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Bilinear multipliers of weighted Lebesgue spaces and variable exponent Lebesgue spaces

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Abstract

Let $1 \leq p_1, p_2 < \infty$, $0 < p_3 \leq \infty$ and ω_1, ω_2, ω_3 be weight functions on $\mathbb{R}^n$. Assume that ω_1, ω_2 are slowly increasing functions.

We say that a bounded function $m(\xi,\eta)$ defined on $\mathbb{R}^n \times \mathbb{R}^n$ is a bilinear multiplier on $\mathbb{R}^n$ of type $(p_1,\omega_1;p_2,\omega_2;p_3,\omega_3)$ (shortly $(\omega_1,\omega_2,\omega_3)$) if

$$B_m(f,g)(x) = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \hat{f}(\xi)\hat{g}(\eta)m(\xi,\eta)e^{2\pi i\langle \xi+\eta,x\rangle}\,d\xi\,d\eta$$

is a bounded bilinear operator from $L^{p_1}_{\omega_1}(\mathbb{R}^n) \times L^{p_2}_{\omega_2}(\mathbb{R}^n)$ to $L^{p_3}_{\omega_3}(\mathbb{R}^n)$. We denote by $\mathrm{BM}(p_1,\omega_1;p_2,\omega_2;p_3,\omega_3)$ (shortly $\mathrm{BM}(\omega_1,\omega_2,\omega_3)$) the vector space of bilinear multipliers of type $(\omega_1,\omega_2,\omega_3)$.

In this paper first we discuss some properties of the space $\mathrm{BM}(\omega_1,\omega_2,\omega_3)$. Furthermore, we give some examples of bilinear multipliers.

At the end of this paper, by using variable exponent Lebesgue spaces $L^{p_1(x)}(\mathbb{R}^n)$, $L^{p_2(x)}(\mathbb{R}^n)$ and $L^{p_3(x)}(\mathbb{R}^n)$, we define the space of bilinear multipliers from $L^{p_1(x)}(\mathbb{R}^n) \times L^{p_2(x)}(\mathbb{R}^n)$ to $L^{p_3(x)}(\mathbb{R}^n)$ and discuss some properties of this space.

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1 Introduction

Throughout this paper $C_c^\infty(\mathbb{R}^n)$, $C_c(\mathbb{R}^n)$ and $S(\mathbb{R}^n)$ denote the space of infinitely differentiable complex-valued functions with compact support on $\mathbb{R}^n$, the space of all continuous, complex-valued functions with compact support on $\mathbb{R}^n$ and the space of infinitely differentiable complex-valued functions on $\mathbb{R}^n$ rapidly decreasing at infinity, respectively. For $1 \leq p \leq \infty$, $L^p(\mathbb{R}^n)$ denotes the usual Lebesgue space. A continuous function ω satisfying $1 \leq \omega(x)$ and $\omega(x+y) \leq \omega(x)\omega(y)$ for $x, y \in \mathbb{R}^n$ will be called a weight function on $\mathbb{R}^n$. If $\omega_1(x) \leq \omega_2(x)$ for all $x \in \mathbb{R}^n$, we say that $\omega_1 \leq \omega_2$. We say that a weight function υ_s is of polynomial type if $\upsilon_s(x) = (1+|x|)^s$ for $s \geq 0$. Let f be a measurable function on $\mathbb{R}^n$. If there exist $C > 0$ and $N \in \mathbb{N}$ such that

$$|f(x)| \leq C(1+x^2)^N$$

for all $x \in \mathbb{R}^n$, then f is said to be a slowly increasing function. It is easy to see that polynomial-type weight functions are slowly increasing.

For $1 \leq p \leq \infty$, we set

$$L^p_\omega(\mathbb{R}^n) = \{f : f\omega \in L^p(\mathbb{R}^n)\}.$$

It is known that $L^p_\omega(\mathbb{R}^n)$ is a Banach space under the norm

$$\|f\|_{p,\omega} = \|f\omega\|_p = \left\{\int_{\mathbb{R}^n} |f(x)\omega(x)|^p \, dx\right\}^{\frac{1}{p}}, \quad 1 \leq p < \infty$$

or

$$\|f\|_{\infty,\omega} = \|f\omega\|_\infty = \operatorname*{ess\,sup}_{x\in\mathbb{R}^n} |f(x)\omega(x)|, \quad p = \infty, [1, 2].$$

The dual of the space $L^p_\omega(\mathbb{R}^n)$ is the space $L^q_{\omega^{-1}}(\mathbb{R}^n)$, where $\frac{1}{p} + \frac{1}{q} = 1$ and $\omega^{-1}(x) = \frac{1}{\omega(x)}$. For $f \in L^1(\mathbb{R}^n)$, the Fourier transform of f is denoted by $\hat{f}$. We know that $\hat{f}$ is a continuous function on $\mathbb{R}^n$, which vanishes at infinity, and it has the inequality $\|\hat{f}\|_\infty \leq \|f\|_1$ [3, 4]. Let f be a measurable function on $\mathbb{R}^n$. The translation, character and dilation operators T_x, M_x and D_s are defined by $T_x f(y) = f(y - x)$, $M_x f(y) = e^{2\pi i\langle x,y\rangle} f(y)$ and $D^p_t f(y) = t^{-\frac{n}{p}} f(\frac{y}{t})$ respectively for $x, y \in \mathbb{R}^n$, $0 < p, t < \infty$. With this notation out of the way one has, for $1 \leq p \leq \infty$ and $\frac{1}{p} + \frac{1}{p'} = 1$,

$$(T_x f)\hat{}(\xi) = M_{-x}\hat{f}(\xi), \qquad (M_x f)\hat{}(\xi) = T_x\hat{f}(\xi), \qquad \left(D^p_t f\right)\hat{}(\xi) = D^{p'}_{t^{-1}}\hat{f}(\xi).$$

We denote by $M(\mathbb{R}^n)$ the space of bounded regular Borel measures, by $M(\omega)$ the space of μ in $M(\mathbb{R}^n)$ such that

$$\|\mu\|_\omega = \int_{\mathbb{R}^n} \omega \, d|\mu| < \infty.$$

If $\mu \in M(\mathbb{R}^n)$, the Fourier-Stieltjes transform of μ is denoted by $\hat{\mu}$ [5]. In this paper, $P(\mathbb{R}^n)$ denotes the family of all measurable functions $p : \mathbb{R}^n \to [1, \infty)$. We put

$$p_* = \operatorname*{ess\,inf}_{x\in\mathbb{R}^n} p(x), \qquad p^* = \operatorname*{ess\,sup}_{x\in\mathbb{R}^n} p(x).$$

We shall also use the notation

$$\Omega_\infty = \{x \in \mathbb{R}^n : p(x) = \infty\}.$$

The generalized Lebesgue space (or the variable exponent Lebesgue space) $L^{p(x)}(\mathbb{R}^n)$ is defined to be a space of (equivalence classes) measurable functions f such that

$$\varrho_p(\lambda f) = \int_{\mathbb{R}^n\setminus\Omega_\infty} |\lambda f(x)|^{p(x)} \, dx + \operatorname*{ess\,sup}_{x\in\Omega_\infty} (\lambda f(x)) < \infty$$

for some $\lambda = \lambda(f) > 0$. If $p^* < \infty$, then

$$\varrho_p(\lambda f) = \int_{\mathbb{R}^n \setminus \Omega_\infty} \left|\lambda f(x)\right|^{p(x)} dx, \quad [6, 7].$$

It is known by Theorem 2.5 in [6] that $L^{p(x)}(\mathbb{R}^n)$ is a Banach space with the Luxemburg norm

$$\|f\|_{p(x)} = \inf\left\{\lambda > 0 : \varrho_p\left(\frac{f}{\lambda}\right) \le 1\right\}.$$

If $p^* < \infty$, then $C_c^\infty(\mathbb{R}^n)$ is dense in $L^{p(x)}(\mathbb{R}^n)$. Also, if $p(x) = p$ is a constant function, then the above norm $\|\cdot\|_{p(x)}$ coincides with the usual norm $\|\cdot\|_p$. The vector space of locally integrable functions on $\mathbb{R}^n$ is denoted by $L^1_{\mathrm{loc}}(\mathbb{R}^n)$. The space $L^{p(x)}(\mathbb{R}^n)$ is a solid space, that is, if $f \in L^{p(x)}(\mathbb{R}^n)$ is given and $g \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ satisfies $|g(x)| \le |f(x)|$ a.e., then $g \in L^{p(x)}(\mathbb{R}^n)$ and $\|g\|_{p(x)} \le \|f\|_{p(x)}$ by [8]. In this paper we assume that $p^* < \infty$.

2 The bilinear multipliers space $\mathrm{BM}(p_1, \omega_1; p_2, \omega_2; p_3, \omega_3)$

Lemma 2.1 *Let $1 \le p < \infty$ and let ω be a slowly increasing weight function. Then $S(\mathbb{R}^n)$ is dense in $L^p_\omega(\mathbb{R}^n)$.*

Proof Let $f \in S(\mathbb{R}^n)$ be given. Since ω is a slowly increasing weight function, there exist $C > 0$ and $N \in \mathbb{N}$ such that

$$\left|\omega(x)\right| \le C\left(1 + x^2\right)^N = m(x) \tag{2.1}$$

for all $x \in \mathbb{R}^n$. Also, since m is a polynomial, then by Proposition 19.2.2 in [9], we have $S(\mathbb{R}^n) \subset L^p_m(\mathbb{R}^n)$. Hence, by (2.1), we obtain $S(\mathbb{R}^n) \subset L^p_m(\mathbb{R}^n) \subset L^p_\omega(\mathbb{R}^n)$.

Now, we show that $C_c^\infty(\mathbb{R}^n)$ is dense $L^p_m(\mathbb{R}^n)$. Let $f \in L^p_w(\mathbb{R}^n)$ be given. Then $fm \in L^p(\mathbb{R}^n)$. Since $C_c^\infty(\mathbb{R}^n)$ is dense $L^p(\mathbb{R}^n)$ by [6], for given $\varepsilon > 0$, there exists $g \in C_c^\infty(\mathbb{R}^n)$ such that

$$\|fm - g\|_p < \varepsilon. \tag{2.2}$$

Therefore, by using the inequality (2.2), we write

$$\|fm - g\|_p = \left\|f - gm^{-1}\right\|_{p,m} < \varepsilon.$$

Also, since $m \neq 0$ and m is a polynomial, we have $gm^{-1} \in C_c^\infty(\mathbb{R}^n)$. Thus, we have $\overline{C_c^\infty(\mathbb{R}^n)} = L^p_m(\mathbb{R}^n)$. By using the inclusion $C_c^\infty(\mathbb{R}^n) \subset S(\mathbb{R}^n) \subset L^p_m(\mathbb{R}^n)$, we obtain $\overline{S(\mathbb{R}^n)} = L^p_m(\mathbb{R}^n)$.

Now, take any $f \in L^p_\omega(\mathbb{R}^n)$. Since $\overline{C_c(\mathbb{R}^n)} = \overline{L^p_m(\mathbb{R}^n)} = L^p_\omega(\mathbb{R}^n)$, there exists $g \in L^p_m(\mathbb{R}^n)$ such that

$$\|f - g\|_{p,\omega} < \frac{\varepsilon}{2}. \tag{2.3}$$

Furthermore, since $S(\mathbb{R}^n)$ is dense $L^p_m(\mathbb{R}^n)$, there exists $h \in S(\mathbb{R}^n)$ such that

$$\|g - h\|_{p,m} < \frac{\varepsilon}{2}. \tag{2.4}$$

Combining the inequalities (2.3) and (2.4), we have

$$\|f-g\|_{p,\omega} \leq \|f-g\|_{p,\omega} + \|h-g\|_{p,\omega} \leq \|f-g\|_{p,\omega} + \|h-g\|_{p,m} < \varepsilon,$$

which means $\overline{S(\mathbb{R}^n)} = L^p_\omega(\mathbb{R}^n)$. □

Definition 2.1 Let $1 \leq p_1, p_2 < \infty$, $0 < p_3 \leq \infty$ and ω_1, ω_2, ω_3 be weight functions on $\mathbb{R}^n$. Assume that ω_1, ω_2 are slowly increasing functions and $m(\xi,\eta)$ is a bounded function on $\mathbb{R}^n \times \mathbb{R}^n$. Define

$$B_m(f,g)(x) = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \hat{f}(\xi)\hat{g}(\eta)m(\xi,\eta)e^{2\pi i\langle \xi+\eta,x\rangle}\,d\xi\,d\eta$$

for all $f,g \in S(\mathbb{R}^n)$.

m is said to be a bilinear multiplier on $\mathbb{R}^n$ of type $(p_1,\omega_1;p_2,\omega_2;p_3,\omega_3)$ (shortly $(\omega_1,\omega_2,\omega_3)$) if there exists $C > 0$ such that

$$\|B_m(f,g)\|_{p_3,\omega_3} \leq C\|f\|_{p_1,\omega_1}\|g\|_{p_2,\omega_2}$$

for all $f,g \in S(\mathbb{R}^n)$. That means B_m extends to a bounded bilinear operator from $L^{p_1}_{\omega_1}(\mathbb{R}^n) \times L^{p_2}_{\omega_2}(\mathbb{R}^n)$ to $L^{p_3}_{\omega_3}(\mathbb{R}^n)$.

