Ahmadullah et al. *Fixed Point Theory and Applications* (2016) 2016:42
DOI 10.1186/s13663-016-0531-6

Fixed Point Theory and Applications
a SpringerOpen Journal

RESEARCH **Open Access**

Unified relation-theoretic metrical fixed point theorems under an implicit contractive condition with an application

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Abstract

The main purpose of this article is to establish relation-theoretic metrical fixed point theorems via an implicit contractive condition which is general enough to yield a multitude of corollaries corresponding to several well known contraction conditions (*e.g.* Banach (Fundam. Math. 3:133-181, 1922), Kannan (Am. Math. Mon. 76:405-408, 1969), Reich (Can. Math. Bull. 14:121-124, 1971), Bianchini (Boll. Unione Mat. Ital. 5:103-108, 1972), Chatterjea (C. R. Acad. Bulg. Sci. 25:727-730, 1972), Hardy and Rogers (Can. Math. Bull. 16:201-206, 1973), Ćirić (Proc. Am. Math. Soc. 45:267-273, 1974) and several others) wherein even such corollaries are new results on their own. As an example we utilize our main results, to prove a theorem on the existence and uniqueness of the solution of an integral equation besides providing an illustrative example.

MSC: Primary 47H10; secondary 54H25

Keywords: complete metric spaces; binary relations; implicit relation; contraction mappings; fixed point

1 Introduction

In 1920, Banach formulated the classical contraction mapping principle in his Ph.D. thesis which was later published in Banach [1]. It is one of the most fruitful and applicable theorems ever proved in classical functional analysis. In the course of the last century, this theorem has been generalized and improved by numerous researchers chiefly by replacing contraction mappings with a relatively more general contractive mappings and this practice is still going on. Rhoades [8] carried out a comparative study of various classes of utilized mappings which include Kannan [2], Reich [3], Bianchini [4], Chatterjea [5], Sehgal [9], Hardy and Rogers [6], Ćirić [7] besides several other ones. The survey article due to Rhoades [8] is generally consulted by every researcher of this domain and also continues to serve as a standard reference.

In 1997, Popa [10] initiated the idea of an implicit function which is designed to cover several well known contraction conditions of the existing literature in one go besides admitting several new ones. Indeed, the strength of an implicit function lies in their unifying power besides being general enough to yield new contraction conditions. Here, it is fascinating to point out that some of the presented examples (in Section 2) are of nonexpansive type and Lipschitzian type. For further details about implicit functions, one can consult [10–17].

In recent years, a multitude of order-theoretic metrical fixed point theorems have been proved for order-preserving contractions. This trend appears to be initiated (in 1986) by Turinici [18]. In 2004, unknowingly, Ran and Reurings [19] rediscovered a slightly more natural order-theoretic version of the Banach contraction principle and utilized his result well to establish the existence and uniqueness of the solution of a system of linear equations under a suitable set of conditions. In the recent past, this result of Ran and Reurings has been generalized and improved by several researchers and by now there exists a considerable literature around this theorem. Out of all such extensions and generalizations, the results due to Nieto and Rodríguez-López [20, 21] and Jachymski [22] deserve special mention. Thereafter, several authors utilized various variants of binary relations namely: preorder, transitive, tolerance, strict order, symmetric closure *etc.* to prove their respective fixed point theorems. Most recently, Alam and Imdad [23, 24] established a new relation-theoretic version of the Banach contraction principle employing general binary relation which in turn generalizes several well known relevant order-theoretic fixed point theorems.

The aim of this paper is to prove some unified metrical fixed point theorems employing an arbitrary binary relation under an implicit contractive condition which is general enough to cover a multitude of well known contraction conditions in one go besides yielding several new ones. We also provide an example to demonstrate the generality of our results over several well known corresponding results. Finally, we utilize our results to prove the existence and uniqueness of the solution of an integral equation.

