Key Mathematical Concepts in the Transition from Secondary School to University

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This report[1] from the ICME12 Survey Team 4 examines issues in the transition from secondary school to university mathematics with a particular focus on mathematical concepts and aspects of mathematical thinking. It comprises a survey of the recent research related to: calculus and analysis; the algebra of generalised arithmetic and abstract algebra; linear algebra; reasoning, argumentation and proof; and modelling, applications and applied mathematics. This revealed a multi-faceted web of cognitive, curricular and pedagogical issues both within and across the mathematical topics above. In addition we conducted an international survey of those engaged in teaching in university mathematics departments. Specifically, we aimed to elicit perspectives on: what topics are taught, and how, in the early parts of university-level mathematical studies; whether the transition should be smooth; student preparedness for university mathematics studies; and, what university departments do to assist those with limited preparedness. We present a summary of the survey results from 79 respondents from 21 countries.

[1] *A fuller version of this report is available from* http://www.math.auckland.ac.nz/~thomas/ST4.pdf.

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Background

Changing mathematics curricula and their emphases, lower numbers of student enrolments in undergraduate mathematics programmes (Barton and Sheryn 2009; and http://www.mathunion.org/icmi/other-activities/pipeline-project/) and changes due to an enlarged tertiary entrant profile (Hockman 2005; Hoyles et al. 2001), have provoked some international concern about the mathematical ability of students entering university (PCAST 2012; Smith 2004) and the traumatic effect of the transition on some of them (Engelbrecht 2010). Decreasing levels of mathematical competency have been reported with regard to essential technical facility, analytical powers, and perceptions of the place of precision and proof in mathematics (Brandell et al. 2008; Hourigan and O'Donoghue 2007; Kajander and Lovric 2005; Luk 2005; Selden 2005). The shifting profile of students who take service mathematics courses has produced a consequent decline in mathematical standards (Gill et al. 2010; Jennings 2009). However, not all studies agree on the extent of the problem (Engelbrecht and Harding 2008; Engelbrecht et al. 2005) and James et al. (2008) found that standards had been maintained. The recent President's Council of Advisors on Science and Technology (PCAST) (2012) states that in the USA alone there is a need to produce, over the next decade, around 1 million more college graduates in Science, Technology, Engineering, and Mathematics (STEM) fields than currently expected and recommends funding around 200 experiments at an average level of $500,000 each to address mathematics preparation issues. This helps to place the transition situation above in context and emphasises the importance of addressing the issues arising.

We found relatively few papers in the recent literature related directly with our brief to consider the role of mathematical thinking and concepts related to transition. Hence we also reviewed literature analysing the learning of mathematics on one or both sides of the transition boundary. To achieve this we formed the somewhat arbitrary division of this mathematics into: calculus and analysis; abstract algebra; linear algebra; reasoning, argumentation and proof; and modelling, applications and applied mathematics, and report findings related to each of these fields. We were aware that other fields such as geometry and statistics and probability should have been included, but were not able to do so.

The Survey

We considered it important to obtain data on transition from university mathematics departments. We wanted to know what topics are taught and how, if the faculty think the transition should be smooth, or not, their opinions on whether their students are well prepared mathematically, and what university departments do to assist those who are not. Hence, we constructed an anonymous questionnaire on transition using an Adobe Acrobat pdf form and sent it internationally by email to members of mathematics departments. The 79 responses from 21 countries were collected electronically. The sample comprised 56 males and 23 females with a mean of 21.9 years of academic teaching. Of these 45 were at the level of associate professor, reader or full professor, and 30 were assistant professors, lecturers or senior lecturers. There were 5 or more responses from each of South Africa, USA, New Zealand and Brazil.

Clearly the experience for beginning university students varies considerably depending on the country and the university that they attend. For example, while the majority teaches pre-calculus (53, 67.1 %), calculus (76, 96.2 %) and linear algebra (49, 62 %) in their first year, minorities teach complex analysis (1), topology (3), group theory (1), real analysis (5), number theory (9), graph theory (12), logic (15), set theory (17) and geometry (18), among other topics. Further, in response to 'Is the approach in **first** year mathematics at your university: Symbolic, Procedural; Axiomatic, Formal; Either, depending on the course.' 21 (26.6 %) answered that their departments introduce symbolic and procedural approaches in first year mathematics courses, while 6 replied that their departments adapt axiomatic formal approaches. Most of the respondents (50, 63.3 %) replied that their approach depended on the course.

