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Dini Lipschitz functions for the Dunkl transform in the Space $\mathrm{L}^2(\mathbb{R}^d, w_k(x)dx)$

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Abstract Using a generalized spherical mean operator, we obtain an analog of Theorem 5.2 in Younis (J Math Sci 9(2),301–312 1986) for the Dunkl transform for functions satisfying the d-Dunkl Dini Lipschitz condition in the space $\mathrm{L}^2(\mathbb{R}^d, w_k(x)dx)$, where w_k is a weight function invariant under the action of an associated reflection group.

Keywords Dunkl transform · Dunkl kernel · Generalized spherical mean operator

Mathematics Subject Classification 42B37

1 Introduction and preliminaries

Younis Theorem 5.2 [13] characterized the set of functions in $\mathrm{L}^2(\mathbb{R})$ satisfying the Cauchy Lipschitz condition by means of an asymptotic estimate growth of the norm of their Fourier transforms, namely, we have the following

Theorem 1.1 [13] *Let $f \in \mathrm{L}^2(\mathbb{R})$. Then the following are equivalents*

1. $\|f(x+h) - f(x)\|_2 = O\left(\frac{h^\alpha}{(\log\frac{1}{h})^\beta}\right)$ *as* $h \longrightarrow 0,\ 0 < \alpha < 1, \beta \geq 0,$
2. $\int_{|x|\geq r} |\mathcal{F}(f)(x)|^2 dx = O\left(\frac{r^{-2\alpha}}{(\log r)^{2\beta}}\right)$ *as* $r \longrightarrow +\infty,$

where $\mathcal{F}$ stands for the Fourier transform of f.

Dedicated to Professor François Rouvière for his 69's birthday.

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In this paper, we obtain an analog of Theorem 1.1 for the Dunkl transform on $\mathbb{R}^d$. For this purpose, we use a generalized spherical mean operator. We point out that similar results have been established in the Bessel transform [4].

We consider the Dunkl operators D_i; $1 \leq i \leq d$, on $\mathbb{R}^d$, which are the differential-difference operators introduced by Dunkl in [6]. These operators are very important in pure mathematics and in physics. The theory of Dunkl operators provides generalizations of various multivariable analytic structures, among others we cite the exponential function, the Fourier transform and the translation operator. For more details about these operators see [5–7]. The Dunkl Kernel E_k has been introduced by Dunkl in [8]. This Kernel is used to define the Dunkl transform.

Let R be a root system in $\mathbb{R}^d$, W the corresponding reflection group, R_+ a positive subsystem of R (see [5,7,9–11]) and k a non-negative and W-invariant function defined on R.

The Dunkl operators is defined for $f \in C^1(\mathbb{R}^d)$ by

$$D_j f(x) = \frac{\partial f}{\partial x_j}(x) + \sum_{\alpha \in R_+} k(\alpha)\alpha_j \frac{f(x) - f(\sigma_\alpha(x))}{\langle \alpha, x\rangle}, \quad x \in \mathbb{R}^d \ (1 \leq j \leq d)$$

Here $\langle , \rangle$ is the usual euclidean scalar product on $\mathbb{R}^d$ with the associated norm $|.|$ and σ_α the reflection with respect to the hyperplane H_α orthogonal to α, and $\alpha_j = \langle \alpha, e_j\rangle$, $(e_1, e_2, \ldots, e_d)$ being the canonical basis of $\mathbb{R}^d$.

The weight function w_k defined by

$$w_k(x) = \prod_{\zeta \in R_+} |\langle \zeta, x\rangle|^{2k(\alpha)}, \quad x \in \mathbb{R}^d,$$

where w_k is W-invariant and homogeneous of degree 2γ where

$$\gamma = \gamma(R) = \sum_{\zeta \in R_+} k(\zeta) \geq 0.$$

The Dunkl Kernel E_k on $\mathbb{R}^d \times \mathbb{R}^d$ has been introduced by Dunkl in [8]. For $y \in \mathbb{R}^d$ the function $x \mapsto E_k(x, y)$ is the unique solution on $\mathbb{R}^d$ of

$$\begin{cases} D_j u(x, y) = y_j u(x, y) & \text{for} \quad 1 \leq j \leq d \\ u(0, y) = 1 & \text{for all } y \in \mathbb{R}^d \end{cases}$$

E_k is called the Dunkl Kernel.