2 Implicit relation

In this section, we consider a suitable implicit function and also furnish a variety of examples which include most of the well known contractions of the existing literature besides several new ones. Here, it can be pointed out that most of the following examples do not meet the requirements of the implicit function due to Popa [17]. In order to describe our implicit function, let $\mathcal{F}$ be the collection of all continuous real valued functions $F:\mathbb{R}_+^6\to\mathbb{R}$ which satisfy the following conditions:

(F_1) F is non-increasing in the fifth variable; and $F(s,t,t,s,s+t,0)\leq 0$ for $s,t\geq 0$ implies that there exists $h\in[0,1)$ such that $s\leq ht$;

(F_2) $F(s,0,s,0,0,s)>0$, for all $s>0$.

Let $\mathcal{G}$ be yet another but relatively smaller collection of all continuous real valued functions $F:\mathbb{R}_+^6\to\mathbb{R}$ which satisfy (F_1) and (F_2) along with the following additional condition:

(F_3) F is non-increasing in the sixth variable; and $F(s,s,0,0,s,s)>0$, for all $s>0$.

Example 1 The function $F:\mathbb{R}_+^6\to\mathbb{R}$ defined by

$$F(s_1,s_2,s_3,s_4,s_5,s_6)=\begin{cases} s_1-ks_2\frac{s_5+s_6}{s_3+s_4}, & \text{if } s_3+s_4\neq 0;\\ s_1-s_2, & \text{if } s_3+s_4=0,\end{cases}$$

where $k\in[0,1)$ satisfies the properties (F_1) and (F_2) with $h=k$ but does not satisfy the property (F_3).

The functions $F : \mathbb{R}_+^6 \to \mathbb{R}$ defined below satisfy the foregoing requirements (see [11, 13, 16, 17]):

(1) $F(s_1, s_2, s_3, s_4, s_5, s_6) = s_1 - ks_2$, where $k \in [0, 1)$;

(2) $F(s_1, s_2, s_3, s_4, s_5, s_6) = s_1 - k(s_3 + s_4)$, where $k \in [0, 1/2)$;

(3) $F(s_1, s_2, s_3, s_4, s_5, s_6) = s_1 - k(s_5 + s_6)$, where $k \in [0, 1/2)$;

(4) $F(s_1, s_2, s_3, s_4, s_5, s_6) = s_1 - a_1 s_2 - a_2(s_3 + s_4) - a_3(s_5 + s_6)$, where $a_1, a_2, a_3 \in [0, 1)$ and $a_1 + 2a_2 + 2a_3 < 1$;

(5) $F(s_1, s_2, s_3, s_4, s_5, s_6) = s_1 - ks_2 - L\min\{s_3, s_4, s_5, s_6\}$, where $k \in [0, 1)$ and $L \geq 0$;

(6) $F(s_1, s_2, s_3, s_4, s_5, s_6) = s_1 - (a_1 s_2 + a_2 s_3 + a_3 s_4 + a_4(s_5 + s_6))$, where $a_1, a_2, a_3, a_4 \geq 0$ and $a_1 + a_2 + a_3 + 2a_4 < 1$;

(7) $F(s_1, s_2, s_3, s_4, s_5, s_6) = s_1 - k\max\{s_2, s_3, s_4, \frac{s_5+s_6}{2}\} - L\min\{s_3, s_4, s_5, s_6\}$, where $k \in [0, 1)$ and $L \geq 0$;

(8) $F(s_1, s_2, s_3, s_4, s_5, s_6) = s_1 - k\max\{s_2, s_3, s_4, s_5, s_6\}$, where $k \in [0, 1/2)$;

(9) $F(s_1, s_2, s_3, s_4, s_5, s_6) = s_1 - (a_1 s_2 + a_2 s_3 + a_3 s_4 + a_4 s_5 + a_5 s_6)$, where a_i's > 0 (for $i = 1, 2, 3, 4, 5$) and sum of them is strictly less than 1;