We denote by $\mathrm{BM}(p_1,\omega_1;p_2,\omega_2;p_3,\omega_3)$ (shortly $\mathrm{BM}(\omega_1,\omega_2,\omega_3)$) the space of all bilinear multipliers of type $(\omega_1,\omega_2,\omega_3)$ and $\|m\|_{(\omega_1,\omega_2,\omega_3)} = \|B_m\|$.

Theorem 2.1 *Let* $\frac{1}{p_1}+\frac{1}{p_2}=\frac{1}{p_3}$ *and* $\omega_3 \leq \omega_1$. *If* $K \in L^1_{\omega_3}(\mathbb{R}^n)$, *then* $m(\xi,\eta) = \hat{K}(\xi-\eta)$ *defines a bilinear multiplier and* $\|m\|_{(\omega_1,\omega_2,\omega_3)} \leq \|K\|_{1,\omega_3}$.

Proof For $f,g \in S(\mathbb{R}^n)$, we have $f(x-y) = \int_{\mathbb{R}^n}\hat{f}(\xi)e^{2\pi i\langle x-y,\xi\rangle}\,d\xi$ and $g(x+y) = \int_{\mathbb{R}^n}\hat{g}(\eta) \times e^{2\pi i\langle x+y,\eta\rangle}\,d\eta$. Thus, by the Fubini theorem, we write

$$\begin{aligned}
B_m(f,g)(x) &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \hat{f}(\xi)\hat{g}(\eta)\hat{K}(\xi-\eta)e^{2\pi i\langle \xi+\eta,x\rangle}\,d\xi\,d\eta \\
&= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \hat{f}(\xi)\hat{g}(\eta)\left(\int_{\mathbb{R}^n} K(y)e^{-2\pi i\langle \xi-\eta,y\rangle}\,dy\right)e^{2\pi i\langle \xi+\eta,x\rangle}\,d\xi\,d\eta \\
&= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \hat{f}(\xi)\hat{g}(\eta)K(y)e^{-2\pi i\langle \xi-\eta,y\rangle}e^{2\pi i\langle \xi+\eta,x\rangle}\,d\xi\,d\eta\,dy. \qquad (2.5)
\end{aligned}$$

Since $f,g \in S(\mathbb{R}^n)$, we have $\hat{f},\hat{g} \in S(\mathbb{R}^n) \subset L^1(\mathbb{R}^n)$. Hence, by (2.5), we obtain

$$\begin{aligned}
B_m(f,g)(x) &= \int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n}\hat{f}(\xi)e^{2\pi i\langle x-y,\xi\rangle}\right)\left(\int_{\mathbb{R}^n}\hat{g}(\eta)e^{2\pi i\langle x+y,\eta\rangle}\,d\eta\right)K(y)\,dy \\
&= \int_{\mathbb{R}^n} f(x-y)g(x+y)K(y)\,dy. \qquad (2.6)
\end{aligned}$$

Since $\omega_3 \leq \omega_1$, then

$$\|f(x-y)\omega_3\|_{p_1} \leq \omega_3(y)\|f\|_{p_1,\omega_1} \qquad (2.7)$$

and hence $f(x-y)\omega_3 \in L^{p_1}(\mathbb{R}^n)$. Therefore from (2.6) and the Minkowski inequality, we write

$$\begin{aligned}\|B_m(f,g)\|_{p_3,\omega_3} &\le \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \|f(x-y)g(x+y)\|_{p_3,\omega_3} |K(y)|\,dy \\ &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \|f(x-y)g(x+y)\omega_3\|_{p_3} |K(y)|\,dy. \end{aligned} \tag{2.8}$$

Hence, using the generalized Hölder inequality and combining (2.7), (2.8), we have

$$\begin{aligned}\|B_m(f,g)\|_{p_3,\omega_3} &\le \int_{\mathbb{R}^n} \|f(x-y)\omega_3\|_{p_1} \|g(x+y)\|_{p_2}\omega_3(y)|K(y)|\,dy \\ &\le \int_{\mathbb{R}^n} \|f\|_{p_1}\|g\|_{p_2}\omega_3(y)|K(y)|\,dy \\ &\le \int_{\mathbb{R}^n} \|f\|_{p_1,\omega_1}\|g\|_{p_2,\omega_2}\omega_3(y)|K(y)|\,dy \\ &= \|f\|_{p_1,\omega_1}\|g\|_{p_2,\omega_2}\|K\|_{1,\omega_3}. \end{aligned} \tag{2.9}$$

If we set $C = \|K\|_{1,\omega_3}$, we obtain

$$\|B_m(f,g)\|_{p_3,\omega_3} \le C\|f\|_{p_1,\omega_1}\|g\|_{p_2,\omega_2}.$$

Then $m \in \mathrm{BM}(\omega_1,\omega_2,\omega_3)$. Consequently, using (2.9), we have

$$\begin{aligned}\|m\|_{(\omega_1,\omega_2,\omega_3)} &= \sup\left\{\frac{\|B_m(f,g)\|_{p_3,\omega_3}}{\|f\|_{p_1,\omega_1}\|g\|_{p_2,\omega_2}} : \|f\|_{p_1,\omega_1}\le 1, \|g\|_{p_2,\omega_2}\le 1\right\} \\ &\le \|K\|_{1,\omega_3}. \end{aligned}$$

□

Definition 2.2 Let $1 \le p_1, p_2 < \infty$, $0 < p_3 \le \infty$ and ω_1, ω_2, ω_3 be weight functions on $\mathbb{R}^n$. Suppose that ω_1, ω_2 are slowly increasing functions. We denote by $\tilde{M}(\omega_1,\omega_2,\omega_3)$ the space of measurable functions $M : \mathbb{R}^n \to \mathbb{C}$ such that $m(\xi,\eta) = M(\xi-\eta) \in \mathrm{BM}(\omega_1,\omega_2,\omega_3)$, that is to say,

$$B_M(f,g)(x) = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \hat{f}(\xi)\hat{g}(\eta)M(\xi-\eta)e^{2\pi i\langle\xi+\eta,x\rangle}\,d\xi\,d\eta$$

extends to a bounded bilinear map from $L^{p_1}_{\omega_1}(\mathbb{R}^n) \times L^{p_2}_{\omega_2}(\mathbb{R}^n)$ to $L^{p_3}_{\omega_3}(\mathbb{R}^n)$. We denote $\|M\|_{(\omega_1,\omega_2,\omega_3)} = \|B_M\|$.

Theorem 2.2 *Let $p_3 \ge 1$ and $\omega_3(-x) = \omega_3(x)$. Then $m \in \mathrm{BM}(\omega_1,\omega_2,\omega_3)$ if and only if there exists $C > 0$ such that*

$$\left|\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \hat{f}(\xi)\hat{g}(\eta)\hat{h}(\xi+\eta)m(\xi,\eta)\,d\xi\,d\eta\right| \le C\|f\|_{p_1,\omega_1}\|g\|_{p_2,\omega_2}\|h\|_{p_3',\omega_3^{-1}}$$

for all $f, g, h \in S(\mathbb{R}^n)$, where $\frac{1}{p_3} + \frac{1}{p_3'} = 1$.

Proof Let $m \in \mathrm{BM}(\omega_1,\omega_2,\omega_3)$. We take any $f,g,h \in S(\mathbb{R}^n)$. From the Fubini theorem, we write

$$\begin{aligned}
&\left|\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\hat{f}(\xi)\hat{g}(\eta)\hat{h}(\xi+\eta)m(\xi,\eta)\,d\xi\,d\eta\right| \\
&\quad=\left|\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\hat{f}(\xi)\hat{g}(\eta)\left\{\int_{\mathbb{R}^n}h(y)e^{-2\pi i\langle\xi+\eta,y\rangle}\,dy\right\}m(\xi,\eta)\,d\xi\,d\eta\right| \\
&\quad=\left|\int_{\mathbb{R}^n}h(y)B_m(f,g)(-y)\,dy\right|=\left|\int_{\mathbb{R}^n}h(y)\tilde{B}_m(f,g)(y)\,dy\right| \\
&\quad\le\int_{\mathbb{R}^n}\left|h(y)\right|\left|\tilde{B}_m(f,g)(y)\right|\,dy, \qquad (2.10)
\end{aligned}$$

where $\tilde{B}_m(f,g)(y)=B_m(f,g)(-y)$. On the other hand, since $m \in \mathrm{BM}(\omega_1,\omega_2,\omega_3)$, then $B_m(f,g) \in L^{p_3}_{\omega_3}(\mathbb{R}^n)$. Thus we obtain $\tilde{B}_m(f,g) \in L^{p_3}_{\omega_3}(\mathbb{R}^n)$. Also, $h \in S(\mathbb{R}^n) \subset L^{p_3'}(\mathbb{R}^n) \subset L^{p_3'}_{\omega_3^{-1}}(\mathbb{R}^n)$. Hence, using the Hölder inequality and the inequality (2.10), we write

$$\begin{aligned}
&\left|\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\hat{f}(\xi)\hat{g}(\eta)\hat{h}(\xi+\eta)m(\xi,\eta)\,d\xi\,d\eta\right| \\
&\quad\le\int_{\mathbb{R}^n}\left|h(y)\omega_3^{-1}(y)\right|\left|\tilde{B}_m(f,g)(y)\omega_3(y)\right|\,dy \\
&\quad\le\left\|\tilde{B}_m(f,g)\right\|_{p_3,\omega_3}\|h\|_{p_3',\omega_3^{-1}}. \qquad (2.11)
\end{aligned}$$

Moreover, since $m \in \mathrm{BM}(\omega_1,\omega_2,\omega_3)$, there exists $C>0$ such that

$$\left\|B_m(f,g)\right\|_{p_3,\omega_3}\le C\|f\|_{p_1,\omega_1}\|g\|_{p_2,\omega_2}. \qquad (2.12)$$

If we combine (2.11) and (2.12), we write

$$\left|\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\hat{f}(\xi)\hat{g}(\eta)\hat{h}(\xi+\eta)m(\xi,\eta)\,d\xi\,d\eta\right|\le C\|f\|_{p_1,\omega_1}\|g\|_{p_2,\omega_2}\|h\|_{p_3',\omega_3^{-1}}.$$

For the proof of converse, assume that there exists a constant $C>0$ such that

$$\left|\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\hat{f}(\xi)\hat{g}(\eta)\hat{h}(\xi+\eta)m(\xi,\eta)\,d\xi\,d\eta\right|\le C\|f\|_{p_1,\omega_1}\|g\|_{p_2,\omega_2}\|h\|_{p_3',\omega_3^{-1}}$$

for all $f,g,h \in S(\mathbb{R}^n)$. From the assumption and (2.10), we write

$$\left|\int_{\mathbb{R}^n}h(y)\tilde{B}_m(f,g)(y)\,dy\right|\le C\|f\|_{p_1,\omega_1}\|g\|_{p_2,\omega_2}\|h\|_{p_3',\omega_3^{-1}}. \qquad (2.13)$$

Define a function l from $S(\mathbb{R}^n) \subset L^{p_3'}_{\omega_3^{-1}}(\mathbb{R}^n)$ to $\mathbb{C}$ such that

$$\ell(h)=\int_{\mathbb{R}^n}h(y)\tilde{B}_m(f,g)(y)\,dy.$$

It is clear that the function ℓ is linear and bounded by (2.13). By using $\overline{C_c(\mathbb{R}^n)}=L^{p_3'}_{\omega_3^{-1}}(\mathbb{R}^n)$ in [10], it is easy to show that $\overline{C_c^\infty(\mathbb{R}^n)}=L^{p_3'}_{\omega_3^{-1}}(\mathbb{R}^n)$. So, by the inclusion $C_c^\infty(\mathbb{R}^n) \subset S(\mathbb{R}^n) \subset$

