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Homoclinic orbits for second order Hamiltonian systems with asymptotically linear terms at infinity

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Abstract

In this paper, by using some different asymptotically linear conditions from those previously used in Hamiltonian systems, we obtain the existence of nontrivial homoclinic orbits for a class of second order Hamiltonian systems by the variational method.

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Keywords: homoclinic orbits; second order Hamiltonian systems; asymptotically linear; variational method

1 Introduction and main result

We consider the following second order Hamiltonian system:

$$u''(t) - A(t)u(t) + \nabla H\bigl(t, u(t)\bigr) = 0, \quad t \in \mathbb{R}, \tag{1.1}$$

where $H \in C^1(\mathbb{R} \times \mathbb{R}^N, \mathbb{R})$ is T-periodic in t, $\nabla H(t,x)$ denotes its gradient with respect to the x variable, and $A(t)$ is the T-periodic $N \times N$ matrix that satisfies

$$A(t) \in C\bigl(\mathbb{R}, \mathbb{R}^{N^2}\bigr), \tag{1.2}$$

and it is symmetric and positive definite uniformly for $t \in [0,T]$. We say that a solution $u(t)$ of (1.1) is homoclinic (with 0) if $u(t) \in C^2(\mathbb{R}, \mathbb{R}^N)$ such that $u(t) \to 0$ and $u'(t) \to 0$ as $|t| \to \infty$. If $u(t) \not\equiv 0$, then $u(t)$ is called a nontrivial homoclinic solution.

Let $G(t,u) := \frac{1}{2}(\nabla H(t,u), u) - H(t,u)$. We assume:

(H_1) $H \in C^1(\mathbb{R} \times \mathbb{R}^N, \mathbb{R})$ is T-periodic in t, and $H(t,u) \geq 0$, $\forall (t,u) \in \mathbb{R} \times \mathbb{R}^N$.

(H_2) There are some constants $c, R_1 > 0$ and $\mu > 2$ such that $|\nabla H(t,u)| \leq c|u|^{\mu-1}$ if $|u| \leq R_1$.

(H_3) There is a constant $V > 0$ such that

$$H(t,u) = \frac{1}{2}V|u|^2 + F(t,u), \qquad \bigl|\nabla F(t,u)\bigr| = o\bigl(|u|\bigr) \quad \text{as } |u| \to \infty.$$

(H_4) $G(t,u) \geq 0$, $\forall (t,u) \in \mathbb{R} \times \mathbb{R}^N$, and there exist $\alpha \in (1,2)$, $c_1, c_2 > 0$, and $R_2 > R_1$ such that

$$G(t,u) \geq c_1|u|^\mu \quad \text{if } |u| \leq R_1, \qquad G(t,u) \geq c_2|u|^\alpha \quad \text{if } |u| \geq R_2.$$

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Let

$$a_0 := \max_{t\in\mathbb{R}} \sup_{u\in\mathbb{R}^N, |u|=1} \big(A(t)u, u\big).$$

Now, our main result reads as follows.

Theorem 1.1 *If* (1.2) *and* (H_1)-(H_4) *with* $V > a_0$ *hold, then* (1.1) *has a nontrivial homoclinic orbit.*

Example 1.1 Let

$$H(t,u) = \begin{cases} (\frac{1}{2}V - h(t))|u|^{\mu} & \text{if } |u| \le 1, \\ \frac{1}{2}V|u|^2 - h(t)|u|^{\alpha} & \text{if } |u| \ge 1, \end{cases}$$

where $h \in C^1(\mathbb{R},\mathbb{R})$ is T-periodic in t, $0 < \inf_{t\in\mathbb{R}} h(t) \le \sup_{t\in\mathbb{R}} h(t) < \frac{1}{2}V$ and $\mu > 2 > \alpha > 1$. It is not hard to check that the above function satisfies (H_1)-(H_4).

We will use the following theorem to prove our main result.