(10) $F(s_1, s_2, s_3, s_4, s_5, s_6) = s_1 - k\max\{s_2, s_3, s_4, \frac{s_5}{2}, \frac{s_6}{2}\}$, where $k \in [0, 1)$;

(11) $F(s_1, s_2, s_3, s_4, s_5, s_6) = s_1 - k\max\{s_2, s_3, s_4\} - (1 - k)(as_5 + bs_6)$, where $k \in [0, 1)$ and $0 \leq a, b < 1/2$;

(12) $F(s_1, s_2, s_3, s_4, s_5, s_6) = s_1^2 - s_1(a_1 s_2 + a_2 s_3 + a_3 s_4) - a_4 s_5 s_6$, where $a_1 > 0$; $a_2, a_3, a_4 \geq 0$; $a_1 + a_2 + a_3 < 1$ and $a_1 + a_4 < 1$;

(13)

$$F(s_1, s_2, s_3, s_4, s_5, s_6) = \begin{cases} s_1 - ks_2 \frac{s_5+s_6}{s_1+s_2}, & \text{if } s_1 + s_2 \neq 0; \\ s_1, & \text{if } s_1 + s_2 = 0, \end{cases}$$

where $k \in [0, 1)$;

(14) $F(s_1, s_2, s_3, s_4, s_5, s_6) = s_1^2 - a_1\max\{s_2^2, s_3^2, s_4^2\} - a_2\max\{s_3 s_5, s_4 s_6\} - a_3 s_5 s_6$, where a_i's ≥ 0 (for $i = 1, 2, 3$); $a_1 + 2a_2 < 1$ and $a_1 + a_3 < 1$;

(15) $F(s_1, s_2, s_3, s_4, s_5, s_6) = s_1^3 - k(s_2^3 + s_3^3 + s_4^3 + s_5^3 + s_6^3)$, where $k \in [0, 1/11)$;

(16)

$$F(s_1, s_2, s_3, s_4, s_5, s_6) = \begin{cases} s_1 - a_1 \frac{s_2 s_4}{s_2+s_4} - a_2 \frac{s_3 s_6}{s_5+s_6+1}, & \text{if } s_2 + s_4 \neq 0; \\ s_1, & \text{if } s_2 + s_4 = 0, \end{cases}$$

where $a_1, a_2 > 0$ and $a_1 < 2$.

3 Relevant relation-theoretic notions

In this section, we present some basic definitions, propositions and relevant relation-theoretic variants of some metrical notions namely: completeness and continuity.

Definition 1 [25] A binary relation on a non-empty set X is defined as a subset of $X \times X$, which will be denoted by $\mathcal{R}$. We say that 'x relates to y under $\mathcal{R}$' iff $(x, y) \in \mathcal{R}$.

In the following, $\mathcal{R}$ stands for a non-empty binary relation while $\mathbb{N}_0$ denotes the set of whole numbers, *i.e.*, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. In this presentation, we always employ a non-empty binary relation (*i.e.*, $\mathcal{R} \neq \emptyset$).

Definition 2 [23] Let $\mathcal{R}$ be a binary relation defined on a non-empty set X. Then any pair of points x, y in X is said to be $\mathcal{R}$-comparative if either $(x,y) \in \mathcal{R}$ or $(y,x) \in \mathcal{R}$, which is together written as $[x,y] \in \mathcal{R}$.

Definition 3 [26] A binary relation $\mathcal{R}$ is called complete if every elements are comparable under that relation (*i.e.*, $[x,y] \in \mathcal{R}$ $\forall x,y \in X$).

Definition 4 [25] Let $\mathcal{R}$ be a binary relation defined on a non-empty set X. Then

(i) the inverse (or transpose or dual) relation of $\mathcal{R}$, is defined as

$$\mathcal{R}^{-1} = \{(x,y) \in X^2 : (y,x) \in \mathcal{R}\} \text{ which is denoted by } \mathcal{R}^{-1};$$

(ii) the symmetric closure of $\mathcal{R}$ is defined as the smallest symmetric relation containing $\mathcal{R}$ (*i.e.*, $\mathcal{R}^s := \mathcal{R} \cup \mathcal{R}^{-1}$). Often, it is denoted by $\mathcal{R}^s$.