Proposition 1.2 [5] *Let $z, w \in \mathbb{C}$ and $\lambda \in \mathbb{C}$. Then*

1. $E_k(z, 0) = 1$.
2. $E_k(z, w) = E_k(w, z)$.
3. $E_k(\lambda z, w) = E_k(z, \lambda w)$.
4. *For all $\nu = (\nu_1, \ldots, \nu_d) \in \mathbb{N}^d$, $x \in \mathbb{R}^d$, $z \in \mathbb{C}^d$, we have*

$$|\partial_z^\nu E_k(x, z)| \leq |x|^{|\nu|} exp(|x||Re(z)|),$$

where

$$\partial_z^\nu = \frac{\partial^{|\nu|}}{\partial z_1^{\nu_1} \ldots \partial z_d^{\nu_d}}, \quad |\nu| = \nu_1 + \cdots + \nu_d.$$

In particular

$$|\partial_z^\nu E_k(ix, z)| \leq |x|^\nu,$$

for all $x, z \in \mathbb{R}^d$.

The Dunkl transform is defined for $f \in \mathrm{L}^1_k(\mathbb{R}^d) = \mathrm{L}^1(\mathbb{R}^d, w_k(x)dx)$ by

$$\widehat{f}(\xi) = c_k^{-1} \int_{\mathbb{R}^d} f(x) E_k(-i\xi, x) w_k(x) dx,$$

'where the constant c_k is given by

$$c_k = \int_{\mathbb{R}^d} e^{\frac{-|z|^2}{2}} w_k(z) dz.$$

The Dunkl transform shares several properties with its counterpart in the classical case, we mention here in particular that Parseval Theorem holds in $\mathrm{L}^2_k = \mathrm{L}^2_k(\mathbb{R}^d) = \mathrm{L}^2_k(\mathbb{R}^d, w_k(x)dx)$, when both f and $\widehat{f}$ are in $\mathrm{L}^1_k(\mathbb{R}^d)$, we have the inversion formula

$$f(x) = \int_{\mathbb{R}^d} \widehat{f}(\xi) E_k(ix, \xi) w_k(\xi) d\xi, \quad x \in \mathbb{R}^d.$$

In $\mathrm{L}^2_k(\mathbb{R}^d)$, consider the generalized spherical mean operator defined by

$$\mathrm{M}_h f(x) = \frac{1}{d_k} \int_{\mathbb{S}^{d-1}} \tau_x(f)(hy) d\eta_k(y), \quad (x \in \mathbb{R}^d, h > 0)$$

where τ_x Dunkl translation operator (see [11,12]), η is the normalized surface measure on the unit sphere $\mathbb{S}^{d-1}$ in $\mathbb{R}^d$ and set $d\eta_k(y) = w_k(x) d\eta(y)$, η_k is a W-invariant measure on S^{d-1} and $d_k = \eta_k(\mathrm{S}^{d-1})$.

We see that $\mathrm{M}_h f \in \mathrm{L}^2_k(\mathbb{R}^d)$ whenever $f \in \mathrm{L}^2_k(\mathbb{R}^d)$ and

$$\|\mathrm{M}_h f\|_{\mathrm{L}^2_k} \leq \|f\|_{\mathrm{L}^2_k},$$

for all $h > 0$.

For $p \geq -\frac{1}{2}$, we introduce the normalized Bessel function of the first kind j_p defined by

$$j_p(z) = \Gamma(p+1) \sum_{n=0}^{\infty} \frac{(-1)^n (\frac{z}{2})^{2n}}{n! \Gamma(n+p+1)}, \quad z \in \mathbb{C}. \tag{1}$$

Lemma 1.3 [1] *The following inequalities are fulfilled*

1. $|j_p(x)| \leq 1,$
2. $1 - j_p(x) = O(1), \ x \geq 1.$
3. $1 - j_p(x) = O(x^2); \ 0 \leq x \leq 1.$

From lemma 1.3, we have

$$|1 - j_p(x)| \leq C_p x, \quad \forall x \in \mathbb{R}^+ \tag{2}$$

Lemma 1.4 *The following inequality is true*

$$|1 - j_p(x)| \geq c,$$

with $|x| \geq 1$, *where* $c > 0$ *is a certain constant which depends only on* p.