Theorem A ([1]) *Let E be a Banach space equipped with the norm $\|\cdot\|$ and let $J \subset \mathbb{R}^+$ be an interval. We consider a family $(I_\lambda)_{\lambda\in J}$ of C^1-functionals on E of the form*

$$I_\lambda(u) = A(u) - \lambda B(u),$$

where $B(u) \ge 0$, $\forall u \in E$ and such that either $A(u) \to +\infty$ or $B(u) \to +\infty$ as $\|u\| \to +\infty$. We assume there are two points (v_1, v_2) in E such that setting

$$\Gamma = \big\{\gamma \in C\big([0,1], E\big), \gamma(0) = v_1, \gamma(1) = v_2\big\}$$

we have, $\forall\lambda \in J$,

$$c_\lambda := \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} I_\lambda\big(\gamma(t)\big) > \max\big\{I_\lambda(v_1), I_\lambda(v_2)\big\}.$$

Then, for almost every $\lambda \in J$, there is a sequence $\{v_n\} \subset E$ such that

$$\{v_n\} \text{ is bounded}, \quad I_\lambda(v_n) \to c_\lambda \quad \text{and} \quad I'_\lambda(v_n) \to 0 \quad \text{in the dual } E^{-1} \text{ of } E.$$

In recent decades, many authors are devoted to the existence and multiplicity of homoclinic orbits for second order Hamiltonian systems with super or asymptotically linear terms by critical point theory, see [2–22] and the references therein. Many authors [2–7, 9, 11–13, 15, 17–19] have studied the existence of homoclinic orbits of (1.1) by considering the following so-called global Ambrosetti-Rabinowitz condition on H due to Ambrosetti and Rabinowitz (*e.g.*, [3]): there exists a constant $\mu > 2$ such that

$$0 < \mu H(t,u) \le \big(\nabla H(t,u), u\big), \quad u \in \mathbb{R}^N\backslash\{0\}, \tag{1.3}$$

where $(\cdot,\cdot)$ denotes the inner product in $\mathbb{R}^N$, and the corresponding norm is denoted by $|\cdot|$. Roughly speaking the role of (1.3) is to insure that all Palais-Smale sequences for the corresponding function of (1.1) at the mountain-pass level are bounded. By removing or weakening the condition (1.3), some authors studied the homoclinic orbits of (1.1). For example, Zou and Li [22] proved that the system (1.1) has infinitely many homoclinic orbits by using the variant fountain theorem; Chen [8] obtained the existence of a ground state homoclinic orbit for (1.1) by a variant generalized weak linking theorem due to Schechter and Zou. Ou and Tang [16] obtained the existence of a homoclinic solution of (1.1) by the minimax methods in the critical point theory. For second order Hamiltonian systems without periodicity, we refer the readers to [20–22] and so on.

The rest of our paper is organized as follows. In Section 2, we give some preliminary lemmas, which are useful in the proof of our result. In Section 3, we give the detailed proof of our result.

2 Preliminary lemmas

Throughout this paper we denote by $\|\cdot\|_{L^q}$ the usual $L^q(\mathbb{R},\mathbb{R}^N)$ norm and C for generic constants.

In what follows, we always assume (1.2) and (H_1)-(H_4) with $V > a_0$ hold. Let $E := H^1(\mathbb{R},\mathbb{R}^N)$ under the usual norm

$$\|u\|_E^2 = \int_{-\infty}^{+\infty} \left(|u|^2 + \left|u'\right|^2\right) dt.$$

Thus E is a Hilbert space and it is not difficult to show that $E \subset C^0(\mathbb{R},\mathbb{R}^N)$, the space of continuous functions u on $\mathbb{R}$ such that $u(t) \to 0$ as $|t| \to \infty$ (see, *e.g.*, [18]). We will seek solutions of (1.1) as critical points of the functional I associated with (1.1) and given by

$$I(u) := \frac{1}{2}\int_{-\infty}^{+\infty} \left(\left|u'\right|^2 + \left(A(t)u,u\right)\right) dt - \int_{-\infty}^{+\infty} H(t,u)\, dt.$$