Proposition 1 [23] *If $\mathcal{R}$ is a binary relation defined on a non-empty set X, then*

$$(x,y) \in \mathcal{R}^s \quad \Longleftrightarrow \quad [x,y] \in \mathcal{R}.$$

Definition 5 [23] Let $\mathcal{R}$ be a binary relation defined on a non-empty set X. Then a sequence $\{x_n\} \subset X$ is called $\mathcal{R}$-preserving if

$$(x_n, x_{n+1}) \in \mathcal{R}, \quad \forall n \in \mathbb{N}_0.$$

Definition 6 [23] Let T be a self-mapping defined on a non-empty set X. Then a binary relation $\mathcal{R}$ defined on X is called T-closed if

$$(x,y) \in \mathcal{R} \quad \Rightarrow \quad (Tx, Ty) \in \mathcal{R}, \quad \text{for all } x,y \in X.$$

Alam and Imdad [24] introduced relation-theoretic variants of some metrical notions namely: completeness and continuity.

Definition 7 Let $(X,d,\mathcal{R})$ be a metric space equipped with a binary relation $\mathcal{R}$ defined on X. We say that (X,d) is $\mathcal{R}$-complete if every $\mathcal{R}$-preserving Cauchy sequence in X converges to a point in X.

Remark 1 Under any binary relation $\mathcal{R}$, every complete metric space is $\mathcal{R}$-complete. Particularly, under the universal relation the notion of $\mathcal{R}$-completeness coincides with usual completeness.

Definition 8 Let $(X,d,\mathcal{R})$ be a metric space equipped with a binary relation $\mathcal{R}$ defined on X. Then a mapping $T : X \to X$ is called $\mathcal{R}$-continuous at x if for any $\mathcal{R}$-preserving sequence $\{x_n\}$ with $x_n \xrightarrow{d} x$, we have $T(x_n) \xrightarrow{d} T(x)$. As usual, T is called $\mathcal{R}$-continuous if it is $\mathcal{R}$-continuous on the whole of X.

Remark 2 Under any binary relation $\mathcal{R}$, every continuous mapping is $\mathcal{R}$-continuous. Particularly, under the universal relation the notion of $\mathcal{R}$-continuity coincides with usual continuity.

Definition 9 [23] Let $(X, d, \mathcal{R})$ be a metric space equipped with a binary relation $\mathcal{R}$ defined on X. Then $\mathcal{R}$ is called d-self-closed if for any $\mathcal{R}$-preserving sequence $\{x_n\}$ with $x_n \xrightarrow{d} x$, there is a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $[x_{n_k}, x] \in \mathcal{R}$, $\forall k \in \mathbb{N}_0$.

Definition 10 [27] Let $(X, d, \mathcal{R})$ be a metric space equipped with a binary relation $\mathcal{R}$ defined on X. Then a subset D of X is called $\mathcal{R}$-directed if for every pair of points x, y in D, there is z in X such that $(x, z) \in \mathcal{R}$ and $(y, z) \in \mathcal{R}$.

Definition 11 [28] Let $\mathcal{R}$ be a binary relation defined on a non-empty set X and a pair of points x, y in X. If there is a finite sequence $\{z_0, z_1, z_2, \ldots, z_l\} \subset X$ such that $z_0 = x$, $z_l = y$ and $(z_i, z_{i+1}) \in \mathcal{R}$ for each $i \in \{0, 1, 2, \ldots, l-1\}$, then this finite sequence is called a path of length l (where $l \in \mathbb{N}$) joining x to y in $\mathcal{R}$.

Observe that a path of length l involves $(l + 1)$ elements of X that need not be distinct in general.

