{aligned}$$

where we have used the Schwarz inequalities for $L^2(X)$ and $\mathbb{R}^{\ell+1}$ in the second and third inequalities respectively. Now, square both sides of the inequality and integrate over X^ℓ with respect to μ_ℓ and use Fubini's theorem, arriving at the following bound to finish the proof:

$$\|\partial_\ell g\|_{L^2(X^\ell)} \le \|K\|_\infty^{(\ell-1)/2} (\ell+1) \|g\|_{L^2(X^{\ell+1})}. \qquad \square$$

Remark 1 As in Proposition 1, we can replace L^2 by L^p, for $p \ge 1$.

We now show that (for $p = 2$) ∂_ℓ is actually the adjoint of $\delta_{\ell-1}$ (which gives a second proof of Proposition 3).

Proposition 4 *$\delta_{\ell-1}^* = \partial_\ell$. That is, $\langle \delta_{\ell-1} f, g\rangle_{L^2(X^{\ell+1})} = \langle f, \partial_\ell g\rangle_{L^2(X^\ell)}$ for all $f \in L^2(X^\ell)$ and $g \in L^2(X^{\ell+1})$.*

Proof For $f \in L^2(X^\ell)$ and $g \in L^2(X^{\ell+1})$ we have, by Fubini's theorem,

$$\begin{aligned}
\langle \delta_{\ell-1} f, g\rangle &= \sum_{i=0}^{\ell} (-1)^i \int_{X^{\ell+1}} \prod_{j \ne i} \sqrt{K(x_i, x_j)}\, f\big(x_0, \ldots, \hat{x}_i, \ldots, x_\ell\big) g(x_0, \ldots, x_\ell)\, \mathrm{d}\mu_{\ell+1} \\
&= \sum_{i=0}^{\ell} (-1)^i \int_{X^\ell} f\big(x_0, \ldots, \hat{x}_i, \ldots, x_\ell\big) \\
&\qquad \times \int_X \prod_{j \ne i} \sqrt{K(x_i, x_j)}\, g(x_0, \ldots, x_\ell)\, \mathrm{d}\mu(x_i)\, \mathrm{d}\mu(x_0) \cdots \widehat{\mathrm{d}\mu(x_i)} \cdots \mathrm{d}\mu(x_\ell).
\end{aligned}$$

In the ith term on the right, relabeling the variables $x_0, \ldots, \hat{x}_i, \ldots x_\ell$ with $y = (y_0, \ldots, y_{\ell-1})$ (that is, $y_j = x_{j+1}$ for $j \ge i$) and putting the sum inside the integral gives us

$$\int_{X^\ell} f(y) \sum_{i=0}^{\ell} (-1)^i \int_X \prod_{j=0}^{\ell-1} \sqrt{K(x_i, y_j)}\, g(y_0, \ldots, y_{i-1}, x_i, y_i, \ldots, y_{\ell-1})\, \mathrm{d}\mu(x_i)\, \mathrm{d}\mu_\ell(y)$$

which is just $\langle f, \partial_\ell g\rangle$. $\square$

We note, as a corollary, that $\partial_{\ell-1} \circ \partial_\ell = 0$, and thus (1) and (4) define a chain complex. We can thus define the homology and cohomology spaces (real coefficients) of (1) and (2) as follows. Since $\operatorname{Im} \partial_\ell \subset \operatorname{Ker} \partial_{\ell-1}$ and $\operatorname{Im} \delta_{\ell-1} \subset \operatorname{Ker} \delta_\ell$ we define the quotient spaces

$$H_\ell(X) = H_\ell(X, K, \mu) = \frac{\operatorname{Ker} \partial_{\ell-1}}{\operatorname{Im} \partial_\ell}, \qquad H^\ell(X) = H^\ell(X, K, \mu) = \frac{\operatorname{Ker} \delta_\ell}{\operatorname{Im} \delta_{\ell-1}} \tag{5}$$

which will be referred to as the L^2-homology and cohomology of degree ℓ, respectively. In later sections, with additional assumptions on X and K, we will investigate the relation between these spaces and the topology of X, for example, the Alexander–Spanier cohomology. In order to proceed with the Hodge theory, we consider δ to be the analogue of the exterior derivative d on ℓ-forms from differential topology, and $\partial = \delta^*$ as the analogue of d^*. We then define the Laplacian (in analogy with the Hodge Laplacian) to be $\Delta_\ell = \delta_\ell^* \delta_\ell + \delta_{\ell-1} \delta_{\ell-1}^*$. Clearly, $\Delta_\ell \colon L^2(X^{\ell+1}) \to L^2(X^{\ell+1})$ is a bounded, self-adjoint, positive semi-definite operator, since for $f \in L^2(X^{\ell+1})$,

$$\langle \Delta f, f \rangle = \langle \delta^* \delta f, f \rangle + \langle \delta \delta^* f, f \rangle = \|\delta f\|^2 + \|\delta^* f\|^2, \tag{6}$$

where we have omitted the subscripts on the operators. The Hodge theorem will give a decomposition of $L^2(X^{\ell+1})$ in terms of the image spaces under δ, δ^* and the kernel of Δ, and also identify the kernel of Δ with $H^\ell(X, K, \mu)$. Elements of the kernel of Δ will be referred to as harmonic. For $\ell = 0$, one easily computes that

$$\frac{1}{2} \Delta_0 f(x) = D(x) f(x) - \int_X K(x, y) f(y)\, \mathrm{d}\mu(y) \quad \text{where } D(x) = \int_X K(x, y)\, \mathrm{d}\mu(y)$$

which, when K is a positive definite kernel on X, is the Laplacian defined in Smale and Zhou [34] (see Sect. 5 below).

Remark 2 It follows from (6) that $\Delta f = 0$ if and only if $\delta_\ell f = 0$ and $\delta_{\ell-1}^* f = 0$, and so $\operatorname{Ker} \Delta_\ell = \operatorname{Ker} \delta_\ell \cap \operatorname{Ker} \delta_{\ell-1}^*$; in other words, a form is harmonic if and only if it is both closed and coclosed.

The main goal of this section is the following L^2-Hodge theorem.

Theorem 2 *Assume that* $0 < \sigma \le K(x, y) \le \|K\|_\infty < \infty$ *almost everywhere. Then we have trivial* L^2*-cohomology in the following sense*:

$$\operatorname{im}(\delta_\ell) = \ker(\delta_{\ell+1}) \quad \forall \ell \ge 0.$$

In particular, $H^\ell(X) = 0$ *for* $\ell > 0$ *and we have by Lemma* 1 *the* (*trivial*) *orthogonal, direct sum decomposition*

$$L^2(X^{\ell+1}) = \operatorname{Im} \delta_{\ell-1} \oplus \operatorname{Im} \delta_\ell^* \oplus \operatorname{Ker} \Delta_\ell$$

and the cohomology space $H^\ell(X, K, \mu)$ *is isomorphic to* $\operatorname{Ker} \Delta_\ell$, *with each equivalence class in the former having a unique representative in the latter.*

For $\ell > 0$, *of course* $\operatorname{Ker} \Delta_\ell = \{0\}$. *For* $\ell = 0$, $\operatorname{Ker} \Delta_0 = \ker \delta_0 \cong \mathbb{R}$ *consists precisely of the constant functions.*

In subsequent sections we will have occasion to use the L^2-spaces of alternating functions:

$$L_a^2\left(X^{\ell+1}\right) = \left\{f \in L^2\left(X^{\ell+1}\right) : f(x_0, \dots, x_\ell) = (-1)^{\operatorname{sign}\sigma} f(x_{\sigma(0)}, \dots, x_{\sigma(\ell)}),\right.$$
$$\left.\sigma \text{ a permutation}\right\}.$$

Due to the symmetry of K, it is easy to check that the coboundary δ preserves the alternating property, and thus Propositions 1 through 4, as well as formulas (1), (2), (5), and (6) hold with L_a^2 in place of L^2. We note that the alternating map

$$\mathrm{Alt}\colon L^2\left(X^{\ell+1}\right) \to L_a^2\left(X^{\ell+1}\right)$$

defined by

$$\mathrm{Alt}(f)(x_0, \dots, x_\ell) := \frac{1}{(\ell+1)!} \sum_{\sigma \in S_{\ell+1}} (-1)^{\operatorname{sign}\sigma} f(x_{\sigma(0)}, \dots, x_{\sigma(\ell)})$$

is a projection relating the two definitions of ℓ-forms. It is easy to compute that this is actually an orthogonal projection; its inverse is just the inclusion map.

Remark 3 It follows from homological algebra that these maps induce isomorphisms, inverse to each other, of the cohomology groups we have defined. Indeed, there is a standard chain homotopy between a variant of the projection Alt and the identity, given by $hf(x_0, \dots, x_n) = \frac{1}{n}\sum_{i=0}^n f(x_i, x_0, \dots, x_n)$. Because many formulas simplify, from now on we will usually work with the subcomplex of alternating functions.

We first recall some relevant facts in a more abstract setting.

Lemma 1 (Hodge Lemma) *Suppose we have the cochain and corresponding dual chain complexes*

$$0 \longrightarrow V_0 \xrightarrow{\delta_0} V_1 \xrightarrow{\delta_1} \cdots \xrightarrow{\delta_{\ell-1}} V_\ell \xrightarrow{\delta_\ell}$$

$$\cdots \xrightarrow{\delta_\ell^*} V_\ell \xrightarrow{\delta_{\ell-1}^*} V_{\ell-1} \xrightarrow{\delta_{\ell-2}^*} \cdots \xrightarrow{\delta_0^*} V_0 \longrightarrow 0$$

where for $\ell = 0, 1, \dots,$ $V_\ell, \langle\,,\,\rangle_\ell$ *is a Hilbert space, and* δ_ℓ (*and thus* δ_ℓ^*, *the adjoint of* δ_ℓ) *is a bounded linear map with* $\delta^2 = 0$. *Let* $\Delta_\ell = \delta_\ell^*\delta_\ell + \delta_{\ell-1}\delta_{\ell-1}^*$. *Then the following are equivalent*:

(1) δ_ℓ *has closed range for all* ℓ.
(2) δ_ℓ^* *has closed range for all* ℓ.
(3) $\Delta_\ell = \delta_\ell^*\delta_\ell + \delta_{\ell-1}\delta_{\ell-1}^*$ *has closed range for all* ℓ.

Furthermore, *if one of these conditions holds*, *we have the orthogonal*, *direct sum decomposition into closed subspaces*

$$V_\ell = \operatorname{Im}\delta_{\ell-1} \oplus \operatorname{Im}\delta_\ell^* \oplus \operatorname{Ker}\Delta_\ell \tag{7}$$

and the quotient space $\frac{\operatorname{Ker}\delta_\ell}{\operatorname{Im}\delta_{\ell-1}}$ *is isomorphic to* $\operatorname{Ker}\Delta_\ell$, *with each equivalence class in the former having a unique representative in the latter.*

Proof We first assume conditions (1) and (2) above and prove the decomposition. For all $f \in V_{\ell-1}$ and $g \in V_{\ell+1}$ we have

$$\left\langle \delta_{\ell-1} f, \delta_\ell^* g\right\rangle_\ell = \langle \delta_\ell \delta_{\ell-1} f, g\rangle_{\ell+1} = 0.$$

Also, as in (6), $\Delta_\ell f = 0$ if and only if $\delta_\ell f = 0$ and $\delta_{\ell-1}^* f = 0$. Therefore, if $f \in \operatorname{Ker}\Delta_\ell$, then for all $g \in V_{\ell-1}$ and $h \in V_{\ell+1}$

$$\langle f, \delta_{\ell-1} g\rangle_\ell = \left\langle \delta_{\ell-1}^* f, g\right\rangle_{\ell-1} = 0 \quad \text{and} \quad \left\langle f, \delta_\ell^* h\right\rangle_\ell = \langle \delta_\ell f, h\rangle_{\ell+1} = 0$$

and thus $\operatorname{Im}\delta_{\ell-1}$, $\operatorname{Im}\delta_\ell^*$, and $\operatorname{Ker}\Delta_\ell$ are mutually orthogonal. We now show that $\operatorname{Ker}\Delta_\ell \supseteq (\operatorname{Im}\delta_{\ell-1} \oplus \operatorname{Im}\delta_\ell^*)^\perp$. This implies the orthogonal decomposition

$$V_\ell = \ker(\Delta_\ell) \oplus \overline{\operatorname{Im}(\delta_{\ell-1})} \oplus \overline{\operatorname{Im}(\delta_\ell^*)}. \tag{8}$$

If (1) and (2) hold, this implies the Hodge decomposition (7). Let $v \in (\operatorname{Im}\delta_{\ell-1} \oplus \operatorname{Im}\delta_\ell^*)^\perp$. Then, for all $w \in V_\ell$,

$$\langle \delta_\ell v, w\rangle = \left\langle v, \delta_\ell^* w\right\rangle = 0 \quad \text{and} \quad \left\langle \delta_{\ell-1}^* v, w\right\rangle = \langle v, \delta_{\ell-1} w\rangle = 0,$$

which implies that $\delta_\ell v = 0$ and $\delta_{\ell-1}^* v = 0$. As noted above, this implies that $\Delta_\ell v = 0$, proving the decomposition.

We define an isomorphism

$$\tilde{P}: \frac{\operatorname{Ker}\delta_\ell}{\operatorname{Im}\delta_{\ell-1}} \to \operatorname{Ker}\Delta_\ell$$

as follows. Let $P: V_\ell \to \operatorname{Ker}\Delta_\ell$ be the orthogonal projection. Then, for an equivalence class $[f] \in \frac{\operatorname{Ker}\delta_\ell}{\operatorname{Im}\delta_{\ell-1}}$ define $\tilde{P}([f]) = P(f)$. Note that if $[f] = [g]$ then $f = g + h$ with $h \in \operatorname{Im}\delta_{\ell-1}$, and therefore $P(f) - P(g) = P(h) = 0$ by the orthogonal decomposition, and so $\tilde{P}$ is well defined, and linear as P is linear. If $\tilde{P}([f]) = 0$ then $P(f) = 0$ and so $f \in \operatorname{Im}\delta_{\ell-1} \oplus \operatorname{Im}\delta_\ell^*$. But $f \in \operatorname{Ker}\delta_\ell$, and so, for all $g \in V_{\ell+1}$ we have $\langle \delta_\ell^* g, f\rangle = \langle g, \delta_\ell f\rangle = 0$, and thus $f \in \operatorname{Im}\delta_{\ell-1}$ and therefore $[f] = 0$ and $\tilde{P}$ is injective. On the other hand, $\tilde{P}$ is surjective because, if $w \in \operatorname{Ker}\Delta_\ell$, then $w \in \operatorname{Ker}\delta_\ell$ and so $\tilde{P}([w]) = P(w) = w$.

Finally, the equivalence of conditions (1), (2), and (3) is a general fact about Hilbert spaces and Hilbert cochain complexes. If $\delta: V \to H$ is a bounded linear map between Hilbert spaces, and δ^* is its adjoint, and if $\operatorname{Im}\delta$ is closed in H, then $\operatorname{Im}\delta^*$ is closed in V. We include the proof for completeness. Since $\operatorname{Im}\delta$ is closed, the bijective map

$$\delta: (\operatorname{Ker}\delta)^\perp \to \operatorname{Im}\delta$$

is an isomorphism by the open mapping theorem. It follows that the norm of δ^-1,

$$\inf\left\{\|\delta(v)\| : v \in (\operatorname{Ker}\delta)^\perp,\ \|v\| = 1\right\} > 0.$$

Since $\operatorname{Im}\delta \subset (\operatorname{Ker}\delta^*)^\perp$, it suffices to show that

$$\delta^*\delta\colon (\operatorname{Ker}\delta)^\perp \to (\operatorname{Ker}\delta)^\perp$$

is an isomorphism, for then $\operatorname{Im}\delta^* = (\operatorname{Ker}\delta)^\perp$, which is closed. However, this is established by noting that $\langle \delta^*\delta v, v\rangle = \|\delta v\|^2$ and the above inequality imply that

$$\inf\bigl\{\bigl\langle \delta^*\delta v, v\bigr\rangle : v \in (\operatorname{Ker}\delta)^\perp,\ \|v\| = 1\bigr\} > 0.$$

The general Hodge decomposition (8) implies that $\Delta_\ell = \delta_\ell^*\delta_\ell$ acts on $\ker(\Delta_\ell)$ as the zero operator (trivially), as $\delta_\ell^*\delta_\ell\colon \overline{\operatorname{im}(\delta_\ell^*)} \to \operatorname{im}(\delta_\ell^*)$ (preserving this subspace) and as $\delta_{\ell-1}\delta_{\ell-1}^*$ on $\overline{\operatorname{im}(\delta_{\ell-1})}$, also mapping this subspace to itself.

Now the image of an operator on a Hilbert space is closed if and only if it maps the complement of its kernel isomorphically (with bounded inverse) to its image. As the kernel of δ_ℓ is the complement of the image of δ_ℓ^* and the kernel of $\delta_{\ell-1}^*$ is the complement of the image of δ_ℓ, this implies indeed that $\operatorname{Im}(\Delta_\ell)$ is closed if and only if (1) and (2) are satisfied.

This finishes the proof of the lemma. □

Corollary 1 *For all $\ell \geq 0$ the following are isomorphisms, provided $\operatorname{Im}(\delta)$ is closed*:

$$\delta_\ell\colon\ \operatorname{Im}\delta_\ell^* \to \operatorname{Im}\delta_\ell \quad \textit{and} \quad \delta_\ell^*\colon\ \operatorname{Im}\delta_\ell \to \operatorname{Im}\delta_\ell^*.$$

Proof The first map is injective because if $\delta(\delta^* f) = 0$ then $0 = \langle \delta\delta^* f, f\rangle = \|\delta^* f\|^2$ and so $\delta^* f = 0$. It is surjective because of the decomposition (omitting the subscripts)

$$\delta(V) = \delta\bigl(\operatorname{Im}\delta \oplus \operatorname{Im}\delta^* \oplus \operatorname{Ker}\Delta\bigr) = \delta\bigl(\operatorname{Im}\delta^*\bigr)$$

since δ is zero on the first and third summands of the left side of the second equality. The argument for the second map is the same. □

The difficulty in applying the Hodge Lemma is in verifying that either δ or δ^* has closed range. A sufficient condition is the following, first pointed out to us by Shmuel Weinberger.

Proposition 5 *Suppose that, in the context of Lemma 1, the L^2-cohomology space $\operatorname{Ker}\delta_\ell/\operatorname{Im}\delta_{\ell-1}$ is finite dimensional. Then $\delta_{\ell-1}$ has closed range.*

Proof We show more generally that if $T\colon B \to V$ is a bounded linear map of Banach spaces, with $\operatorname{Im} T$ having finite codimension in V, then $\operatorname{Im} T$ is closed in V. We can assume without loss of generality that T is injective, by replacing B with $B/\operatorname{Ker} T$ if necessary. Thus $T\colon B \to \operatorname{Im} T \oplus F = V$ where $\dim F < \infty$. Now define $G\colon B \oplus F \to V$ by $G(x, y) = Tx + y$. G is bounded, surjective and injective, and thus an isomorphism by the open mapping theorem. Therefore, $G(B) = T(B)$ is closed in V. □

Consider the special case where $K(x, y) = 1$ for all x, y in X. Let ∂_ℓ^0 be the corresponding operator in (4). We have

Lemma 2 *For $\ell > 1$, $\operatorname{Im} \partial_\ell^0 = \operatorname{Ker} \partial_{\ell-1}^0$, and $\operatorname{Im} \partial_1^0 = \{1\}^\perp$ is the orthogonal complement of the constants in $L^2(X)$.*

Under the assumption $K \equiv 1$, we can already finish the proof of Theorem 2; the general case is proven later. Indeed, Lemma 2 implies that $\operatorname{Im} \partial_\ell$ is closed for all ℓ since null spaces and orthogonal complements are closed, and in fact shows that the homology (5) in this case is trivial for $\ell > 0$ and one dimensional for $\ell = 0$.