Proof (Analog of lemma 2.9 in [3]) □

Moreover, from (1) we see that

$$\lim_{z \to 0} \frac{\left(j_{\gamma + \frac{d}{2} - 1}(z) - 1\right)}{z^2} \neq 0 \tag{3}$$

Proposition 1.5 *Let $f \in \mathrm{L}_k^2(\mathbb{R}^d)$. Then*

$$\widehat{(\mathrm{M}_h f)}(\xi) = j_{\gamma+\frac{d}{2}-1}(h|\xi|)\widehat{f}(\xi).$$

i.e

$$\mathrm{M}_h f(x) = \int_{\mathbb{R}^d} j_{\gamma+\frac{d}{2}-1}(h|\xi|)\widehat{f}(\xi)E_k(ix,\xi)w_k(\xi)d\xi$$

and

$$f(x) = \int_{\mathbb{R}^d} \widehat{f}(\xi)E_k(ix,\xi)w_k(\xi)d\xi.$$

We have

$$\mathrm{M}_h f(x) - f(x) = \int_{\mathbb{R}^d} (j_{\gamma+\frac{d}{2}-1}(h|\xi|) - 1)\widehat{f}(\xi)E_k(ix,\xi)w_k(\xi)d\xi. \tag{4}$$

Invoking Parseval's identity (4) gives

$$\|\mathrm{M}_h f(x) - f(x)\|^2_{\mathrm{L}_k^2} = \int_{\mathbb{R}^d} |j_{\gamma+\frac{d}{2}-1}(h|\xi|) - 1|^2|\widehat{f}(\xi)|^2 w_k(\xi)d\xi.$$

2 Dini Lipschitz condition

Definition 2.1 Let $f(x) \in \mathrm{L}_k^2(\mathbb{R}^d)$, and let

$$\|\mathrm{M}_h f(x) - f(x)\|_{\mathrm{L}_k^2} \leq C\frac{h^\alpha}{(\log\frac{1}{h})^\gamma}, \quad \gamma \geq 0$$

i.e

$$\|\mathrm{M}_h f(x) - f(x)\|_{\mathrm{L}_k^2} = O\left(\frac{h^\alpha}{(\log\frac{1}{h})^\gamma}\right),$$

for all x in $\mathbb{R}^d$ and for all sufficiently small h, C being a positive constant. Then we say that f satisfies a d-Dunkl Dini Lipschitz of order α, or f belongs to $Lip(\alpha,\gamma)$.

Definition 2.2 If however

$$\frac{\|\mathrm{M}_h f(x) - f(x)\|_{\mathrm{L}_k^2}}{\frac{h^\alpha}{(\log\frac{1}{h})^\gamma}} \to 0 \quad as\, h \to 0$$

i.e

$$\|\mathrm{M}_h f(x) - f(x)\|_{\mathrm{L}_k^2} = o\left(\frac{h^\alpha}{(\log\frac{1}{h})^\gamma}\right) \quad as\ h \to 0,\ \gamma \geq 0$$

then f is said to be belong to the little d-Dunkl Dini Lipschitz class $lip(\alpha,\gamma)$.

Remark It follows immediately from these definitions that

$$lip(\alpha,\gamma) \subset Lip(\alpha,\gamma).$$

Theorem 2.3 *Let $\alpha > 1$. If $f \in Lip(\alpha, \gamma)$, then $f \in lip(1, \gamma)$.*

Proof For $x \in \mathbb{R}^d$ and h small, $f \in Lip(\alpha, \gamma)$ we have

$$\|\mathrm{M}_h f(x) - f(x)\|_{\mathrm{L}^2_k} \leq C \frac{h^\alpha}{(\log \frac{1}{h})^\gamma}.$$

Then

$$\left(\log \frac{1}{h}\right)^\gamma \|\mathrm{M}_h f(x) - f(x)\|_{\mathrm{L}^2_k} \leq C h^\alpha$$

Therefore

$$\frac{(\log \frac{1}{h})^\gamma}{h} \|\mathrm{M}_h f(x) - f(x)\|_{\mathrm{L}^2_k} \leq C h^{\alpha-1},$$

which tends to zero with $h \to 0$. Thus

$$\frac{(\log \frac{1}{h})^\gamma}{h} \|\mathrm{M}_h f(x) - f(x)\|_{\mathrm{L}^2_k} \to 0 \quad as\, h \to 0$$