Involving a non-empty set X, a binary relation $\mathcal{R}$ on X, a self-mapping T on X and a $\mathcal{R}$-directed subset D of X, we use the following notations:

- $F(T)$: the collection of all fixed points of T;
- $X(T, \mathcal{R})$: the set of all points x in X such that $(x, Tx) \in \mathcal{R}$;
- $\Delta(D, \mathcal{R}) := \bigcup_{x,y\in D}\{z \in X : (x, z) \in \mathcal{R} \text{ and } (y, z) \in \mathcal{R}\}$;
- $\Upsilon(x, y, \mathcal{R})$: the collection of all paths joining x to y in $\mathcal{R}$ where $x, y \in X$;
- $\Upsilon_T(x, y, \mathcal{R})$: the collection of all paths $\{z_0, z_1, z_2, \ldots, z_l\}$ joining x to y in $\mathcal{R}$ such that $[z_i, Tz_i] \in \mathcal{R}$ for each $i \in \{1, 2, 3, \ldots, l-1\}$.

4 Fixed point theorems

Now, we are equipped to prove the main result of this paper.

Theorem 1 *Let $(X, d, \mathcal{R})$ be a metric space equipped with a binary relation $\mathcal{R}$ defined on X and T a self-mapping on X. Assume that the following conditions hold*:

(a) *(X, d) is $\mathcal{R}$-complete,*
(b) *$X(T, \mathcal{R})$ is non-empty,*
(c) *$\mathcal{R}$ is T-closed,*
(d) *either T is $\mathcal{R}$-continuous or $\mathcal{R}$ is d-self-closed,*
(e) *there exists an implicit function $F \in \mathcal{F}$ with*

$$F\bigl(d(Tx, Ty), d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\bigr) \leq 0,$$

for all $x, y \in X$ such that $(x, y) \in \mathcal{R}$.

Then T has a fixed point.

Proof Choose $x_0 \in X(T, \mathcal{R})$. Construct a Picard sequence (iterates) $\{x_n\}$, *i.e.*, $x_n = T^n(x_0)$, $\forall n \in \mathbb{N}_0$. Since $(x_0, Tx_0) \in \mathcal{R}$ and $\mathcal{R}$ is T-closed (hypothesis (c)), we have

$$\bigl(Tx_0, T^2x_0\bigr), \bigl(T^2x_0, T^3x_0\bigr), \ldots, \bigl(T^nx_0, T^{n+1}x_0\bigr), \ldots \in \mathcal{R}.$$

Notice that

$$(x_n, x_{n+1}) \in \mathcal{R}, \quad \forall n \in \mathbb{N}_0, \tag{1}$$

so that the sequence $\{x_n\}$ is $\mathcal{R}$-preserving. On using condition (e), we have (for all $n \in \mathbb{N}_0$)

$$F\bigl(d(Tx_n, Tx_{n+1}), d(x_n, x_{n+1}), d(x_n, Tx_n), d(x_{n+1}, Tx_{n+1}), d(x_n, Tx_{n+1}), d(x_{n+1}, Tx_n)\bigr) \leq 0,$$

or

$$F\bigl(d(x_{n+1}, x_{n+2}), d(x_n, x_{n+1}), d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2}), d(x_n, x_{n+2}), d(x_{n+1}, x_{n+1})\bigr) \leq 0.$$

Putting $s = d(x_{n+1}, x_{n+2})$ and $t = d(x_n, x_{n+1})$ in the above inequality, we get

$$F\bigl(s, t, t, s, d(x_n, x_{n+2}), 0\bigr) \leq 0.$$

On using the triangular inequality, we have

$$d(x_n, x_{n+2}) \leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) = s + t,$$

so that (owing to non-increasing property of F in the fifth variable)

$$F(s, t, t, s, s + t, 0) \leq 0,$$

implying thereby (using (F_1)) the existence of some $h \in [0, 1)$ such that $s \leq ht$, *i.e.*,

$$d(x_{n+1}, x_{n+2}) \leq hd(x_n, x_{n+1}),$$

which inductively gives rise to

$$d(x_{n+1}, x_{n+2}) \leq h^{n+1} d(x_0, x_1), \quad \forall n \in \mathbb{N}_0. \tag{2}$$