$L^{p_3'}_{\omega_3^{-1}}(\mathbb{R}^n)$, we have $\overline{S(\mathbb{R}^n)} = L^{p_3'}_{\omega_3^{-1}}(\mathbb{R}^n)$. Thus ℓ extends to a bounded function from $L^{p_3'}_{\omega_3^{-1}}(\mathbb{R}^n)$ to $\mathbb{C}$. Then $\ell \in (L^{p_3'}_{\omega_3^{-1}}(\mathbb{R}^n))^* = L^{p_3}_{\omega_3}(\mathbb{R}^n)$ and by (2.13), we have

$$\begin{aligned}\|B_m(f,g)\|_{p_3,\omega_3} &= \|\tilde{B}_m(f,g)\|_{p_3,\omega_3} = \|\ell\| = \sup_{\|h\|_{p_3',\omega_3^{-1}}\le 1} \frac{|l(h)|}{\|h\|_{p_3',\omega_3^{-1}}} \\ &\le \sup_{\|h\|_{p_3',\omega_3^{-1}}\le 1} \frac{C\|f\|_{p_1,\omega_1}\|g\|_{p_2,\omega_2}\|h\|_{p_3',\omega_3^{-1}}}{\|h\|_{p_3',\omega_3^{-1}}} \le C\|f\|_{p_1,\omega_1}\|g\|_{p_2,\omega_2}.\end{aligned}$$

Hence, we obtain $m \in \mathrm{BM}(\omega_1,\omega_2,\omega_3)$. □

Theorem 2.3 *Let* $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3}$, $p_3 \ge 1$ *and* $\upsilon_s(x) = (1+|x|)^s$, $s \ge 0$ *be a weight function of polynomial type such that* $\upsilon_s \le \omega_1$. *If* $\mu \in M(\upsilon_s)$ *and* $m(\xi,\eta) = \hat{\mu}(\alpha\xi + \beta\eta)$ *for* $\alpha, \beta \in \mathbb{R}$, *then* $m \in \mathrm{BM}(\omega_1,\omega_2,\upsilon_s)$. *Moreover,*

$$\begin{aligned}&\|m\|_{(\omega_1,\omega_2,\upsilon_s)} \le \|\mu\|_{\upsilon_s} \quad \text{if } |\alpha| \le 1, \\ &\|m\|_{(\omega_1,\omega_2,\upsilon_s)} \le |\alpha|^s \|\mu\|_{\upsilon_s} \quad \text{if } |\alpha| > 1.\end{aligned}$$

Proof Let $f, g, \in S(\mathbb{R}^n)$. Then

$$\begin{aligned}B_m(f,g)(x) &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \hat{f}(\xi)\hat{g}(\eta)\hat{\mu}(\alpha\xi+\beta\eta)e^{2\pi i\langle \xi+\eta,x\rangle}\,d\xi\,d\eta \\ &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \hat{f}(\xi)\hat{g}(\eta)\left\{\int_{\mathbb{R}^n} e^{-2\pi i\langle\alpha\xi+\beta\eta,t\rangle}\,d\mu(t)\right\}e^{2\pi i\langle\xi+\eta,x\rangle}\,d\xi\,d\eta \\ &= \int_{\mathbb{R}^n}\left\{\int_{\mathbb{R}^n}\hat{f}(\xi)e^{2\pi i\langle x-\alpha t,\xi\rangle}\,d\xi\right\}\left\{\int_{\mathbb{R}^n}\hat{g}(\eta)e^{2\pi i\langle x-\beta t,\eta\rangle}\,d\eta\right\}d\mu(t) \\ &= \int_{\mathbb{R}^n} f(x-\alpha t)g(x-\beta t)\,d\mu(t). \end{aligned} \tag{2.14}$$

On the other hand, by the assumption $\upsilon_s \le \omega_1$, it is easy to see that $f(x-\alpha t)\upsilon_s \in L^{p_1}(\mathbb{R}^n)$ and

$$\|f(x-\alpha t)\upsilon_s\|_{p_1} \le \upsilon_s(\alpha t)\|f\|_{p_1,\omega_1}. \tag{2.15}$$

Also, $g(x-\beta t) \in L^{p_2}(\mathbb{R}^n)$. Then, by (2.14), (2.15) and the generalized Hölder inequality, we have

$$\begin{aligned}\|B_m(f,g)\|_{p_3,\upsilon_s} &\le \int_{\mathbb{R}^n}\|f(x-\alpha t)g(x-\beta t)\|_{p_3,\upsilon_s}\,d|\mu|(t) \\ &\le \int_{\mathbb{R}^n}\|f(x-\alpha t)\upsilon_s\|_{p_1}\|g(x-\beta t)\|_{p_2}\,d|\mu|(t) \\ &\le \int_{\mathbb{R}^n}\upsilon_s(\alpha t)\|f\|_{p_1,\omega_1}\|g\|_{p_2}\,d|\mu|(t) \\ &\le \|f\|_{p_1,\omega_1}\|g\|_{p_2,\omega_2}\int_{\mathbb{R}^n}\upsilon_s(\alpha t)\,d|\mu|(t).\end{aligned} \tag{2.16}$$

Now, suppose that $|\alpha| \leq 1$. Then we write

$$\begin{aligned}\int_{\mathbb{R}^n} \upsilon_s(\alpha t)\, d|\mu|(t) &= \int_{\mathbb{R}^n} \left(1 + |\alpha t|\right)^s d|\mu|(t) \\ &\leq \int_{\mathbb{R}^n} \left(1 + |t|\right)^s d|\mu|(t) = \|\mu\|_{\upsilon_s}.\end{aligned}$$

Hence by (2.16)

$$\left\|B_m(f,g)\right\|_{p_3,\upsilon_s} \leq \|f\|_{p_1,\omega_1}\|g\|_{p_2,\omega_2}\|\mu\|_{\upsilon_s}. \tag{2.17}$$

Thus $m \in \mathrm{BM}(\omega_1, \omega_2, \upsilon_s)$ and by (2.17), we have

$$\|m\|_{(\omega_1,\omega_2,\upsilon_s)} = \sup\left\{ \frac{\|B_m(f,g)\|_{p_3,\upsilon_s}}{\|f\|_{p_1,\omega_1}\|g\|_{p_2,\omega_2}} : \|f\|_{p_1,\omega_1} \leq 1, \|g\|_{p_2,\omega_2} \leq 1\right\} \leq \|\mu\|_{\upsilon_s}.$$

Similarly, if $|\alpha| > 1$, then we write

$$\begin{aligned}\int_{\mathbb{R}^n} \upsilon_s(\alpha t)\, d|\mu|(t) &< \int_{\mathbb{R}^n} \left(|\alpha| + |\alpha||t|\right)^s d|\mu|(t) \\ &= |\alpha|^s \int_{\mathbb{R}^n} \upsilon_s(t)\, d|\mu|(t) = |\alpha|^s\|\mu\|_{\upsilon_s}.\end{aligned}$$

Again, by (2.16) we have

$$\left\|B_m(f,g)\right\|_{p_3,\upsilon_s} \leq |\alpha|^s\|f\|_{p_1,\omega_1}\|g\|_{p_2,\omega_2}\|\mu\|_{\upsilon_s}. \tag{2.18}$$

Hence, we obtain $m \in \mathrm{BM}(\omega_1, \omega_2, \upsilon_s)$ and by (2.18)

$$\|m\|_{(\omega_1,\omega_2,\upsilon_s)} = \sup\left\{ \frac{\|B_m(f,g)\|_{p_3,\upsilon_s}}{\|f\|_{p_1,\omega_1}\|g\|_{p_2,\omega_2}} : \|f\|_{p_1,\omega_1} \leq 1, \|g\|_{p_2,\omega_2} \leq 1\right\} \leq |\alpha|^s\|\mu\|_{\upsilon_s}. \qquad \square$$

Theorem 2.4 *Let* $m \in \mathrm{BM}(\omega_1, \omega_2, \omega_3)$.

(a) $T_{(\xi_0,\eta_0)}m \in \mathrm{BM}(\omega_1, \omega_2, \omega_3)$ *for each* $(\xi_0, \eta_0) \in \mathbb{R}^{2n}$ *and*

$$\|T_{(\xi_0,\eta_0)}m\|_{(\omega_1,\omega_2,\omega_3)} = \|m\|_{(\omega_1,\omega_2,\omega_3)}.$$

(b) $M_{(\xi_0,\eta_0)}m \in \mathrm{BM}(\omega_1, \omega_2, \omega_3)$ *for each* $(\xi_0, \eta_0) \in \mathbb{R}^{2n}$ *and*

$$\|M_{(\xi_0,\eta_0)}m\|_{(\omega_1,\omega_2,\omega_3)} \leq \omega_1(-\xi_0)\omega_2(-\eta_0)\|m\|_{(\omega_1,\omega_2,\omega_3)}.$$

Proof (a) Let us take any $f \in L^{p_1}_{\omega_1}(\mathbb{R}^n)$ and $g \in L^{p_2}_{\omega_2}(\mathbb{R}^n)$. If we say that $\xi - \xi_0 = u$ and $\eta - \eta_0 = \nu$, then

$$\begin{aligned}B_{T_{(\xi_0,\eta_0)}m}(f,g)(x) &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \hat{f}(\xi)\hat{g}(\eta)T_{(\xi_0,\eta_0)}m(\xi,\eta)e^{2\pi i\langle\xi+\eta,x\rangle}\, d\xi\, d\eta \\ &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \hat{f}(u+\xi_0)\hat{g}(\nu+\eta_0)m(u,\nu)e^{2\pi i\langle u+\xi_0,x\rangle}e^{2\pi i\langle \nu+\eta_0,x\rangle}\, du\, d\nu \\ &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} T_{-\xi_0}\hat{f}(u)T_{-\eta_0}\hat{g}(\nu)m(u,\nu)e^{2\pi i\langle\xi_0+\eta_0,x\rangle}e^{2\pi i\langle u+\nu,x\rangle}\, du\, d\nu. \end{aligned} \tag{2.19}$$

By (2.19), we have

$$\begin{aligned} B_{T_{(\xi_0,\eta_0)}m}(f,g)(x) &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} T_{-\xi_0}\hat{f}(u)T_{-\eta_0}\hat{g}(v)e^{2\pi i\langle \xi_0+\eta_0,x\rangle}e^{2\pi i\langle u+v,x\rangle}m(u,v)\,du\,dv \\ &= e^{2\pi i\langle \xi_0+\eta_0,x\rangle}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}(M_{-\xi_0}f)\hat{}\,(u)(M_{-\eta_0}g)\hat{}\,(v)m(u,v)e^{2\pi i\langle u+v,x\rangle}\,du\,dv \\ &= e^{2\pi i\langle \xi_0+\eta_0,x\rangle}B_m(M_{-\xi_0}f,M_{-\eta_0}g)(x). \end{aligned} \tag{2.20}$$

Since $m\in \mathrm{BM}(\omega_1,\omega_2,\omega_3)$, $\|M_{-\xi_0}f\|_{p_1,\omega_1}=\|f\|_{p_1,\omega_1}$ and $\|M_{-\eta_0}g\|_{p_2,\omega_2}=\|g\|_{p_2,\omega_2}$ are satisfied for all $f\in L^{p_1}_{\omega_1}(\mathbb{R}^n)$ and $g\in L^{p_2}_{\omega_2}(\mathbb{R}^n)$. Hence, by (2.20), we have

$$\begin{aligned} \big\|B_{T_{(\xi_0,\eta_0)}m}(f,g)\big\|_{p_3,\omega_3} &= \big\|e^{2\pi i\langle \xi_0+\eta_0,x\rangle}B_m(M_{-\xi_0}f,M_{-\eta_0}g)\big\|_{p_3,\omega_3} \\ &\le C\|f\|_{p_1,\omega_1}\|g\|_{p_2,\omega_2} \end{aligned}$$

for some $C>0$. Thus $T_{(\xi_0,\eta_0)m}\in \mathrm{BM}(\omega_1,\omega_2,\omega_3)$. Also, we obtain

$$\begin{aligned} &\|T_{(\xi_0,\eta_0)}m\|_{(\omega_1,\omega_2,\omega_3)} \\ &\quad= \|B_{T_{(\xi_0,\eta_0)}m}\| \\ &\quad= \sup\left\{\frac{\|B_{T_{(\xi_0,\eta_0)}m}(f,g)\|_{p_3,\omega_3}}{\|f\|_{p_1,\omega_1}\|g\|_{p_2,\omega_2}} : \|f\|_{p_1,\omega_1}\le 1, \|g\|_{p_2,\omega_2}\le 1\right\} \\ &\quad= \sup\left\{\frac{\|B_m(M_{-\xi_0}f,M_{-\eta_0}g)\|_{p_3,\omega_3}}{\|M_{-\xi_0}f\|_{p_1,\omega_1}\|M_{-\eta_0}g\|_{p_2,\omega_2}} : \|M_{-\xi_0}f\|_{p_1,\omega_1}\le 1, \|M_{-\eta_0}g\|_{p_2,\omega_2}\le 1\right\} \\ &\quad= \|B_m\| = \|m\|_{(\omega_1,\omega_2,\omega_3)}. \end{aligned}$$