Then $f \in lip(1, \gamma)$. □

Theorem 2.4 *If $\alpha < \beta$, then $Lip(\alpha, 0) \supset Lip(\beta, 0)$ and $lip(\alpha, 0) \supset lip(\beta, 0)$.*

Proof We have $0 \leq h \leq 1$ and $\alpha < \beta$, then $h^\beta \leq h^\alpha$.
Then the proof of this theorem. □

Theorem 2.5 *Let $f, g \in \mathrm{L}^2_k(\mathbb{R}^d)$ such that $\mathrm{M}_h(fg)(x) = \mathrm{M}_h f(x) \mathrm{M}_h g(x)$. If $f, g \in Lip(\alpha, \gamma)$, then $fg \in Lip(\alpha, \gamma)$.*

Proof Since $f, g \in Lip(\alpha, \gamma)$, we have for all x in $\mathbb{R}^d$

$$\|\mathrm{M}_h f(x) - f(x)\|_{\mathrm{L}^2_k} \leq C_f \frac{h^\alpha}{(\log \frac{1}{h})^\gamma}$$

and

$$\|\mathrm{M}_h g(x) - g(x)\|_{\mathrm{L}^2_k} \leq C_g \frac{h^\alpha}{(\log \frac{1}{h})^\gamma}$$

It is clear that

$$\begin{aligned}
&\|\mathrm{M}_h(fg)(x) - f(x)g(x)\|_{\mathrm{L}^2_k} \\
&\quad = \|\mathrm{M}_h(fg)(x) - f(x)\mathrm{M}_h g(x) + f(x)\mathrm{M}_h g(x) - f(x)g(x)\|_{\mathrm{L}^2_k} \\
&\quad = \|\mathrm{M}_h f(x)\mathrm{M}_h g(x) - f(x)\mathrm{M}_h g(x) + f(x)\mathrm{M}_h g(x) - f(x)g(x)\|_{\mathrm{L}^2_k} \\
&\quad = \|\mathrm{M}_h g(x)(\mathrm{M}_h f(x) - f(x)) + f(x)(\mathrm{M}_h g(x) - g(x))\|_{\mathrm{L}^2_k} \\
&\quad \leq \|\mathrm{M}_h g(x)\|_{\mathrm{L}^2_k} \|\mathrm{M}_h f(x) - f(x)\|_{\mathrm{L}^2_k} + \|f(x)\|_{\mathrm{L}^2_k} \|\mathrm{M}_h g(x) - g(x)\|_{\mathrm{L}^2_k} \\
&\quad \leq K_1 C_f \frac{h^\alpha}{(\log \frac{1}{h})^\gamma} + K_2 C_g \frac{h^\alpha}{(\log \frac{1}{h})^\gamma} \\
&\quad \leq M \frac{h^\alpha}{(\log \frac{1}{h})^\gamma},
\end{aligned}$$

where $M = \max(K_1 C_f, K_2 C_g)$. Then $fg \in Lip(\alpha, \gamma)$ □

3 New results on Dini Lipschitz class

Theorem 3.1 *Let $\alpha > 2$. If f belong to the d-Dunkl Dini Lipschitz class, i.e*

$$f \in Lip(\alpha, \gamma), \quad \alpha > 2, \ \gamma \geq 0.$$

Then f is equal to the null function in $\mathbb{R}^d$.

Proof Assume that $f \in Lip(\alpha, \gamma)$. Then

$$\|\mathrm{M}_h f(x) - f(x)\|_{\mathrm{L}^2_k} \leq C_f \frac{h^\alpha}{(\log \frac{1}{h})^\gamma}.$$

We have to recall that the Dunkl transform of $f(x)$ satisfies the Parseval's identity $\|f\|_{\mathrm{L}^2_k} = \|\widehat{f}\|_{\mathrm{L}^2_k}$.

So

$$\|\mathrm{M}_h f(x) - f(x)\|_{\mathrm{L}^2_k} = \|\widehat{\mathrm{M}_h f - f}\|_{\mathrm{L}^2_k}$$

i.e

$$\left\|(1 - j_{\gamma+\frac{d}{2}-1}(h|\xi|))\widehat{f}(\xi)\right\|_{\mathrm{L}^2_k} \leq C_f \frac{h^\alpha}{(\log \frac{1}{h})^\gamma}.$$

it follows that

$$\int_{\mathbb{R}^d} |1 - j_{\gamma+\frac{d}{2}-1}(h|\xi|)|^2 |\widehat{f}(\xi)|^2 w_k(\xi) d\xi \leq C_f^2 \frac{h^{2\alpha}}{(\log \frac{1}{h})^{2\gamma}}.$$