Proof of Lemma 2 Let $h \in \{1\}^\perp \subset L^2(X)$. Define $g \in L^2(X^2)$ by $g(x, y) = h(y)$. Then from (4),

$$\partial_1^0 g(x_0) = \int_X \big(g(t, x_0) - g(x_0, t)\big)\, \mathrm{d}\mu(t) = \int_X \big(h(x_0) - h(t)\big)\, \mathrm{d}\mu(t) = h(x_0)$$

since $\mu(X) = 1$ and $\int_X h \, \mathrm{d}\mu = 0$. It can be easily checked that ∂_1^0 maps $L^2(X^2)$ into $\{1\}^\perp$, thus proving the lemma for $\ell = 1$. For $\ell > 1$ let $h \in \operatorname{Ker} \partial_{\ell-1}^0$. Define $g \in L^2(X^{\ell+1})$ by $g(x_0, \dots, x_\ell) = (-1)^\ell h(x_0, \dots, x_{\ell-1})$. Then by (4),

$$\begin{aligned}
\partial_\ell^0 g(x_0, \dots, x_{\ell-1}) &= \sum_{i=0}^{\ell} (-1)^i \int_X g(x_0, \dots, x_{i-1}, t, x_i, \dots, x_{\ell-1})\, \mathrm{d}\mu(t) \\
&= (-1)^\ell \sum_{i=0}^{\ell-1} (-1)^i \int_X h(x_0, \dots, x_{i-1}, t, x_i, \dots, x_{\ell-2})\, \mathrm{d}\mu(t) \\
&\quad + (-1)^{2\ell} h(x_0, \dots, x_{\ell-1}) \\
&= (-1)^\ell \partial_{\ell-1}^0 h(x_0, \dots, x_{\ell-2}) + h(x_0, \dots, x_{\ell-1}) \\
&= h(x_0, \dots, x_{\ell-1})
\end{aligned}$$

since $\partial_{\ell-1}^0 h = 0$, finishing the proof. □

The next lemma give some general conditions on K that guarantee ∂_ℓ has closed range.

Lemma 3 *Assume that $K(x, y) \ge \sigma > 0$ for all $x, y \in X$. Then $\operatorname{Im} \partial_\ell$ is closed for all ℓ. In fact, $\operatorname{Im} \partial_\ell = \operatorname{Ker} \partial_{\ell-1}$ for $\ell > 1$ and has codimension one in $L^2(X)$ for $\ell = 1$.*

Proof Let $M_\ell \colon L^2(X^\ell) \to L^2(X^\ell)$ be the multiplication operator

$$M_\ell(f)(x_0, \dots, x_\ell) = \prod_{j \ne k} \sqrt{K(x_j, x_k)}\, f(x_0, \dots, x_\ell).$$

Since $K \in L^\infty(X^2)$ and is bounded below by σ, M_ℓ clearly defines an isomorphism. The lemma then follows from Lemma 2 and the observation that

$$\partial_\ell = M_{\ell-1}^{-1} \circ \partial_\ell^0 \circ M_\ell.$$ □

Theorem 2 now follows from the Hodge Lemma and Lemma 3. We note that Lemmas 2, 3, and Theorem 2 hold in the alternating setting, when $L^2(X^\ell)$ is replaced with $L^2_a(X^\ell)$; so the cohomology is also trivial in that setting.

For background, one could refer to Munkres [30] for the algebraic topology, Lang [24] for the analysis, and Warner [37] for the geometry.

3 Metric Spaces

For the rest of the paper, we assume that X is a complete, separable metric space, and that μ is a Borel probability measure on X, and K is a continuous function on X^2 (as well as symmetric, non-negative, and bounded as in Sect. 2). We will also assume throughout the rest of the paper that $\mu(U) > 0$ for U any non-empty open set.

The goal of this section is obtaining a Hodge decomposition for continuous alternating functions. Let $C(X^{\ell+1})$ denote the continuous functions on $X^{\ell+1}$. We will use the following notation:

$$C^{\ell+1} = C\left(X^{\ell+1}\right) \cap L^2_a\left(X^{\ell+1}\right) \cap L^\infty\left(X^{\ell+1}\right).$$

Note that

$$\delta\colon C^{\ell+1} \to C^{\ell+2} \quad \text{and} \quad \partial\colon C^{\ell+1} \to C^\ell$$

are well-defined linear maps. The only thing to check is that $\delta(f)$ and $\partial(f)$ are continuous and bounded if $f \in C^{\ell+1}$. In the case of $\delta(f)$ this is obvious from (3). The following proposition from analysis, (4), and the fact that μ is Borel imply that $\partial(f)$ is bounded and continuous.

Proposition 6 *Let Y and X be metric spaces, μ a Borel measure on X, and let $M, g \in C(Y \times X) \cap L^\infty(Y \times X)$. Then $dg \in C(X) \cap L^\infty(X)$, where*

$$dg(x) = \int_X M(x,t)g(x,t)\,\mathrm{d}\mu(t).$$

Proof The fact that dg is bounded follows easily from the definition and properties of M and g, and continuity follows from a simple application of the dominated convergence theorem, proving the proposition. □

Therefore, we have the chain complexes

$$\cdots \xrightarrow{\partial_{\ell+1}} C^{\ell+1} \xrightarrow{\partial_\ell} C^\ell \xrightarrow{\partial_{\ell-1}} \cdots \xrightarrow{\partial_1} C^1 \xrightarrow{\partial_0} 0$$

and

$$0 \longrightarrow C^1 \xrightarrow{\delta_0} C^2 \xrightarrow{\delta_1} \cdots \xrightarrow{\delta_{\ell-1}} C^{\ell+1} \xrightarrow{\delta_\ell} \cdots.$$

In this setting we will prove the following.

Theorem 3 *Assume that K satisfies the hypotheses of Theorem 2, and is continuous. Then we have the orthogonal (with respect to L^2) direct sum decomposition*

$$C^{\ell+1} = \delta\big(C^\ell\big) \oplus \partial\big(C^{\ell+2}\big) \oplus \operatorname{Ker}_C \Delta,$$

where $\operatorname{Ker}_C \Delta$ denotes the subspace of elements in $\operatorname{Ker} \Delta$ that are in $C^{\ell+1}$.

As in Theorem 2, the third summand is trivial except when $\ell = 0$, in which case it consists of the constant functions. We first assume that $K \equiv 1$. The proof follows from a few propositions. In the remainder of the section, $\operatorname{Im}\delta$ and $\operatorname{Im}\partial$ will refer to the image spaces of δ and ∂ as operators on L^2_a. The next proposition gives formulas for ∂ and Δ on alternating functions.

Proposition 7 *For $f \in L^2_a(X^{\ell+1})$ we have*

$$\partial f(x_0, \dots, x_{\ell-1}) = (\ell+1) \int_X f(t, x_0, \dots, x_{\ell-1})\, \mathrm{d}\mu(t)$$

and

$$\Delta f(x_0, \dots, x_\ell) = (\ell+2) f(x_0, \dots, x_\ell) - \frac{1}{\ell+1} \sum_{i=0}^{\ell} \partial f\big(x_0, \dots, \hat{x}_i, \dots, x_\ell\big).$$

Proof The first formula follows immediately from (4) and the fact that f is alternating. The second follows from a simple calculation using (3), (4), and the fact that f is alternating. □

Let P_1, P_2, and P_3 be the orthogonal projections implicit in Theorem 2,

$$P_1 \colon L^2_a(X^{\ell+1}) \to \operatorname{Im}\delta, \quad P_2 \colon L^2_a(X^{\ell+1}) \to \operatorname{Im}\partial, \quad \text{and} \quad P_3 \colon L^2_a(X^{\ell+1}) \to \operatorname{Ker}\Delta.$$

Proposition 8 *Let $f \in C^{\ell+1}$. Then $P_1(f) \in C^{\ell+1}$.*

Proof It suffices to show that $P_1(f)$ is continuous and bounded. Let $g = P_1(f)$. It follows from Theorem 2 that $\partial f = \partial g$, and therefore ∂g is continuous and bounded. Since $\delta g = 0$, we have, for $t, x_0, \dots, x_\ell \in X$,

$$0 = \delta g(t, x_0, \dots, x_\ell) = g(x_0, \dots, x_\ell) - \sum_{i=0}^{\ell} (-1)^i g\big(t, x_0, \dots, \hat{x}_i, \dots, x_\ell\big).$$

Integrating over $t \in X$ gives us

$$\begin{aligned} g(x_0, \dots, x_\ell) &= \int_X g(x_0, \dots, x_\ell)\, \mathrm{d}\mu(t) = \sum_{i=0}^{\ell} (-1)^i \int_X g\big(t, x_0, \dots, \hat{x}_i, \dots, x_\ell\big)\, \mathrm{d}\mu(t) \\ &= \frac{1}{\ell+1} \sum_{i=0}^{\ell} (-1)^i \partial g\big(x_0, \dots, \hat{x}_i, \dots, x_\ell\big). \end{aligned}$$

As ∂g is continuous and bounded, this implies g is continuous and bounded. □

Corollary 2 *If $f \in C^{\ell+1}$, then $P_2(f) \in C^{\ell+1}$.*

This follows from the Hodge decomposition (Theorem 2) and the fact that $P_3(f)$ is continuous and bounded (being a constant).

The following proposition can be thought of as analogous to a regularity result in elliptic PDEs. It states that solutions to $\Delta u = f$, f continuous, which are *a priori* in L^2 are actually continuous.

Proposition 9 *If $f \in C^{\ell+1}$ and $\Delta u = f$, $u \in L^2_a(X^{\ell+1})$, then $u \in C^{\ell+1}$.*

Proof From Proposition 7 (with u in place of f), we have

$$\begin{aligned}\Delta u(x_0,\dots,x_\ell) &= (\ell+2)u(x_0,\dots,x_\ell) - \frac{1}{\ell+1}\sum_{i=0}^{\ell}\partial u\bigl(x_0,\dots,\hat{x}_i,\dots,x_\ell\bigr)\\ &= f(x_0,\dots,x_\ell)\end{aligned}$$

and solving for u, we get

$$u(x_0,\dots,x_\ell) = \frac{1}{\ell+2}f(x_0,\dots,x_\ell) + \frac{1}{(\ell+2)(\ell+1)}\sum_{i=0}^{\ell}\partial u\bigl(x_0,\dots,\hat{x}_i,\dots,x_\ell\bigr).$$

It therefore suffices to show that ∂u is continuous and bounded. However, it is easy to check that $\Delta \circ \partial = \partial \circ \Delta$ and thus

$$\Delta(\partial u) = \partial \Delta u = \partial f$$

is continuous and bounded. But then, again using Proposition 7,

$$\begin{aligned}\Delta(\partial u)(x_0,\dots,x_{\ell-1}) &= (\ell+1)\partial u(x_0,\dots,x_{\ell-1})\\ &\quad - \frac{1}{\ell}\sum_{i=0}^{\ell-1}(-1)^i\partial(\partial u)\bigl(x_0,\dots,\hat{x}_i,\dots,x_{\ell-1}\bigr)\end{aligned}$$

and so, using $\partial^2 = 0$, we get

$$(\ell+1)\partial u = \partial f$$

which implies that ∂u is continuous and bounded, finishing the proof. □

Proposition 10 *If $g \in C^{\ell+1} \cap \operatorname{Im}\delta$, then $g = \delta h$ for some $h \in C^{\ell}$.*

Proof From the corollary of the Hodge lemma, let h be the unique element in $\operatorname{Im}\partial$ with $g = \delta h$. Now ∂g is continuous and bounded, and

$$\partial g = \partial\delta h = \partial\delta h + \delta\partial h = \Delta h$$

since $\partial h = 0$. But now h is continuous and bounded from Proposition 9. □

Proposition 11 *If $g \in C^{\ell+1} \cap L_a^2(X^{\ell+1})$, then $g = \partial h$ for some $h \in C^{\ell+2}$.*

The proof is identical to the one for Proposition 10.

Theorem 3, in the case $K \equiv 1$, now follows from Propositions 8 through 11. The proof easily extends to general K which is bounded below by a positive constant.

4 Hodge Theory at Scale α

As seen in Sects. 2 and 3, the chain and cochain complexes constructed on the whole space yield trivial cohomology groups. In order to have a theory that gives us topological information about X, we define our complexes on a neighborhood of the diagonal, and restrict the boundary and coboundary operator to these complexes. The corresponding cohomology can be considered a cohomology of X at a scale, with the scale being the size of the neighborhood. We will assume throughout this section that (X, d) is a compact metric space. For $x, y \in X^\ell$, $\ell > 1$, this induces a metric compatible with the product topology,

$$d_\ell(x, y) = \max\big\{d(x_0, y_0), \dots, d(x_{\ell-1}, y_{\ell-1})\big\}.$$

The diagonal D_ℓ of X^ℓ is just $\{x \in X^\ell : x_i = x_j,\ i, j = 0, \dots, \ell - 1\}$. For $\alpha > 0$ we let U_α^ℓ be the α-neighborhood of the diagonal in X^ℓ, namely

$$\begin{aligned} U_\alpha^\ell &= \big\{x \in X^\ell : d_\ell(x, D_\ell) \le \alpha\big\} \\ &= \big\{x \in X^\ell : \exists t \in X \text{ such that } d(x_i, t) \le \alpha,\ i = 0, \dots, \ell - 1\big\}. \end{aligned}$$

Observe that U_α^ℓ is closed and that for $\alpha \ge$ diameter X, $U_\alpha^\ell = X^\ell$.

Alternatively, one could have defined neighborhoods V_α^ℓ as those $x \in X^\ell$ such that $d(x_i, x_j) \le \alpha$ whenever $0 \le i, j < \ell$. This definition appears in the Vietoris-Rips complex; see Remark 7. Both definitions are very close, in the sense that $V_\alpha^\ell \subseteq U_\alpha^\ell \subseteq V_{2\alpha}^\ell$.

The measure μ_ℓ induces a Borel measure on U_α^ℓ which we will simply denote by μ_ℓ (not a probability measure). For simplicity, we will take $K \equiv 1$ throughout this section and consider only alternating functions in our complexes. We first discuss the L^2-theory, and thus our basic spaces will be $L_a^2(U_\alpha^\ell)$, the space of alternating functions on U_α^ℓ that are in L^2 with respect to μ_ℓ, $\ell > 0$. Note that if $(x_0, \dots, x_\ell) \in U_\alpha^{\ell+1}$, then $(x_0, \dots, \hat{x}_i, \dots, x_\ell) \in U_\alpha^\ell$ for $i = 0, \dots, \ell$. It follows that if $f \in L_a^2(U_\alpha^\ell)$, then $\delta f \in L_a^2(U_\alpha^{\ell+1})$. We therefore have the well-defined cochain complex

$$0 \longrightarrow L_a^2\big(U_\alpha^1\big) \xrightarrow{\ \delta\ } L_a^2\big(U_\alpha^2\big) \xrightarrow{\ \delta\ } \cdots$$

$$\xrightarrow{\ \delta\ } L_a^2\big(U_\alpha^\ell\big) \xrightarrow{\ \delta\ } L_a^2\big(U_\alpha^{\ell+1}\big) \cdots .$$

Since $\partial = \delta^*$ depends on the integral, its expression will be different from the one in (4). We define a "slice" by

$$S_{x_0 \cdots x_{\ell-1}} = \big\{t \in X : (x_0, \dots, x_{\ell-1}, t) \in U_\alpha^{\ell+1}\big\}.$$

We note that, for $S_{x_0\cdots x_{\ell-1}}$ to be non-empty, $(x_0,\dots,x_{\ell-1})$ must be in U_α^ℓ. Furthermore,

$$U_\alpha^{\ell+1}=\left\{(x_0,\dots,x_\ell):(x_0,\dots,x_{\ell-1})\in U_\alpha^\ell,\ \text{and } x_\ell\in S_{x_0\cdots x_{\ell-1}}\right\}.$$

It follows from the proof of Proposition 1 of Sect. 2 and the fact that $K\equiv 1$, that $\delta\colon L_a^2(U_\alpha^\ell)\to L_a^2(U_\alpha^{\ell+1})$ is bounded and that $\|\delta\|\le \ell+1$, and therefore δ^* is bounded. The adjoint of the operator $\delta\colon L_a^2(U_\alpha^\ell)\to L_a^2(U_\alpha^{\ell+1})$ will be denoted, as before, by either ∂ or δ^* (without the subscript ℓ).

Proposition 12 *For $f\in L_a^2(U_\alpha^{\ell+1})$ we have*

$$\partial f(x_0,\dots,x_{\ell-1})=(\ell+1)\int_{S_{x_0\cdots x_{\ell-1}}} f(t,x_0,\dots,x_{\ell-1})\,\mathrm{d}\mu(t).$$

Proof The proof is essentially the same as the proof of Proposition 4, using the fact that $K\equiv 1$, f is alternating, and the above remark. □

Note that the domain of integration depends on $x\in U_\alpha^\ell$, and this makes the subsequent analysis more difficult than that in Sect. 3. We thus have the corresponding chain complex

$$\cdots\xrightarrow{\ \partial\ }L_a^2\big(U_\alpha^{\ell+1}\big)\xrightarrow{\ \partial\ }L_a^2\big(U_\alpha^\ell\big)\xrightarrow{\ \partial\ }\cdots\xrightarrow{\ \partial\ }L_a^2\big(U_\alpha^1\big)\xrightarrow{\ \partial\ }0.$$

Of course, $U_\alpha^1=X$. The corresponding Hodge Laplacian is the operator $\Delta\colon L_a^2(U_\alpha^\ell)\to L_a^2(U_\alpha^\ell)$, $\Delta=\partial\delta+\delta\partial$, where all of these operators depend on ℓ and α. When we want to emphasize this dependence, we will list ℓ and (or) α as subscripts. We will use the following notation for the cohomology and harmonic functions of the above complexes:

$$H_{L^2,\alpha}^\ell(X)=\frac{\operatorname{Ker}\delta_{\ell,\alpha}}{\operatorname{Im}\ \delta_{\ell-1,\alpha}}\quad\text{and}\quad \operatorname{Harm}_\alpha^\ell(X)=\operatorname{Ker}\Delta_{\ell,\alpha}.$$

Remark 4 If $\alpha\ge\operatorname{diam}(X)$, then $U_\alpha^\ell=X^\ell$, so the situation is as in Theorem 2 of Sect. 2, and thus $H_{L^2,\alpha}^\ell(X)=0$ for $\ell>0$ and $H_{L^2,\alpha}^0(X)=\mathbb{R}$. Also, if X is a finite union of connected components $X_1,\dots,X_k$, and $\alpha<d(X_i,X_j)$ for all $i\neq j$, then $H_{L^2,\alpha}^\ell(X)=\bigoplus_{i=1}^k H_{L^2,\alpha}^\ell(X_i)$.