When asked 'Do you think students have any problems in moving from school to university mathematics?' 72 (91.1 %) responded "Yes" and 6 responded "No". One third of those who answered "Yes" described these problems as coming from a lack of preparation in high school, supported by comments such as "They don't have a sufficiently good grasp of the expected school-mathematics skills that they need." Further, two thirds of those who answered "Yes" described the problems as arising from the differences, such as class size and work load, between high school classes and university, with many specifically citing the conceptual nature of university mathematics as being different from the procedural nature of high school mathematics. Comments here included "university is much more theoretical" and "Move from procedural to formal and rigourous [sic], introduction to proof, importance of definitions and conditions of theorems/rules/statements/formulas." There is also a need to "…deal with misconceptions which students developed in secondary school…We also have to review secondary school concepts and procedures from an adequate mathematical point of view." Other responses cited: students' weak algebra skills (12.5 %); that university classes are harder (5 %); personal difficulties in adjusting (10 %); poor placement (3 %); and, poor teaching at university (1 %).

Looking at specific mathematical knowledge, we enquired 'How would you rate first year students' mathematical understanding of each of the following on entry to university?' With a maximum score of 5 for high, the mean scores of the responses were algebra or generalised arithmetic (3.0), functions (2.8), real numbers (2.7), differentiation (2.5), complex numbers (1.9), definitions (1.9), vectors (1.9), sequences and series (1.9), Riemann integration (1.8), matrix algebra (1.7), limits (1.7) and proof (1.6). The mathematicians were specifically asked whether students were well prepared for calculus study. Those whose students did not study calculus at school rated their students' preparation for calculus at 2.1 out of 5. Those whose students did, rated secondary school calculus as preparation to study calculus at university at 2.4, and as preparation to study analysis at university at 1.5. These results suggest that there is some room for improvement in school preparation for university study of calculus and analysis.

Since the view has been expressed (e.g., Clark and Lovric 2009) that, rather than being ‘smooth’, the transition to university should require some measure of struggle by students, we asked ‘Do you think the transition from secondary to university education in mathematics should be smooth?’ Here, 54 (68.4 %) responded “Yes” and 22 (27.8 %) responded “No”. Of those who responded “No”, many of the comments were similar to the following, expressing the belief that change is a necessary part of the transition: “Not necessarily smooth, because it is for most students a huge change to become more independent as learners.” and “To learn mathematics is sometimes hard.” Those who answered yes were then asked ‘what could be done to make the transition from secondary to university education in mathematics smoother?’ The majority of responses mentioned changes that could be made at the high school level, such as: encourage students to think independently and abstractly; change the secondary courses; have better trained secondary teachers; and, have less focus in secondary school on standardised tests and procedures. A few mentioned changes that could be made at the university, such as: better placement of students in classes; increasing the communication between secondary and tertiary teachers; and, addressing student expectations at each level. This lack of communication between the two sectors was highlighted as a major area requiring attention by the two-year study led by Thomas (Hong et al. 2009).

Since one would expect that, seeing students with difficulties in transition, universities would respond in an appropriate manner (see e.g. Hockman 2005), we asked ‘Does your department periodically change the typical content of your first year programme?’ 33 (41.8 %) responded “Yes” and 44 (55.7 %) “No”. The responses to the question ‘How does your department decide on appropriate content for the first year mathematics programme for students?’ by those who answered yes to the previous question showed that departments change the content of the first year programme based on the decision of committees on a university or department level. Some respondents said that they change the course content for the first year students based on a decision by an individual member of faculty who diagnoses student needs and background. 15 of the 35 responded that their universities try to integrate student, industry, and national needs into first year mathematics courses. The follow-up question ‘How has the content of your first year mathematics courses changed in the last 5 years?’ showed that 35 had changed their courses in the last 5 years, but 10 of these said that the change was not significant. 17 out of the 35 respondents reported that their departments changed first year mathematics courses by removing complex topics, or by introducing practical mathematical topics. In some of the courses, students were encouraged to use tools for calculation and visualisation. In contrast, six departments *increased* the complexity and the rigour of their first year mathematics courses.