(b) Let us rewrite the value $B_m(f,g)$ as follows:

$$\begin{aligned} B_{M_{(\xi_0,\eta_0)}m}(f,g)(x) &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\hat{f}(\xi)\hat{g}(\eta)M_{(\xi_0,\eta_0)}m(\xi,\eta)e^{2\pi i\langle \xi+\eta,x\rangle}\,d\xi\,d\eta \\ &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\hat{f}(\xi)\hat{g}(\eta)e^{2\pi i\langle(\xi_0,\eta_0),(\xi,\eta)\rangle}m(\xi,\eta)e^{2\pi i\langle \xi+\eta,x\rangle}\,d\xi\,d\eta \\ &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}M_{\xi_0}\hat{f}(\xi)M_{\eta_0}\hat{g}(\eta)m(\xi,\eta)e^{2\pi i\langle \xi+\eta,x\rangle}\,d\xi\,d\eta \\ &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}(T_{-\xi_0}f)\hat{}\,(\xi)(T_{-\eta_0}g)\hat{}\,(\eta)m(\xi,\eta)e^{2\pi i\langle \xi+\eta,x\rangle}\,d\xi\,d\eta \\ &= B_m(T_{-\xi_0}f,T_{-\eta_0}g)(x). \end{aligned} \tag{2.21}$$

Also, the inequalities $\|T_{-\xi_0}f\|_{p_1,\omega_1}\le \omega_1(-\xi_0)\|f\|_{p_1,\omega_1}$ and $\|T_{-\eta_0}g\|_{p_2,\omega_2}\le \omega_2(-\eta_0)\|g\|_{p_2,\omega_2}$ are satisfied for all $f\in L^{p_1}_{\omega_1}(\mathbb{R}^n)$, $g\in L^{p_2}_{\omega_2}(\mathbb{R}^n)$. Hence, since $m\in \mathrm{BM}(\omega_1,\omega_2,\omega_3)$, by (2.21) we have

$$\begin{aligned} \big\|B_{M_{(\xi_0,\eta_0)}m}(f,g)\big\|_{p_3,\omega_3} &= \big\|B_m(T_{-\xi_0}f,T_{-\eta_0}g)\big\|_{p_3,\omega_3} \le \|B_m\|\|T_{-\xi_0}f\|_{p_1,\omega_1}\|T_{-\eta_0}g\|_{p_2,\omega_2} \\ &\le \omega_1(-\xi_0)\omega_2(-\eta_0)\|B_m\|\|f\|_{p_1,\omega_1}\|g\|_{p_2,\omega_2}. \end{aligned} \tag{2.22}$$

Then $M_{(\xi_0,\eta_0)}m \in \mathrm{BM}(\omega_1,\omega_2,\omega_3)$, and by (2.22) we obtain

$$\begin{aligned}\|M_{(\xi_0,\eta_0)}m\|_{(\omega_1,\omega_2,\omega_3)} &= \sup\left\{\frac{\|B_{M_{(\xi_0,\eta_0)}m}(f,g)\|_{p_3,\omega_3}}{\|f\|_{p_1,\omega_1}\|g\|_{p_2,\omega_2}} : \|f\|_{p_1,\omega_1}\le 1, \|g\|_{p_2,\omega_2}\le 1\right\}\\ &\le \omega_1(-\xi_0)\omega_2(-\eta_0)\|m\|_{(\omega_1,\omega_2,\omega_3)}.\end{aligned}$$

□

Lemma 2.2 *If υ_s is a polynomial-type weight function and $f \in L^p_{\upsilon_s}(\mathbb{R}^n)$, then $D^p_t f \in L^p_{\upsilon_s}(\mathbb{R}^n)$. Moreover,*

$$\begin{aligned}&\left\|D^p_t f\right\|_{p,\upsilon_s} \le \|f\|_{p,\upsilon_s} \quad \text{if } t\le 1,\\ &\left\|D^p_t f\right\|_{p,\upsilon_s} < t^s\|f\|_{p,\upsilon_s} \quad \text{if } t> 1.\end{aligned}$$

Proof Let υ_s be a polynomial-type weight function and $f \in L^p_{\upsilon_s}(\mathbb{R}^n)$. Assume that $t \le 1$. If we get $\frac{x}{t} = u$,

$$\begin{aligned}\left\|D^p_t f\right\|_{p,\upsilon_s} &= \left\{\int_{\mathbb{R}^n}\left|D^p_t f(x)\right|^p \upsilon_s(x)^p\,dx\right\}^{\frac{1}{p}}\\ &= \left\{\int_{\mathbb{R}^n}\left|t^{-\frac{n}{p}}f\left(\frac{x}{t}\right)\right|^p\left(1+|x|\right)^{sp}dx\right\}^{\frac{1}{p}} = \left\{\int_{\mathbb{R}^n}\left|f(u)\right|^p\left(1+|ut|\right)^{sp}du\right\}^{\frac{1}{p}}\\ &\le \left\{\int_{\mathbb{R}^n}\left|f(u)\right|^p\left(1+|u|\right)^{sp}du\right\}^{\frac{1}{p}}\\ &= \|f\|_{p,\upsilon_s} < \infty.\end{aligned} \tag{2.23}$$

Thus we have $D^p_t f \in L^p_{\upsilon_s}(\mathbb{R}^n)$ and $\|D^p_t f\|_{p,\upsilon_s} \le \|f\|_{p,\upsilon_s}$.

Now, assume that $t > 1$. Similarly by (2.23)

$$\begin{aligned}\left\|D^p_t f\right\|_{p,\upsilon_s} &= \left\{\int_{\mathbb{R}^n}\left|f(u)\right|^p\left(1+|ut|\right)^{sp}du\right\}^{\frac{1}{p}}\\ &< \left\{\int_{\mathbb{R}^n}\left|f(u)\right|^p\left(t+|ut|\right)^{sp}du\right\}^{\frac{1}{p}} = t^s\left\{\int_{\mathbb{R}^n}\left|f(u)\right|^p\left(1+|u|\right)^{sp}du\right\}^{\frac{1}{p}}\\ &= t^s\|f\|_{p,\upsilon_s} < \infty.\end{aligned}$$

Hence $D^p_t f \in L^p_{\upsilon_s}(\mathbb{R}^n)$, and we also have $\|D^p_t f\|_{p,\upsilon_s} < t^s\|f\|_{p,\upsilon_s}$. □

Theorem 2.5 *Let υ_{s_1}, υ_{s_2}, υ_{s_3} be weight functions of polynomial type and let $m \in \mathrm{BM}(\upsilon_{s_1},\upsilon_{s_2},\upsilon_{s_3})$. If $\frac{2}{q} = \frac{1}{p_1} + \frac{1}{p_2} - \frac{1}{p_3}$ and $0 < t < \infty$, then $D^q_t m \in \mathrm{BM}(\upsilon_{s_1},\upsilon_{s_2},\upsilon_{s_3})$. Moreover, then*

$$\begin{aligned}&\left\|D^q_t m\right\|_{(\upsilon_{s_1},\upsilon_{s_2},\upsilon_{s_3})} \le \left(\frac{1}{t}\right)^{s_3}\|m\|_{(\upsilon_{s_1},\upsilon_{s_2},\upsilon_{s_3})} \quad \text{if } t\le 1,\\ &\left\|D^q_t m\right\|_{(\upsilon_{s_1},\upsilon_{s_2},\upsilon_{s_3})} < t^{s_1+s_2}\|m\|_{(\upsilon_{s_1},\upsilon_{s_2},\upsilon_{s_3})} \quad \text{if } t> 1.\end{aligned}$$

Proof Let $f \in L^{p_1}_{\upsilon_{s_1}}(\mathbb{R}^n)$ and $g \in L^{p_2}_{\upsilon_{s_2}}(\mathbb{R}^n)$ be given. We know by Lemma 2.2 that $D^{p_1}_t f \in L^{p_1}_{\upsilon_{s_1}}(\mathbb{R}^n)$ and $D^{p_2}_t g \in L^{p_2}_{\upsilon_{s_2}}(\mathbb{R}^n)$. If we get $\frac{\xi}{t} = u$ and $\frac{\eta}{t} = v$, we obtain

$$\begin{aligned} B_{D^q_t m}(f,g)(x) &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \hat{f}(\xi)\hat{g}(\eta) D^q_t m(\xi,\eta) e^{2\pi i\langle \xi+\eta,x\rangle}\,d\xi\,d\eta \\ &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \hat{f}(tu)\hat{g}(tv) t^{-\frac{2n}{q}} m(u,v) e^{2\pi i\langle u+v,tx\rangle} t^{2n}\,du\,dv. \end{aligned}$$

Hence, from the equality $\frac{2}{q} = \frac{1}{p_1} + \frac{1}{p_2} - \frac{1}{p_3}$, we have

$$\begin{aligned} B_{D^q_t m}(f,g)(x) &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \hat{f}(tu)\hat{g}(tv) t^{-n(\frac{1}{p_1}+\frac{1}{p_2}-\frac{1}{p_3})} m(u,v) e^{2\pi i\langle u+v,tx\rangle} t^{2n}\,du\,dv \\ &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} t^{-n(1-\frac{1}{p'_1})}\hat{f}(tu) t^{-n(1-\frac{1}{p'_1})}\hat{g}(tv) t^{\frac{n}{p_3}} m(u,v) e^{2\pi i\langle u+v,tx\rangle} t^{2n}\,du\,dv \\ &= t^{\frac{n}{p_3}} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} D^{p'_1}_{t^{-1}}\hat{f}(u) D^{p'_2}_{t^{-1}}\hat{g}(v) m(u,v) e^{2\pi i\langle u+v,tx\rangle}\,du\,dv \\ &= t^{\frac{n}{p_3}} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \big(D^{p_1}_t f\big)\hat{}\,(u)\big(D^{p_2}_t g\big)\hat{}\,(v) m(u,v) e^{2\pi i\langle u+v,tx\rangle}\,du\,dv \\ &= D^{p_3}_{t^{-1}} B_m\big(D^{p_1}_t f, D^{p_2}_t g\big)(x). \end{aligned} \tag{2.24}$$

Assume that $t \le 1$. Since $m \in B_m(\upsilon_{s_1},\upsilon_{s_2},\upsilon_{s_3})$, by Lemma 2.2 and using equality (2.24), we obtain

$$\begin{aligned} \big\|B_{D^q_t m}(f,g)\big\|_{p_3,\upsilon_{s_3}} &= \big\|D^{p_3}_{t^{-1}} B_m\big(D^{p_1}_t f, D^{p_2}_t g\big)(x)\big\|_{p_3,\upsilon_{s_3}} \\ &\le \left(\frac{1}{t}\right)^{s_3} \big\|B_m\big(D^{p_1}_t f, D^{p_2}_t g\big)(x)\big\|_{p_3,\upsilon_{s_3}} \\ &\le \left(\frac{1}{t}\right)^{s_3} \|B_m\| \big\|D^{p_1}_t f\big\|_{p,\upsilon_{s_1}} \big\|D^{p_2}_t g\big\|_{p,\upsilon_{s_2}} \\ &\le \left(\frac{1}{t}\right)^{s_3} \|m\|_{(\upsilon_{s_1},\upsilon_{s_2},\upsilon_{s_3})} \|f\|_{p_1,\upsilon_{s_1}} \|g\|_{p_2,\upsilon_{s_2}}. \end{aligned} \tag{2.25}$$

Then $D^q_t m \in \mathrm{BM}(\upsilon_{s_1},\upsilon_{s_2},\upsilon_{s_3})$, and by (2.25)

$$\big\|D^q_t m\big\|_{(\upsilon_{s_1},\upsilon_{s_2},\upsilon_{s_3})} \le \left(\frac{1}{t}\right)^{s_3} \|m\|_{(\upsilon_{s_1},\upsilon_{s_2},\upsilon_{s_3})}.$$