Using (2) and the triangular inequality, for all $n, m \in \mathbb{N}_0$ with $m > n$, we have

$$\begin{aligned} d(x_n, x_m) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{m-1}, x_m) \\ &\leq \bigl(h^n + h^{n+1} + \cdots + h^{m-1}\bigr) d(x_0, x_1) \\ &= h^n d(x_0, x_1) \sum_{j=0}^{m-n-1} h^j \\ &\leq \frac{h^n}{1-h} d(x_0, x_1) \\ &\to 0 \quad \text{as } n \to \infty, \end{aligned}$$

which amounts to saying that the sequence $\{x_n\}$ is Cauchy in X. Hence, $\{x_n\}$ is an $\mathcal{R}$-preserving Cauchy sequence in X. By assumption (a) (*i.e.*, (X, d) is $\mathcal{R}$-complete), $\exists x \in X$ such that $x_n \xrightarrow{d} x$.

In the assumption (d), first of all, assume that T is $\mathcal{R}$-continuous so that

$$x_{n+1} = T(x_n) \xrightarrow{d} T(x).$$

Appealing to uniqueness of the limit, we have $T(x) = x$, so that x is the fixed point of T.

Next, suppose that $\mathcal{R}$ is d-self-closed. Since $\{x_n\}$ is an $\mathcal{R}$-preserving sequence and $x_n \xrightarrow{d} x$, there is a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ with $[x_{n_k}, x] \in \mathcal{R}$, $\forall k \in \mathbb{N}_0$. Notice that $[x_{n_k}, x] \in \mathcal{R}$, $\forall k \in \mathbb{N}_0$ implies that either $(x_{n_k}, x) \in \mathcal{R}$, $\forall k \in \mathbb{N}_0$ or $(x, x_{n_k}) \in \mathcal{R}$, $\forall k \in \mathbb{N}_0$. Applying condition (e) to $(x_{n_k}, x) \in \mathcal{R}$, $\forall k \in \mathbb{N}_0$, we have

$$F\bigl(d(Tx_{n_k}, Tx), d(x_{n_k}, x), d(x_{n_k}, Tx_{n_k}), d(x, Tx), d(x_{n_k}, Tx), d(x, Tx_{n_k})\bigr) \le 0,$$

or

$$F\bigl(d(x_{n_k+1}, Tx), d(x_{n_k}, x), d(x_{n_k}, x_{n_k+1}), d(x, Tx), d(x_{n_k}, Tx), d(x, x_{n_k+1})\bigr) \le 0.$$

Passing $n \to \infty$, and using $x_{n_k} \xrightarrow{d} x$ and the continuity of F and d, we obtain

$$F\bigl(d(x, Tx), 0, 0, d(x, Tx), d(x, Tx), 0\bigr) \le 0.$$

Hence, owing to (F_1), we obtain $d(x, Tx) = 0$, so that $Tx = x$, *i.e.*, x is the fixed point of T.

Similarly, if $(x, x_{n_k}) \in \mathcal{R}$, $\forall k \in \mathbb{N}_0$, then owing to (F_2), we obtain $d(Tx, x) = 0$, so that $Tx = x$, *i.e.*, x is the fixed point of T.

Thus, in all the cases T has a fixed point. □

Theorem 2 *In addition to the hypotheses* (a)-(e) *of Theorem* 1, *suppose that the following condition holds*:

$$\text{(f)}: \Upsilon_T\bigl(x, y, \mathcal{R}^s\bigr) \textit{ is non-empty, for each } x, y \in X,$$

wherein F also enjoys (F_3). *Then T has a unique fixed point.*

Proof Observe that (in view of Theorem 1) $F(T)$ is non-empty. In the case $F(T)$ is singleton then there is nothing to prove. Otherwise, to accomplish the proof, take two arbitrary but distinct elements x, y in $F(T)$, so that

$$Tx = x \quad \text{and} \quad Ty = y. \tag{3}$$

Now, we are required to show that $x = y$.