Definition 1 We say that Hodge theory for X at scale α holds if we have the orthogonal direct sum decomposition into closed subspaces

$$L_a^2\big(U_\alpha^\ell\big)=\operatorname{Im}\delta_{\ell-1}\oplus\operatorname{Im}\ \delta_\ell^*\oplus\operatorname{Harm}_\alpha^\ell(X)\quad\text{for all }\ell$$

and furthermore, $H_{\alpha,L^2}^\ell(X)$ is isomorphic to $\operatorname{Harm}_\alpha^\ell(X)$, with each equivalence class in the former having a unique representative in the latter.

Remark 5 Hodge theory is functorial in the sense that, for any $s \geq 1$, the inclusion $U_\alpha^\ell \subseteq U_{s\alpha}^\ell$ induces corestriction maps $H_{s\alpha}^\ell \to H_\alpha^\ell$. In seeking a robust notion of cohomology, it will make sense to consider the images of these maps at a sufficiently large separation s, rather than at individual cohomology groups H_α^ℓ.

More generally, let $f : X \to Y$ be an L-Lipschitz, measure-class-preserving map. Then there are induced maps $U_\alpha^n(X) \to U_{L\alpha}^n(Y)$, and therefore homomorphisms $f^* : H_{L\alpha}^n(Y) \to H_\alpha^n(X)$ for all n.

Theorem 4 *If X is a compact metric space*, $\alpha > 0$, *and the L^2-cohomology spaces* $\operatorname{Ker}\delta_{\ell,\alpha}/\operatorname{Im}\delta_{\ell-1,\alpha}$, $\ell \geq 0$ *are finite dimensional*, *then Hodge theory for X at scale α holds*.

Proof This is immediate from the Hodge lemma (Lemma 1), using Proposition 5 from Sect. 2. □

We record the formulas for $\delta\partial f$ and $\partial\delta f$ for $f \in L_a^2(U_\alpha^{\ell+1})$:

$$\begin{aligned} \delta(\partial f)(x_0,\dots,x_\ell) &= (\ell+1)\sum_{i=0}^{\ell}(-1)^i \int_{S_{x_0,\dots,\hat{x}_i,\dots,x_\ell}} f\big(t,x_0,\dots,\hat{x}_i,\dots,x_\ell\big)\,\mathrm{d}\mu(t) \\ \partial(\delta f)(x_0,\dots,x_\ell) &= (\ell+2)\mu(S_{x_0,\dots,x_\ell})f(x_0,\dots,x_\ell) \\ &\quad + (\ell+2)\sum_{i=0}^{\ell}(-1)^{i+1}\int_{S_{x_0,\dots,x_\ell}} f\big(t,x_0,\dots,\hat{x}_i,\dots,x_\ell\big)\,\mathrm{d}\mu(t). \end{aligned} \tag{9}$$

Of course, the formula for Δf is found by adding these two.

Remark 6 Harmonic forms are solutions of the optimization problem: Minimize the "Dirichlet norm" $\|\delta f\|^2 + \|\partial f\|^2 = \langle \Delta f, f\rangle = \langle \Delta^{1/2} f, \Delta^{1/2} f\rangle$ over $f \in L_a^2(U_\alpha^{\ell+1})$.

Remark 7 The alternative neighborhoods $V_\alpha^{\ell+1}$ giving rise to the Vietoris-Rips complex (see Chazal and Oudot [7]) were defined by $(x_0,\dots,x_\ell) \in U_\alpha^{\ell+1}$ if and only if $d(x_i,x_j) \leq \alpha$ for all i, j. This corresponds to the theory developed in Sect. 2 with $K(x,y)$ equal to the characteristic function of V_α^2. A version of Theorem 4 holds in this case. We prefer our (Čech) version to the Vietoris-Rips development, since the Čech theory is so standard in the topological literature.

5 L^2-Theory of α-Harmonic 0-Forms

In this section we assume that we are in the setting of Sect. 4, with $\ell = 0$. Thus X is a compact metric space with a probability measure and with a fixed scale $\alpha > 0$.

Recall that $f \in L^2(X)$ is α-harmonic if $\Delta_\alpha f = 0$. Moreover, if $\delta\colon L^2(X) \to L_a^2(U_\alpha^2)$ denotes the coboundary, then $\Delta_\alpha f = 0$ if and only if $\delta f = 0$; also $\delta f(x_0,x_1) = f(x_1) - f(x_0)$ for all pairs $(x_0,x_1) \in U_\alpha^2$.

Recall that for any $x \in X$, the slice $S_{x,\alpha} = S_x \subset X^2$ is the set

$$S_x = S_{x,\alpha} = \{t \in X : \exists p \in X \text{ such that } x, t \in B_\alpha(p)\}.$$

Note that $B_\alpha(x) \subset S_{x,\alpha} \subset B_{2\alpha}(x)$. It follows that $x_1 \in S_{x_0,\alpha}$ if and only if $x_0 \in S_{x_1,\alpha}$. We conclude the following.

Proposition 13 *Let $f \in L^2(X)$. Then $\Delta_\alpha f = 0$ if and only if f is locally constant in the sense that f is constant on $S_{x,\alpha}$ for every $x \in X$. Moreover, if $\Delta_\alpha f = 0$, then*

(a) *If X is connected, then f is constant.*
(b) *If α is greater than the maximum distance between components of X, then f is constant.*
(c) *For any $x \in X$, $f(x) =$ average of f on $S_{x,\alpha}$ and on $B_\alpha(x)$.*
(d) *Harmonic functions are continuous.*

We note that continuity of f follows from the fact that f is constant on each slice $S_{x,\alpha}$, and thus locally constant.

Remark 8 We will show that (d) is also true for harmonic 1-forms with an additional assumption on μ (Sect. 8), but are unable to prove it for harmonic 2-forms.

Consider next an extension of (d) to the Poisson regularity problem. If $\Delta_\alpha f = g$ is continuous, is f continuous? In general the answer is no, and we will give an example.

Since ∂_0 on $L^2(X)$ is zero, the L^2-α-Hodge theory (Sect. 9) takes the form

$$L^2(X) = \operatorname{Im} \partial \oplus \operatorname{Harm}_\alpha,$$

where $\partial : L^2(U_\alpha^2) \to L^2(X)$ and $\Delta f = \partial \delta f$. Thus, for $f \in L^2(X)$, by (9),

$$\Delta_\alpha f(x) = 2\mu(S_{x,\alpha}) f(x) - 2 \int_{S_{x,\alpha}} f(t)\, \mathrm{d}\mu(t). \tag{10}$$

The following example shows that an additional assumption is needed for the Poisson regularity problem to have an affirmative solution. Let X be the closed interval $[-1, 1]$ with the usual metric d and let μ be the Lebesgue measure on X with an atom at 0, $\mu(\{0\}) = 1$. Fix any $\alpha < 1/4$. We will define a piecewise linear function on X with discontinuities at -2α and 2α as follows. Let a and b be any real numbers $a \neq b$, and define

$$f(x) = \begin{cases} \frac{a-b}{8\alpha} + a, & -1 \le x < -2\alpha, \\ \frac{b-a}{4\alpha}(x - 2\alpha) + b, & -2\alpha \le x \le 2\alpha, \\ \frac{a-b}{8\alpha} + b, & 2\alpha < x \le 1. \end{cases}$$

Using (10) above one readily checks that $\Delta_\alpha f$ is continuous by computing left-hand and right-hand limits at $\pm 2\alpha$. (The constant values of f outside $[-2\alpha, 2\alpha]$ are chosen precisely so that the discontinuities of the two terms on the right side of (10) cancel out.)

With an additional "regularity" hypothesis imposed on μ, the Poisson regularity property holds. In the rest of this section assume that $\mu(S_x \cap A)$ is a continuous function of $x \in X$ for each measurable set A. One can show that if μ is Borel regular, then this will hold, provided $\mu(S_x \cap A)$ is continuous for all closed sets A (or all open sets A).

Proposition 14 *Assume that $\mu(S_x \cap A)$ is a continuous function of $x \in X$ for each measurable set A. If $\Delta_\alpha f = g$ is continuous for $f \in L^2(X)$, then f is continuous.*

Proof From (10) we have

$$f(x) = \frac{g(x)}{2\mu(S_x)} + \frac{1}{\mu(S_x)} \int_{S_x} f(t)\, \mathrm{d}\mu(t).$$

The first term on the right is clearly continuous by our hypotheses on μ and the fact that g is continuous. It suffices to show that the function $h(x) = \int_{S_x} f(t)\,\mathrm{d}\mu(t)$ is continuous. If $f = \chi_A$ is the characteristic function of any measurable set A, then $h(x) = \mu(S_x \cap A)$ is continuous, and therefore h is continuous for f any simple function (linear combination of characteristic functions of measurable sets). From general measure theory, if $f \in L^2(X)$, we can find a sequence of simple functions f_n such that $f_n(t) \to f(t)$ a.e., and $|f_n(t)| \leq |f(t)|$ for all $t \in X$. Thus $h_n(x) = \int_{S_x} f_n(t)\,\mathrm{d}\mu(t)$ is continuous and

$$\bigl|h_n(x) - h(x)\bigr| \leq \int_{S_x} \bigl|f_n(t) - f(t)\bigr|\,\mathrm{d}\mu(t) \leq \int_X \bigl|f_n(t) - f(t)\bigr|\,\mathrm{d}\mu(t).$$

Since $|f_n - f| \to 0$ a.e., and $|f_n - f| \leq 2|f|$ with f being in $L^1(X)$, it follows from the dominated convergence theorem that $\int_X |f_n - f|\,\mathrm{d}\mu \to 0$. Thus h_n converges uniformly to h, and so continuity of h follows from continuity of h_n. □

We don't have a similar result for 1-forms.

Partly to relate our framework of α-harmonic theory to some previous work, we combine the setting of Sect. 2 with Sect. 4. Thus we now put back the function K. Assume $K > 0$ is a symmetric and continuous function $K\colon X \times X \to \mathbb{R}$, and δ and ∂ are defined as in Sect. 2, but use a similar extension to general $\alpha > 0$, of Sect. 4, all in the L^2-theory.

Let $D\colon L^2(X) \to L^2(X)$ be the operator defined as multiplication by the function

$$D(x) = \int_X G(x,y)\,\mathrm{d}\mu(y) \quad \text{where } G(x,y) = K(x,y)\chi_{U_\alpha^2}$$

using the characteristic function $\chi_{U_\alpha^2}$ of U_α^2. So $\chi_{U_\alpha^2}(x_0, x_1) = 1$ if $(x_0, x_1) \in U_\alpha^2$ and 0 otherwise. Furthermore, let $L_G\colon L^2(X) \to L^2(X)$ be the integral operator defined by

$$L_G f(x) = \int_X G(x,y) f(y)\,\mathrm{d}\mu(y).$$

Note that $L_G(1) = D$ where 1 is the constant function. When X is compact L_G is a Hilbert-Schmidt operator (this was first noted to us by Ding-Xuan Zhou). Thus L_G is trace class and self-adjoint. It is not difficult to see now that (10) takes the form

$$\frac{1}{2}\Delta_\alpha f = Df - L_G f. \tag{11}$$

(For the special case $\alpha = \infty$, i.e. α is irrelevant as in Sect. 2, this is the situation as in Smale and Zhou [34] for the case where K is a reproducing kernel.) As in the previous proposition, we have the following.

Proposition 15 *The Poisson regularity property holds for the operator of* (11).

To get a better understanding of (11), it is useful to define a normalization of the kernel G and the operator L_G as follows. Let $\hat{G}: X \times X \to \mathbb{R}$ be defined by

$$\hat{G}(x, y) = \frac{G(x, y)}{(D(x)D(y))^{1/2}}$$

and $L_{\hat{G}}: L^2(X) \to L^2(X)$ be the corresponding integral operator. Then $L_{\hat{G}}$ is trace class and self-adjoint, and has a complete orthonormal system of continuous eigenfunctions.

A normalized α-Laplacian may be defined on $L^2(X)$ by

$$\frac{1}{2}\hat{\Delta} = I - L_{\hat{G}}$$

so that the spectral theory of $L_{\hat{G}}$ may be transferred to $\hat{\Delta}$. (Also, one might consider $\frac{1}{2}\Delta^* = I - D^{-1}L_G$ as in Belkin, De Vito, and Rosasco [3].)

In Smale and Zhou [34], for $\alpha = \infty$, error estimates are given (reproducing kernel case) for the spectral theory of $L_{\hat{G}}$ in terms of finite-dimensional approximations. See especially Belkin and Niyogi [2] for limit theorems as $\alpha \to 0$.

6 Harmonic Forms on Constant Curvature Manifolds

In this section we will give an explicit description of harmonic forms in a special case. Let X be a compact, connected, oriented manifold of dimension $n > 0$, with a Riemannian metric g of constant sectional curvature. Also, assume that g is normalized so that $\mu(X) = 1$ where μ is the measure induced by the volume form associated with g, and let d be the metric on X induced by g. Let $\alpha > 0$ be sufficiently small so that, for all $p \in X$, the ball $B_{2\alpha}(p)$ is geodesically convex. That is, for $x, y \in B_{2\alpha}(p)$ there is a unique, length-minimizing geodesic γ from x to y, and γ lies in $B_{2\alpha}(p)$. Note that if $(x_0, \dots, x_n) \in U_\alpha^{n+1}$, then $d(x_i, x_j) \le 2\alpha$ for all i, j, and thus all x_i lie in a common geodesically convex ball. Such a point defines an n-simplex with vertices $x_0, \dots, x_n$ whose faces are totally geodesic submanifolds, which we will denote by $\sigma(x_0, \dots, x_n)$. We will also denote the k-dimensional faces by $\sigma(x_{i_0}, \dots, x_{i_k})$ for $k < n$. Thus $\sigma(x_i, x_j)$ is the geodesic segment from x_i to x_j, $\sigma(x_i, x_j, x_k)$ is the union of geodesic segments from x_i to points on $\sigma(x_j, x_k)$, and higher dimensional simplices are defined inductively. (Since X has constant curvature, this construction is symmetric in $x_0, \dots, x_n$.) A k-dimensional face will be called degenerate if one of its vertices is contained in one of its $(k-1)$-dimensional faces.

Note that the cohomology of the Vietoris-Rips complex has already been considered by Hausmann [21], but his construction is quite different from ours. He considers the limit, as $\epsilon \to 0$, of the simplicial cohomology of X_ϵ. First, we contend that important information is visible in X_α at particular scales α, possibly determined by the problem at hand, and not tending to 0. Second, Hausmann considers simplicial homology, with arbitrary coefficients, while we consider ℓ^2 cohomology, with real or complex coefficients.

For $(x_0, \dots, x_n) \in U_\alpha^{n+1}$, the orientation on X induces an orientation on $\sigma(x_0, \dots, x_n)$ (assuming it is non-degenerate). For example, if $v_1, \dots, v_n$ denote the tangent vectors at x_0 to the geodesics from x_0 to $x_1, \dots, x_n$, we can define $\sigma(x_0, \dots, x_n)$ to be positive (negative) if $\{v_1, \dots, v_n\}$ is a positive (respectively negative) basis for the tangent space at x_0. Of course, if τ is a permutation, the orientation of $\sigma(x_0, \dots, n)$ is equal to $(-1)^{\operatorname{sign}\tau}$ times the orientation of $\sigma(x_{\tau(0)}, \dots, x_{\tau(n)})$. We now define $f \colon U_\alpha^{\ell+1} \to \mathbb{R}$ by

$$\begin{aligned} f(x_0, \dots, x_n) &= \mu\bigl(\sigma(x_0, \dots, x_n)\bigr) \quad \text{for } \sigma(x_0, \dots, x_n) \text{ positive} \\ &= -\mu\bigl(\sigma(x_0, \dots, x_n)\bigr) \quad \text{for } \sigma(x_0, \dots, x_n) \text{ negative} \\ &= 0 \quad \text{for } \sigma(x_0, \dots, x_n) \text{ degenerate.} \end{aligned}$$

Thus f is the signed volume of oriented geodesic n-simplices. Clearly, f is continuous, as non-degeneracy is an open condition and the volume of a simplex varies continuously in the vertices.

Recall that, in classical Hodge theory, every de Rham cohomology class has a unique harmonic representative. In particular, the volume form is harmonic, and generates top-dimensional cohomology. In our more elaborate context, we can also pinpoint the "form" generating top-dimensional cohomology. (See Remark 9 below on relaxing the constant curvature hypothesis.) The main result of this section is the following theorem.

Theorem 5 *Let X be an oriented Riemannian n-manifold of constant sectional curvature and f, α as above. Then f is harmonic. In fact, f is the unique harmonic n-form in $L_a^2(U_\alpha^{n+1})$ up to scaling.*

Proof Uniqueness follows from Sect. 9. We will show that $\partial f = 0$ and $\delta f = 0$. Let $(x_0, \dots, x_{n-1}) \in U_\alpha^n$. To show $\partial f = 0$, it suffices to show, by Proposition 12, that

$$\int_{S_{x_0 \cdots x_{n-1}}} f(t, x_0, \dots, x_{n-1})\, \mathrm{d}\mu(t) = 0. \tag{12}$$

We may assume that $\sigma(x_0, \dots, x_{n-1})$ is non-degenerate; otherwise the integrand is identically zero. Recall that $S_{x_0 \cdots x_{n-1}} = \{t \in X : (t, x_0, \dots, x_{n-1}) \in U_\alpha^{n+1}\} \subset B_{2\alpha}(x_0)$

where $B_{2\alpha}(x_0)$ is the geodesic ball of radius 2α centered at x_0. Let Γ be the intersection of the totally geodesic $n-1$ dimensional submanifold containing $x_0, \dots, x_{n-1}$ with $B_{2\alpha}(x_0)$. Thus Γ divides $B_{2\alpha}(x_0)$ into two pieces B^+ and B^-. For $t \in \Gamma$, the simplex $\sigma(t, x_0, \dots, x_{n-1})$ is degenerate; therefore, the orientation is constant on each of B^+ and B^-, and we can assume that the orientation of $\sigma(t, x_0, \dots, x_{n-1})$ is positive on B^+ and negative on B^-. For $x \in B_{2\alpha}(x_0)$ define $\phi(x)$ to be the reflection of x across Γ. Thus the geodesic segment from x to $\phi(x)$ intersects Γ perpendicularly at its midpoint. Because X has constant curvature, ϕ is a local isometry and since $x_0 \in \Gamma$, $d(x, x_0) = d(\phi(x), x_0)$. Therefore, $\phi\colon B_{2\alpha}(x_0) \to B_{2\alpha}(x_0)$ is an isometry which maps B^+ isometrically onto B^- and B^- onto B^+. Denote $S_{x_0 \cdots x_{n-1}}$ by S. It is easy to see that $\phi\colon S \to S$, and so defining $S^{\pm} = S \cap B^{\pm}$ it follows that $\phi\colon S^+ \to S^-$ and $\phi\colon S^- \to S^+$ are isometries. Now

$$\begin{aligned}
&\int_{S_{x_0 \cdots x_{n-1}}} f(t, x_0, \dots, x_{n-1})\, \mathrm{d}\mu(t) \\
&\quad = \int_{S^+} f(t, x_0, \dots, x_{n-1})\, \mathrm{d}\mu(t) + \int_{S^-} f(t, x_0, \dots, x_{n-1})\, \mathrm{d}\mu(t) \\
&\quad = \int_{S^+} \mu\bigl(\sigma(t, x_0, \dots, x_{n-1})\bigr)\, \mathrm{d}\mu(t) - \int_{S^-} \mu\bigl(\sigma(t, x_0, \dots, x_{n-1})\bigr)\, \mathrm{d}\mu(t).
\end{aligned}$$

Since $\mu(\sigma(t, x_0, \dots, x_{n-1})) = \mu(\sigma(\phi(t)t, x_0, \dots, x_{n-1}))$ for $t \in S^+$, the last two terms on the right side cancel, establishing (12).