We define a new norm

$$\|u\|^2 := \int_{-\infty}^{+\infty} \left(\left(A(t)u,u\right) + \left|u'\right|^2\right) dt,$$

and its corresponding inner product is denoted by $\langle\cdot,\cdot\rangle$. By (1.2), $\|\cdot\|$ can and will be taken as an equivalent norm on E. Hence I can be written as

$$I(u) := \frac{1}{2}\|u\|^2 - \int_{-\infty}^{+\infty} H(t,u)\, dt. \tag{2.1}$$

The assumptions on H imply that $I \in C^1(E,\mathbb{R})$. Moreover, critical points of I are classical solutions of (1.1) satisfying $u'(t) \to 0$ as $|t| \to \infty$. Thus u is a homoclinic solution of (1.1). Let us show that I has a mountain-pass geometry. Since $I(0) = 0$ this is a consequence of the two following results.

Lemma 2.1 $I(u) = \frac{1}{2}\|u\|^2 + o(\|u\|^2)$ *as* $u \to 0$.

Proof By (H_2) and (H_3), we know for any $\varepsilon > 0$ there exists a $C_\varepsilon > 0$ such that

$$\left|\nabla H(t,u)\right| \le \varepsilon|u| + C_\varepsilon|u|^{p-1}, \quad \forall(t,u) \in \mathbb{R} \times \mathbb{R}^N, \tag{2.2}$$

where $p > 2$. It follows from $\frac{1}{2}(\nabla H(t,u),u) \ge H(t,u)$ (see (H_4)) that

$$\left|H(t,u)\right| \le \frac{\varepsilon}{2}|u|^2 + \frac{C_\varepsilon}{2}|u|^p, \quad \forall(t,u) \in \mathbb{R} \times \mathbb{R}^N. \tag{2.3}$$

By (2.3) and the Sobolev embedding theorem, we deduce that

$$\int_{-\infty}^{+\infty} \left|H(t,u)\right| dt \le \frac{\varepsilon}{2}\|u\|^2 + C\|u\|^p,$$

which implies the conclusion. □

Lemma 2.2 *There is a function $v \in E$ with $v \ne 0$ satisfying $I(v) \le 0$.*

Proof Let

$$d^2 := N\int_{-\infty}^{+\infty} e^{-2\alpha t^2}\,dt, \quad 0 < \alpha < V - a_0,$$

$$w_{\alpha,i}(t) := \frac{1}{d}e^{-\alpha t^2}, \quad i = 1,\dots,N \quad \text{and} \quad w_\alpha(t) := \big(w_{\alpha,1}(t),\dots,w_{\alpha,N}(t)\big).$$

Obviously, $w'_{\alpha,i}(t) := -\frac{1}{d}2\alpha t e^{-\alpha t^2}$, $i = 1,\dots,N$. Straightforward calculations show that

$$\|w_\alpha\|_{L^2} = 1 \quad \text{and} \quad \left\|w'_\alpha\right\|_{L^2}^2 = \alpha. \tag{2.4}$$

For every $t \in \mathbb{R}$, $|sw_\alpha| \to +\infty$ as $s \to \infty$. It follows from (H_3) that

$$\lim_{s\to\infty} \frac{H(t,sw_\alpha)}{s^2} = \lim_{s\to\infty} \frac{H(t,sw_\alpha)}{s^2|w_\alpha|^2}|w_\alpha|^2 = \frac{1}{2}V|w_\alpha|^2, \quad \text{a.e. } t \in \mathbb{R},$$

which together with (2.4), the definition of a_0 (above Theorem 1.1) and the Fatou lemma implies

$$\begin{aligned}
\lim_{s\to\infty} \frac{I(sw_\alpha)}{s^2} &= \frac{1}{2}\left\|w'_\alpha\right\|_{L^2}^2 + \frac{1}{2}\int_{-\infty}^{+\infty} \big(A(t)w_\alpha, w_\alpha\big)\,dt - \lim_{s\to\infty}\int_{-\infty}^{+\infty} \frac{H(t,sw_\alpha)}{s^2}\,dt \\
&\le \frac{1}{2}\alpha + \frac{a_0}{2}\|w_\alpha\|_{L^2}^2 - \lim_{s\to\infty}\int_{-\infty}^{+\infty} \frac{H(t,sw_\alpha)}{s^2}\,dt \\
&= \frac{1}{2}\alpha + \frac{a_0}{2} - \frac{1}{2}V < 0.
\end{aligned}$$