Now let $t > 1$. Again, since $m \in \mathrm{BM}(\upsilon_{s_1},\upsilon_{s_2},\upsilon_{s_3})$, by Lemma 2.2 and using equality (2.24), we obtain

$$\begin{aligned} \big\|B_{D^q_t m}(f,g)\big\|_{p_3,\upsilon_{s_3}} &< \big\|B_m\big(D^{p_1}_t f, D^{p_2}_t g\big)\big\|_{p_3,\upsilon_{s_3}} \\ &\le \|B_m\| \big\|D^{p_1}_t f\big\|_{p,\upsilon_{s_1}} \big\|D^{p_2}_t g\big\|_{p,\upsilon_{s_2}} \\ &< t^{s_1+s_2} \|B_m\| \|f\|_{p_1,\upsilon_{s_1}} \|g\|_{p_2,\upsilon_{s_2}} \\ &= t^{s_1+s_2} \|m\|_{(\upsilon_{s_1},\upsilon_{s_2},\upsilon_{s_3})} \|f\|_{p_1,\upsilon_{s_1}} \|g\|_{p_2,\upsilon_{s_2}}. \end{aligned} \tag{2.26}$$

Thus $D_t^q m \in \mathrm{BM}(\upsilon_{s_1}, \upsilon_{s_2}, \upsilon_{s_3})$ and by (2.26)

$$\|D_t^q m\|_{(\upsilon_{s_1},\upsilon_{s_2},\upsilon_{s_3})} < t^{s_1+s_2} \|m\|_{(\upsilon_{s_1},\upsilon_{s_2},\upsilon_{s_3})}. \qquad \square$$

Theorem 2.6 *Let υ_{s_1}, υ_{s_2}, υ_{s_3} be weight functions of polynomial type and let $m \in \mathrm{BM}(\upsilon_{s_1}, \upsilon_{s_2}, \upsilon_{s_3})$ such that $m(t\xi, t\eta) = m(\xi, \eta)$ for any $t > 0$, where $\frac{2}{q} = \frac{1}{p_1} + \frac{1}{p_2} - \frac{1}{p_3}$. Then*

$$\frac{2}{q} < \frac{s_3}{n} \quad \text{if } t < 1,$$
$$\frac{2}{q} > -\frac{s_1 + s_2}{n} \quad \text{if } t > 1.$$

Proof Take any $f \in L_{\upsilon_{s_1}}^{p_1}(\mathbb{R}^n)$, $g \in L_{\upsilon_{s_2}}^{p_2}(\mathbb{R}^n)$. It is known by Theorem 2.5 that

$$B_{D_t^q m}(f,g)(x) = D_{t^{-1}}^{p_3} B_m\big(D_t^{p_1} f, D_t^{p_2} g\big)(x), \quad x \in \mathbb{R}^n. \tag{2.27}$$

On the other hand, using $m(t\xi, t\eta) = m(\xi, \eta)$ and changing the variables $tu = \xi$, $tv = \eta$, we note that

$$\begin{aligned}
&D_{t^{-1}}^{p_3} B_m\big(D_t^{p_1} f, D_t^{p_2} g\big)(x) \\
&\quad = t^{\frac{n}{p_3}} \int_{\mathbb{R}^n} \big(D_t^{p_1} f\big)^\wedge(u) \big(D_t^{p_2} g\big)^\wedge(v) m(u,v) e^{2\pi i \langle u+v, tx\rangle}\, du\, dv \\
&\quad = t^{\frac{n}{p_3}} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} D_{t^{-1}}^{p_1'} \hat{f}(u) D_{t^{-1}}^{p_2'} \hat{g}(v) m(u,v) e^{2\pi i \langle u+v, tx\rangle}\, du\, dv \\
&\quad = t^{\frac{n}{p_3}} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \hat{f}(\xi) \hat{g}(\eta) m\big(t^{-1}\xi, t^{-1}\eta\big) e^{2\pi i \langle t^{-1}\xi + t^{-1}\eta, tx\rangle} t^{n(\frac{1}{p_1'} + \frac{1}{p_2'})} t^{-2n}\, d\xi\, d\eta \\
&\quad = t^{\frac{n}{p_3}} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \hat{f}(\xi) \hat{g}(\eta) m(\xi,\eta) e^{2\pi i \langle u+v, x\rangle} t^{-n(\frac{1}{p_1} + \frac{1}{p_2})}\, d\xi\, d\eta \\
&\quad = t^{n(\frac{1}{p_3} - \frac{1}{p_1} - \frac{1}{p_2})} B_m(f,g)(x).
\end{aligned} \tag{2.28}$$

Hence by (2.27) and (2.28), we have

$$B_m(f,g)(x) = t^{-n(\frac{1}{p_3} - \frac{1}{p_1} - \frac{1}{p_2})} B_{D_t^q m}(f,g)(x).$$

Since $D_t^q m = m$ for $t = 1$, we let $t \neq 1$. Assume first that $t < 1$. Also, since $m \in \mathrm{BM}(\upsilon_{s_1}, \upsilon_{s_2}, \upsilon_{s_3})$, by Theorem 2.5 we have $D_t^q m \in \mathrm{BM}(\upsilon_{s_1}, \upsilon_{s_2}, \upsilon_{s_3})$ and $\|D_t^q m\|_{(\upsilon_{s_1},\upsilon_{s_2},\upsilon_{s_3})} < (\frac{1}{t})^{s_3} \times \|m\|_{(\upsilon_{s_1},\upsilon_{s_2},\upsilon_{s_3})}$. Then by (2.28)

$$\begin{aligned}
\big\|B_m(f,g)(x)\big\|_{p_3,\upsilon_{s_3}} &= t^{-n(\frac{1}{p_3} - \frac{1}{p_1} - \frac{1}{p_2})} \big\|B_{D_t^q}(f,g)(x)\big\|_{p_3,\upsilon_{s_3}} \\
&\leq t^{-n(\frac{1}{p_3} - \frac{1}{p_1} - \frac{1}{p_2})} \big\|D_t^q m\big\|_{(\upsilon_{s_1},\upsilon_{s_2},\upsilon_{s_3})} \|f\|_{p_1,\upsilon_{s_1}} \|g\|_{p_2,\upsilon_{s_2}} \\
&< t^{-n(\frac{1}{p_3} - \frac{1}{p_1} - \frac{1}{p_2}) - s_3} \|m\|_{(\upsilon_{s_1},\upsilon_{s_2},\upsilon_{s_3})} \|f\|_{p_1,\upsilon_{s_1}} \|g\|_{p_2,\upsilon_{s_2}} \\
&= t^{-n(\frac{1}{p_3} - \frac{1}{p_1} - \frac{1}{p_2}) - s_3} \|B_m\| \|f\|_{p_1,\upsilon_{s_1}} \|g\|_{p_2,\upsilon_{s_2}}.
\end{aligned}$$

Thus,

$$\|B_m\| < t^{-n(\frac{1}{p_3}-\frac{1}{p_1}-\frac{1}{p_2})-s_3}\|B_m\| = t^{\frac{2n}{q}-s_3}\|B_m\|.$$

Hence $1 < t^{\frac{2n}{q}-s_3}$. Since $t < 1$, we have $\frac{2n}{q} - s_3 < 0$. Thus, we write $\frac{2}{q} < \frac{s_3}{n}$.

Assume now that $t > 1$. Again, by Theorem 2.5, we have $D_t^q m \in \mathrm{BM}(\upsilon_{s_1}, \upsilon_{s_2}, \upsilon_{s_3})$ and $\|D_t^q m\|_{(\upsilon_{s_1},\upsilon_{s_2},\upsilon_{s_3})} < t^{s_1+s_2}\|m\|_{(\upsilon_{s_1},\upsilon_{s_2},\upsilon_{s_3})}$. Similarly,

$$\left\|B_m(f,g)(x)\right\|_{p_3,\upsilon_{s_3}} < t^{-n(\frac{1}{p_3}-\frac{1}{p_1}-\frac{1}{p_2})+s_1+s_2}\|B_m\|\|f\|_{p_1,\upsilon_{s_1}}\|g\|_{p_2,\upsilon_{s_2}}.$$

Thus, we have

$$\|B_m\| < t^{-n(\frac{1}{p_3}-\frac{1}{p_1}-\frac{1}{p_2})+s_1+s_2}\|B_m\| = t^{\frac{2n}{q}+s_1+s_2}\|B_m\|.$$

Hence $1 < t^{\frac{2n}{q}+s_1+s_2}$ Since $t > 1$, we have $\frac{2n}{q} + s_1 + s_2 > 0$. Thus, we write $\frac{2}{q} > -\frac{s_1+s_2}{n}$. □

Theorem 2.7 *Let* $m \in \mathrm{BM}(\omega_1, \omega_2, \omega_3)$ *and* $p_3 \geq 1$.

(a) *If* $\Phi \in L^1(\mathbb{R}^n)$, *then* $\Phi * m \in \mathrm{BM}(\omega_1, \omega_2, \omega_3)$ *and*

$$\|\Phi * m\|_{(\omega_1,\omega_2,\omega_3)} \leq \|\Phi\|_1\|m\|_{(\omega_1,\omega_2,\omega_3)}.$$

(b) *If* $\Phi \in L^1_\omega(\mathbb{R}^n)$ *such that* $\omega(u,\upsilon) = \omega_1(u)\omega_2(\upsilon)$, *then* $\hat{\Phi}m \in \mathrm{BM}(\omega_1, \omega_2, \omega_3)$ *and*

$$\|\hat{\Phi}m\|_{(\omega_1,\omega_2,\omega_3)} \leq \|\Phi\|_{1,\omega}\|m\|_{(\omega_1,\omega_2,\omega_3)}.$$

Proof (a) Let $f \in L^{p_1}_{\omega_1}(\mathbb{R}^n)$ and $g \in L^{p_2}_{\omega_2}(\mathbb{R}^n)$. Since $L^{p_1}_{\omega_1}(\mathbb{R}^n) \subset L^{p_1}(\mathbb{R}^n)$ and $L^{p_2}_{\omega_2}(\mathbb{R}^n) \subset L^{p_2}(\mathbb{R}^n)$, then by Proposition 2.5 in [11]

$$B_{\Phi*m}(f,g)(x) = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\Phi(u,v)B_{T_{(u,v)}m}(f,g)(x)\,du\,dv.$$

Also, since $m \in \mathrm{BM}(\omega_1, \omega_2, \omega_3)$, we have $T_{(u,v)}m \in \mathrm{BM}(\omega_1, \omega_2, \omega_3)$ by Theorem 2.4. So, we write

$$\begin{aligned}\left\|B_{\Phi*m}(f,g)\right\|_{p_3,\omega_3} &\leq \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\left\|\Phi(u,v)B_{T_{(u,v)}m}(f,g)\right\|_{p_3,\omega_3}du\,dv \\ &\leq \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\left|\Phi(u,v)\right|\|T_{(u,v)}m\|_{(\omega_1,\omega_2,\omega_3)}\|f\|_{p_1,\omega_1}\|g\|_{p_2,\omega_2}\,du\,dv \\ &= \|m\|_{(\omega_1,\omega_2,\omega_3)}\|\Phi\|_1\|f\|_{p_1,\omega_1}\|g\|_{p_2,\omega_2} < \infty. \end{aligned} \tag{2.29}$$

Hence $\Phi * m \in \mathrm{BM}(\omega_1, \omega_2, \omega_3)$. Finally, by (2.29), we obtain

$$\|\Phi * m\|_{(\omega_1,\omega_2,\omega_3)} \leq \|\Phi\|_1\|m\|_{(\omega_1,\omega_2,\omega_3)}.$$

(b) Let $\Phi \in L^1_\omega(\mathbb{R}^n)$. Take any $f \in L^{p_1}_{\omega_1}(\mathbb{R}^n)$ and $g \in L^{p_2}_{\omega_2}(\mathbb{R}^n)$. It is known by Proposition 2.5 in [11] that the equality

$$B_{\hat{\Phi}m}(f,g)(x) = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\Phi(u,v)B_{M_{(-u,-v)}m}(f,g)(x)\,du\,dv.$$

Since $m \in \mathrm{BM}(\omega_1, \omega_2, \omega_3)$, by Theorem 2.4 we have $M_{(-u,-v)}m \in \mathrm{BM}(\omega_1, \omega_2, \omega_3)$ and

$$\|M_{(-u,-v)}m\|_{(\omega_1,\omega_2,\omega_3)} \leq \omega_1(u)\omega_2(\upsilon)\|m\|_{(\omega_1,\omega_2,\omega_3)}.$$