We now show that $\delta f = 0$. Let $(t, x_0, \dots, x_n) \in U_\alpha^{n+2}$. Thus

$$\delta f(t, x_0, \dots, x_n) = f(x_0, \dots, x_n) + \sum_{i=0}^{n} (-1)^{i+1} f\bigl(t, x_0, \dots, \hat{x}_i, \dots, x_n\bigr)$$

and we must show that

$$f(x_0, \dots, x_n) = \sum_{i=0}^{n} (-1)^i f\bigl(t, x_0, \dots, \hat{x}_i, \dots, x_n\bigr). \tag{13}$$

Without loss of generality, we will assume that $\sigma(x_0, \dots, x_n)$ is positive. The demonstration of (13) depends on the location of t. Suppose that t is in the interior of the simplex $\sigma(x_0, \dots, x_n)$. Then, for each i, the orientation of $\sigma(x_0, \dots, x_{i-1}, t, x_{i+1}, \dots, x_n)$ is the same as the orientation of $\sigma(x_0, \dots, x_n)$, since t and x_i lie on the same side of the face $\sigma(x_0, \dots, \hat{x}_i, \dots, x_n)$, and is thus positive. On the other hand, the orientation of $\sigma(t, x_0, \dots, \hat{x}_i, \dots, x_n)$ is $(-1)^i$ times the orientation of $\sigma(x_0, \dots, x_{i-1}, t, x_{i+1}, \dots, x_n)$. Therefore, the right side of (13) becomes

$$\sum_{i=0}^{n} \mu\bigl(\sigma(x_0, \dots, x_{i-1}, t, x_{i+1}, \dots, x_n)\bigr).$$

However, this equals $\mu(\sigma(x_0,\dots,x_n))$, which is the left side of (13), since

$$\sigma(x_0,\dots,x_n)=\bigcup_{i=0}^{n}\sigma(x_0,\dots,x_{i-1},t,x_{i+1},\dots,x_n)$$

when t is interior to $\sigma(x_0,\dots,x_n)$.

There are several cases when t is exterior to $\sigma(x_0,\dots,x_n)$ (or on one of the faces), depending on which side of the various faces it lies. We just give the details for one of these, the others being similar. Simplifying notation, let F_i denote the face "opposite" x_i, $\sigma(x_0,\dots,\hat{x}_i,\dots,x_n)$, and suppose that t is on the opposite side of F_0 from x_0, but on the same side of F_i as x_i for $i\neq 0$. As in the above argument, the orientation of $\sigma(x_0,\dots,x_{i-1},t,x_{i+1},\dots,x_n)$ is positive for $i\neq 0$ and is negative for $i=0$. Therefore, the right side of (13) is equal to

$$\sum_{i=1}^{n}\mu\bigl(\sigma(x_0,\dots,x_{i-1},t,x_{i+1},\dots,x_n)\bigr)-\mu\bigl(\sigma(t,x_1,\dots,x_n)\bigr). \tag{14}$$

Let s be the point where the geodesic from x_0 to t intersects F_0. Then, for each $i>0$,

$$\begin{aligned}&\sigma(x_0,\dots,x_{i-1},t,x_{i+1},\dots,x_n)\\&\quad=\sigma(x_0,\dots,x_{i-1},s,x_{i+1},\dots,x_n)\cup\sigma(s,\dots,x_{i-1},t,x_{i+1},\dots,x_n).\end{aligned}$$

Taking μ of both sides and summing over i gives

$$\begin{aligned}&\sum_{i=1}^{n}\mu\bigl(\sigma(x_0,\dots,x_{i-1},t,x_{i+1},\dots,x_n)\bigr)\\&\quad=\sum_{i=1}^{n}\mu\bigl(\sigma(x_0,\dots,x_{i-1},s,x_{i+1},\dots,x_n)\bigr)\\&\qquad+\sum_{i=1}^{n}\mu\bigl(\sigma(s,\dots,x_{i-1},t,x_{i+1},\dots,x_n)\bigr).\end{aligned}$$

However, the first term on the right is just $\mu(\sigma(x_0,\dots,x_n))$ and the second term is $\mu(\sigma(t,x_1,\dots,x_n))$. Combining this with (14) gives us (13), finishing the proof of $\delta f=0$. □

Remark 9 The proof that $\partial f=0$ strongly used the fact that X has constant curvature. In the case where X is an oriented Riemannian surface of variable curvature, totally geodesic n simplices do not generally exist, although geodesic triangles $\sigma(x_0,x_1,x_2)$ are well defined for $(x_0,x_1,x_2)\in U_\alpha^3$. In this case, the proof above shows that $\delta f=0$. More generally, for an n-dimensional connected oriented Riemannian manifold, using the order of a tuple $(x_0,\dots,x_n)$ one can iteratively form convex combinations and in this way assign an oriented n-simplex to $(x_0,\dots,x_n)$ and then define the volume cocycle as above (if α is small enough).

Using a chain map to simplicial cohomology which evaluates at the vertices' points, it is easy to check that these cocycles represent a generator of the cohomology in degree n (which by the results of Sect. 9 is exactly one dimensional).

7 Cohomology

Traditional cohomology theories on general spaces are typically defined in terms of limits as in Čech theory, with nerves of coverings. However, an algorithmic approach suggests a development via a scaled theory, at a given scale $\alpha > 0$. Then, as $\alpha \to 0$ one recovers the classical setting. A closely related point of view is that of persistent homology; see Edelsbrunner, Letscher, and Zomorodian [17], Zomorodian and Carlsson [40], and Carlsson [5].

We give a setting for such a scaled theory, with a fixed scaling parameter $\alpha > 0$.

Let X be a separable, complete metric space with metric d, and $\alpha > 0$ a "scale." We will define a (generally infinite) simplicial complex $C_{X,\alpha}$ associated to (X, d, α). Toward that end let $X^{\ell+1}$, for $\ell \geq 0$, be the $(\ell+1)$-fold Cartesian product, with metric still denoted by d, $d\colon X^{\ell+1} \times X^{\ell+1} \to \mathbb{R}$ where $d(x, y) = \max_{i=0,\dots,\ell} d(x_i, y_i)$. As in Sect. 4, let

$$U_\alpha^{\ell+1}(X) = U_\alpha^{\ell+1} = \left\{x \in X^{\ell+1} : d(x, D_{\ell+1}) \leq \alpha\right\},$$

where $D_{\ell+1} \subset X^{\ell+1}$ is the diagonal, so $D_{\ell+1} = \{(t, \dots, t)\ \ell + 1 \text{ times}\}$. Then let $C^\ell_{X,\alpha} = U_\alpha^{\ell+1}$. This has the structure of a simplicial complex whose ℓ-simplices consist of points of $U_\alpha^{\ell+1}$. This is well defined since if $x \in U_\alpha^{\ell+1}$, then $y = (x_0, \dots, \hat{x}_i, \dots, x_\ell) \in U_\alpha^\ell$, for each $i = 0, \dots, \ell$. We will write $\alpha = \infty$ to mean that $U_\alpha^\ell = X^\ell$. Following e.g. Munkres [30], there is a well-defined cohomology theory, simplicial cohomology, for this simplicial complex, with cohomology vector spaces (always over $\mathbb{R}$) denoted by $H_\alpha^\ell(X)$. We especially note that $C_{X,\alpha}$ is not necessarily a finite simplicial complex. For example, if X is an open non-empty subset of Euclidean space, the vertices of $C_{X,\alpha}$ are the points of X and of course infinite in number. The complex $C_{X,\alpha}$ will be called the simplicial complex at scale α associated to X.

Example 1 X is finite. Fix $\alpha > 0$. In this case, for each ℓ, the set of ℓ-simplices is finite, the ℓ-chains form a finite-dimensional vector space, and the α-cohomology groups (i.e. vector spaces) $H_\alpha^\ell(X)$ are all finite dimensional. One can check that for $\alpha = \infty$ one has $\dim H_\alpha^0(X) = 1$ and $H_\alpha^i(X)$ are trivial for all $i > 0$. Moreover, for α sufficiently small ($\alpha < \min\{d(x, y) : x, y \in X,\ x \neq y\}$) $\dim H_\alpha^0(X) =$cardinality of X, with $H_\alpha^i(X) = 0$ for all $i > 0$. For intermediate α, the α-cohomology can be rich in higher dimensions, but $C_{X,\alpha}$ is a finite simplicial complex.

Example 2 First let $A \subset \mathbb{R}^2$ be the annulus $A = \{x \in \mathbb{R}^2 : 1 \leq \|x\| \leq 2\}$. Form A^* by deleting the finite set of points with rational coordinates $(p/q, r/s)$, with $|q|$, $|s| \leq 10^{10}$. Then one may check that for $\alpha > 4$, $H_\alpha^\ell(A^*)$ has the cohomology of a point, for certain intermediate values of α, $H_\alpha^\ell(A^*) = H_\alpha^\ell(A)$, and for α small enough $H_\alpha^\ell(A^*)$ has enormous dimension. Thus the scale is crucial to see the features of A^* clearly.

Returning to the case of general X, note that if $0 < \beta < \alpha$ one has a natural inclusion $J\colon U_\beta^\ell \to U_\alpha^\ell$, $J\colon C_{X,\beta} \to C_{X,\alpha}$ and the restriction $J^*\colon L_a^2(U_\alpha^\ell) \to L_a^2(U_\beta^\ell)$ commuting with δ (a chain map).

Now assume X is compact. For fixed scale α, consider the covering $\{B_\alpha(x) : x \in X\}$, where $B_\alpha(x)$ is the ball $B_\alpha(x) = \{y \in X : d(x,y) < \alpha\}$, and the nerve of the covering is $C_{X,\alpha}$, giving the "Čech construction at scale α." Thus from Čech cohomology theory, we see that the limit as $\alpha \to 0$ of $H_\alpha^\ell(X) = H^\ell(X) = H_{\text{Čech}}^\ell(X)$ is the ℓth Čech cohomology group of X.

The next observation is to note that our construction of the scaled simplicial complex $C_{X,\alpha}$ of X follows the same path as Alexander–Spanier theory (see Spanier [36]). Thus the scaled cohomology groups $H_\alpha^\ell(X)$ will have the direct limit as $\alpha \to 0$ which maps to the Alexander–Spanier group $H_{\text{Alex-Sp}}^\ell(X)$ (and in many cases will be isomorphic). Thus $H^\ell(X) = H_{\text{Alex-Sp}}^\ell(X) = H_{\text{Čech}}^\ell(X)$. In fact, in much of the literature this is recognized by the use of the term Alexander–Spanier–Čech cohomology. What we have done is describe a finite scale version of the classical cohomology.

Now that we have defined the scale α cohomology groups, $H_\alpha^\ell(X)$ for a metric space X, our Hodge theory suggests this modification. From Theorem 4, we have considered instead of arbitrary cochains (i.e. arbitrary functions on $U_\alpha^{\ell+1}$ which give the definition here of $H_\alpha^\ell(X)$), cochains defined by L^2-functions on $U_\alpha^{\ell+1}$. Thus we have constructed cohomology groups at scale α from L^2-functions on $U_\alpha^{\ell+1}$, $H_{\alpha,L^2}^\ell(X)$, when $\alpha > 0$, and X is a metric space equipped with Borel probability measure.

Question 1 (Cohomology Identification Problem (CIP)) *To what extent are $H_{L^2,\alpha}^\ell(X)$ and $H_\alpha^\ell(X)$ isomorphic?*

This is important via Theorem 4, which asserts that $H_{\alpha,L^2}^\ell(X) \to \text{Harm}_\alpha^\ell(X)$ is an isomorphism, in the case where $H_{\alpha,L^2}^\ell(X)$ is finite dimensional.

One may replace L^2-functions in the construction of the α-scale cohomology theory by continuous functions. As in the L^2-theory, this gives rise to cohomology groups $H_{\alpha,\text{cont}}^\ell(X)$. Analogous to CIP we have the simpler question: To what extent is the natural map $H_{\alpha,\text{cont}}^\ell(X) \to H_\alpha^\ell(X)$ an isomorphism?

We will give answers to these questions for special X in Sect. 9.

Note that in the case where X is finite, or $\alpha = \infty$, we have an affirmative answer to this question, as well as to CIP (see Sects. 2 and 3).

Proposition 16 *There is a natural injective linear map*

$$\text{Harm}_{\text{cont},\alpha}^\ell(X) \to H_{\text{cont},\alpha}^\ell(X).$$

Proof The inclusion, which is injective

$$J\colon \operatorname{Im}_{\text{cont},\alpha}\delta \oplus \text{Harm}_{\text{cont},\alpha}^\ell(X) \to \text{Ker}_{\text{cont},\alpha}$$

induces an injection

$$J^*\colon \operatorname{Harm}^\ell_{\mathrm{cont},\alpha}(X) = \frac{\operatorname{Im}_{\mathrm{cont},\alpha}\delta \oplus \operatorname{Harm}^\ell_{\mathrm{cont},\alpha}(X)}{\operatorname{Im}_{\mathrm{cont},\alpha}\delta} \to \frac{\operatorname{Ker}_{\mathrm{cont},\alpha}}{\operatorname{Im}_{\mathrm{cont},\alpha}} = H^\ell_{\mathrm{cont},\alpha}(X)$$

and the proposition follows. □

8 Continuous Hodge Theory on the Neighborhood of the Diagonal

As in the last section, (X, d) will denote a compact metric space equipped with a Borel probability measure μ. For topological reasons (see Sect. 6) it would be nice to have a Hodge decomposition for continuous functions on $U^{\ell+1}_\alpha$, analogous to the continuous theory on the whole space (Sect. 4). We will use the following notation. $C^{\ell+1}_\alpha$ will denote the continuous alternating real-valued functions on $U^{\ell+1}_\alpha$, $\operatorname{Ker}_{\alpha,\mathrm{cont}}\Delta_\ell$ will denote the functions in $C^{\ell+1}_\alpha$ that are harmonic, and $\operatorname{Ker}_{\alpha,\mathrm{cont}}\delta_\ell$ will denote those elements of $C^{\ell+1}_\alpha$ that are closed. Also, $H^\ell_{\alpha,\mathrm{cont}}(X)$ will denote the quotient space (cohomology space) $\operatorname{Ker}_{\alpha,\mathrm{cont}}\delta_\ell/\delta(C^\ell_\alpha)$. We raise the following question, analogous to Theorem 4.

Question 2 (Continuous Hodge Decomposition) *Under what conditions on X and $\alpha > 0$ is it true that there is the following orthogonal* (*with respect to the L^2-inner product*) *direct sum decomposition*

$$C^{\ell+1}_\alpha = \delta\bigl(C^\ell_\alpha\bigr) \oplus \partial\bigl(C^{\ell+2}_\alpha\bigr) \oplus \operatorname{Ker}_{\alpha,\mathrm{cont}}\Delta_\ell,$$

where $\operatorname{Ker}_{\mathrm{cont},\alpha}\Delta_\ell$ *is isomorphic to $H^\ell_{\alpha,\mathrm{cont}}(X)$, with every element in $H^\ell_{\alpha,\mathrm{cont}}(X)$ having a unique representative in* $\operatorname{Ker}_{\alpha,\mathrm{cont}}\Delta_\ell$*?*

There is a related analytical problem that is analogous to elliptic regularity for partial differential equations; in fact, elliptic regularity features prominently in classical Hodge theory.

Question 3 (The Poisson Regularity Problem) *For $\alpha > 0$ and $\ell > 0$, suppose that $\Delta f = g$ where $g \in C^{\ell+1}_\alpha$ and $f \in L^2_a(U^{\ell+1}_\alpha)$. Under what conditions on (X, d, μ) is f continuous?*

Theorem 6 *An affirmative answer to the Poisson Regularity Problem, together with closed image $\delta(L^2_a(U^\ell_\alpha))$, implies an affirmative solution to the continuous Hodge decomposition question.*

Proof Assume that the Poisson regularity property holds, and let $f \in C^{\ell+1}_\alpha$. From Theorem 4 we have the L^2-Hodge decomposition

$$f = \delta f_1 + \partial f_2 + f_3,$$

where $f_1 \in L^2_a(U^\ell_\alpha)$, $f_2 \in L^2_a(U^{\ell+2}_\alpha)$, and $f_3 \in L^2_a(U^{\ell+1}_\alpha)$ with $\Delta f_3 = 0$. It suffices to show that f_1 and f_2 can be taken to be continuous, and f_3 is continuous. Since $\Delta f_3 =$

0 is continuous, f_3 is continuous by Poisson regularity. We will show that $\partial f_2 = \partial(\delta h_2)$ where δh_2 is continuous (and thus f_2 can be taken to be continuous). Recall (corollary of the Hodge Lemma in Sect. 2) that the following maps are isomorphisms:

$$\delta\colon \partial\bigl(L_a^2\bigl(U_\alpha^{\ell+2}\bigr)\bigr) \to \delta\bigl(L_a^2\bigl(U_\alpha^{\ell+1}\bigr)\bigr) \quad \text{and} \quad \partial\colon \delta\bigl(L_a^2\bigl(U_\alpha^{\ell}\bigr)\bigr) \to \partial\bigl(L_a^2\bigl(U_\alpha^{\ell+1}\bigr)\bigr)$$

for all $\ell \geq 0$. Thus

$$\partial f_2 = \partial(\delta h_2) \quad \text{for some } h_2 \in L_a^2\bigl(U_\alpha^{\ell+1}\bigr).$$

Now,

$$\Delta\bigl(\delta(h_2)\bigr) = \delta\bigl(\partial(\delta(h_2))\bigr) + \partial\bigl(\delta\bigl(\delta(h_2)\bigr)\bigr) = \delta\bigl(\partial\bigl(\delta(h_2)\bigr)\bigr) = \delta\bigl(\partial(f_2)\bigr) \tag{15}$$

since $\delta^2 = 0$. However, from the decomposition for f we have, since $\delta f_3 = 0$,

$$\delta f = \delta(\partial f_2)$$

and since f is continuous δf is continuous, and therefore $\delta(\partial f_2)$ is continuous. It then follows from Poisson regularity and (15) that δh_2 is continuous, as will be shown. A dual argument shows that $\delta f_1 = \delta(\partial h_1)$ where ∂h_1 is continuous, completing the proof. □

Notice that a somewhat weaker result than Poisson regularity would imply that f_3 above is continuous, namely regularity of harmonic functions.

Question 4 (Harmonic Regularity Problem) *For $\alpha > 0$ and $\ell > 0$, suppose that $\Delta f = 0$ where $f \in L_a^2(U_\alpha^{\ell+1})$. What conditions on (X, d, μ) would imply f is continuous?*

Under some additional conditions on the measure, we have answered this for $\ell = 0$ (see Sect. 5) and can do so for $\ell = 1$, which we now consider.