Therefore, we can choose $v := sw_\alpha$ with s big enough such that $v \in E$ with $v \ne 0$ satisfying $I(v) \le 0$. □

We define on E the family of functionals

$$I_\lambda(u) := \frac{1}{2}\|u\|^2 - \lambda\int_{-\infty}^{+\infty} H(t,u)\,dt, \quad \lambda \in [1,2]. \tag{2.5}$$

Lemma 2.3 *The family* (I_λ) *with* $\lambda \in [1,2]$ *satisfies the hypotheses of Theorem* A. *In particular for almost every* $\lambda \in [1,2]$ *there is a bounded sequence* $\{v_j\} \subset E$ *satisfying*

$$I_\lambda(v_j) \to c_\lambda \quad \textit{and} \quad I'_\lambda(v_j) \to 0.$$

Proof For the $v \in E$ obtained in Lemma 2.2, we have $I_\lambda(v) \leq 0$. It follows from (H_1) that $I_\lambda(v) \leq I(v) \leq 0$, $\forall \lambda \in [1,2]$. By the proof in Lemma 2.1, we have

$$\int_{-\infty}^{+\infty} H(t,u)\,dt = o\bigl(\|u\|^2\bigr) \quad \text{as } u \to 0. \tag{2.6}$$

Let

$$\Gamma := \bigl\{\gamma \in C\bigl([0,1],E\bigr) : \gamma(0) = 0 \text{ and } \gamma(1) = v\bigr\},$$

then it follows from (2.5) and (2.6) that

$$c_\lambda := \inf_{\gamma \in \Gamma} \max_{s \in [0,1]} I_\lambda\bigl(\gamma(s)\bigr) > 0, \quad \forall \lambda \in [1,2].$$

An application of Theorem A now completes the proof. □

Lemma 2.4 *If* $\{u_j\} \subset E$ *vanishes and is bounded, then*

$$\lim_{j\to\infty} \int_{-\infty}^{+\infty} G(t,u_j)\,dt = 0.$$

Proof It is known that if $\{u_j\}$ vanishes, then $u_j \to 0$ in $L^q(\mathbb{R},\mathbb{R}^N)$ for all $q \in (2,\infty)$, which together with (2.2), (2.3), and the Hölder inequality implies

$$\int_{-\infty}^{+\infty} \bigl(\nabla H(t,u_j),u_j\bigr) \leq \varepsilon \|u_j\|_{L^2}^2 + C_\varepsilon \|u_j\|_{L^p}^p \to 0$$

and

$$\int_{-\infty}^{+\infty} H(t,u_j)\,dt \leq \frac{\varepsilon}{2}\|u_j\|_{L^2}^2 + \frac{C_\varepsilon}{2}\|u_j\|_{L^p}^p \to 0.$$

Therefore, the proof follows from the definition of G. □

Lemma 2.5 *If* $\{u_j\}$ *is bounded in* E *and satisfies*

$$0 < \lim_{j\to\infty} I_\lambda(u_j) \leq c_\lambda \quad \textit{and} \quad I'_\lambda(u_j) \to 0,$$

then up to a subsequence, $u_j \rightharpoonup u_\lambda \neq 0$ *with* $I_\lambda(u_\lambda) \leq c_\lambda$ *and* $I'_\lambda(u_\lambda) = 0$.