Then we write

$$\begin{aligned}\|B_{\hat{\Phi}m}(f,g)\|_{p_3,\omega_3} &\leq \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \|\Phi(u,v)B_{M_{(-u,-v)}m}(f,g)\|_{p_3,\omega_3}\,du\,dv \\ &\leq \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |\Phi(u,v)|\,\|M_{(-u,-v)}m\|_{(\omega_1,\omega_2,\omega_3)}\|f\|_{p_1,\omega_1}\|g\|_{p_2,\omega_2}\,du\,dv \\ &\leq \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |\Phi(u,v)|\omega_1(u)\omega_2(\upsilon)\|m\|_{(\omega_1,\omega_2,\omega_3)}\|f\|_{p_1,\omega_1}\|g\|_{p_2,\omega_2}\,du\,dv \\ &= \|m\|_{(\omega_1,\omega_2,\omega_3)}\|f\|_{p_1,\omega_1}\|g\|_{p_2,\omega_2}\|\Phi\|_{1,\omega}. \end{aligned} \tag{2.30}$$

Thus from (2.30), we obtain $\hat{\Phi}m \in \mathrm{BM}(\omega_1, \omega_2, \omega_3)$ and

$$\|\hat{\Phi}m\|_{(\omega_1,\omega_2,\omega_3)} \leq \|\Phi\|_{1,\omega}\|m\|_{(\omega_1,\omega_2,\omega_3)}. \qquad \square$$

Theorem 2.8 *Let υ_{s_1}, υ_{s_2}, υ_{s_3} be weight functions of polynomial type and let $m \in \mathrm{BM}(\upsilon_{s_1}, \upsilon_{s_2}, \upsilon_{s_3})$. If $\Psi \in L^1(\mathbb{R}^+, t^{-\frac{2n}{q}}\,dt)$ such that $\frac{2}{q} = \frac{1}{p_1} + \frac{1}{p_2} - \frac{1}{p_3}$, then $m_\Psi(\xi,\eta) = \int_0^\infty m(t\xi, t\eta)\Psi(t)\,dt \in \mathrm{BM}(\upsilon_{s_1}, \upsilon_{s_2}, \upsilon_{s_3})$. Moreover,*

$$\|m_\Psi\|_{(\upsilon_{s_1},\upsilon_{s_2},\upsilon_{s_3})} < \|\Psi\|_{L^1(\mathbb{R}^+, t^{-\frac{2n}{q}}\,dt)}\|m\|_{(\upsilon_{s_1},\upsilon_{s_2},\upsilon_{s_3})}.$$

Proof Let us take $f, g \in S(\mathbb{R}^n)$. Then

$$\begin{aligned} B_{m_\Psi}(f,g)(x) &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \hat{f}(\xi)\hat{g}(\eta)m_\Psi(\xi,\eta)e^{2\pi i\langle u+v,x\rangle}\,d\xi\,d\eta \\ &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \hat{f}(\xi)\hat{g}(\eta)\left\{\int_0^\infty m(t\xi,t\eta)\Psi(t)\,dt\right\}e^{2\pi i\langle u+v,x\rangle}\,d\xi\,d\eta \\ &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \hat{f}(\xi)\hat{g}(\eta)\left\{\int_0^\infty D^q_{t^{-1}}m(\xi,\eta)\Psi(t)t^{-\frac{2n}{q}}\,dt\right\}e^{2\pi i\langle u+v,x\rangle}\,d\xi\,d\eta \\ &= \int_0^\infty B_{D^q_{t^{-1}}m}(f,g)\Psi(t)t^{-\frac{2n}{q}}\,dt. \end{aligned}$$

Since $m \in \mathrm{BM}(\upsilon_{s_1}, \upsilon_{s_2}, \upsilon_{s_3})$, $D^q_{t^{-1}}m \in \mathrm{BM}(\upsilon_{s_1}, \upsilon_{s_2}, \upsilon_{s_3})$ by Theorem 2.5, thus we observe that

$$\begin{aligned} \|B_{m_\Psi}(f,g)(x)\|_{p_3,\upsilon_{s_3}} &\leq \int_0^\infty \|B_{D^q_{t^{-1}}m}(f,g)\|_{p_3,\upsilon_{s_3}}|\Psi(t)|t^{-\frac{2n}{q}}\,dt \\ &\leq \int_0^\infty \|B_{D^q_{t^{-1}}m}\|\,\|f\|_{p_1,\upsilon_{s_1}}\|g\|_{p_2,\upsilon_{s_2}}|\Psi(t)|t^{-\frac{2n}{q}}\,dt \\ &= \int_0^\infty \|D^q_{t^{-1}}m\|_{(\upsilon_{s_1},\upsilon_{s_2},\upsilon_{s_3})}\|f\|_{p_1,\upsilon_{s_1}}\|g\|_{p_2,\upsilon_{s_2}}|\Psi(t)|t^{-\frac{2n}{q}}\,dt \\ &< \int_0^1 t^{s_3}\|m\|_{(\upsilon_{s_1},\upsilon_{s_2},\upsilon_{s_3})}\|f\|_{p_1,\upsilon_{s_1}}\|g\|_{p_2,\upsilon_{s_2}}|\Psi(t)|t^{-\frac{2n}{q}}\,dt \\ &\quad + \int_1^\infty t^{-s_1-s_2}\|m\|_{(\upsilon_{s_1},\upsilon_{s_2},\upsilon_{s_3})}\|f\|_{p_1,\upsilon_{s_1}}\|g\|_{p_2,\upsilon_{s_2}}|\Psi(t)|t^{-\frac{2n}{q}}\,dt \end{aligned}$$

$$= \|m\|_{(\upsilon_{s_1},\upsilon_{s_2},\upsilon_{s_3})}\|f\|_{p_1,\upsilon_{s_1}}\|g\|_{p_2,\upsilon_{s_2}} \times \left\{\int_0^1 t^{s_3}\left|\Psi(t)\right|t^{-\frac{2n}{q}}\,dt + \int_1^\infty t^{-s_1-s_2}\left|\Psi(t)\right|t^{-\frac{2n}{q}}\,dt\right\}. \quad (2.31)$$

Also, since $t^{s_3} \le 1$ for $s_3 \ge 0$, $t \le 1$ and $t^{-s_1-s_2} < 1$ for $-s_1 - s_2 \le 0$, $t > 1$, by (2.31)

$$\left\|B_{m_\Psi}(f,g)(x)\right\|_{p_3,\upsilon_{s_3}} < \|\Psi\|_{L^1(\mathbb{R}^+,t^{-\frac{2n}{q}}dt)}\|m\|_{(\upsilon_{s_1},\upsilon_{s_2},\upsilon_{s_3})}\|f\|_{p_1,\upsilon_{s_1}}\|g\|_{p_2,\upsilon_{s_2}}.$$

Hence, $m_\Psi \in \mathrm{BM}(\upsilon_{s_1},\upsilon_{s_2},\upsilon_{s_3})$ and

$$\|m_\Psi\|_{(\upsilon_{s_1},\upsilon_{s_2},\upsilon_{s_3})} < \|\Psi\|_{L^1(\mathbb{R}^+,t^{-\frac{2n}{q}}dt)}\|m\|_{(\upsilon_{s_1},\upsilon_{s_2},\upsilon_{s_3})}.$$

□

Theorem 2.9 *Let $p_3 \ge 1$ and $m \in \mathrm{BM}(\omega_1,\omega_2,\omega_3)$. If Q_1, Q_2 are bounded measurable sets in $\mathbb{R}^n$, then*

$$h(\xi,\eta) = \frac{1}{\mu(Q_1\times Q_2)}\int\int_{Q_1\times Q_2} m(\xi+u,\eta+v)\,du\,dv \in \mathrm{BM}(\omega_1,\omega_2,\omega_3).$$

Proof Take any $f,g \in S(\mathbb{R}^n)$. Then we write

$$\begin{aligned} &B_h(f,g)(x) \\ &\quad= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\hat{f}(\xi)\hat{g}(\eta)h(\xi,\eta)e^{2\pi i\langle\xi+\eta,x\rangle}\,d\xi\,d\eta \\ &\quad= \frac{1}{\mu(Q_1\times Q_2)}\int\int_{Q_1\times Q_2}\left\{\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\hat{f}(\xi)\hat{g}(\eta)m(\xi+u,\eta+v)e^{2\pi i\langle\xi+\eta,x\rangle}\,d\xi\,d\eta\right\}du\,dv \\ &\quad= \frac{1}{\mu(Q_1\times Q_2)}\int\int_{Q_1\times Q_2} B_{T_{(-u,-v)}m}(f,g)(x)\,du\,dv. \end{aligned}$$

By using Theorem 2.4, we have

$$\begin{aligned} \left\|B_h(f,g)\right\|_{p_3,\omega_3} &\le \frac{1}{\mu(Q_1\times Q_2)}\int\int_{Q_1\times Q_2}\left\|B_{T_{(-u,-v)}m}(f,g)\right\|_{p_3,\omega_3}du\,dv \\ &\le \frac{1}{\mu(Q_1\times Q_2)}\int\int_{Q_1\times Q_2}\|T_{(-u,-v)}m\|_{(\omega_1,\omega_2,\omega_3)}\|f\|_{p_1,\omega_1}\|g\|_{p_2,\omega_2}\,du\,dv \\ &= \frac{1}{\mu(Q_1\times Q_2)}\|m\|_{(\omega_1,\omega_2,\omega_3)}\|f\|_{p_1,\omega_1}\|g\|_{p_2,\omega_2}\mu(Q_1\times Q_2) \\ &= \|m\|_{(\omega_1,\omega_2,\omega_3)}\|f\|_{p_1,\omega_1}\|g\|_{p_2,\omega_2}. \end{aligned}$$

Hence, we obtain $h(\xi,\eta) \in \mathrm{BM}(\omega_1,\omega_2,\omega_3)$. □

Theorem 2.10 *Let $\omega(u,v) = \omega_1(u)\omega_2(v)$, $\omega_3 \le \omega_1$, $\omega_3(-u) = \omega_3(u)$ and $\frac{1}{p_1}+\frac{1}{p_2}=\frac{1}{p_3}\le 1$. Assume that $\Phi \in L^1_\omega(\mathbb{R}^{2n})$, $\Psi_1 \in L^{p_1}_{\omega_1}(\mathbb{R}^n)$ and $\Psi_2 \in L^{p_2}_{\omega_2}(\mathbb{R}^n)$. If $m(\xi,\eta) = \hat{\Psi}_1(\xi)\hat{\Phi}(\xi,\eta)\hat{\Psi}_2(\eta)$, then $m \in \mathrm{BM}(1,\omega_1;1,\omega_2;p_3,\omega_3)$.*

Proof For the proof we will use Theorem 2.2. Take any $f, g, h \in S(\mathbb{R}^n)$. Then

$$\begin{aligned}
&\left|\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\hat{f}(\xi)\hat{g}(\eta)\hat{h}(\xi+\eta)m(\xi,\eta)\,d\xi\,d\eta\right| \\
&\quad = \left|\int_{\mathbb{R}^n} h(y)\left\{\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\hat{f}(\xi)\hat{g}(\eta)\hat{\Psi}_1(\xi)\hat{\Phi}(\xi,\eta)\hat{\Psi}_2(\eta)e^{-2\pi i\langle\xi+\eta,x\rangle}\,d\xi\,d\eta\right\}dy\right| \\
&\quad \le \int_{\mathbb{R}^n}\left|h(y)\omega_3^{-1}(y)B_{\hat{\Phi}}(f*\Psi_1, g*\Psi_2)(-y)\omega_3(y)\right|dy.
\end{aligned} \tag{2.32}$$

Since the spaces $L^{p_1}_{\omega_1}(\mathbb{R}^n)$ and $L^{p_2}_{\omega_2}(\mathbb{R}^n)$ are Banach convolution module over the spaces $L^1_{\omega_1}(\mathbb{R}^n)$, $L^1_{\omega_2}(\mathbb{R}^n)$ respectively, we write $f*\Psi_1 \in L^{p_1}_{\omega_1}(\mathbb{R}^n)$ and $g*\Psi_2 \in L^{p_2}_{\omega_2}(\mathbb{R}^n)$. Also, by Theorem 2.7, $\hat{\Phi} \in \mathrm{BM}(p_1,\omega_1;p_2,\omega_2;p_3,\omega_3)$. Therefore we obtain $B_{\hat{\Phi}}(f*\Psi_1, g*\Psi_2) \in L^{p_3}_{\omega_3}(\mathbb{R}^n)$. By using the Hölder inequality and the inequality (2.32), we find

$$\begin{aligned}
&\left|\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\hat{f}(\xi)\hat{g}(\eta)\hat{h}(\xi+\eta)m(\xi,\eta)\,d\xi\,d\eta\right| \\
&\quad \le \|h\|_{p_3^{-1},\omega_3^{-1}}\left\|B_{\hat{\Phi}}(f*\Psi_1, g*\Psi_2)\right\|_{p_3,\omega_3} \\
&\quad \le \|h\|_{p_3^{-1},\omega_3^{-1}}\|B_{\hat{\Phi}}\|\|f\|_{1,\omega_1}\|\Psi_1\|_{p_1,\omega_1}\|g\|_{1,\omega_2}\|\Psi_2\|_{p_2,\omega_2}.
\end{aligned}$$

If we say $C = \|B_{\hat{\Phi}}\|\|\Psi_1\|_{p_1,\omega_1}\|\Psi_2\|_{p_2,\omega_2}$, then we obtain

$$\left|\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\hat{f}(\xi)\hat{g}(\eta)\hat{h}(\xi+\eta)m(\xi,\eta)\,d\xi\,d\eta\right| \le C\|f\|_{1,\omega}\|g\|_{1,\omega_2}\|h\|_{p_3^{-1},\omega_3^{-1}},$$

which means $m \in \mathrm{BM}(1,\omega_1;1,\omega_2;p_3,\omega_3)$. □

The following theorem can be proved easily by using the technique of the proof in Theorem 2.10.