We assume in addition that the inclusion of continuous functions into L^2-functions induces an epimorphism of the associated Alexander–Spanier–Čech cohomology groups, i.e. that every cohomology class in the L^2-theory has a continuous representative. In Sect. 9 we will see that this is often the case.

Let now $f \in L_a^2(U_\alpha^2)$ be harmonic. Let g be a continuous function in the same cohomology class. Then there is $x \in L_a^2(U_\alpha^1)$ such that $f = g + dx$. As $\delta^* f = 0$ it follows that $\delta^* dx = -\delta^* g$ is continuous. If the Poisson regularity property in degree zero holds (compare Proposition 14 of Sect. 5), then x is continuous and therefore $f = g + dx$ is also continuous.

Thus we have the following proposition.

Proposition 17 *Assume that $\mu(S_x \cap A)$ are continuous for $x \in X$ and all A measurable. Assume that every cohomology class of degree 1 has a continuous representative. If f is an α-harmonic 1-form in $L_a^2(U_\alpha^2)$, then f is continuous.*

As in Sect. 5, if μ is Borel regular, it suffices that the hypotheses hold for all A closed (or all A open).

9 Finite-Dimensional Cohomology

In this section, we will establish conditions on X and $\alpha > 0$ that imply that the α cohomology is finite dimensional. In particular, in the case of the L^2-α cohomology, they imply that the image of δ is closed, and that Hodge theory for X at scale α holds. Along the way, we will compute the α-cohomology in terms of ordinary Čech cohomology of a covering and that the different variants of our Alexander-Spanier-Čech cohomology at fixed scale (L^2, continuous, ...) are all isomorphic. We then show that the important class of metric spaces, Riemannian manifolds satisfy these conditions for α small. In particular, in this case the α-cohomology will be isomorphic to ordinary cohomology with $\mathbb{R}$-coefficients.

Note that in [31, Sect. 4], a Rips version of the L^2-Alexander–Spanier complex on a finite scale is introduced which is similar to ours. It is then sketched how, for sufficiently small scales on a manifold or a simplicial complex, its cohomology should be computable in terms of the L^2-simplicial or L^2-de Rham cohomology, without giving detailed arguments. These results are rather similar to our results. The fact that we work with the α-neighborhood of the diagonal causes some additional difficulties we have to overcome.

Throughout this section, (X,d) will denote a compact metric space, μ a Borel probability measure on X such that $\mu(U) > 0$ for all non-empty open sets $U \subset X$, and $\alpha > 0$. As before U_α^ℓ will denote the closed α-neighborhood of the diagonal in X^ℓ. We will denote by $F_a(U_\alpha^\ell)$ the space of all alternating real-valued functions on U_α^ℓ, by $C_a(U_\alpha^\ell)$ the continuous alternating real-valued functions on U_α^ℓ, and by $L_a^p(U_\alpha^\ell)$ the L^p alternating real-valued functions on U_α^ℓ for $p \geq 1$ (in particular, the case $p = 2$ was discussed in the preceding sections). If X is a smooth Riemannian manifold, $C_a^\infty(U_\alpha^\ell)$ will be the smooth alternating real-valued functions on U_α^ℓ. We will be interested in the following cochain complexes:

$$0 \longrightarrow L_a^p(X) \xrightarrow{\delta_0} L_a^p\bigl(U_\alpha^2\bigr) \xrightarrow{\delta_1} \cdots \xrightarrow{\delta_{\ell-1}} L_a^p\bigl(U_\alpha^{\ell+1}\bigr) \xrightarrow{\delta_\ell} \cdots$$

$$0 \longrightarrow C_a(X) \xrightarrow{\delta_0} C_a\bigl(U_\alpha^2\bigr) \xrightarrow{\delta_1} \cdots \xrightarrow{\delta_{\ell-1}} C_a\bigl(U_\alpha^{\ell+1}\bigr) \xrightarrow{\delta_\ell} \cdots$$

$$0 \longrightarrow F_a(X) \xrightarrow{\delta_0} F_a\bigl(U_\alpha^2\bigr) \xrightarrow{\delta_1} \cdots \xrightarrow{\delta_{\ell-1}} F_a\bigl(U_\alpha^{\ell+1}\bigr) \xrightarrow{\delta_\ell} \cdots$$

And if X is a smooth Riemannian manifold,

$$0 \longrightarrow C_a^\infty(X) \xrightarrow{\delta_0} C_a^\infty\bigl(U_\alpha^2\bigr) \xrightarrow{\delta_1} \cdots \xrightarrow{\delta_{\ell-1}} C_a^\infty\bigl(U_\alpha^{\ell+1}\bigr) \xrightarrow{\delta_\ell} \cdots$$

The corresponding cohomology spaces $\operatorname{Ker}\delta_\ell / \operatorname{Im}\delta_{\ell-1}$ will be denoted by $H^\ell_{\alpha,L^p}(X)$, or briefly H^ℓ_{α,L^p}, $H^\ell_{\alpha,\mathrm{cont}}$, H^ℓ_α, and $H^\ell_{\alpha,\mathrm{smooth}}$ respectively. The proof of finite dimensionality of these spaces, under certain conditions, involves the use of bicomplexes, some facts about which we collect here.

A bicomplex $C^{*,*}$ will be a rectangular array of vector spaces $C^{j,k}$, $j,k \geq 0$, and linear maps (coboundary operators) $c_{j,k}\colon C^{j,k} \to C^{j+1,k}$ and $d_{j,k}\colon C^{j,k} \to C^{j,k+1}$ such that the rows and columns are chain complexes, that is $c_{j+1,k}c_{j,k} = 0$,

$d_{j,k+1}d_{j,k} = 0$, and $c_{j,k+1}d_{j,k} = d_{j+1,k}c_{j,k}$. Given such a bicomplex, we associate the total complex E^*, a chain complex

$$0 \longrightarrow E^0 \xrightarrow{D_0} E^1 \xrightarrow{D_1} \cdots \xrightarrow{D_{\ell-1}} E^\ell \xrightarrow{D_\ell} \cdots$$

where $E^\ell = \bigoplus_{j+k=\ell} C^{j,k}$ and where on each term $C^{j,k}$ in E^ℓ, $D_\ell = c_{j,k} + (-1)^k d_{j,k}$. Using commutativity of c and d, one can easily check that $D_{\ell+1}D_\ell = 0$, and thus the total complex is a chain complex. We recall a couple of definitions from homological algebra. If E^* and F^* are cochain complexes of vector spaces with coboundary operators e and f respectively, then a chain map $g\colon E^* \to F^*$ is a collection of linear maps $g_j\colon E^j \to F^j$ that commute with e and f. A chain map induces a map on cohomology. A cochain complex E^* is said to be exact at the kth term if the kernel of $e_k\colon E_k \to E_{k+1}$ is equal to the image of $e_{k-1}\colon E_{k-1} \to_k$. Thus the cohomology at that term is zero. E^* is defined to be exact if it is exact at each term. A chain contraction $h\colon E^* \to E^*$ is a family of linear maps $h_j\colon E^j \to E^{j-1}$ such that $e_{j-1}h_j + h_{j+1}e_j = \mathrm{Id}$. The existence of a chain contraction on E^* implies that E^* is exact. The following fact from homological algebra is fundamental in proving the finite dimensionality of our cohomology spaces.

Lemma 4 *Suppose that $C^{*,*}$ is a bicomplex as above, and E^* is the associated total complex. Suppose that we augment the bicomplex with a column on the left which is a chain complex $C^{-1,*}$,*

$$C^{-1,0} \xrightarrow{d_{-1,0}} C^{-1,1} \xrightarrow{d_{-1,1}} \cdots \xrightarrow{d_{-1,\ell-1}} C^{-1,\ell} \xrightarrow{d_{-1,\ell}} \cdots$$

and with linear maps $c_{-1,k}\colon C^{-1,k} \to C^{0,k}$, such that the augmented rows

$$0 \longrightarrow C^{-1,k} \xrightarrow{c_{-1,k}} C^{0,k} \xrightarrow{c_{0,k}} \cdots \xrightarrow{c_{\ell-1,k}} C^{\ell,k} \xrightarrow{c_{\ell,k}} \cdots$$

are chain complexes with $d_{0,k}c_{-1,k} = c_{-1,k+1}d_{-1,k}$. Then, the maps $c_{-1,k}$ induce a chain map $c_{-1,*}\colon C^{-1,*} \to E^*$. Furthermore, if the first K rows of the augmented complex are exact, then $c_{-1,*}$ induces an isomorphism on the homology of the complexes $c^*_{-1,*}\colon H^k(C^{-1,*}) \to H^k(E^*)$ for $k \le K$ and an injection for $k = K+1$. In fact, one only needs exactness of the first K rows up to the Kth term $C^{K,j}$.

A simple proof of this is given in Bott and Tu [4, pp. 95–97], for the Čech-de Rham complex, but the proof generalizes to the abstract setting. Of course, if we augmented the bicomplex with a row $C^{*,-1}$ with the same properties, the conclusions would hold. In fact, we will show that the cohomologies of two chain complexes are isomorphic by augmenting a bicomplex as above with one such row and one such column.

Corollary 3 *Suppose that $C^{*,*}$ is a bicomplex as in Lemma* 4, *and that $C^{*,*}$ is augmented with a column $C^{-1,*}$ as in the lemma, and with a row $C^{*,-1}$ that is also a chain complex with coboundary operators $c_{j,-1}\colon C^{j,-1} \to C^{j+1,-1}$ and linear maps $d_{j,-1}\colon C^{j,-1} \to C^{j,0}$ such that the augmented columns*

$$0 \longrightarrow C^{j,-1} \xrightarrow{d_{j,-1}} C^{0,k} \xrightarrow{d_{j,-1}} \cdots \xrightarrow{d_{j,\ell-1}} C^{j,\ell} \xrightarrow{d_{j,\ell}} \cdots$$

are chain complexes, and $c_{j,0}d_{j,-1} = d_{j+1,-1}c_{j,-1}$. *Then, if the first* K *rows are exact and the first* $K+1$ *columns are exact, up to the* $K+1$ *term, it follows that the cohomology* $H^\ell(C^{-1,*})$ *of* $C^{-1,*}$ *and* $H^\ell(C^{*,-1})$ *of* $C^{*,-1}$ *are isomorphic for* $0 \le K$, *and* $H^{K+1}(C^{-1,*})$ *is isomorphic to a subspace of* $H^{K+1}(C^{*,-1})$.

Proof This follows immediately from the lemma, as the cohomology up to order K of both $C^{-1,*}$ and $C^{*,-1}$ is isomorphic to the cohomology of the total complex. Also, $H^{K+1}(C^{-1,*})$ is isomorphic to a subspace of $H^{K+1}(E^*)$ which is isomorphic to $H^{K+1}(C^{*,-1})$. □

Remark 10 If all of the spaces $C^{j,k}$ in Lemma 4 and Corollary 3 are Banach spaces, and the coboundaries $c_{j,k}$ and $d_{j,k}$ are bounded, then the isomorphisms of cohomology can be shown to be topological isomorphisms, where the topologies on the cohomology spaces are induced by the quotient semi-norms.

Let $\{V_i, i \in S\}$ be a finite covering of X by Borel sets (usually taken to be balls). We construct the corresponding Čech-L^p-Alexander bicomplex at scale α as follows:

$$C^{k,\ell} = \bigoplus_{I \in S^{k+1}} L^p_a\big(U^{\ell+1}_\alpha \cap V^{\ell+1}_I\big) \quad \text{for } k, \ell \ge 0$$

where we use the abbreviation $V_I = V_{i_0,\dots,i_k} = \bigcap_{j=0}^k V_{i_j}$. The vertical coboundary $d_{k,\ell}$ is just the usual coboundary δ_ℓ as in Sect. 4, acting on each $L^p_a(U^{\ell+1}_\alpha \cap V_{I^{\ell+1}})$. The horizontal coboundary $c_{k,\ell}$ is the "Čech differential." More explicitly, if $f \in C^{k,\ell}$, then it has components f_I which are functions on $U^{\ell+1}_\alpha \cap V^{\ell+1}_I$ for each $(k+1)$-tuple I, and for any $k+2$ tuple $J = (j_0, \dots, j_{k+1})$, cf is defined on $U^{\ell+1}_\alpha \cap V^{\ell+1}_J$ by

$$(c_{k,\ell} f)_J = \sum_{i=0}^{k+1} (-1)^i f_{j_0,\dots,\hat{j}_i,\dots,j_{k+1}} \quad \text{restricted to } V^{\ell+1}_J.$$

It is not hard to check that the coboundaries commute, $c\delta = \delta c$. We augment the complex on the left with the column (chain complex) $C^{-1,\ell} = L^p_a(U^{\ell+1}_\alpha)$ with the horizontal map $c_{-1,\ell}$ equal to restriction on each V_i and the vertical map the usual coboundary. We augment the complex on the bottom with the chain complex $C^{*,-1}$, which is the Čech complex of the cover $\{V_i\}$. That is, an element $f \in C^{k,-1}$ is a function that assigns to each V_I a real number or equivalently $C^{k,-1} = \bigoplus_{I \in S^{k+1}} \mathbb{R} V_I$. The vertical maps are just inclusions into $C^{*,0}$, and the horizontal maps are the Čech differential as defined above.

Remark 11 We can similarly define the Čech–Alexander bicomplex, the Čech-continuous Alexander bicomplex, and the Čech-smooth Alexander bicomplex (when X is a smooth Riemannian manifold) by replacing L^p_a everywhere in the above complex with F_a, C_a, and C^∞_a respectively.

Remark 12 The cohomology spaces of $C^{*,-1}$ are finite dimensional since the cover $\{V_i\}$ is finite. This is called the Čech cohomology of the cover, and it is the same as the simplicial cohomology of the simplicial complex that is the nerve of the cover $\{V_i\}$.

We will use the above complex to show, under some conditions, that H^ℓ_{α,L^p}, H^ℓ_α, and $H^\ell_{\alpha,\mathrm{cont}}$ are isomorphic to the Čech cohomology of an appropriate finite open cover of X and thus finite dimensional.

Theorem 7 *Let $\{V_i\}_{i\in S}$ be a finite cover of X by Borel sets as above, and assume that $\{V_i^{K+1}\}_{i\in S}$ is a cover for U_α^{K+1} for some $K \geq 0$. Assume also that the first $K+1$ columns of the corresponding Čech-L^p-Alexander complex are exact up to the $K+1$ term. Then H^ℓ_{α,L^p} is isomorphic to $H^\ell(C^{*,-1})$ for $\ell \leq K$ and is thus finite dimensional. Also, H^{K+1}_{α,L^p} is isomorphic to a subspace of $H^{K+1}(C^{*,-1})$. If $\{V_i\}_{i\in S}$ is an open cover, then the same conclusion holds for H^ℓ_α, $H^\ell_{\alpha,\mathrm{cont}}$, and $H^\ell_{\alpha,\mathrm{smooth}}$ (when X is a smooth Riemannian manifold), and hence all are isomorphic to each other. Those isomorphisms are induced by the natural inclusion maps of smooth functions into continuous functions into L^q-functions into L^p-functions ($q \geq p$) into arbitrary real-valued functions.*

Proof In light of Corollary 3, it suffices to show that the first K rows of the bicomplex are exact. Indeed, we are computing the sheaf cohomology of U_α^{k+1} for a flabby sheaf (the sheaf of smooth or continuous or L^p or arbitrary functions) which vanishes. We write out the details: Note that for $\ell \leq K$, $\{V_i^{\ell+1}\}$ covers $U_\alpha^{\ell+1}$ and therefore $c_{-1,\ell}\colon L^p_a(U_\alpha^{\ell+1}) \to \bigoplus_{i\in S} L^p_a(U_\alpha^{\ell+1} \cap V_i^{\ell+1})$ is injective (as $c_{-1,\ell}$ is a restriction); therefore, we have exactness at the first term. In general, we construct a chain contraction h on the ℓth row. Let $\{\phi_i\}$ be a measurable partition of unity for $U_\alpha^{\ell+1}$ subordinate to the cover $\{U_\alpha^{\ell+1} \cap V_i^{\ell+1}\}$ (and thus support $\phi_i \subset U_\alpha^{\ell+1} \cap V_i^{\ell+1}$ and $\sum_i \phi_i(x) = 1$ for all x). Then define

$$h\colon \bigoplus_{I\in S^{k+1}} L^p_a\bigl(U_\alpha^{\ell+1} \cap V_I^{\ell+1}\bigr) \to \bigoplus_{I\in S^k} L^p_a\bigl(U_\alpha^{\ell+1} \cap V_I^{\ell+1}\bigr)$$

for each k by $(hf)_{i_0,\dots,i_{k-1}} = \sum_{j\in S} \phi_j f_{j,i_0,\dots,i_{k-1}}$. We show that h is a chain contraction, that is $ch + hc = \mathrm{Id}$:

$$\bigl(c(hf)\bigr)_{i_0,\dots,i_k-1} = \sum_{n=0}^{k-1} (-1)^n (hf)_{i_0,\dots,\hat{i}_n,\dots,i_{k-1}} = \sum_{j,n} (-1)^n \phi_j f_{j,i_0,\dots,\hat{i}_n,\dots,i_{k-1}}.$$

Now,

$$\begin{aligned}
\bigl(h(cf)\bigr)_{i_0,\dots,i_k-1} &= \sum_{j\in S} \phi_j (cf)_{j,i_0,\dots,i_{k-1}} \\
&= \sum_j \phi_j \left(f_{i_0,\dots,i_{k-1}} - \sum_n^{k-1} (-1)^n f_{j,i_0,\dots,\hat{i}_n,\dots,i_{k-1}} \right) \\
&= f_{i_0,\dots,i_{k-1}} - \bigl(c(hf)\bigr)_{i_0,\dots,i_{k-1}}.
\end{aligned}$$

Thus h is a chain contraction for the ℓth row, proving exactness (note that exactness follows, since if $cf = 0$ then from above $c(hf) = f$). If $\{V_i\}$ is an open cover, then the partition of unity $\{\phi_i\}$ can be chosen to be continuous, or even smooth in the case where X is a smooth Riemannian manifold. Then h as defined above is a chain contraction on the corresponding complexes with L_a^p replaced by F_a, C_a, or C_a^∞.

Observe that the inclusions $C^\infty \hookrightarrow C^0 \hookrightarrow L^q \hookrightarrow L^p \hookrightarrow F$ (where F stands for arbitrary real-valued functions) extend to inclusions of the augmented bicomplexes, whose restriction to the Čech column $C^{*,-1}$ is the identity. As the identity clearly induces an isomorphism in cohomology, and the inclusion of this augmented bottom row into the (non-augmented) bicomplex also does, then by naturality the various inclusions of the bicomplexes induce isomorphisms in cohomology. The same argument applied backwards to the inclusions of the Alexander–Spanier–Čech rows into the bicomplexes shows that the inclusions of the smaller function spaces into the larger function spaces induce isomorphisms in α-cohomology.

This finishes the proof of the theorem. □

We can use Theorem 7 to prove finite dimensionality of the cohomologies in general, for $\ell = 0$ and 1.