Proof Note that $\{u_j\}$ is bounded and

$$\int_{-\infty}^{+\infty} G(t,u_j)\,dt = I_\lambda(u_j) - \frac{1}{2}I'_\lambda(u_j)u_j \to \lim_{j\to\infty} I_\lambda(u_j) > 0,$$

it follows from Lemma 2.4 that $\{u_j\}$ does not vanish, *i.e.*, there are $r, \delta > 0$ and a sequence $\{s_j\} \subset \mathbb{R}$ such that

$$\lim_{j\to\infty} \int_{B_r(s_j)} u_j^2 \, dt \geq \delta, \tag{2.7}$$

where $B_r(s_j) := [s_j - r, s_j + r]$. The fact that $\{u_j\}$ is bounded implies that $u_j \rightharpoonup u_\lambda$ in E and $u_j \to u_\lambda$ in $L^2_{\text{loc}}(\mathbb{R}, \mathbb{R}^N)$ (see [23]) after passing to a subsequence, thus we get $u_\lambda \neq 0$ by (2.7). By $I'_\lambda(u_j) \to 0$ and the fact I'_λ is weakly sequentially continuous [24], we have

$$I'_\lambda(u_\lambda)v = \lim_{j\to\infty} I'_\lambda(u_j)v = 0, \quad \forall v \in E.$$

It implies that $I'_\lambda(u_\lambda) = 0$.

Observe that (H_4) implies $G(t,u) \geq 0$ for all $(t,u) \in \mathbb{R} \times \mathbb{R}^N$, which together with the Fatou lemma and $I'_\lambda(u_\lambda) = 0$ implies

$$\begin{aligned} c_\lambda &\geq \lim_{j\to\infty} \left(I_\lambda(u_j) - \frac{1}{2} I'_\lambda(u_j)u_j \right) \\ &= \lim_{j\to\infty} \lambda \int_{-\infty}^{+\infty} G(t,u_j) \, dt \\ &\geq \lambda \int_{-\infty}^{+\infty} G(t,u_\lambda) \, dt \\ &= I_\lambda(u_\lambda) - \frac{1}{2} I'_\lambda(u_\lambda)u_\lambda = I_\lambda(u_\lambda). \end{aligned}$$

Therefore, the proof is finished. □

By Lemmas 2.3 and 2.5, we deduce the existence of a sequence $\{(\lambda_j, u_j)\} \subset [1,2] \times E$ such that:

$$\begin{aligned} &\bullet\ \lambda_j \to 1 \text{ and } \{\lambda_j\} \text{ is decreasing,} \\ &\bullet\ u_j \neq 0, I_{\lambda_j}(u_j) \leq c_{\lambda_j} \text{ and } I'_{\lambda_j}(u_j) = 0. \end{aligned} \tag{2.8}$$

Since

$$\frac{1}{2}\|u_j\|^2 - \lambda_j \int_{-\infty}^{+\infty} H(t,u_j) \, dt \leq c_{\lambda_j} \quad \text{and} \quad \|u_j\|^2 = \lambda_j \int_{-\infty}^{+\infty} \big(\nabla H(t,u_j), u_j\big) \, dt,$$

we have

$$\int_{-\infty}^{+\infty} G(t,u_j) \, dt \leq \frac{c_{\lambda_j}}{\lambda_j}.$$

Clearly $\frac{c_{\lambda_j}}{\lambda_j}$ is increasing and bounded by $c = c_1$, and it follows that

$$\int_{-\infty}^{+\infty} G(t,u_j) \, dt \leq c, \quad \forall j \in \mathbb{N}. \tag{2.9}$$