The survey considered the notion of proof in several questions. In response to ‘How important do you think definitions are in **first** year mathematics?’ 52 (65.8 %)

replied that definitions are important in first year mathematics, while 15 presented their responses as neutral. Only 8 respondents replied that definitions are not important in first year mathematics. Responses to the question 'Do you have a course that explicitly teaches methods of proof construction?' were evenly split with 49.4 % answering each of "Yes" and "No". Of those who responded "Yes", 15 (38.4 %) replied that they teach methods of proof construction during the first year, 23 (58.9 %) during the second year and 5 (12.8 %) in either third or fourth year. While some had separate courses (e.g. proof method and logic course) for teaching methods of proofs, many departments teach methods of proofs traditionally, by introducing examples of proof and exercises in mathematics class. Some respondents replied that they teach methods of proof construction in interactive contexts, citing having the course taught as a seminar, with students constructing proofs, presenting them to the class, and discussing/critiquing them in small size class. One respondent used the modified Moore method in interactive lectures. Looking at some specific methods of introducing students to proof construction was the question 'How useful do you think that a course that includes assistance with the following would be for students?' Four possibilities were listed, with mean levels of agreement out of 5 (high) being: Learning how to read a proof, 3.7; Working on counterexamples, 3.8; Building conjectures, 3.7; Constructing definitions, 3.6. These responses appear to show a good level of agreement with employing the suggested approaches as components of a course on proof construction. It may be that these are ideas that the 49.4 % of universities that currently do not have a course explicitly teaching proof construction could consider implementing as a way to assist transition.

Mathematical modelling in universities was another topic our survey addressed. In response to the questions "Does your university have a mathematical course/activity dedicated to mathematical modeling and applications?" and "Are mathematical modelling and applications contents/activities integrated into other mathematical courses?", 44 replied that their departments offer dedicated courses for modelling, while 41 said they integrate teaching of modelling into mathematics courses such as calculus, differential equations, statistics, etc. and 7 answered that their university does not offer mathematics courses for mathematical modelling and applications. Reasons given for choosing dedicated courses include: the majority of all mathematics students will end up doing something other than mathematics so applications are far more important to them than are detailed theoretical developments; most of the mathematics teaching is service teaching for students not majoring in mathematics so it is appropriate to provide a relevant course of modelling and applications that meets the needs of the target audience; if modelling is treated as an add-on then students may not learn mathematical modelling methods. Those who chose integrated courses did so because students need to be equipped with a wide array of mathematical techniques and solid knowledge base. Hence, it is appropriate for earlier mathematics courses to contain some theory, proofs, concepts and skills, as well as applications.

Considering what happens in upper secondary schools, 26 (33 %) reported that secondary schools in their location have mathematical modelling and applications integrated into other mathematical courses, with only 4 having dedicated courses. 44 (56 %) said that there were no such modelling courses in their area. When asked for their opinion on how modelling should be taught in schools, most of the answers stated that it should be integrated into other mathematical courses. The main reasons presented for this were: the many facets of mathematics; topics too specialised to form dedicated courses; to allow cross flow of ideas, avoid compartmentalization; and students need to see the connection between theory and practice, build meaning, appropriate knowledge. The question ‘What do you see as the key differences between the teaching and learning of modelling and applications in secondary schools and university, if any?’ was answered by 33 (42 %) of respondents. The key differences pointed out by those answering this question were: at school, modelling is poor, too basic and mechanical, often close implementation of simple statistics tests; students have less understanding of application areas; university students are more independent; they have bigger range of mathematical tools, more techniques; they are concerned with rigour and proof. Asked ‘What are the key difficulties for student transition from secondary school to university in the field of mathematical modelling and applications, if any?’ the 35 (44 %) university respondents cited: lack of knowledge (mathematical theory, others subjects such as physics, chemistry, biology, ecology); difficulties in formulating precise mathematical problems/ interpreting word problems/understanding processes, representations, use of parameters; poor mathematical skills, lack of logical thinking; no experience from secondary schools; and lack of support. One message for transition is to construct more realistic modeling applications for students to study in schools.