Theorem 8 *For any compact X and any $\alpha > 0$, H^ℓ_{α,L^p}, H^ℓ_α, $H^\ell_{\alpha,\text{cont}}$, and $H^\ell_{\alpha,\text{smooth}}$ (X a smooth manifold) are finite dimensional and are isomorphic, for $\ell = 0, 1$.*

Let $\{V_i\}$ be a covering of X by open balls of radius $\alpha/3$. Then the first row ($\ell = 0$) of the Čech-L^p-Alexander complex is exact from the proof of Theorem 7 (taking $K = 0$). It suffices to show that the columns are exact. Note that $V_I^{\ell+1} \subset U_\alpha^{\ell+1}$ trivially for each ℓ and $I \in S^{k+1}$ because $\operatorname{diam}(V_I) < \alpha$. For k fixed, and $I \in S^{k+1}$ we define $g\colon L_a^p(V_I^{\ell+1}) \to L_a^p(V_I^\ell)$ by

$$gf(x_0, \dots, x_{\ell-1}) = \frac{1}{\mu(V_I)} \int_{V_I} f(t, x_0, \dots, x_{\ell-1})\, \mathrm{d}\mu(t).$$

We check that g defines a chain contraction:

$$\begin{aligned}\delta(gf)(x_0, \dots, x_\ell) &= \sum_i (-1)^i (gf)\bigl(x_0, \dots, \hat{x}_i, \dots, x_\ell\bigr) \\ &= \sum_i (-1)^i \frac{1}{\mu(V_I)} \int_{V_I} f\bigl(t, x_0, \dots, \hat{x}_i, \dots, x_\ell\bigr)\, \mathrm{d}\mu(t).\end{aligned}$$

But,

$$\begin{aligned}&g(\delta f)(x_0, \dots, x_\ell) \\ &\quad= \frac{1}{\mu(V_I)} \int_{V_I} \delta f(t, x_0, \dots, x_\ell)\, \mathrm{d}\mu(t) \\ &\quad= \frac{1}{\mu(V_I)} \left(\int_{V_I} f\biggl((x_0, \dots, x_\ell)\, \mathrm{d}\mu(t) - \sum_i (-1)^i \int_{V_I} f\bigl(t, \dots, \hat{x}_i, \dots, x_\ell\bigr)\, \mathrm{d}\mu)t \biggr) \right) \\ &\quad= f(x_0, \dots, x_\ell) - \delta(gf)(x_0, \dots, x_\ell).\end{aligned}$$

Thus g defines a chain contraction on the kth column, and the columns are exact. For the corresponding Alexander, continuous and smooth, bicomplexes, a chain contraction can be defined by fixing for each V_I, $I \in S^{k+1}$ a point $p \in V_I$ and setting $gf(x_0, \dots, x_{\ell-1}) = f(p, x_0, \dots, x_{\ell-1})$. This is easily verified to be a chain contraction, finishing the proof of the theorem.

Recall that for $x = (x_0, \dots, x_{\ell-1}) \in U_\alpha^\ell$ we define the slice $S_x = \{t \in X : (t, x_0, \dots, x_{\ell-1}) \in U_\alpha^{\ell+1}\}$. We consider the following hypothesis on X, $\alpha > 0$, and non-negative integer K:

Hypothesis $(*)$ There exists a $\delta > 0$ such that, whenever $V = \bigcap_i V_i$ is a non-empty intersection of finitely many open balls of radius $\alpha + \delta$, then there is a Borel set W of positive measure such that, for each $\ell \leq K + 1$,

$$W \subset V \cap \Bigg(\bigcap_{x \in U_\alpha^\ell \cap V^\ell} S_x \Bigg).$$

Theorem 9 *Assume that* X, $\alpha > 0$, *and* K *satisfy Hypothesis* $(*)$. *Then, for* $\ell \leq K$, H_{α,L^p}^ℓ, H_α^ℓ, $H_{\alpha,\mathrm{cont}}^\ell$, *and* $H_{\alpha,\mathrm{smooth}}^\ell$ (*when* X *is a smooth Riemannian manifold*) *are all finite dimensional, and are isomorphic to the Čech cohomology of some finite covering of* X *by open balls of radius* $\alpha + \delta$. *Furthermore, the Hodge theorem for* X *at scale* α *holds* (*Theorem* 4 *of Sect*. 4).

Proof Let $\{V_i\}$, $i \in S$ be a finite open cover of X by balls of radius $\alpha + \delta$ such that $\{V_i^{K+1}\}$ is a covering for U_α^{K+1}. This can always be done since U_α^{K+1} is compact. We first consider the case of the Čech-L^p-Alexander bicomplex corresponding to the cover. By Theorem 7, it suffices to show that there is a chain contraction of the columns up to the Kth term. For each $I \in S^{k+1}$ and $\ell \leq K + 1$, let W be the Borel set of positive measure assumed to exist in $(*)$ with V_I playing the role of V in $(*)$. Then we define $g\colon L_a^p(U_\alpha^{\ell+1} \cap V_I^{\ell+1}) \to L_a^p(U_\alpha^\ell \cap V_I^\ell)$ by

$$gf(x_0, \dots, x_{\ell-1}) = \frac{1}{\mu(W)} \int_W f(t, x_0, \dots, x_{\ell-1}) \, \mathrm{d}\mu(t).$$

The Hypothesis $(*)$ implies that g is well defined. The proof that g defines a chain contraction on the kth column (up to the Kth term) is identical to the one in the proof of Theorem 8. As in the proof of Theorem 8, the chain contraction for the case when L_a^p is replaced by F_a, C_a, and C_a^∞ can be taken to be $gf(x_0, \dots, x_{\ell-1}) = f(p, x_0, \dots, x_{\ell-1})$ for some fixed $p \in W$. Note that, in these cases, we don't require that $\mu(W) > 0$, only that $W \neq \emptyset$. □

Remark 13 If X satisfies certain local conditions as in Wilder [38], then the Čech cohomology of the cover, for small α, is isomorphic to the Čech cohomology of X.

Our next goal is to give somewhat readily verifiable conditions on X and α that will imply $(*)$. This involves the notion of the midpoint and radius of a closed set in X.

Let $\Lambda \subset X$ be closed. We define the radius $r(\Lambda)$ by $r(\Lambda) = \inf\{\beta : \bigcap_{x\in\Lambda} B_\beta(x) \neq \emptyset\}$ where $B_\beta(x)$ denotes the closed ball of radius β centered at x.

Proposition 18 *$\bigcap_{x\in\Lambda} B_{r(\Lambda)}(x) \neq \emptyset$. Furthermore, if $p \in \bigcap_{x\in\Lambda} B_{r(\Lambda)}(x)$, then $\Lambda \subset B_{r(\Lambda)}(p)$, and if $\Lambda \subset B_\beta(q)$ for some $q \in \Lambda$, then $r(\Lambda) \leq \beta$.*

Such a p is called a midpoint of Λ.

Proof Let $J = \{\beta \in \mathbb{R}\colon \bigcap_{x\in\Lambda} B_\beta(x) \neq \emptyset\}$. For $\beta \in J$ define $R_\beta = \bigcap_{x\in\Lambda} B_\beta(x)$. Note that if $\beta \in J$ and $\beta < \beta'$, then $\beta' \in J$, and $R_\beta \subset R_{\beta'}$. R_β is compact, and therefore $\bigcap_{\beta\in J} R_\beta \neq \emptyset$. Let $p \in \bigcap_{\beta\in J} R_\beta$. Then, for $x \in \Lambda$, $p \in B_\beta(x)$ for all $\beta \in J$ and so $d(p,x) \leq \beta$. Taking the infimum of this over $\beta \in J$ yields $d(p,x) \leq r(\Lambda)$ or $p \in R_{r(\Lambda)}$, proving the first assertion of the proposition. Now, if $x \in \Lambda$ then $p \in B_{r(\Lambda)}(x)$, which implies $x \in B_{r(\Lambda)}(p)$ and thus $\Lambda \subset B_{r(\Lambda)}(p)$. Now suppose that $\Lambda \subset B_\beta(q)$ for some $q \in \Lambda$. Then for every $x \in \Lambda$, $q \in B_\beta(x)$ and thus $\bigcap_{x\in\Lambda} B_\beta(x) \neq \emptyset$ which implies $\beta \geq r(\Lambda)$, finishing the proof. □

We define $\mathcal{K}(X) = \{\Lambda \subset X : \Lambda \text{ is compact}\}$, and we endow $\mathcal{K}(X)$ with the Hausdorff metric $D(A,B) = \max\{\sup_{t\in B} d(t,A), \sup_{t\in A} d(t,B)\}$. We also define, for $x = (x_0,\ldots,x_\ell) \in U_\alpha^{\ell+1}$, the witness set of x by $w_\alpha(x) = \bigcap_i B_\alpha(x_i)$ (we are suppressing the dependence of w_α on ℓ). Thus $w_\alpha\colon U_\alpha^{\ell+1} \to \mathcal{K}(X)$.

Theorem 10 *Let X be compact, and $\alpha > 0$. Suppose that $w_\alpha\colon U_\alpha^{\ell+1} \to \mathcal{K}(X)$ is continuous for $\ell \leq K+1$, and suppose there exists $\delta_0 > 0$ such that whenever $\Lambda = \bigcap_{i=0}^k B_i$ is a finite intersection of closed balls of radius $\alpha+\delta$, $\delta \in (0,\delta_0]$, then $r(\Lambda) \leq \alpha+\delta$. Then Hypothesis $(*)$ holds.*

The proof will follow from the next proposition.

Proposition 19 *Under the hypotheses of Theorem 10, given $\epsilon > 0$, there exists $\delta > 0$, $\delta \leq \delta_0$ such that for all $\beta \in [\alpha,\alpha+\delta]$ we have $D(w_\alpha(\sigma), w_\beta(\sigma)) \leq \epsilon$ for all simplices $\sigma \in U_\alpha^{\ell+1} \subset U_\beta^{\ell+1}$.*

Proof of Theorem 10 Fix $\epsilon < \alpha$, and let $\delta > 0$ be as in Proposition 19. Let $\{V_i\}$ be a finite collection of open balls of radius $\alpha+\delta$ such that $\bigcap_i V_i \neq \emptyset$, and let $\{B_i\}$ be the corresponding collection of closed balls of radius $\alpha+\delta$. Define Λ to be the closure of $\bigcap_i V_i$ and thus

$$\Lambda = \overline{\bigcap_i V_i} \subset \bigcap_i \overline{V_i} \subset \bigcap_i B_i.$$

Let p be a midpoint of Λ. We will show that $d(p, w_\alpha(\sigma)) \leq \epsilon$ for any $\sigma = (x_0,\ldots,x_{\ell+1}) \in \Lambda^{\ell+1}$. We have

$$p \in \bigcap_{x\in\Lambda} B_{r(\Lambda)}(x) \subset \bigcap_{i=0}^{\ell+1} B_{r(\Lambda)}(x_i) = w_{r(\Lambda)}(\sigma) \subset w_{\alpha+\delta}(\sigma)$$

since $r(\Lambda) \leq \alpha + \delta$. But $D(w_\alpha(\sigma), w_{\alpha+\delta}(\sigma)) \leq \epsilon$ from Proposition 19, and so $d(p, w_\alpha(\sigma)) \leq \epsilon$. In particular, there exists $q \in w_\alpha(\sigma)$ with $d(p,q) \leq \epsilon$. Now, if $x \in B_{\alpha-\epsilon}(p) \cap \Lambda$, then $d(x,q) \leq d(x,p) + d(p,q) \leq \alpha - \epsilon + \epsilon = \alpha$. Therefore, $(x, x_0, \ldots, x_\ell) \in U_\alpha^{\ell+2}$ and so $x \in S_\sigma \cap \Lambda$. Thus $B_{\alpha-\epsilon}(p) \cap \Lambda \subset \bigcap_{\sigma \in U^{\ell+1} \cap \Lambda^{\ell+1}} S_\sigma$. Let $B'_s(p)$ denote the open ball of radius s, and let $V = \bigcap_i V_i$. Then define $W = B'_{\alpha-\epsilon}(p) \cap V$. Then W is a non-empty open set (since $p \in \overline{V}$), $\mu(W) > 0$, and $W \subset \bigcap_{\sigma \in U_\alpha^{\ell+1} \cap V^{\ell+1}} S_\alpha$. Thus Hypothesis $(*)$ is satisfied, finishing the proof of Theorem 10. □

Proof of Proposition 19 Let $\epsilon > 0$. Note that for $\beta \geq \alpha$ and $\sigma \in U_\alpha^{\ell+1}$, $w_\alpha(\sigma) \subset w_\beta(\sigma)$. It thus suffices to show that there exists $\delta > 0$ such that

$$\sup_{x \in w_\beta(\sigma)} d\bigl(x, w_\alpha(\sigma)\bigr) \leq \epsilon \quad \text{for all } \beta \in [\alpha, \alpha + \delta].$$

Suppose that this is not the case. Then there exists $\beta_j \downarrow \alpha$ and $\sigma_j \in U_\alpha^{\ell+1}$ such that

$$\sup_{x \in w_{\beta_j}(\sigma)} d\bigl(x, w_\alpha(\sigma_j)\bigr) > \epsilon$$

and thus there exists $x_n \in w_{\beta_n}(\sigma_n)$ with $d(x_n, w_\alpha(\sigma_n)) \geq \epsilon$. Let $\sigma_n = (y_0^n, \ldots, y_\ell^n)$. Thus $d(x_n, y_i^n) \leq \beta_n$ for all i. By compactness, after taking a subsequence, we can assume $\sigma_n \to \sigma = (y_0, \ldots, y_\ell)$ and $x_n \to x$. Thus $d(x, y_i) \leq \alpha$ for all i and $\sigma \in U_\alpha^{\ell+1}$, and $x \in w_\alpha(\sigma)$. However, by continuity of w_α, $w_\alpha(\sigma_n) \to w_\alpha(\sigma)$, which implies $d(x, w_\alpha(\sigma)) \geq \epsilon$ (since $d(x_n, w_\alpha(\sigma_n)) \geq \epsilon$), a contradiction. We thus finish the proof of the proposition. □

We now turn to the case where X is a compact Riemannian manifold of dimension n, with Riemannian metric g. We will always assume that the metric d on X is induced from g. Recall that a set $\Lambda \subset X$ is strongly convex if, given $p, q \in \Lambda$, the length-minimizing geodesic from p to q is unique, and lies in Λ. The strong convexity radius at a point $x \in X$ is defined by $\rho(x) = \sup\{r : B_r(x) : \text{is strongly convex}\}$. The strong convexity radius of X is defined as $\rho(X) = \inf\{\rho(x) : x \in X\}$. It is a basic fact of Riemannian geometry that, for X compact, $\rho(X) > 0$. Thus for any $x \in X$ and $r < \rho(X)$, $B_r(x)$ is strongly convex.

Theorem 11 *Assume as above that X is a compact Riemannian manifold. Let $k > 0$ be an upper bound for the sectional curvatures of X, and let $\alpha < \min\{\rho(X), \frac{\pi}{2\sqrt{k}}\}$. Then Hypothesis $(*)$ holds.*

Corollary 4 *In the situation of Theorem 11, the cohomology groups H^ℓ_{α,L^p}, H^ℓ_α, $H^\ell_{\alpha,\mathrm{cont}}$, and $H^\ell_{\alpha,\mathrm{smooth}}$ are finite dimensional and isomorphic to each other and to the ordinary cohomology of X with real coefficients (and the natural inclusions induce the isomorphisms). Moreover, Hodge theory for X at scale α holds.*

Proof of Theorem 11 From Theorem 10, it suffices to prove the following propositions.

Proposition 20 *Let $\alpha < \min\{\rho(X), \frac{\pi}{2\sqrt{k}}\}$. Then $w_\alpha\colon U_\alpha^{\ell+1} \to \mathcal{K}(X)$ is continuous for $\ell \leq K$.*

Proposition 21 *Let $\delta > 0$ such that $\alpha + \delta < \min\{\rho(X), \frac{\pi}{2\sqrt{k}}\}$. Whenever Λ is a closed, convex set in some $B_{\alpha+\delta}(z)$, then $r(\Lambda) \leq \alpha + \delta$.*

Of course, the conclusion of Proposition 21 is stronger than the second hypothesis of Theorem 10, since the finite intersection of balls of radius $\alpha + \delta$ is convex and $\alpha + \delta < \rho(X)$.

Proof of Proposition 20 We start with

Claim 1 Let $\sigma = (x_0, \dots, x_\ell) \in U_\alpha^{\ell+1}$, and suppose that $p, q \in w_\alpha(\sigma)$ and that x is on the minimizing geodesic from p to q (but not equal to p or q). Then $B_\epsilon(x) \subset w_\alpha(\sigma)$ for some $\epsilon > 0$.

Proof of Claim For points r, s, t in a strongly convex neighborhood in X, we define $\angle rst$ to be the angle that the minimizing geodesic from s to r makes with the minimizing geodesic from s to t. Let γ be the geodesic from p to q, and for fixed i let ϕ be the geodesic from x to x_i. Now, the angles that ϕ makes with γ at x satisfy $\angle pxx_i + \angle x_i xq = \pi$, and therefore one of these angles is greater than or equal to $\pi/2$. Assume, without loss of generality, that $\theta = \angle pxx_i \geq \pi/2$. Let $c = d(x, x_i)$, $r = d(p, x)$, and $d = d(p, x_i) \leq \alpha$ (since $p \in w_\alpha(\sigma)$). Now consider a geodesic triangle in the sphere of curvature k with vertices p', x', and x_i' such that

$$d\bigl(p', x'\bigr) = d(p, x) = r, \quad d\bigl(x', x_i'\bigr) = d(x, x_i) = c \quad \text{and} \quad \angle p'x'x_i' = \theta,$$

and let $d' = d(p', x_i')$. Then, the hypotheses on α imply that the Rauch comparison theorem (see e.g. do Carmo [12]) holds, and we can conclude that $d' \leq d$. However, with $\theta \geq \pi/2$, it follows that on a sphere, where p', x', x_i' are inside a ball of radius less than the strong convexity radius, that $c' < d'$. Therefore, we have $c = c' < d' \leq d \leq \alpha$ and there is an $\epsilon > 0$ such that $y \in B_\epsilon(x)$ implies $d(y, x_i) \leq \alpha$. Taking the smallest $\epsilon > 0$ so that this is true for each $i = 0, \dots, \ell$ finishes the proof of the claim. □

Corollary 5 (Corollary of Claim) *For $\sigma \in U_\alpha^{\ell+1}$, either $w_\alpha(\sigma)$ consists of a single point, or every point of $w_\alpha(\sigma)$ is an interior point or the limit of interior points.*

Now suppose that $\sigma_j \in U_\alpha^{\ell+1}$ and $\sigma_j \to \sigma$ in $U_\alpha^{\ell+1}$. We must show $w_\alpha(\sigma_j) \to w_\alpha(\sigma)$, that is,

(a) $\sup_{x \in w_\alpha(\sigma_j)} d(x, w_\alpha(\sigma)) \to 0$,
(b) $\sup_{x \in w_\alpha(\sigma)} d(x, w_\alpha(\sigma_j)) \to 0$.