Lemma 2.6 *The sequence* $\{u_j\}$ *obtained in* (2.8) *is bounded.*

Proof Since $H \in C^1(\mathbb{R} \times \mathbb{R}^N, \mathbb{R})$ is T-periodic in t, by (H_4) and (H_3), respectively,

$$\begin{aligned} \int_{\{t\in\mathbb{R}:|u_j|\geq R_1\}} G(t,u_j)\,dt &= \int_{\{t\in\mathbb{R}:R_1\leq|u_j|\leq R_2\}\cup\{t\in\mathbb{R}:|u_j|\geq R_2\}} G(t,u_j)\,dt \\ &\geq c_1' \int_{\{t\in\mathbb{R}:|u_j|\geq R_1\}} |u_j|^\alpha\,dt \end{aligned} \tag{2.10}$$

and

$$\int_{\{t\in\mathbb{R}:|u_j|\geq R_1\}} \left|\nabla H(t,u_j)\right| \cdot |u_j|\,dt \leq c_2' \int_{\{t\in\mathbb{R}:|u_j|\geq R_1\}} |u_j|^2\,dt \tag{2.11}$$

for some positive constants c_1' and c_2'. Note that (2.8) implies

$$\frac{I_{\lambda_j}(u_j) - \frac{1}{2}I'_{\lambda_j}(u_j)u_j}{\lambda_j} \leq C,$$

thus it follows from (H_4) and (2.10) that

$$\begin{aligned} C &\geq \frac{I_{\lambda_j}(u_j) - \frac{1}{2}I'_{\lambda_j}(u_j)u_j}{\lambda_j} \\ &= \int_{\mathbb{R}} G(t,u_j)\,dt \\ &= \int_{\{t\in\mathbb{R}:|u_j|\leq R_1\}} G(t,u_j)\,dt + \int_{\{t\in\mathbb{R}:|u_j|\geq R_1\}} G(t,u_j)\,dt \\ &\geq c_1 \int_{\{t\in\mathbb{R}:|u_j|\leq R_1\}} |u_j|^\mu\,dt + c_1' \int_{\{t\in\mathbb{R}:|u_j|\geq R_1\}} |u_j|^\alpha\,dt. \end{aligned} \tag{2.12}$$

Take $s \in (0, \frac{\alpha}{2})$, then by (2.12), the Hölder inequality, and the Sobolev imbedding theorem,

$$\begin{aligned} &\int_{\{t\in\mathbb{R}:|u_j|\geq R_1\}} |u_j|^2\,dt \\ &\quad= \int_{\{t\in\mathbb{R}:|u_j|\geq R_1\}} |u_j|^{2s}|u_j|^{2(1-s)}\,dt \\ &\quad\leq \left(\int_{\{t\in\mathbb{R}:|u_j|\geq R_1\}} |u_j|^\alpha\right)^{\frac{2s}{\alpha}} \left(\int_{\{t\in\mathbb{R}:|u_j|\geq R_1\}} |u_j|^{\frac{2\alpha(1-s)}{\alpha-2s}}\right)^{\frac{\alpha-2s}{\alpha}} \\ &\quad\leq C_1\|u_j\|^{2(1-s)} \end{aligned} \tag{2.13}$$

for some positive constant C_1, where $\frac{2\alpha(1-s)}{\alpha-2s} \geq 2$. Note that $I'_{\lambda_j}(u_j)u_j = 0$, it follows from (H_2), (2.11)-(2.13), the Hölder inequality and the Sobolev imbedding theorem that

$$\begin{aligned} \|u_j\|^2 &= \lambda_j \int_{\mathbb{R}} \left(\nabla H(t,u_j), u_j\right) dt \\ &\leq C_2 \int_{\{t\in\mathbb{R}:|u_j|\leq R_1\}} \left|\nabla H(t,u_j)\right| \cdot |u_j|\,dt + C_2 \int_{\{t\in\mathbb{R}:|u_j|\geq R_1\}} \left|\nabla H(t,u_j)\right| \cdot |u_j|\,dt \\ &\leq C_3 \int_{\{t\in\mathbb{R}:|u_j|\leq R_1\}} |u_j|^{\mu-1} \cdot |u_j|\,dt + C_3 \int_{\{t\in\mathbb{R}:|u_j|\geq R_1\}} |u_j|^2\,dt \\ &\leq C_4 + C_3C_1\|u_j\|^{2(1-s)} \end{aligned} \tag{2.14}$$

for some positive constants C_2, C_3, and C_4, where $0 < 2(1-s) < 2$. Therefore, (2.14) implies that $\{u_j\}$ is bounded and the proof is finished. □

3 Proof of main result

We are now in a position to prove our main result.