In order to investigate how universities respond to assist students with transition problems we enquired “Do you have any academic support structures to assist students in the transition from school to university? (e.g., workshops, bridging courses, mentoring, etc.).”, and 56 (71 %) replied ‘Yes’ and 22 ‘No’. Of those saying yes, 34 % have a bridging course, 25 % some form of tutoring arrangement, while 23 % mentioned mentoring, with one describing it as a “Personal academic mentoring program throughout degree for all mathematics students” and another saying “We tried a mentoring system once, but there was almost no uptake by students.” Other support structures mentioned included ‘study skills courses’, ‘maths clinics’, ‘support workshops’, ‘pre-course’, ‘remedial mathematics unit’, and a ‘Mathematics Learning Service (centrally situated), consulting & assignment help room (School of Maths). The MLS has a drop-in help room, and runs a series of seminars on Maths skills. These are also available to students on the web.’ Others talked of small group peer study, assisted study sessions, individual consultations, daily help sessions, orientation programmes and remedial courses. There is some evidence that bridging courses can assist in transition (Varsavsky 2010), by addressing skill deficiencies in basic mathematical topics (Tempelaar et al. 2012) and building student confidence (Carmichael and Taylor 2005). Other successful transition courses (e.g., Leviatan 2008) introduce students to the mathematical “culture” and its typical activities (generalizations, deductions, definitions, proofs, etc.), as well as central concepts and tools.

Overall the survey confirmed that students do have some difficulties in transition and these are occasionally related to a deficit in student preparation or mathematical knowledge. However, there are also a number of areas that universities could address to assist students, such as adjusting the content of first year courses, and instituting a course on proving and proof (where this doesn't already exist) and constructing appropriate bridging courses.

Literature Review

A number of different lenses have been used to analyse the mathematical transition from school to university. Some have been summarised well elsewhere (see e.g., Winsløw 2010) but we preface our discussion with a brief list of the major theoretical perspectives we found in the transition-related literature. One theory that is in common use is the Anthropological Theory of Didactics (ATD) based on the ideas of Chevallard (1985), with its concept of a *praxeology* comprising task, technique, technology, theory. ATD focuses on analysis of the organisation of praxeologies relative to institutions and the diachronic development of didactic systems. A second common perspective is the Theory of Didactical Situations (TDS) of Brousseau (1997), where *didactical situations* are constructed in which the teacher orchestrates elements of the didactical milieu under the constraints of a dynamic didactical contract. Other research uses the action-process-object-schema (APOS) framework of Dubinsky (e.g. Dubinsky and McDonald 2001) for studying learning. This describes how a process can be constructed from actions by reflective abstraction, and subsequently an object is formed by encapsulation of the process. The Three Worlds of Mathematics (TWM) framework of Tall (2008) is also considered useful by some. This describes thinking and learning as taking place in three worlds: the embodied; the symbolic; and the formal. In the embodied world we build mental conceptions using visual and physical attributes of concepts and enactive sensual experiences. In the symbolic world symbolic representations of concepts are acted upon, or manipulated, and the formal world is where properties of objects are formalized as axioms, with learning comprising building and proving of theorems by logical deduction from these axioms. We use the acronyms above to refer to each of these frameworks in the text below.

Calculus and Analysis

A number of epistemological and mathematical obstacles have been identified in the study of the transition from calculus to analysis. These include:

Functions: Students have a limited understanding of the concept of function (Junior 2006) and need to be able to switch between local and global perspectives (Artigue 2009; Rogalski 2008; Vandebrouck 2011). Using a TWM lens Vandebrouck

(2011) suggests a need to reconceptualise the concept of function in terms of its multiple registers and process-object duality. The formal axiomatic world of university mathematics requires students to adopt a local perspective on functions, whereas only pointwise (functions considered as a correspondence between two sets of numbers) and global points of view (representations are tables of variation) are constructed at secondary school. An ATD-based study of the transition from concrete to abstract perspectives in real analysis by Winsløw (2008) suggests that in secondary schools the focus is on practical-theoretical blocks of concrete analysis, while at university level the focus is on more complex praxeologies of concrete analysis and on abstract analysis.