Then

$$\frac{\int_{\mathbb{R}^d} |1 - j_{\gamma+\frac{d}{2}-1}(h|\xi|)|^2 |\widehat{f}(\xi)|^2 w_k(\xi) d\xi}{h^4} \leq C_f^2 \frac{h^{2\alpha-4}}{(\log \frac{1}{h})^{2\gamma}}.$$

Since $\alpha > 2$ we have

$$\lim_{h \to 0} \frac{h^{2\alpha-4}}{(\log \frac{1}{h})^{2\gamma}} = 0$$

Then

$$\lim_{h \to 0} \int_{\mathbb{R}^d} \left(\frac{|1 - j_{\gamma+\frac{d}{2}-1}(h|\xi|)|}{|\xi|^2 h^2} \right)^2 |\xi|^4 |\widehat{f}(\xi)|^2 w_k(\xi) d\xi = 0.$$

and also from the formula (3) and Fatou's theorem, we obtain $\||\xi|^2 \widehat{f}(\xi)\|_{\mathrm{L}^2_k} = 0$. Thus $|\xi|^2 \widehat{f}(\xi) = 0$ for all $\xi \in \mathbb{R}^d$, then $f(x)$ is the null function. □

Analog of the theorem 3.1, we obtain this theorem

Theorem 3.2 *Let $f \in \mathrm{L}^2_k(\mathbb{R}^d)$. If f belong to $lip(2, 0)$, i.e*

$$\|\mathrm{M}_h f(x) - f(x)\|_{\mathrm{L}^2_k} = o(h^2) \quad as\ h \to 0.$$

Then f is equal to null function in $\mathbb{R}^d$.

Now, we give another the main result of this paper analog of theorem 1.1.

Theorem 3.3 *Let $\alpha \in (0, 1)$, $\gamma \geq 0$ and $f \in \mathrm{L}_k^2(\mathbb{R}^d)$. Then the following are equivalents*

1. $f \in Lip(\alpha, \gamma)$
2. $\int_{|\xi|\geq s} |\widehat{f}(\xi)|^2 w_k(\xi) d\xi = O\left(\frac{s^{-2\alpha}}{(\log s)^{2\gamma}}\right)$ *as* $s \to +\infty$

Proof 1) $\Longrightarrow$ 2) Assume that $f \in Lip(\alpha, \gamma)$. Then we have

$$\|\mathrm{M}_h f(x) - f(x)\|_{\mathrm{L}_k^2} = O\left(\frac{h^\alpha}{(\log \frac{1}{h})^\gamma}\right) \quad as\, h \longrightarrow 0,$$

Proposition 1.5 and Parseval's identity give

$$\|\mathrm{M}_h f(x) - f(x)\|_{\mathrm{L}_k^2}^2 = \int_{\mathbb{R}^d} |1 - j_{\gamma+\frac{d}{2}-1}(h|\xi|)|^2 |\widehat{f}(\xi)|^2 w_k(\xi) d\xi.$$

If $|\xi| \in [\frac{1}{h}, \frac{2}{h}]$ then $h|\xi| \geq 1$ and lemma 1.4 implies that

$$1 \leq \frac{1}{c^2} |1 - j_{\gamma+\frac{d}{2}-1}(h|\xi|)|^2.$$

Then

$$\begin{aligned}\int_{\frac{1}{h}\leq|\xi|\leq\frac{2}{h}} |\widehat{f}(\xi)|^2 w_k(\xi) d\xi &\leq \frac{1}{c^2} \int_{\frac{1}{h}\leq|\xi|\leq\frac{2}{h}} |1 - j_{\gamma+\frac{d}{2}-1}(h|\xi|)|^2 |\widehat{f}(\xi)|^2 w_k(\xi) d\xi \\ &\leq \frac{1}{c^2} \int_{\mathbb{R}^d} |1 - j_{\gamma+\frac{d}{2}-1}(h|\xi|)|^2 |\widehat{f}(\xi)|^2 w_k(\xi) d\xi \\ &= O\left(\frac{h^{2\alpha}}{(\log \frac{1}{h})^{2\gamma}}\right).\end{aligned}$$

We obtain

$$\int_{s\leq|\xi|\leq 2s} |\widehat{f}(\xi)|^2 w_k(\xi) d\xi \leq C \frac{s^{-2\alpha}}{(\log s)^{2\gamma}}.$$

where C is a positive constant.