In view of the hypothesis (f), there exists a path (say, $\{z_0, z_1, z_2, \ldots, z_l\}$) of finite length l in $\mathcal{R}^s$ from x to y, with

$$z_0 = x, \qquad z_l = y, \quad [z_i, z_{i+1}] \in \mathcal{R}, \text{ for each } i \in \{0, 1, 2, \ldots, l-1\}, \tag{4}$$

and

$$[z_i, Tz_i] \in \mathcal{R}, \quad \text{for each } i \in \{1, 2, \ldots, l-1\}. \tag{5}$$

Construct two constant sequences

$$z_n^0 = x \quad \text{and} \quad z_n^l = y.$$

Then, by using (3),

$$Tz_n^0 = Tx = x \quad \text{for all } n \in \mathbb{N}_0, \quad \text{and}$$
$$Tz_n^l = Ty = y \quad \text{for all } n \in \mathbb{N}_0.$$

Setting

$$z_0^i = z_i \quad \text{for each } i \in \{0, 1, 2, \ldots, l\}, \tag{6}$$

we can construct Picard sequence $\{z_n^i\}$, *i.e.*, $Tz_n^i = z_{n+1}^i$ corresponding to each z_i. Hence, $Tz_n^i = z_{n+1}^i$ for $n \in \mathbb{N}_0$ and for each $i \in \{0, 1, 2, \ldots, l\}$. Since $[z_0^i, z_1^i] \in \mathcal{R}$ (owing to (5) and (6)) and $\mathcal{R}$ is T closed, on using (2), we get

$$\lim_{n\to\infty} d\bigl(z_n^i, z_{n+1}^i\bigr) = 0, \quad \text{for each } i \in \{1, 2, \ldots, l-1\}. \tag{7}$$

By using $[z_0^i, z_0^{i+1}] \in \mathcal{R}$ (due to (4) and (6)) and $\mathcal{R}$ is T-closed, we obtain

$$\bigl[T^n z_0^i, T^n z_0^{i+1}\bigr] \in \mathcal{R}, \quad \text{for each } i \in \{0, 1, 2, \ldots, l-1\} \text{ and for all } n \in \mathbb{N}_0$$
$$\Rightarrow \quad \bigl[z_n^i, z_n^{i+1}\bigr] \in \mathcal{R}, \quad \text{for each } i \in \{0, 1, 2, \ldots, l-1\} \text{ and for all } n \in \mathbb{N}_0.$$

Define $d_n^i := d(z_n^i, z_n^{i+1})$, for all $n \in \mathbb{N}_0$ and for each $i \in \{0, 1, 2, \ldots, l-1\}$.
We assert that

$$\lim_{n\to\infty} d_n^i = 0.$$

Let $\lim_{n\to\infty} d_n^i = r > 0$. Since $[z_n^i, z_n^{i+1}] \in \mathcal{R}$, either $(z_n^i, z_n^{i+1}) \in \mathcal{R}$ or $(z_n^{i+1}, z_n^i) \in \mathcal{R}$. If $(z_n^i, z_n^{i+1}) \in \mathcal{R}$, then applying condition (e) to it, we obtain

$$F\bigl(d\bigl(Tz_n^i, Tz_n^{i+1}\bigr), d\bigl(z_n^i, z_n^{i+1}\bigr), d\bigl(z_n^i, Tz_n^i\bigr), d\bigl(z_n^{i+1}, Tz_n^{i+1}\bigr), d\bigl(z_n^i, Tz_n^{i+1}\bigr), d\bigl(z_n^{i+1}, Tz_n^i\bigr)\bigr) \le 0,$$

or

$$F\bigl(d\bigl(z_{n+1}^i, z_{n+1}^{i+1}\bigr), d\bigl(z_n^i, z_n^{i+1}\bigr), d\bigl(z_n^i, z_{n+1}^i\bigr), d\bigl(z_n^{i+1}, z_{n+1}^{i+1}\bigr), d\bigl(z_n^i, z_{n+1}^{i+1}\bigr), d\bigl(z_n^{i+1}, z_{n+1}^i\bigr)\bigr) \le 0.$$