Limits: Students need to work with limits, especially of infinite sequences or series. Two obstacles regarding the concept of infinite sum are the intuitive and natural idea that the sum of infinity of terms should also be infinite, and the conception that an infinite process must go through each step, one after the other and without stopping, which leads to the potential infinity concept (González-Martín 2009; González-Martín et al. 2011). According to Oehrtman (2009), students' reasoning about limit concepts appears to be influenced by metaphorical application of experiential conceptual domains, including collapse, approximation, proximity, infinity as number and physical limitation metaphors. However, only physical limitation metaphors were consistently detrimental to students' understanding. One approach to building thinking about limits, suggested by Mamona-Downs (2010), is the set-oriented characterization of convergence behaviour of sequences of that supports the metaphor of 'arbitrary closeness' to a point. Another, employing a TDS framework (Ghedamsi 2008) developed situations that allowed students to connect productively the intuitive, perceptual and formal dimensions of the limit concept.

Institutional factors: An aspect of transition highlighted by the ATD is that praxeologies exist in relation to institutions. Employing the affordances of ATD, Praslon (2000) showed that by the end of high school in France a substantial institutional relationship with the concept of derivative is already established. Hence, for this concept, he claims that the secondary-tertiary transition is not about intuitive and proceptual perspectives moving towards formal perspectives, as TWM might suggest, but is more complex, involving an accumulation of micro-breaches and changes in balance according several dimensions (tool/object dimensions, particular/general objects, autonomy given in the solving process, role of proofs, etc.). Building on this work Bloch and Ghedamsi (2004) identified nine factors contributing to a discontinuity between high school and university in analysis and Bosch et al. (2004) show the existence of strong discontinuities in the praxeological organization between high school and university, and build specific tools for qualifying and quantifying these. Also employing an institutional approach, Dias et al. (2008; see also Artigue 2008) conducted a comparative ATD study of the secondary-tertiary transition in Brazil and France, using the concept of function as a filter. They conclude that although contextual influences tend to remain invisible there is a need for those inside a given educational system to become aware of them in order to envisage productive collaborative work and evolution of the system.

Other areas: One TDS-based research project examined a succession of situations for introducing the notions of interior and closure of a set and open and closed set (Bridoux 2010), using meta-mathematical discourse and graphical representations to assist students to develop an intuitive insight that allowed the teacher to characterise them in a formal language. Another examined the notion of completeness (Bergé 2008), analysing whether students have an operational or conceptual view, or if it is taken for granted. The conclusion was that many students have a weak understanding of ideas such as the suprema of bounded subsets, convergence of Cauchy sequences and the completeness of **R**.

Some possible ways to assist the calculus-analysis transition have been considered. For example, Gyöngyösi et al. (2011) report an experiment using Maple CAS-based work to ease the transition from calculus to real analysis. A similar use of graphing calculator technology in consideration of the Fundamental Theorem of Calculus by Scucuglia (2006) made it possible for the students to become gradually engaged in deductive mathematical discussions based on results obtained from experiments. In addition, Biehler et al. (2011) propose that blending traditional courses with systematic e-learning can facilitate bridging of school and university mathematics.

Abstract Algebra

Understanding the constructs, principles, and eventually axioms, of the algebra of generalised arithmetic could be a way to assist students in the transition to study of more general algebraic structures. Focusing on students' work on solving a parametric system of simultaneous equations and the difficulties they experience with working with variables, parameters and unknowns, Stadler (2011) describes their experience of the transition from school to university mathematics as an often perplexing re-visiting of content and ways of working. The study showed that constructs of number, symbolic literals, operators, the '=' symbol itself, and the formal equivalence relation, as well as the principles of arithmetic, all contribute to building a deep understanding of equation. This agrees with the observations of Godfrey and Thomas (2008), who, using the TWM framework, provided evidence that many students have a surface structure view of equation and fail to integrate the properties of the object with that surface structure.