Theorem 2.11 *Let $\omega(u,v) = \omega_1(u)\omega_2(v)$, $\omega_3 \le \omega_1$, $\omega_3(-u) = \omega_3(u)$ and $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} \le 1$. If $m(\xi,\eta) = \hat{\Psi}_1(\xi)\hat{\Phi}(\xi,\eta)\hat{\Psi}_2(\eta)$ such that $\Phi \in L^1_\omega(\mathbb{R}^{2n})$, $\Psi_1 \in L^1_{\omega_1}(\mathbb{R}^n)$ and $\Psi_2 \in L^1_{\omega_2}(\mathbb{R}^n)$, then $m \in \mathrm{BM}(p_1,\omega_1;p_2,\omega_2;p_3,\omega_3)$.*

3 The bilinear multipliers space BM($p_1(x), p_2(x), p_3(x)$)

Definition 3.1 Let $p_1(x), p_2(x), p_3(x) \in P(\mathbb{R}^n)$ and let $p_1^* < \infty$, $p_2^* < \infty$, $p_3^* < \infty$. Assume that $m(\xi,\eta)$ is a bounded function on $\mathbb{R}^n \times \mathbb{R}^n$. Define

$$B_m(f,g)(x) = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\hat{f}(\xi)\hat{g}(\eta)m(\xi,\eta)e^{2\pi i\langle\xi+\eta,x\rangle}\,d\xi\,d\eta$$

for all $f, g \in C_c^\infty(\mathbb{R}^n)$.

m is said to be a bilinear multiplier on $\mathbb{R}^n$ of type $(p_1(x), p_2(x), p_3(x))$ if there exists $C > 0$ such that

$$\left\|B_m(f,g)\right\|_{p_3(x)} \le C\|f\|_{p_1(x)}\|g\|_{p_2(x)}$$

for all $f, g \in C_c^\infty(\mathbb{R}^n)$, *i.e.*, B_m extends to a bounded bilinear operator from $L^{p_1(x)}(\mathbb{R}^n) \times L^{p_2(x)}(\mathbb{R}^n)$ to $L^{p_3(x)}(\mathbb{R}^n)$. We denote by $\mathrm{BM}(p_1(x), p_2(x), p_3(x))$ the space of bilinear multipliers of type $(p_1(x), p_2(x), p_3(x))$ and $\|m\|_{(p_1(x),p_2(x),p_3(x))} = \|B_m\|$.

The following theorem can be proved easily by using the technique of the proof in Theorem 2.2.

Theorem 3.1 *Let $p_3(-x) = p_3(x)$ and $\frac{1}{p_3(x)} + \frac{1}{q(x)} = 1$ for all $x \in \mathbb{R}^n$. Then $m \in \mathrm{BM}(p_1(x), p_2(x), p_3(x))$ if and only if there exists $C > 0$ such that*

$$\left|\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\hat{f}(\xi)\hat{g}(\eta)\hat{h}(\xi+\eta)m(\xi,\eta)\,d\xi\,d\eta\right| \le C\|f\|_{p_1(x)}\|g\|_{p_2(x)}\|h\|_{q(x)}$$

for all $f, g, h \in C_c^\infty(\mathbb{R}^n)$.

Theorem 3.2 *Let $\frac{1}{p(x)} + \frac{1}{q(x)} = \frac{1}{r}$. If $\Phi \in L^1(\mathbb{R}^n)$, then $m(\xi,\eta) = \hat{\Phi}(\xi+\eta) \in \mathrm{BM}(p(x), q(x), r)$.*

Proof Take any $f, g, h \in C_c^\infty(\mathbb{R}^n)$. Then

$$\begin{aligned}
&\left|\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\hat{f}(\xi)\hat{g}(\eta)\hat{h}(\xi+\eta)\hat{\Phi}(\xi+\eta)\,d\xi\,d\eta\right| \\
&\quad= \left|\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\hat{f}(\xi)\hat{g}(\eta)(h*\Phi)\hat{}\,(\xi+\eta)\,d\xi\,d\eta\right| \\
&\quad= \left|\int_{\mathbb{R}^n}(h*\Phi)(x)\left\{\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\hat{f}(\xi)\hat{g}(\eta)e^{-2\pi i\langle\xi+\eta,x\rangle}\,d\xi\,d\eta\right\}dx\right| \\
&\quad\le \int_{\mathbb{R}^n}\left|(h*\Phi)(x)\right|\left|\tilde{B}_1(f,g)(x)\right|dx.
\end{aligned} \tag{3.1}$$

Since the space $L^{r'}(\mathbb{R}^n)$ is the Banach convolution module over $L^1(\mathbb{R}^n)$ such that $\frac{1}{r} + \frac{1}{r'} = 1$, we write $h * \Phi \in L^{r'}(\mathbb{R}^n)$. Also, we have $1 \in \mathrm{BM}(p(x), q(x), r)$. Then by (3.1), we find $C_1 > 0$ such that

$$\left|\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\hat{f}(\xi)\hat{g}(\eta)\hat{h}(\xi+\eta)\hat{\Phi}(\xi+\eta)\,d\xi\,d\eta\right| \le C_1\|h\|_{r'}\|\Phi\|_1\|f\|_{p(x)}\|g\|_{q(x)}.$$

If we set $C = C_1\|\Phi\|_1$, we obtain

$$\left|\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\hat{f}(\xi)\hat{g}(\eta)\hat{h}(\xi+\eta)\hat{\Phi}(\xi+\eta)\,d\xi\,d\eta\right| \le C\|f\|_{p(x)}\|g\|_{q(x)}\|h\|_{r'}$$

and $m(\xi,\eta) = \hat{\Phi}(\xi+\eta) \in \mathrm{BM}(p(x), q(x), r)$. □

Theorem 3.3 *If $m \in \mathrm{BM}(p_1(x), p_2(x), p_3(x))$, then $T_{(\xi_0,\eta_0)}m \in \mathrm{BM}(p_1(x), p_2(x), p_3(x))$ and*

$$\|T_{(\xi_0,\eta_0)}m\|_{(p_1(x),p_2(x),p_3(x))} = \|m\|_{(p_1(x),p_2(x),p_3(x))}$$

for all $(\xi_0,\eta_0) \in \mathbb{R}^{2n}$.

Proof Let us take any $f, g \in C_c^\infty(\mathbb{R}^n)$. By the proof of (a) Theorem 2.4, we know that

$$B_{T_{(\xi_0,\eta_0)}m}(f,g)(x) = e^{2\pi i\langle\xi_0+\eta_0,x\rangle}B_m(M_{-\xi_0}f, M_{-\eta_0}g)(x), \quad x \in \mathbb{R}^n. \tag{3.2}$$

By Lemma 5 in [8], we know $\|M_{-\xi_0}f\|_{p_1(x)} = \|f\|_{p_1(x)}$and $\|M_{-\eta_0}g\|_{p_2(x)} = \|g\|_{p_2(x)}$. Since $m \in \mathrm{BM}(p_1(x), p_2(x), p_3(x))$, by (3.2), there exists $C > 0$ such that

$$\|B_{T_{(\xi_0,\eta_0)}m}(f,g)\|_{p_3(x)} = \|B_m(M_{-\xi_0}f, M_{-\eta_0}g)\|_{p_3(x)} \le C\|f\|_{p_1(x)}\|g\|_{p_2(x)}.$$

Thus $T_{(\xi_0,\eta_0)}m \in \mathrm{BM}(p_1(x), p_2(x), p_3(x))$. Moreover, by using the same technique as in the proof of Theorem 2.4, we obtain

$$\|T_{(\xi_0,\eta_0)}m\|_{(p_1(x),p_2(x),p_3(x))} = \|m\|_{(p_1(x),p_2(x),p_3(x))}. \qquad \square$$

Theorem 3.4 *Let* $\frac{1}{p(x)} + \frac{1}{q(x)} = \frac{1}{r(x)}$. *If* $m \in \mathrm{BM}(p(x), q(x), r(x))$, *then* $\Phi * m \in \mathrm{BM}(p(x), q(x), r(x))$ *and there exists* $C > 0$ *such that*

$$\|\Phi * m\|_{(p(x),q(x),r(x))} \le C\|\Phi\|_1\|m\|_{(p(x),q(x),r(x))}$$

for all $\Phi \in L^1(\mathbb{R}^{2n})$.

Proof Take any $f, g \in C_c^\infty(\mathbb{R}^n)$. By Proposition 2.5 in [11], we know that

$$B_{\Phi*m}(f,g)(x) = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \Phi(u,v)B_{T_{(\xi_0,\eta_0)}m}(f,g)(x)\,du\,dv. \tag{3.3}$$

Since $m \in \mathrm{BM}(p(x), q(x), r(x))$, then $T_{(\xi_0,\eta_0)}m \in \mathrm{BM}(p(x), q(x), r(x))$ and

$$\|T_{(\xi_0,\eta_0)}m\|_{(p(x),q(x),r(x))} = \|m\|_{(p(x),q(x),r(x))}$$

by Theorem 3.3. Using (3.3) and the Minkowski inequality for a variable exponent Lebesgue space [12], we find $C > 0$ such that

$$\begin{aligned}\|B_{\Phi*m}(f,g)\|_{r(x)} &\le C\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}|\Phi(u,v)|\,\|B_{T_{(\xi_0,\eta_0)}m}(f,g)\|_{r(x)}\,du\,dv\\ &\le C\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}|\Phi(u,v)|\,\|B_{T_{(\xi_0,\eta_0)}m}\|\,\|f\|_{p(x)}\|g\|_{q(x)}\,du\,dv\\ &= C\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}|\Phi(u,v)|\,\|m\|_{(p(x),q(x),r(x))}\|f\|_{p(x)}\|g\|_{q(x)}\,du\,dv\\ &= C\|m\|_{(p(x),q(x),r(x))}\|f\|_{p(x)}\|g\|_{q(x)}\|\Phi\|_1.\end{aligned} \tag{3.4}$$

Hence $\Phi * m \in \mathrm{BM}(p(x), q(x), r(x))$ and by (3.4), we have

$$\|\Phi * m\|_{(p(x),q(x),r(x))} \le C\|\Phi\|_1\|m\|_{(p(x),q(x),r(x))}. \qquad \square$$

Theorem 3.5 *Let* $r(-x) = r(x)$.

(a) *If* $\Psi_1 \in L^p(\mathbb{R}^n)$, $\Psi_2 \in L^q(\mathbb{R}^n)$ *and* $m \in \mathrm{BM}(p, q, r(x))$, *then* $\hat{\Psi}_1(\xi)m(\xi,\eta)\hat{\Psi}_2(\eta) \in \mathrm{BM}(1, 1, r(x))$.
(b) *If* $\Psi_1, \Psi_2 \in L^1(\mathbb{R}^n)$ *and* $m \in \mathrm{BM}(p, q, r(x))$, *then* $\hat{\Psi}_1(\xi)m(\xi,\eta)\hat{\Psi}_2(\eta) \in \mathrm{BM}(p, q, r(x))$.
(c) *If* $\Psi_1 \in L^p(\mathbb{R}^n)$ *and* $m \in \mathrm{BM}(p, q, r(x))$, *then* $\hat{\Psi}_1(\xi)m(\xi,\eta) \in \mathrm{BM}(1, q(x), r(x))$.
(d) *If* $\Psi_1 \in L^1(\mathbb{R}^n)$ *and* $m \in \mathrm{BM}(p, q, r(x))$, *then* $\hat{\Psi}_1(\xi)m(\xi,\eta) \in \mathrm{BM}(p, q(x), r(x))$.