Proof of Theorem 1.1 $\{u_j\}$ is bounded by Lemma 2.6, so we can assume $u_j \rightharpoonup u$ and $u_j \to u$ a.e. $t \in \mathbb{R}$, up to a subsequence. By (2.8), we have

$$\lim_{j\to\infty} I'(u_j)\varphi = \lim_{j\to\infty}\left(I'_{\lambda_j}(u_j)\varphi + (\lambda_j - 1)\int_{-\infty}^{+\infty}\big(\nabla H(t,u_j),\varphi\big)\,dt\right) = 0, \quad \forall \varphi \in E.$$

Note that

$$\lim_{j\to\infty} I(u_j) = \lim_{j\to\infty}\left(I_{\lambda_j}(u_j) + (\lambda_j - 1)\int_{-\infty}^{+\infty} H(t,u_j)\,dt\right).$$

We distinguish two cases: either $\limsup_{j\to\infty} I_{\lambda_j}(u_j) > 0$ or $\limsup_{j\to\infty} I_{\lambda_j}(u_j) \le 0$. If the first case holds, we get $\limsup_{j\to\infty} I(u_j) > 0$ and the result follows from Lemma 2.5. If $\limsup_{j\to\infty} I_{\lambda_j}(u_j) \le 0$, we define the sequence $\{z_j\} \in E$ by $z_j = t_j u_j$ with $t_j \in [0,1]$ satisfying

$$I_{\lambda_j}(z_j) = \max_{t\in[0,1]} I_{\lambda_j}(tu_j). \tag{3.1}$$

(If for a $j \in \mathbb{N}$, t_j defined by (3.1) is not unique we choose the smaller possible value.) Since $\{u_j\}$ is bounded, $\{z_j\}$ is bounded. Note that $I'_{\lambda_j}(z_j)z_j = 0$, $\forall j \in \mathbb{N}$, thus

$$\lambda_j \int_{-\infty}^{+\infty} G(t,z_j)\,dt = I_{\lambda_j}(z_j) - \frac{1}{2} I'_{\lambda_j}(z_j)z_j = I_{\lambda_j}(z_j). \tag{3.2}$$

On the other hand it is easily seen, following the proof of Lemma 2.1, that $I'_{\lambda_j}(u)u = \|u\|^2 + o(\|u\|^2)$ as $u \to 0$, uniformly in $j \in \mathbb{N}$. Therefore, since $I'_{\lambda_j}(u_j) = 0$, there is $\theta > 0$ such that $\|u_j\| \ge \theta$, $\forall j \in \mathbb{N}$. Recording that $\limsup_{j\to\infty} I_{\lambda_j}(u_j) \le 0$, then we obtain from Lemma 2.1 and (3.1) $\liminf_{j\to\infty} I_{\lambda_j}(z_j) > 0$, and from (3.2) it follows that

$$\liminf_{j\to\infty} \int_{-\infty}^{+\infty} G(t,z_j)\,dt = \liminf_{j\to\infty} I_{\lambda_j}(z_j) > 0.$$

It follows from the fact $\{z_j\}$ is bounded and Lemma 2.4 that $\{z_j\}$ does not vanish, so $\{u_j\}$ does not vanish. The proof of $u \neq 0$ and $I'(u) = 0$ is similar to the proof of Lemma 2.5. □

Competing interests

The author declares that they have no competing interests.