Students' encounter with abstract algebra at university marks a significant point in the transition to advanced mathematical formalism and abstraction, with concepts introduced abstractly, defined and presented by their properties, and deduction of facts from these properties alone. The role of verbalisation in this process, as a semantic mediator between symbolic and visual mathematical expression, may require a level of verbalisation skills that Nardi (2008, 2011) notes is often lacking in first year undergraduates.

Studies that focus on the student experience in their first encounters with key concepts in abstract algebra describe a number of difficulties. While some have suggested that an over-reliance on concrete examples of groups leading to a lack of skills in proof production, others, such as Burn (1996), recommend reversing the order of presentation, using examples and applications to stimulate the discovery of definitions and theorems through permutation and symmetry. An example of reducing group theory's high levels of abstraction (Hazzan 2001) is to ask students to construct the operation table for low order groups. This was also implemented by Larsen (2009) as a series of tasks exploring symmetries of an equilateral triangle, constructing low order group multiplication tables and culminating in negotiating preliminary understandings of group structure, the order of a group and isomorphism.

In an analysis of student responses to introductory group theory problem sheets, Nardi (2000) identified student difficulties with the order of an element, group operation, and the notions of coset and isomorphism. The duality underlying the concept of group and its binary operation, were also discussed by Iannone and Nardi (2002). They offer evidence of a student tendency to ignore the binary operation, consider the group axioms as properties of the group elements and omit checking axioms perceived as obvious, such as associativity. In addition, research by Ioannou (see Ioannou and Nardi 2009, 2010; Ioannou and Iannone 2011) considers students' first encounter with abstract algebra, focusing on the Subgroup Test, symmetries of a cube, equivalence relations, and employing the notions of kernel and image in the First Isomorphism Theorem. Provisional conclusions are that students' overall problematic experience of the transition to abstract algebra is characterised by the strong interplay between strictly conceptual matters, affective issues and those germane to first year students' wider study skills and coping strategies.

Linear Algebra

A sizeable amount of research in linear algebra has documented students' transition difficulties, particularly as these relate to students' intuitive or geometric ways of reasoning and the formal mathematics of linear algebra (e.g. Dogan-Dunlap 2010). The theoretical framework of Hillel (2000) for understanding student reasoning in linear algebra that identified geometric, algebraic, and abstract modes of description is valuable. For example, the relationship between linear algebra and geometry were at the core of Gueudet's research programme (2004, 2008; Gueudet-Chartier, 2004) that identified specific views on student difficulties. She claims that the epistemological view leads to a focus on linear algebra as an axiomatic theory, which is very abstract for the students and identifies a need for various forms of flexibility, in particular between dimensions. Further work at the geometry-formalism boundary by Portnoy et al. (2006) and Britton and Henderson (2009) has demonstrated some difficulties. First, pre-service teachers who engaged with transformations as geometric processes still had difficulty writing proofs involving linear transformations, and second, students experienced problems moving between a formal understanding of subspace and algebraic problem statements due to an insufficient understanding of the symbols used in the questions and in the formal definition of subspace.

Employing a framework using APOS theory in conjunction with TWM, Stewart and Thomas (2009, 2010; Thomas and Stewart 2011) analysed student understanding of various concepts in linear algebra, including linear independence, eigenvectors, span and basis. The authors found that generally students do not think of these concepts from an embodied standpoint, but instead rely upon a symbolic, process-oriented matrix manipulation manner of reasoning. However, employing a course that introduced students to embodied, geometric representations in linear algebra, along with the formal and the symbolic, appeared to enrich student understanding of the concepts and allowed them to bridge between them more effectively than with just symbolic processes.

Another aspect that has been investigated is students' intuitive thinking in linear algebra. Working with modelling and APOS frameworks Possani et al. (2010) leveraged students' intuitive ways of thinking through a genetic composition of linear independence and systems of equations. Student use of different modes of representation in making sense of the formal notion of subspace was analysed by Wawro, Sweeney and Rabin (2011a), and their results suggest that in generating explanations for the definition, students rely on their intuitive understandings of subspace, which can be problematic but can also help develop a more comprehensive understanding of subspace.