In fact, (a) holds for any metric space and any $\alpha > 0$. Suppose that (a) were not true. Then there exists a subsequence (still denoted by σ_j) and $\eta > 0$ such that

$$\sup_{x \in w_\alpha(\sigma_j)} d\bigl(x, w_\alpha(\sigma)\bigr) \geq \eta$$

and therefore there exists $y_j \in w_\alpha(\sigma_j)$ with $d(y_j, w_\alpha(\sigma)) \geq \eta/2$. After taking another subsequence, we can assume $y_j \to y$. But if $\sigma_j = (x_0^j, \dots, x_\ell^j)$ and $\sigma = (x_0, \dots, x_\ell)$, then $d(y_j, x_i^j) \leq \alpha$, which implies $d(y, x_i) \leq \alpha$ for each i and thus $y \in w_\alpha(\sigma)$. But this is impossible given $d(y_j, w_\alpha(\sigma)) \geq \eta/2$.

We use the corollary to Claim 1 to establish (b). First, suppose that $w_\alpha(\sigma)$ consists of a single point p. We show that $d(p, w_\alpha(\sigma_j)) \to 0$. Let $p_j \in w_\alpha(\sigma_j)$ such that $d(p, p_j) = d(p, w_\alpha(\sigma_j))$. If $d(p, p_j)$ does not converge to 0, then, after taking a subsequence, we can assume $d(p, p_j) \geq \eta > 0$ for some η. But after taking a further subsequence, we can also assume $p_j \to y$ for some y. However, as in the argument above, it is easy to see that $y \in w_\alpha(\sigma)$ and therefore $y = p$, a contradiction, and so (b) holds in this case.

Now suppose that every point in $w_\alpha(\sigma)$ is either an interior point or the limit of interior points. If (b) did not hold, there would be a subsequence (still denoted by σ_j) such that

$$\sup_{x \in w_\alpha(\sigma)} d\big(x, w_\alpha(\sigma_j)\big) \geq \eta > 0$$

and thus there exists $p_j \in w_\alpha(\sigma)$ such that $d(p_j, w_\alpha(\sigma_j)) \geq \eta/2$. After taking another subsequence, we can assume $p_j \to p$ and $p \in w_\alpha(\sigma)$, and, for j sufficiently large, $d(p, w_\alpha(\sigma_j)) \geq \eta/4$. If p is an interior point of $w_\alpha(\sigma)$, then $d(p, x_i) < \alpha$ for $i = 0, \dots, \ell$. But then, for all j sufficiently large, $d(p, x_i^j) \leq \alpha$ for each i. But this implies $p \in w_\alpha(\sigma_j)$, a contradiction. If p is not an interior point, then p is a limit point of interior points q_m. But then, from above, $q_m \in w_\alpha(\sigma_{j_m})$ for j_m large, which implies $d(p, w_\alpha(\sigma_{j_m})) \to 0$, a contradiction. Thus we have established (b) and finished the proof of Proposition 20. □

Proof of Proposition 21 Let δ be such that $\alpha + \delta < \min\{\rho(X), \frac{\pi}{2\sqrt{k}}\}$, and let Λ be any closed convex set in $B_{\alpha+\delta}(z)$. We will show $r(\Lambda) \leq \alpha + \delta$. If $z \in \Lambda$, we are done, for then $\Lambda \subset B_{\alpha+\delta}(z)$ implies $r(\Lambda) \leq \alpha + \delta$ by Proposition 18. If $z \notin \Lambda$ let $z_0 \in \Lambda$ such that $d(z, z_0) = d(z, \Lambda)$ (the closest point in Λ to z). Now let $y_0 \in \Lambda$ such that $d(z_0, y_0) = \max_{y \in \Lambda} d(z_0, y)$. Let γ be the minimizing geodesic from z_0 to y_0, and ϕ the minimizing geodesic from z_0 to z. Since Λ is convex γ lies on Λ. If θ is the angle between γ and ϕ, $\theta = \angle z z_0 y_0$, then, by the first variation formula of arc length [12], $\theta \geq \pi/2$; otherwise the distance from z to points on γ would be initially decreasing. Let $c = d(z, z_0)$, $d = d(z_0, y_0)$, and $R = d(z, y_0)$. In the sphere of constant curvature k, let z', z_0', y_0' be the vertices of a geodesic triangle such that $d(z', z_0') = d(z, z_0) = c$, $d(z_0', y_0') = d(z_0, y_0) = d$, and $\angle z' z_0' y_0' = \theta$. Let $R' = d(z', y_0')$. Then by the Rauch comparison theorem, $R' \leq R$. However, it can easily be checked that on the sphere of curvature k it holds that $d' < R'$, since z', z_0', and y_0' are all within a strongly convex ball and $\theta \geq \pi/2$. Therefore, $d = d' < R' \leq R \leq \alpha + \delta$. Thus $\Lambda \subset B_{\alpha+\delta}(z_0)$ with $z_0 \in \Lambda$, which implies $r(\Lambda) \leq \alpha + \delta$ by Proposition 18. This finishes the proof of Proposition 21. □

The proof of Theorem 11 is finished. □

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Appendix A: An Example Whose Codifferential Does Not Have Closed Range

For convenience, we fix the scale $\alpha = 10$; any large enough value is suitable for our construction. We consider a compact metric measure space X of the following type.

As a metric space, it has three cluster points $x_\infty, y_\infty, z_\infty$ and discrete points $(x_n)_{n\in\mathbb{N}}$, $(y_n)_{n\in\mathbb{N}}$, $(z_n)_{n\in\mathbb{N}}$ converging to x_∞, y_∞, z_∞, respectively.

We set $K_x := \{x_k : k \in \mathbb{N} \cup \{\infty\}\}$, $K_y := \{y_k : k \in \mathbb{N} \cup \{\infty\}\}$, and $K_z := \{z_k : k \in \mathbb{N} \cup \{\infty\}\}$. Then X is the disjoint union of the three "clusters" K_x, K_y, K_z.

We require

$$d(x_\infty, y_\infty) = d(y_\infty, z_\infty) = \alpha \quad \text{and} \quad d(x_\infty, z_\infty) = 2\alpha.$$

We also require

$$d(x_k, y_n) < \alpha \quad \text{precisely when } n \in \{2k, 2k+1, 2k+2\},\ n \in \mathbb{N},\ k \in \mathbb{N} \cup \{\infty\},$$
$$d(z_k, y_n) < \alpha \quad \text{precisely when } n \in \{2k-1, 2k, 2k+1\},\ n \in \mathbb{N},\ k \in \mathbb{N} \cup \{\infty\}.$$

We finally require that the clusters K_x, K_y, K_z have diameter $< \alpha$, and that the distance between K_x and K_y as well as between K_z and K_y is $\geq \alpha$.

This configuration can easily be found in an infinite-dimensional Banach space such as $l^1(\mathbb{N})$. For example, in $l^1(\mathbb{N})$ consider the canonical basis vectors $e_0, e_1, \dots$, and set

$$x_\infty := -\alpha e_0, \qquad y_\infty := 0, \qquad z_\infty := \alpha e_0.$$

Define then

$$\begin{aligned} x_k &:= -\left(\alpha + \frac{1}{10k} - \frac{1}{2k} - \frac{1}{2k+1} - \frac{1}{2k+2}\right) e_0 \\ &\quad + \frac{1}{2k} e_{2k} + \frac{1}{2k+1} e_{2k+1} + \frac{1}{2k+2} e_{2k+2}, \\ y_k &:= \frac{1}{k} e_k, \\ z_k &:= \left(\alpha + \frac{1}{10k} - \frac{1}{2k-1} - \frac{1}{2k} - \frac{1}{2k+1}\right) e_0 \\ &\quad + \frac{1}{2k-1} e_{2k-1} + \frac{1}{2k} e_{2k} + \frac{1}{2k+1} e_{2k+1}. \end{aligned}$$

We can now give a very precise description of the open α-neighborhood U_d of the diagonal in X^d. It contains all the tuples whose entries

- all belong to $K_x \cup \{y_{2k}, y_{2k+1}, y_{2k+2}\}$ for some $k \in \mathbb{N}$; or
- all belong to $K_y \cup \{x_k, x_{k+1}, z_{k+1}\}$ for some $k \in \mathbb{N}$; or
- all belong to $K_y \cup \{x_k, z_k, z_{k+1}\}$ for some $k \in \mathbb{N}$; or
- all belong to $K_z \cup \{y_{2k-1}, y_{2k}, y_{2k+1}\}$ for some $k \in \mathbb{N}$.

For the closed α-neighborhood, one has to add tuples whose entries all belong to $K_y \cup \{x_\infty\}$ or to $K_y \cup \{z_\infty\}$.

This follows by looking at the possible intersections of α-balls centered at our points.

In this topology, every set is a Borel set. We give $x_\infty, y_\infty, z_\infty$ measure zero. When considering L^2-functions on the U_d, we can therefore ignore all tuples containing one of these points.

We specify $\mu(x_n) := \mu(z_n) := 2^{-n}$ and $\mu(y_n) := 2^{-2^n}$; in this way, the total mass is finite.

We form the L^2-Alexander chain complex at scale α and complement it by $C^{-1} := \mathbb{R}^3 = \mathbb{R}x \oplus \mathbb{R}y \oplus \mathbb{R}z$; the three summands standing for the three clusters. The differential $c^{-1}\colon C^{-1} \to L^2(X)$ is defined by $(\alpha, \beta, \gamma) \mapsto \alpha\chi_{K_x} + \beta\chi_{K_y} + \gamma\chi_{K_z}$, where χ_{K_j} denotes the characteristic function of the cluster K_j.

Restriction to functions supported on K_x^{*+1} defines a bounded surjective cochain map from the L^2-Alexander complex at scale α for X to the one for K_x. Note that $\operatorname{diam}(K_x) < \alpha$; consequently, its Alexander complex at scale α is contractible.

Therefore, looking at the long exact sequence associated to a short exact sequence of Banach cochain complexes, the cohomology of X is isomorphic (as topological vector spaces) to the cohomology of the kernel of this projection, i.e. to the cohomology of the Alexander complex of functions vanishing on K_x^{k+1}.

This can be done two more times (looking at the kernels of the restrictions to K_y and K_z), so that finally we arrive at the chain complex C^* of L^2-functions on X^{k+1} vanishing at $K_x^{k+1} \cup K_y^{k+1} \cup K_z^{k+1}$.

In particular, $C^{-1} = 0$ and $C^0 = 0$.

We now construct a sequence in C^1 whose differentials converge in C^2, but such that the limit point does not lie in the image of c^1.

Following the above discussion, the α-neighborhood of the diagonal in X^2 contains in particular the "1-simplices" $v_k := (x_k, z_k)$ and $v'_k := (x_k, z_{k+1})$ and their "inverses" $\overline{v_k} : -(z_k, x_k)$, $\overline{v'_k} := (z_{k+1}, x_k)$.

We define $f_\lambda \in C^1$ with $f_\lambda(\overline{v_k}) := f_\lambda(\overline{v'_k}) := -f_\lambda(v_k)$, $f_\lambda(v'_k) := f_\lambda(v_k) := b_{\lambda,k} := 2^{\lambda k}$, and $f_n(v) = 0$ for all other simplices.

Note that, for $0 < \lambda < 1$,

$$\begin{aligned}\int_{X^2} |f|^2 &= \sum_{k=1}^{\infty} \left|f(v_k)\right|^2 \mu(v_k) + \left|f\left(v'_k\right)\right|^2 \mu\left(v'_k\right) + \left|f\left(\overline{v_k}\right)\right|^2 \mu\left(v_k\right) + \left|f\left(\overline{v'_k}\right)\right|^2 \mu\left(v'_k\right) \\ &= \sum_{k=1}^{\infty} 2 \cdot \left(2^{2\lambda k} 2^{-2k} + 2^{2\lambda k} 2^{-2k-1}\right)\end{aligned}$$

which is a finite sum, whereas for $\lambda = 1$ the sum is no longer finite.

Let us now consider $g_\lambda := c^1(f_\lambda)$. It vanishes on all "2-simplices" (points in X^2) except those of the form

- $d_k := (x_k, z_k, z_{k+1})$ and more generally $d_k^\sigma := \sigma(x_k, z_k, z_{k+1})$ for $\sigma \in S_3$ a permutation of three entries;
- $e_k := (x_{k-1}, z_k, x_k)$ or more generally d_k^σ as before;
- on degenerate simplices of the form (x_k, z_k, x_k) etc. g vanishes because $f(x_k, z_k) = -f(z_k, x_k)$.

We obtain

$$g_\lambda(d_k) = -f\big(v_k'\big) + f(v_k) = 0,$$
$$g_\lambda(e_k) = f\big(\overline{v_k}\big) + f\big(v_{k-1}'\big) = -2^{\lambda k} + 2^{\lambda(k-1)} = 2^{\lambda k} \cdot \big(2^{-\lambda} - 1\big).$$

Similarly, $g_\lambda(d_k^\sigma) = 0$ and $g_\lambda(e_k^\sigma) = \operatorname{sign}(\sigma) g_\lambda(e_k)$.

We claim that g_1, defined with these formulas, belongs to $L^2(X^3)$ and is the limit in L^2 of g_λ as λ tends to 1.

To see this, we simply compute the L^2-norm

$$\begin{aligned}\int_{X^3} |g_1 - g_\lambda|^2 &= 6\sum_{k=1}^{\infty} \big|2^{k-1} - 2^{\lambda k}\big(1 - 2^{-\lambda}\big)\big|^2 2^{1-3k} \\ &\le 6\Big(\sup_{k\in\mathbb{N}} 2^{-k/2}\big|2^{-1} - 2^{(\lambda-1)k}\big(1 - 2^{-\lambda}\big)\big|^2\Big) \cdot \sum_{k=1}^{\infty} 2^{1-k/2} \\ &\xrightarrow{\lambda\to 1} 0\end{aligned}$$

(the factor 6 comes from the six permutations of each non-degenerate simplex which each contribute equally).

The supremum tends to zero because each individual term does so even without the factor $2^{-k/2}$ and the sequence is bounded.

Now we study which properties the $f \in C^1$ with $c^1(f) = g_1$ must have.

Observe that for an arbitrary $f \in C^1$, $c^1 f(e_k^\sigma)$ is determined by $f(v_k)$, $f(\overline{v_k})$, $f(v_{k-1}')$, $f(\overline{v_{k-1}'})$ (as f vanishes on K_x).

If $c^1 f$ must vanish on degenerate simplices (and this is the case for g_1), then $f(v_k) = -f(\overline{v_k})$ and $f(v_k') = -f(\overline{v_k'})$.

$c^1 f(d_k^\sigma) = 0$ then implies that $f(v_k) = f(v_k')$.

It is now immediate from the formulas for $c^1 f(d_k)$ and $c^1 f(e_k)$ that the values of f at v_k, v_k' are determined by $c^1 f(d_k)$, $c^1 f(e_k)$ up to addition of a constant.

Finally, observe that (in the Alexander cochain complex without growth conditions) f_1 (which is not in L^2) satisfies $c^1(f_1) = g_1$.

As constant functions are in L^2, we observe that $f_1 + K$ is not in L^2 for any $K \in \mathbb{R}$, nor is any function f on X^2 which coincides with $f_1 + K$ on v_k, v_k', $\overline{v_k}$, $\overline{v_k'}$.

But these are the only candidates which could satisfy $c^1(f) = g_1$. It follows that g_1 is not in the image of c^1. On the other hand, we constructed it so that it is in the closure of the image. Therefore, the image is not closed.

A.1 A Modified Example Where Volumes of Open and Closed Balls Coincide

The example given has one drawback: Although at the chosen scale α open and closed balls coincide in volume (and even as sets, except for the balls around $x_\infty, y_\infty, z_\infty$), for other balls this is not the case—and necessarily so, as we construct a zero-dimensional object.

We modify our example as follows, by replacing each of the points x_k, y_k, z_k by a short interval: Inside $X \times [0,1]$, with l^1 metric (that is, $d_Y((x,t),(y,u)) = d_X(x,y) + |t-u|$), consider

$$Y = \bigcup_{k \in \mathbb{N} \cup \{\infty\}} \{x_k, y_k, z_k\} \times \big[0, 1/(12k)\big].$$

For convenience, let us write $I_{x,k}$ for the interval $\{x_k\} \times [0, 1/(12k)]$, and similarly for the y_k and z_k. On each of these intervals we then put the standard Lebesgue measure weighted by a suitable factor, so that $\mu_Y(I_{x,k}) = \mu(x_k)$, and similarly for the y_k and z_k.

Now, if two points x_k, y_n are at distance less than α in X, then they are at distance $< \alpha - 1/k$; the corresponding statement holds for all other pairs of points. On the other hand, because of our choice of metric, $d((x_k,t),(y_n,s)) \geq d(x_k,y_n)$ and again the corresponding statement holds for all other pairs of points in Y. It follows that the α-neighborhood of the diagonal in Y^k is the union of products of the corresponding intervals, and exactly those intervals show up where the corresponding tuple is contained in the 1-neighborhood of the diagonal in X^k.

It is now quite hard to explicitly compute the cohomology of the L^2-Alexander cochain complex at scale α.

However, we do have a projection $Y \to X$, namely the projection on the first coordinate. By the remark about the α-neighborhoods, this projection extends to a map from the α-neighborhoods of Y^k onto those of X^k, which is compatible with the projections onto the factors.

It follows that pullback of functions defines a bounded cochain map (in the reverse direction) between the L^2-Alexander cochain complexes at scale α. Note that this is an isometric embedding by our choice of the measures.

This cochain map has a one-sided inverse given by integration of a function on (the α-neighborhood of the diagonal in) Y^k over a product of intervals (divided by the measure of this product) and assigning this value to the corresponding tuple in X^k. By Cauchy–Schwarz, this is bounded with norm 1.

As pullback along projections commutes with the weighted integral we are using, one checks easily that this local integration map is also a cochain map for our L^2-Alexander complexes at scale α.

Consequently, the induced maps in cohomology give an isometric inclusion with inverse between the cohomology of X and of Y.

We have shown that in $H^2(X)$ there are non-zero classes of norm 0. Their images (under pullback) are non-zero classes (because of the retraction given by the integration map) of norm 0. Therefore, the cohomology of Y is non-Hausdorff, and the first codifferential does not have a closed image, either.

Appendix B: An Example of a Space with Infinite-Dimensional α-Scale Homology

This appendix was contributed by Anthony W. Baker, Mathematics and Computing Technology, The Boeing Company (e-mail: anthony.w.baker@boeing.com).

The work in the main body of this paper has inspired the question of the existence of a separable, compact metric space with infinite-dimensional α-scale homology. This appendix provides one such example and further shows the sensitivity of the homology to changes in α.

Let X be a separable, complete metric space with metric d, and $\alpha > 0$ a "scale." Define an associated (generally infinite) simplicial complex $C_{X,\alpha}$ to (X, d, α). Let $X^{\ell+1}$, for $\ell > 0$, be the $(\ell+1)$-fold Cartesian product, with metric denoted by $d, d : X^{\ell+1} \times X^{\ell+1} \to \mathbb{R}$ where $d(x; y) = \max_{i=0,\dots,\ell} d(x_i; y_i)$. Let

$$U_\alpha^{\ell+1}(X) = U_\alpha^{\ell+1} = \bigl\{x \in X^{\ell+1} : d\bigl(x; D^{\ell+1}\bigr) \le \alpha\bigr\},$$

where $D^{\ell+1} \subset X^{\ell+1}$ is the diagonal, so $D^{\ell+1} = \{t \in X : (t, \dots, t), \ell+1 \text{ times}\}$. Let $C_{X,\alpha} = \bigcup_{t=0}^{\infty} U_\alpha^{\ell+1}$. This has the structure of a simplicial complex whose ℓ simplices consist of points of $U_\alpha^{\ell+1}$.