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References

1. Jeanjean, L: On the existence of bounded Palais-Smale sequences and application to a Landesman-Lazer type problem set on $\mathbf{R}^N$. Proc. R. Soc. Edinb., Sect. A **129**, 787-809 (1999)
2. Alves, CO, Carrião, PC, Miyagaki, OH: Existence of homoclinic orbits for asymptotically periodic systems involving Duffing-like equation. Appl. Math. Lett. **16**(5), 639-642 (2003)
3. Ambrosetti, A, Rabinowitz, PH: Dual variational methods in critical point theory and applications. J. Funct. Anal. **14**(4), 349-381 (1973)
4. Ambrosetti, A, Coti Zelati, V: Multiple homoclinic orbits for a class of conservative systems. Rend. Semin. Mat. Univ. Padova **89**, 177-194 (1993)
5. Caldiroli, P, Montecchiari, P: Homoclinic orbits for second order Hamiltonian systems with potential changing sign. Commun. Appl. Nonlinear Anal. **1**(2), 97-129 (1994)
6. Carrião, PC, Miyagaki, OH: Existence of homoclinic solutions for a class of time-dependent Hamiltonian systems. J. Math. Anal. Appl. **230**(1), 157-172 (1999)
7. Coti Zelati, V, Rabinowitz, PH: Homoclinic orbits for second order Hamiltonian systems possessing superquadratic potentials. J. Am. Math. Soc. **4**(4), 693-727 (1991)
8. Chen, G: Superquadratic or asymptotically quadratic Hamiltonian systems: ground state homoclinic orbits. Ann. Mat. Pura Appl. (2014). doi:10.1007/s10231-014-0403-9
9. Ding, Y, Girardi, M: Periodic and homoclinic solutions to a class of Hamiltonian systems with the potentials changing sign. Dyn. Syst. Appl. **2**(1), 131-145 (1993)
10. Ding, Y, Lee, C: Homoclinics for asymptotically quadratic and superquadratic Hamiltonian systems. Nonlinear Anal. **71**, 1395-1413 (2009)
11. Flavia, A: Periodic and homoclinic solutions to a class of Hamiltonian systems with indefinite potential in sign. Boll. Unione Mat. Ital., B **10**(2), 303-324 (1996)
12. Izydorek, M, Janczewska, J: Homoclinic solutions for a class of the second order Hamiltonian systems. J. Differ. Equ. **219**(2), 375-389 (2005)
13. Korman, P, Lazer, AC: Homoclinic orbits for a class of symmetric Hamiltonian systems. Electron. J. Differ. Equ. **1994**, 1 (1994)
14. Lv, X, Lu, S, Yan, P: Existence of homoclinic solutions for a class of second-order Hamiltonian systems. Nonlinear Anal. **72**, 390-398 (2010)
15. Omana, W, Willem, M: Homoclinic orbits for a class of Hamiltonian systems. Differ. Integral Equ. **5**(5), 1115-1120 (1992)
16. Ou, ZQ, Tang, CL: Existence of homoclinic solution for the second order Hamiltonian systems. J. Math. Anal. Appl. **291**(1), 203-213 (2004)
17. Paturel, E: Multiple homoclinic orbits for a class of Hamiltonian systems. Calc. Var. Partial Differ. Equ. **12**(2), 117-143 (2001)
18. Rabinowitz, PH: Homoclinic orbits for a class of Hamiltonian systems. Proc. R. Soc. Edinb., Sect. A **114**, 33-38 (1990)
19. Rabinowitz, PH, Tanaka, K: Some results on connecting orbits for a class of Hamiltonian systems. Math. Z. **206**(3), 473-499 (1991)
20. Wang, J, Zhang, F, Xu, J: Existence and multiplicity of homoclinic orbits for the second order Hamiltonian systems. J. Math. Anal. Appl. **366**, 569-581 (2010)
21. Wei, J, Wang, J: Infinitely many homoclinic orbits for the second order Hamiltonian systems with general potentials. J. Math. Anal. Appl. **366**, 694-699 (2010)
22. Zou, W, Li, S: Infinitely many homoclinic orbits for the second-order Hamiltonian systems. Appl. Math. Lett. **16**, 1283-1287 (2003)
23. Ding, YH: Variational Methods for Strongly Indefinite Problems. Interdiscip. Math. Sci., vol. 7. World Scientific, Singapore (2007)
24. Willem, M: Minimax Theorems. Birkhäuser, Basel (1996)

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