Some research teams have spearheaded innovations in the teaching and learning of linear algebra. For example, Cooley et al. (2007) developed a linear algebra course combined with learning about APOS theory and found the focus on a theory for how mathematical knowledge is generated enriched understanding of linear algebra. Another group of researchers used a design research approach simultaneously creating instructional sequences and examining students' reasoning about key concepts such as eigenvectors and eigenvalues, linear independence, linear dependence, span, and linear transformation (Henderson et al. 2010; Larson et al. 2008; Sweeney 2011). They argue that knowledge of student thinking prior to formal instruction is essential for developing thoughtful teaching that builds on and extends student thinking. In a study on tasks for developing student reasoning they (Wawro et al. 2011b) report how an innovative instructional sequence beginning with vector equations rather than systems of equations successfully leveraged students' intuitive imagery of vectors as movement to develop formal definitions.

Proof and Proving[2]

The transition to university mathematics includes a requirement for understanding and producing proofs. This requires logical deductive reasoning (Engelbrecht 2010) and rigour (Leviatan 2008). Research highlighting examples of this includes

[2] *At the time of writing the book* Proof and Proving in Mathematics Education: *The 19th ICMI study–Hanna & de Villiers,* 2012*, was still in press.*

conceptualisation related to the use of quantifiers (Chellougui 2004), the relationship between syntax and semantics in the proving process (Barrier 2009; Blossier et al. 2009) and logical competencies (Durand-Guerrier and Njomgang Ngansop 2010).

One recommendation is the need for more explicit teaching of proof, both in school and university (Balacheff 2008; Hanna and de Villiers 2008; Hemmi 2008), with some (e.g., Stylianides and Stylianides 2007; Hanna and Barbeau 2008) arguing for it to be made a central topic in both institutions. A possible introduction to proof, suggested by Harel (2008) and Palla et al. (2012) is proof by mathematical induction. However, they propose that it should be introduced slowly, building on students' own pre-existing epistemological resources (Solomon 2006) valuing both ways of understanding and thinking (Harel 2008), and distinguishing between proof schemes and proofs.

A number of potential difficulties in any attempt to place proving and proof more prominently in the transition years have been identified. These include the role of definitions, and the problem of student met-befores (Tall and Mejia-Ramos 2006). Using definitions as the basis of deductive reasoning in schools is likely to meet serious problems (Harel 2008; Hemmi 2008) since this form of reasoning is generally not available to school students, and Hemmi (2008) advocates the principle of *transparency*, which makes the difference between empirical evidence and deductive argument visible to students. In addition, the influence of student met-before can be strong, with Cartiglia et al. (2004) showing that the most recent met-before for university students, a formal approach, had a strong influence on their reasoning. A further difficulty, highlighted by Iannone and Inglis (2011), is a range of weaknesses in beginning university mathematics students' ability to produce a deductive argument, even when they were aware they should do so.

Some consideration has been given to methods of bridging the gap between the fields of argumentation and proof. One pedagogical strategy that may be an effective way to introduce the learning of proof and proving is student construction and justification of conjectures. The idea of an interconnecting problem was employed by Kondratieva (2011) to get students to construct and justify conjectures. Further, conjectures may also have a role during production of indirect argumentation (Antonini and Mariotti 2008), such as that in contradiction and contraposition, by activating and bridging significant hidden cognitive processes. Another approach discussed by Pedemonte (2007, 2008) employs the construct of *structural distance*, and she argues for an abductive step in the structurant argumentation in order to assist transition by decreasing the gap between argumentation and proof. Another proposition is that pivotal, bridging or counterexamples could assist students with proof ideas (Stylianides and Stylianides 2007; Zazkis and Chernoff 2008). A potential benefit of a counterexample is to produce cognitive conflict in the student, while a pivotal example is designed to create a turning point in the learner's cognitive perception. Counterexamples may also foster deductive reasoning, since deductions are made by building models and looking for counterexamples. For Zazkis and Chernoff (2008) a counterexample is a mathematical concept, while a pivotal example is a pedagogical concept, which is within, but pushing the boundaries of the set of examples students have experienced. The role of examples also arose in research by Weber and Mejia-Ramos (2011) on proof reading by mathematicians. This suggests that students might be taught how to use examples to increase their conviction in, or understanding of, a proof. In order to know what skills to teach students, Alcock and Inglis (2008) maintain that identifying different strategies of proof construction among experts will grow knowledge of what skills to teach students, and how they can be employed.