So that

$$\begin{aligned}\int_{|\xi|\geq s} |\widehat{f}(\xi)|^2 w_k(\xi) d\xi &= \left(\int_{s\leq|\xi|\leq 2s} + \int_{2s\leq|\xi|\leq 4s} + \int_{4s\leq|\xi|\leq 8s} + \cdots\right) |\widehat{f}(\xi)|^2 w_k(\xi) d\xi \\ &\leq C\left(\frac{s^{-2\alpha}}{(\log s)^{2\gamma}} + \frac{(2s)^{-2\alpha}}{(\log 2s)^{2\gamma}} + \frac{(4s)^{-2\alpha}}{(\log 4s)^{2\gamma}} + \cdots\right) \\ &\leq C \frac{s^{-2\alpha}}{(\log s)^{2\gamma}} (1 + 2^{-2\alpha} + (2^{-2\alpha})^2 + (2^{-2\alpha})^3 \cdots) \\ &\leq C K_\alpha \frac{s^{-2\alpha}}{(\log s)^{2\gamma}},\end{aligned}$$

where $K_\alpha = C(1 - 2^{-2\alpha})^{-1}$ since $2^{-2\alpha} < 1$.

This proves that

$$\int_{|\xi|\geq s} |\widehat{f}(\xi)|^2 w_k(\xi) d\xi = O\left(\frac{s^{-2\alpha}}{(\log s)^{2\gamma}}\right) \quad as\, s \longrightarrow +\infty$$

2) $\Longrightarrow$ 1) Suppose now that

$$\int_{|\xi|\geq s}|\widehat{f}(\xi)|^2 w_k(\xi)d\xi = O\left(\frac{s^{-2\alpha}}{(\log s)^{2\gamma}}\right) \quad as\, s \longrightarrow +\infty.$$

We have to show that

$$\int_0^{\infty} r^{2\gamma+d-1}|1-j_{\gamma+\frac{d}{2}-1}(hr)|^2\phi(r)dr = O\left(\frac{h^{2\alpha}}{(\log\frac{1}{h})^{2\gamma}}\right), \quad as\, h \to 0$$

where

$$\phi(r) = \int_{\mathbb{S}^{d-1}}|\widehat{f}(ry)|^2 w_k(y)dy.$$

We write

$$\int_0^{\infty} r^{2\gamma+d-1}|1-j_{\gamma+\frac{d}{2}-1}(hr)|^2\phi(r)dr = \mathrm{I}_1+\mathrm{I}_2,$$

where

$$\mathrm{I}_1 = \int_0^{1/h} r^{2\gamma+d-1}|1-j_{\gamma+\frac{d}{2}-1}(hr)|^2\phi(r)dr,$$

and

$$\mathrm{I}_2 = \int_{1/h}^{\infty} r^{2\gamma+d-1}|1-j_{\gamma+\frac{d}{2}-1}(hr)|^2\phi(r)dr.$$

Firstly, from (1) in lemma 1.3 we see that

$$\mathrm{I}_2 \leq 4\int_{1/h}^{\infty} r^{2\gamma+d-1}\phi(r)dr = O\left(\frac{h^{2\alpha}}{(\log\frac{1}{h})^{2\gamma}}\right) \quad as\, h \longrightarrow 0.$$

Set

$$\psi(r) = \int_r^{\infty} x^{2\gamma+d-1}\phi(x)dx.$$

From formula 2, an integration by parts yields

$$\begin{aligned}
\mathrm{I}_1 &= \int_0^{1/h} r^{2\gamma+d-1}|1-j_{\gamma+\frac{d}{2}-1}(hr)|^2\phi(r)dr \\
&\leq -C_p h^2\int_0^{1/h} r^2\psi'(r)dr \\
&\leq -C_p\psi(1/h) + 2C_p h^2\int_0^{1/h} r\psi(r)dr \\
&\leq 2C_p h^2\int_0^{1/h} r\frac{r^{-2\alpha}}{(\log r)^{2\gamma}}dr \\
&\leq C_1\frac{h^{2\alpha}}{(\log\frac{1}{h})^{2\gamma}}
\end{aligned}$$

where C_1 is a positive constant, and this ends the proof $\square$

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