The α-scale homology is that homology generated by the simplicial complex above.

The original exploration of example compact metric spaces resulted in low-dimensional α-scale homology groups. Missing from the results were any examples with infinite-dimensional homology groups. Also, examination of the first α-scale homology group was less promising for infinite-dimensional results; the examination resulted in the proof that the first homology group is always finite, as shown in Sect. 9.

The infinite-dimensional example in this paper was derived through several failed attempts to prove that the α-scale homology was finite. The difficulty was the inability to slightly perturb vertices and still have the perturbed object remain a simplex. This sensitivity is derived from the "equality" in the definition of $U_\alpha^{\ell+1}$. It is interesting to note the contrast between the first homology group and higher homology groups. For the first homology group all 1-cycles can be represented by relatively short simplices; there is no equivalent statement for higher homology groups.

Lemma 5 *A 1-cycle in α-scale homology can be represented by simplices with length less than or equal to α.*

Proof For any $[x_i, x_j]$ simplex with length greater than α there exists a point p such that $d(x_i, p) \le \alpha$ and $d(x_j, p) \le \alpha$. This indicates that $[x_i, p]$, $[p, x_j]$, and $[x_i, p, x_j]$ are simplices. Since $[x_i, p, x_j]$ is a simplex, we can substitute $[x_i, p] + [p, x_j]$ for $[x_i, x_j]$ and remain in the original equivalence class. □

In the section that follows we present an example that relies on the rigid nature of the definition to produce an infinite-dimensional homology group. The example is a countable set of points in $\mathbb{R}^3$. One noteworthy point is that from this example it is easy to construct a 1-manifold embedded in $\mathbb{R}^3$ with infinite α-scale homology.

In addition to showing that for a fixed α the homology is infinite, we consider the sensitivity of the result around that fixed α.

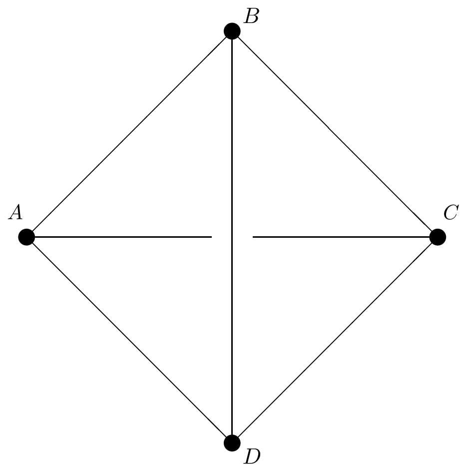
The existence of an infinite-dimensional example leads to the following question for future consideration: Are there necessary and sufficient conditions on (X, d) for the α-scale homology to be finite?

B.1 An Infinite-Dimensional Example

The following example illustrates a space such that the second homology group is infinite. For the discussion below, fix $\alpha = 1$.

Consider the set of points $\{A, B, C, D\}$ in the diagram below such that

$$d(A, B) = d(B, C) = d(C, D) = d(A, D) = 1,$$
$$d(A, C) = d(B, D) = \sqrt{2}.$$



The lines in the diagram suggest the correct structure of the α-simplices for $\alpha = 1$. The 1-simplices are $\{\{A, B\}, \{B, C\}, \{C, D\}, \{A, D\}, \{A, C\}, \{B, D\}\}$. The 2-simplices are the faces $\{\{A, B, C\}, \{A, B, D\}, \{A, C, D\}, \{B, C, D\}\}$. There are no (non-degenerate) 3-simplices. A 3-simplex would imply a point such that all of the points are within $\alpha = 1$, and no such point exists. The chain $[ABC] - [ABD] + [ACD] - [BCD]$ is a cycle with no boundary.

Define R as $R = \{r \in [0, 1, 1/2, 1/3, \ldots]\}$. In this example, note that R acts as an index set, and the dimension of the homology is shown to be at least that of R.

Let $X = \{A, B, C, D\} \times R$. Define $A_r = (A, r)$, $B_r = (B, r)$, $C_r = (C, r)$, and $D_r = (D, r)$.

We can again enumerate the 1-simplices for X. Let $r, s, t, u \in R$. The 1-simplices are

$$\bigl\{\{A_r, B_s\}, \{B_r, C_s\}, \{C_r, D_s\}, \{A_r, D_s\},$$
$$\{A_r, A_s\}, \{B_r, B_s\}, \{C_r, C_s\}, \{D_r, D_s\},$$
$$\{\mathbf{B_r}, \mathbf{D_r}\}, \{\mathbf{A_r}, \mathbf{C_r}\}\bigr\}.$$

The last two 1-simplices (highlighted) must have the same index in R due to the distance constraint.

The 2-simplices are

$$\begin{aligned}\big\{&\{A_r, B_s, C_r\}, \{A_s, B_r, D_r\}, \{A_r, C_r, D_s\}, \{B_r, C_s, D_r\},\\ &\{A_r, B_s, B_r\}, \{B_r, C_s, C_r\}, \{C_r, D_s, D_r\}, \{A_r, D_s, D_r\},\\ &\{A_s, A_r, B_s\}, \{B_s, B_r, C_s\}, \{C_s, C_r, D_s\}, \{A_r, A_r, D_s\},\\ &\{A_r, A_s, A_t\}, \{B_r, B_s, B_t\}, \{C_r, C_s, C_t\}, \{D_r, D_s, D_t\}\big\}.\end{aligned}$$

The 3-simplices are

$$\begin{aligned}\big\{&\{A_r, B_s, B_t, C_r\}, \{A_s, A_t, B_r, D_r\}, \{A_r, C_r, D_s, D_t\}, \{B_r, C_s, C_t, D_r\},\\ &\{A_r, B_t, B_s, B_r\}, \{B_r, C_t, C_s, C_r\}, \{C_r, D_t, D_s, D_r\}, \{A_r, D_t, D_s, D_r\},\\ &\{A_t, A_s, A_r, B_s\}, \{B_t, B_s, B_r, C_s\}, \{C_t, C_s, C_r, D_s\}, \{A_t, A_r, A_r, D_s\},\\ &\{A_r, A_s, A_t, A_u\}, \{B_r, B_s, B_t, B_u\}, \{C_r, C_s, C_t, C_u\}, \{D_r, D_s, D_t, D_u\}\big\}.\end{aligned}$$

Define $\sigma_r := [A_r B_r C_r] - [A_r B_r D_r] + [A_r C_r D_r] - [B_r C_r D_r]$. By calculation, σ_r is shown to be a cycle. Suppose that there existed a chain of 3-simplices such that the σ_r is the boundary; then $\gamma = [A_r A_s B_r D_r]$ must be included in the chain to eliminate $[A_r B_r D_r]$. Since the boundary of γ contains $[A_s B_r D_r]$, there must be a term to eliminate this term. The only term with such a boundary is of the form $[A_s A_t B_r D_r]$. Again, a new simplex to eliminate the extra boundary term is in the same form. Either this goes on *ad infinitum*, impossible since the chain is finite, or it returns to A_r, in which case the boundary of the original chain is 0 (contradicting that the $[A_r B_r D_r]$ term is eliminated). For all $r \in R$, σ_r is a generator for homology.

If $s \neq t$ then σ_s and σ_t are not in the same equivalence class. Suppose they are. The same argument above shows that any term with the face $[A_t B_t D_t]$ will necessarily have a face $[A_u B_t D_t]$ for some $u \in R$. Such a term must be eliminated since it cannot be in the final image, but such an elimination would cause another such term or cancel out the $[A_t B_t D_t]$. In either case the chain would not satisfy the boundary condition necessary to equivalence σ_s and σ_t together.

Each σ_s is a generator of homology and, therefore, the dimension of the homology is at least the cardinality of R, which in this case is infinite.

Theorem 12 *For $\alpha = 1$, the second α-scale homology group for*

$$X = \{A, B, C, D\} \times R$$

is infinite dimensional.

B.2 Consideration for $\alpha < 1$

The example above is tailored for scale $\alpha = 1$. In this metric space the nature of the second α-scale homology group changes significantly as α changes.

Consider when α falls below one. In this case the structure of the simplices collapses to simplices restricted to a line (with simplices of the form $\{\{A_r, A_s, A_t\}, \{B_r, B_s, B_t\}, \{C_r, C_s, C_t\}, \{D_r, D_s, D_t\}\}$). These are clearly degenerate simplices resulting in a trivial second homology group.

In this example the homology was significantly reduced as α decreased. This is not necessarily always the case. The above example could be further enhanced by replicating smaller versions of itself in a fractal-like manner, so that as α decreases the α-scale homology encounters many values with infinite-dimensional homology.

B.3 Consideration for $\alpha > 1$

There are two cases to consider when $\alpha > 1$. The first is the behavior for very large α values. In this case the problem becomes simple, as illustrated by the lemma below.

Define α *large* with respect to d if there exists $\rho \in X$ such that $d(\rho, y) \leq \alpha$ for all $y \in X$.

Lemma 6 *Let X be a separable, compact metric space with metric d. If α is large with respect to d, then the α-scale homology of X is trivial.*

Proof Let $\rho \in X$ satisfy the attribute above. Then $U_\alpha^{\ell+1} = X^{\ell+1}$ since

$$d\big((\rho, \dots, \rho), (x_0, x_1, \dots, x_\ell)\big) \leq \alpha$$

for all values of x_i.

Let $\sigma = \sum_{j=1}^{k} c_j(a_0^j, a_1^j, \dots, a_n^j)$ be an n-cycle. Define

$$\sigma_\rho = \sum_{j=1,k} c_j\big(a_0^j, a_1^j, \dots, a_n^j, \rho\big).$$

The $n+1$-cycle, σ_ρ, acts as a cone with base σ. Since σ is a cycle, the terms in the boundary of σ_ρ containing ρ cancel each other out to produce zero. The terms without ρ are exactly the original σ. Therefore, there exist no cycles without boundaries. This proves that for α large and X infinite the homology of X is trivial. □

In the case where $\alpha > 1$ but is still close to 1, the second homology group changes significantly but does not completely disappear. In the example, simplices that existed only by the equality in the definition of α-scale homology when $\alpha = 1$ now find neighboring 2-simplices joined by higher dimensional 3-simplices. The result is larger equivalence classes of 2-cycles. This reduces the infinite-dimensional homology for $\alpha = 1$ to a finite dimension for α slightly larger than 1. As α gets closer to 1 from above the dimension of the homology increases without bound.

Interestingly, the infinite characteristics for $\alpha = 1$ are tied heavily to the fact that the simplices that determined the structure lived on the bounds of being simplices. As α changes from 1, the rigid restrictions on the simplices no longer occur in this example. The result is a significant reduction in the dimension of the homology.

Appendix C: Open Problems and Remarks

Throughout the text, we have attempted to give indications to promising areas of future research. Here we summarize some of the main points.

- How do the methods of this paper apply to concrete examples, in particular, to the data in Carlsson et al. [6]? Specifically, can we recognize surfaces? Which substitutes for torsion do we have at hand?
- For non-oriented manifolds, can one introduce a twisted version of the coefficients that would make the top-dimensional Hodge cohomology visible?
- Is the Hodge cohomology independent of the choice of neighborhoods (Vietoris-Rips or ours)? Under which properties of metric spaces are the images of the corestriction maps (mentioned in Remark 5) independent of these choices?
- The Cohomology Identification Problem (Question 1): To what extent are $H^{\ell}_{L^2,\alpha}(X)$ and $H^{\ell}_{\alpha}(X)$ isomorphic?
- The Continuous Hodge Decomposition (Question 2): Under what conditions on X and $\alpha > 0$ is it true that there is an orthogonal direct sum decomposition of $C^{\ell+1}_{\alpha}$ in boundaries, coboundaries, and harmonic functions?
- The Poisson Regularity Problem (Question 3): For $\alpha > 0$ and $\ell > 0$, suppose that $\Delta f = g$ where $g \in C^{\ell+1}_{\alpha}$ and $f \in L^2_a(U^{\ell+1}_{\alpha})$. Under what conditions on (X, d, μ) is f continuous?
- The Harmonic Regularity Problem (Question 4): For $\alpha > 0$ and $\ell > 0$, suppose that $\Delta f = 0$ where $f \in L^2_a(U^{\ell+1}_{\alpha})$. What conditions on (X, d, μ) would imply that f is continuous?

References

1. M.F. Atiyah, Elliptic operators, discrete groups and von Neumann algebras, in *Colloque "Analyse et Topologie" en l'Honneur de Henri Cartan*, Orsay, 1974. Astérisque, No. 32–33 (Soc. Math. France, Paris, 1976), pp. 43–72.
2. M. Belkin, P. Niyogi, Convergence of Laplacian eigenmaps. Preprint, May (2008).
3. M. Belkin, E. De Vito, L. Rosasco, Random estimates of operators and their spectral properties for learning. Working paper.
4. R. Bott, L.W. Tu, *Differential Forms in Algebraic Topology*, Graduate Texts in Mathematics, vol. 82. (Springer, New York, 1982).
5. G. Carlsson, Topology and data, *Bull. Am. Math. Soc. (N.S.)* **46**(2), 255–308 (2009).
6. G. Carlsson, T. Ishkhanov, V. de Silva, A. Zomorodian, On the local behavior of spaces of natural images, *Int. J. Comput. Vis.* **76**(1), 1–12 (2008).
7. F. Chazal, S.Y. Oudot, Towards persistence-based reconstruction in Euclidean spaces, in *Computational geometry (SCG'08)* (ACM, New York, 2008), pp. 232–241.
8. J. Cheeger, M. Gromov, L_2-cohomology and group cohomology, *Topology* **25**(2), 189–215 (1986).
9. F.R.K. Chung, *Spectral Graph Theory*, CBMS Regional Conference Series in Mathematics, vol. 92 (Published for the Conference Board of the Mathematical Sciences, Washington, 1997).
10. R.R. Coifman, M. Maggioni, Diffusion wavelets, *Appl. Comput. Harmon. Anal.* **21**(1), 53–94 (2006).
11. W. Dicks, T. Schick, The spectral measure of certain elements of the complex group ring of a wreath product, *Geom. Dedic.* **93**, 121–137 (2002).
12. M.P. do Carmo, *Riemannian Geometry*, Mathematics: Theory & Applications (Birkhäuser, Boston, 1992). Translated from the second Portuguese edition by Francis Flaherty.
13. J. Dodziuk, Finite-difference approach to the Hodge theory of harmonic forms, *Am. J. Math.* **98**(1), 79–104 (1976).
14. J. Dodziuk, De Rham-Hodge theory for L^2-cohomology of infinite coverings, *Topology* **16**(2), 157–165 (1977).
15. J. Dodziuk, P. Linnell, V. Mathai, T. Schick, S. Yates, Approximating L^2-invariants and the Atiyah conjecture, *Commun. Pure Appl. Math.* **56**(7), 839–873 (2003). Dedicated to the memory of Jürgen K. Moser.
16. B. Eckmann, Harmonische Funktionen und Randwertaufgaben in einem Komplex, *Comment. Math. Helv.* **17**, 240–255 (1945).

17. H. Edelsbrunner, D. Letscher, A. Zomorodian, Topological persistence and simplification, *Discrete Comput. Geom.* **28**(4), 511–533 (2002). Discrete and computational geometry and graph drawing (Columbia, SC, 2001).
18. R. Forman, A user's guide to discrete Morse theory. *Sém. Lothar. Combin.*, 48:Art. B48c, 35 pp. (electronic) 2002.
19. J. Friedman, Computing Betti numbers via combinatorial Laplacians, in *Proceedings of the Twenty-eighth Annual ACM Symposium on the Theory of Computing*, Philadelphia, PA, 1996 (ACM, New York, 1996), pp. 386–391.
20. G. Gilboa, S. Osher, Nonlocal operators with applications to image processing, *Multiscale Model. Simul.* **7**(3), 1005–1028 (2008).
21. J.-C. Hausmann, On the Vietoris–Rips complexes and a cohomology theory for metric spaces, in *Prospects in Topology*, Princeton, NJ, 1994, Ann. of Math. Stud., vol. 138 (Princeton Univ. Press, Princeton, 1995), pp. 175–188.
22. W.V.D. Hodge, *The Theory and Applications of Harmonic Integrals* (Cambridge University Press, Cambridge, 1941).
23. X. Jiang, L.-H. Lim, Y. Yao, Y. Ye, Statistical ranking and combinatorial Hodge theory, *Math. Program.* **127**(1), 203–244 (2011).
24. S. Lang, *Analysis II* (Addison-Wesley, Reading, 1969).
25. A.B. Lee, K.S. Pedersen, D. Mumford, The non-linear statistics of high-contrast patches in natural images, *Int. J. Comput. Vis.* **54**, 83–103 (2003).
26. P. Linnell, T. Schick, Finite group extensions and the Atiyah conjecture, *J. Am. Math. Soc.* **20**(4), 1003–1051 (2007).
27. J. Lott, W. Lück, L^2-topological invariants of 3-manifolds, *Invent. Math.* **120**(1), 15–60 (1995).
28. W. Lück, Approximating L^2-invariants by their finite-dimensional analogues, *Geom. Funct. Anal.* **4**(4), 455–481 (1994).
29. W. Lück, *L^2-Invariants: Theory and Applications to Geometry and K-Theory*, Ergebnisse der Mathematik und ihrer Grenzgebiete, vol. 44 (Springer, Berlin, 2002). 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics].
30. J.R. Munkres, *Elements of Algebraic Topology* (Addison-Wesley, Menlo Park, 1984).
31. P. Pansu, Introduction to L^2 Betti numbers, in *Riemannian geometry*, Waterloo, ON, 1993. Fields Inst. Monogr., vol. 4 (Amer. Math. Soc, Providence, 1996), pp. 53–86.
32. T. Schick, Integrality of L^2-Betti numbers, *Math. Ann.* **317**(4), 727–750 (2000).
33. T. Schick, L^2-determinant class and approximation of L^2-Betti numbers, *Trans. Am. Math. Soc.* **353**(8), 3247–3265 (2001).
34. S. Smale, D.-X. Zhou, Geometry on probability spaces. *Constr. Approx.* **30**(3), 311–323 (2009).
35. S. Smale, L. Rosasco, J. Bouvrie, A. Caponnetto, T. Poggio, Mathematics of the neural response. *Found. Comput. Math.* **10**(1), 67–91 (2010).
36. E.H. Spanier, *Algebraic Topology* (Springer, New York, 1981). Corrected reprint.
37. F.W. Warner, *Foundations of Differentiable Manifolds and Lie Groups*, Graduate Texts in Mathematics, vol. 94 (Springer, New York, 1983). Corrected reprint of the 1971 edition.
38. R.L. Wilder, *Topology of Manifolds*, American Mathematical Society Colloquium Publications, vol. 32 (American Mathematical Society, Providence, 1979). Reprint of 1963 edition.
39. D. Zhou, B. Schölkopf, Regularization on discrete spaces, in *Pattern Recognition, Proc. 27th DAGM Symposium*, Berlin (2005), pp. 361–368.
40. A. Zomorodian, G. Carlsson, Computing persistent homology, *Discrete Comput. Geom.* **33**(2), 249–274 (2005).