Proof (a) Let $f, g, h \in C_c^\infty(\mathbb{R}^n)$ be given. Then

$$\begin{aligned}
&\left|\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \hat{f}(\xi)\hat{g}(\eta)\hat{h}(\xi+\eta)\hat{\Psi}_1(\xi)m(\xi,\eta)\hat{\Psi}_2(\eta)\,d\xi\,d\eta\right| \\
&\quad = \left|\int_{\mathbb{R}^n} h(y)\left\{\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \hat{f}(\xi)\hat{g}(\eta)\hat{\Psi}_1(\xi)m(\xi,\eta)\hat{\Psi}_2(\eta)e^{-2\pi i\langle\xi+\eta,y\rangle}\,d\xi\,d\eta\right\}dy\right| \\
&\quad = \left|\int_{\mathbb{R}^n} h(y)\left\{\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} (f*\Psi_1)\hat{}(\xi)(g*\Psi_2)\hat{}(\eta)m(\xi,\eta)e^{-2\pi i\langle\xi+\eta,y\rangle}\,d\xi\,d\eta\right\}dy\right| \\
&\quad \le \int_{\mathbb{R}^n} \left|h(y)\right|\left|\tilde{B}_m(f*\Psi_1, g*\Psi_2)(y)\right|dx. \qquad (3.5)
\end{aligned}$$

Since the spaces $L^p(\mathbb{R}^n)$ and $L^q(\mathbb{R}^n)$ are Banach convolution module over $L^1(\mathbb{R}^n)$, we have $f*\Psi_1 \in L^p(\mathbb{R}^n)$ and $g*\Psi_2 \in L^q(\mathbb{R}^n)$. Also, since $m \in \mathrm{BM}(p,q,r(x))$, we write $B_m(f*\Psi_1, g*\Psi_2)(y) \in L^{r(x)}(\mathbb{R}^n)$. Then, by the equality

$$\left\|\tilde{B}_m(f*\Psi_1, g*\Psi_2)(y)\right\|_{r(x)} = \left\|B_m(f*\Psi_1, g*\Psi_2)(y)\right\|_{r(x)},$$

the Hölder inequality and the inequality (3.5), we have

$$\begin{aligned}
&\left|\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \hat{f}(\xi)\hat{g}(\eta)\hat{h}(\xi+\eta)\hat{\Psi}_1(\xi)m(\xi,\eta)\hat{\Psi}_2(\eta)\,d\xi\,d\eta\right| \\
&\quad \le \|h\|_{r'(x)}\left\|B_m(f*\Psi_1, g*\Psi_2)(y)\right\|_{r(x)} \\
&\quad \le \|h\|_{r'(x)}\|B_m\|\|f\|_1\|\Psi_1\|_p\|g\|_1\|\Psi_2\|_q.
\end{aligned}$$

If we say $C = \|B_m\|\|\Psi_1\|_p\|\Psi_2\|_q$, we obtain

$$\left|\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \hat{f}(\xi)\hat{g}(\eta)\hat{h}(\xi+\eta)\hat{\Psi}_1(\xi)m(\xi,\eta)\hat{\Psi}_2(\eta)\,d\xi\,d\eta\right| \le C\|f\|_1\|g\|_1\|h\|_{r'(x)}.$$

Hence, $\hat{\Psi}_1(\xi)m(\xi,\eta)\hat{\Psi}_2(\eta) \in \mathrm{BM}(1,1,r(x))$.

(b) Take any $f, g, h \in C_c^\infty(\mathbb{R}^n)$. By (a), we know that

$$\begin{aligned}
&\left|\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \hat{f}(\xi)\hat{g}(\eta)\hat{h}(\xi+\eta)\hat{\Psi}_1(\xi)m(\xi,\eta)\hat{\Psi}_2(\eta)\,d\xi\,d\eta\right| \\
&\quad \le \int_{\mathbb{R}^n} \left|h(y)\right|\left|\tilde{B}_m(f*\Psi_1, g*\Psi_2)(y)\right|dx.
\end{aligned}$$

Similarly, if we say $C = \|B_m\|\|\Psi_1\|_1\|\Psi_2\|_1$, we obtain

$$\left|\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \hat{f}(\xi)\hat{g}(\eta)\hat{h}(\xi+\eta)\hat{\Psi}_1(\xi)m(\xi,\eta)\hat{\Psi}_2(\eta)\,d\xi\,d\eta\right| \le C\|f\|_p\|g\|_q\|h\|_{r'(x)},$$

which means $\hat{\Psi}_1(\xi)m(\xi,\eta)\hat{\Psi}_2(\eta) \in \mathrm{BM}(p,q,r(x))$.

In this theorem, (c) and (d) can be proved easily by using the technique of the proof in (a) and (b), respectively. □

Theorem 3.6 *Let $m \in \mathrm{BM}(p_1(x), p_2(x), p_3(x))$. If $Q_1, Q_2 \subset \mathbb{R}^n$ are bounded sets, then*

$$h(\xi,\eta) = \frac{1}{\mu(Q_1 \times Q_2)} \int\int_{Q_1\times Q_2} m(\xi+u, \eta+v)\,du\,dv \in \mathrm{BM}\big(p_1(x), p_2(x), p_3(x)\big).$$

Proof Let $f, g, h \in C_c^\infty(\mathbb{R}^n)$ be given. By using the Fubini theorem, we can easily prove the following equality:

$$B_h(f,g)(x) = \frac{1}{\mu(Q_1 \times Q_2)} \int\int_{Q_1\times Q_2} B_{T_{(-u,-v)}m}(f,g)(x)\,du\,dv.$$

Also, by the Minkowski inequality and Theorem 3.3, we find $C_0 > 0$ such that

$$\begin{aligned}
&\big\|B_h(f,g)\big\|_{p_3(x)} \\
&\quad\le \frac{1}{\mu(Q_1 \times Q_2)} C_0 \int\int_{Q_1\times Q_2} \big\|B_{T_{(-u,-v)}m}(f,g)\big\|_{p_3(x)}\,du\,dv \\
&\quad\le \frac{1}{\mu(Q_1 \times Q_2)} C_0 \int\int_{Q_1\times Q_2} \|T_{(-u,-v)}m\|_{(p_1(x),p_2(x),p_3(x))} \|f\|_{p_1(x)} \|g\|_{p_2(x)}\,du\,dv \\
&\quad= \frac{1}{\mu(Q_1 \times Q_2)} C_0 \|m\|_{(p_1(x),p_2(x),p_3(x))} \|f\|_{p_1(x)} \|g\|_{p_2(x)} \mu(Q_1 \times Q_2) \\
&\quad= C_0 \|m\|_{(p_1(x),p_2(x),p_3(x))} \|f\|_{p_1(x)} \|g\|_{p_2(x)}.
\end{aligned}$$

If we say $C = C_0 \|m\|_{(p_1(x),p_2(x),p_3(x))}$, we obtain

$$\|B_h\|_{(p_1(x),p_2(x),p_3(x))} \le C\|f\|_{p_1(x)}\|g\|_{p_2(x)}.$$

Hence $h(\xi,\eta) \in \mathrm{BM}(p_1(x), p_2(x), p_3(x))$. □

Theorem 3.7

(a) *If $L^{s(x)}(\mathbb{R}^n) \subset L^{r(x)}(\mathbb{R}^n)$ and $m \in \mathrm{BM}(p(x), q(x), s(x))$, then $m \in \mathrm{BM}(p(x), q(x), r(x))$.*

(b) *If $L^{s(x)}(\mathbb{R}^n) \subset L^{r(x)}(\mathbb{R}^n)$, $L^{p(x)}(\mathbb{R}^n) \subset L^{k(x)}(\mathbb{R}^n)$, $L^{q(x)}(\mathbb{R}^n) \subset L^{t(x)}(\mathbb{R}^n)$ and $m \in \mathrm{BM}(k(x), t(x), s(x))$, then $m \in \mathrm{BM}(p(x), q(x), r(x))$.*

Proof (a) Take any $f, g \in C_c^\infty(\mathbb{R}^n)$. Since $m \in \mathrm{BM}(p(x), q(x), s(x))$, there exists $C_1 > 0$ such that

$$\big\|B_m(f,g)\big\|_{s(x)} \le C_1\|f\|_{p(x)}\|g\|_{q(x)}. \tag{3.6}$$

Also, since $L^{s(x)}(\mathbb{R}^n) \subset L^{r(x)}(\mathbb{R}^n)$, there exists $C_2 > 0$ such that

$$\big\|B_m(f,g)\big\|_{r(x)} \le C_2\big\|B_m(f,g)\big\|_{s(x)}. \tag{3.7}$$

If we set $C = C_1C_2$ and combine the inequalities (3.6) and (3.7), we have

$$\big\|B_m(f,g)\big\|_{r(x)} \le C\|f\|_{p(x)}\|g\|_{q(x)}.$$

Therefore $m \in \mathrm{BM}(p(x), q(x), r(x))$.

(b) Let us take any $f, g \in C_c^\infty(\mathbb{R}^n)$. Since $m \in \mathrm{BM}(k(x), t(x), s(x))$, there exists $C_3 > 0$ such that

$$\left\| B_m(f,g) \right\|_{s(x)} \le C_3 \|f\|_{k(x)} \|g\|_{t(x)}. \tag{3.8}$$

By using the inclusions $L^{s(x)}(\mathbb{R}^n) \subset L^{r(x)}(\mathbb{R}^n)$, $L^{p(x)}(\mathbb{R}^n) \subset L^{k(x)}(\mathbb{R}^n)$ and $L^{q(x)}(\mathbb{R}^n) \subset L^{t(x)}(\mathbb{R}^n)$, we find $C_4, C_5, C_6 > 0$ such that

$$\left\| B_m(f,g) \right\|_{r(x)} \le C_4 \left\| B_m(f,g) \right\|_{s(x)}, \tag{3.9}$$

$$\|f\|_{k(x)} \le C_5 \|f\|_{p(x)} \tag{3.10}$$

and

$$\|g\|_{t(x)} \le C_6 \|g\|_{q(x)}. \tag{3.11}$$

If we set $C = C_3 C_4 C_5 C_6$ and combine the inequalities (3.8), (3.9), (3.10) and (3.11), we have

$$\left\| B_m(f,g) \right\|_{r(x)} \le C \|f\|_{p(x)} \|g\|_{q(x)}.$$

Then $m \in \mathrm{BM}(p(x), q(x), r(x))$. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The authors completed the paper together. They also read and approved the final manuscript.

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References
1. Feichtinger, H, Gürkanlı, AT: On a family of weighted convolution algebras. Int. J. Math. Math. Sci. **13**(3), 517-526 (1990)
2. Fischer, RH, Gürkanlı, AT, Liu, TS: On a family of weighted spaces. Math. Slovaca **46**(1), 71-82 (1996)
3. Katznelson, Y: An Introduction to Harmonic Analysis. Dover, New York (1968)
4. Reiter, H: Classical Harmonic Analysis and Locally Compact Groups. Oxford University Press, Oxford (1968)
5. Rudin, W: Fourier Analysis on Groups. Interscience, New York (1962)
6. Kovacik, O, Rakosnik, J: On spaces $L^{p(x)}$ and $W^{k,p(x)}$. Czechoslov. Math. J. **41**(116), 592-618 (1991)
7. Aydın, İ, Gürkanlı, AT: Weighted variable exponent amalgam spaces. Glas. Mat. **47**(67), 165-174 (2012)
8. Aydın, İ, Gürkanlı, AT: On some properties of the spaces $A_\omega^{p(x)}(\mathbb{R}^n)$. Proc. Jangjeon Math. Soc. **12**(2), 141-155 (2009)
9. Gasquet, C, Witomski, P: Fourier Analysis and Applications. Springer, New York (1999)
10. Murthy, GNK, Unni, KR: Multipliers on weighted Spaces. In: Functional Analysis and Its Applications. Lecture Notes in Mathematics, vol. 399, pp. 272-291. Springer, Berlin (1973)
11. Blasco, O: Notes on the spaces of bilinear multipliers. Rev. Unión Mat. Argent. **50**(2), 23-37 (2009)
12. Samko, SG: Convolution type operators in $L^{p(x)}$. Integral Transforms Spec. Funct. **7**, 123-144 (1998)

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