Mathematical Modelling and Applications

Mathematical modelling and applications continues to be a central theme in mathematics education research (Blum et al. 2002), with a primary focus on practice activities. However, it appears that little or no literature exists explicitly discussing these topics with a focus on the ‘transition’ from the secondary to the university levels, possibly because there have been no roadmaps to sustained implementation of modelling education at all levels. Hence, recent literature relevant to the secondary-tertiary transition issue is briefly considered here.

One crucial duality, mentioned by Niss et al. (2007), is the difference between ‘applications and modelling for the learning of mathematics’ and ‘learning mathematics for applications and modelling’. This duality is seldom made explicit in lower secondary school, and instead both orientations are simultaneously insisted on. However, at upper secondary or tertiary level the duality is often a significant one. The close relationship between modelling and problem solving is taken up by a number of authors. For example, English and Sriraman (2010) suggest that mathematical modelling is a powerful option for advancing the development of problem solving in the curriculum. In addition, according to Petocz et al. (2007), there are distinct advantages to using real world tasks in problem solving in order to model the way mathematicians work. This is supported by the research of Perrenet and Taconis (2009), who describe significant shifts in the growth of attention to metacognitive aspects in problem solving related to the change from secondary school mathematics problems to authentic mathematics problems at university. One difficulty outlined by Ärlebäck and Frejd (2010) is that upper secondary students have little experience working with real situations and modelling problems, making the incorporation of real problems from industry problematic. A second possible difficulty (Gainsburg 2008) is that teachers tend not to make many real-world connections in teaching. One possible solution is to bring together combinations of students, teachers and mathematicians to work on modelling problems (Kaiser and Schwarz 2006). This opportunity may be created through a “modelling week” (Göttlich 2010; Heilio 2010; Kaland et al. 2010), during which small groups of school or tertiary students work intensely, in a supported environment, on selected, authentic modelling problems.

There is some agreement that the secondary school curriculum could include more modelling activities, although high-stakes assessment at the secondary-tertiary interface is an unresolved problem in any implementation (Stillman 2007). Other initiatives for embedding modelling in the curriculum proposed by Stillman and Ng

(2010) include a system-wide focus emphasising an applications and modelling approach to teaching and assessing mathematical subjects in the last two years of school and interdisciplinary project work from primary through secondary school, with mathematics as the anchor subject.

Conclusion

The literature review presented here reveals a multi-faceted web of cognitive, curricular and pedagogical issues, some spanning across mathematical topics and some intrinsic to certain topics—and certainly exhibiting variation across the institutional contexts of the many countries our survey focused on. For example, most of the research we reviewed discusses the students' limited cognitive preparedness for the requirements of university-level formal mathematical thinking (whether this concerns the abstraction, for example, within Abstract Algebra courses or the formalism of Analysis). Within other areas, such as discrete mathematics, much of the research we reviewed highlighted that students may arrive at university with little or no awareness of certain mathematical fields.

The review presented in this report, as well as the longer version, is certainly not exhaustive. However we believe it is reasonable to claim that the bulk of research on transition is in a limited number of areas (e.g. calculus, proof) and that there is little research in other areas (e.g. discrete mathematics). While this might simply reflect curricular emphases in the various countries that our survey focused on, it also indicates directions that future research may need to pursue. Furthermore across the preceding sections a pattern seems to emerge with regard to *how*, not merely *what*, students experience in their first encounters with advanced mathematical topics, whether at school or at university. Fundamental to addressing issues of transition seems also to be the coordination and dialogue across educational levels—here mostly secondary and tertiary—and our survey revealed that at the moment this appears largely absent.

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