Taking $n \to \infty$ and using $\lim_{n\to\infty} d_n^i = r$ along with (7), we get

$$F(r, r, 0, 0, r, r) \le 0,$$

which is a contradiction (due to (F_3)) and hence

$$\lim_{n\to\infty} d_n^i = r = 0.$$

Similarly, if $(z_n^{i+1}, z_n^i) \in \mathcal{R}$, then, as earlier, we obtain

$$\lim_{n\to\infty} d_n^i = r = 0.$$

Thus,

$$\lim_{n\to\infty} d_n^i := \lim_{n\to\infty} d\bigl(z_n^i, z_n^{i+1}\bigr) = 0, \quad \text{for each } i \in \{0,1,2,\ldots,l-1\}.$$

On using $\lim_{n\to\infty} d_n^i = 0$ and the triangular inequality, we have

$$\begin{aligned} d(x,y) = d\bigl(z_n^0, z_n^l\bigr) &\le \sum_{i=0}^{l-1} d\bigl(z_n^i, z_n^{i+1}\bigr) \\ &= \sum_{i=0}^{l-1} d_n^i \\ &\to 0 \quad \text{as } n \to \infty, \end{aligned}$$

so that $d(x,y) = 0$ implying thereby $x = y$. Thus, T has a unique fixed point. This completes the proof. □

If $\mathcal{R}$ is complete or X is $\mathcal{R}^s$-directed, then the following corollary is worth recording.

Corollary 1 *Theorem* 2 *remains true if we replace condition* (f) *by one of the following conditions besides retaining the rest of the hypotheses*:

(f′) $\mathcal{R}$ *is complete*;
(f″) X *is* $\mathcal{R}^s$*-directed and* $\Delta(X,\mathcal{R}^s) \subset X(T,\mathcal{R}^s)$.

Proof Suppose that the condition (f′) holds. Then for any pair of points x, y in X, we have $[x,y] \in \mathcal{R}$, which implies that $\{x,y\}$ is a path of length 1 from x to y in $\mathcal{R}^s$, so that $\Upsilon_T(x,y,\mathcal{R}^s)$ is non-empty. Finally, proceeding along the lines of the proof of Theorem 2, we complete the proof.

Alternatively, if (f″) holds, then for any pair of points x, y in X, there is z in X such that $[x,z] \in \mathcal{R}$ and $[y,z] \in \mathcal{R}$ so that $\{x,z,y\}$ is a path of length 2 joining x to y in $\mathcal{R}^s$. As $z \in \Delta(X,\mathcal{R}^s) \subset X(T,\mathcal{R}^s)$, therefore $[z,Tz] \in \mathcal{R}$. Thus, for each x, y in X, $\Upsilon_T(x,y,\mathcal{R}^s)$ is non-empty and hence in view of Theorem 2 the result follows. □

From Theorems 1 and 2, we can deduce a host of corollaries which are embodied in the following.

Corollary 2 *The conclusions of Theorems* 1 *and* 2 *remain true if for all* $x,y \in X$ *with* $(x,y) \in \mathcal{R}$, *the implicit relation* (e) *is replaced by one of the following besides retaining the rest of the hypotheses*:

$$d(Tx,Ty) \le kd(x,y), \quad \textit{where } k \in [0,1); \tag{8}$$

$$d(Tx,Ty) \le k\bigl[d(x,Tx) + d(y,Ty)\bigr], \quad \textit{where } k \in [0,1/2); \tag{9}$$

$$d(Tx,Ty) \le k\bigl[d(x,Ty) + d(y,Tx)\bigr], \quad \textit{where } k \in [0,1/2); \tag{10}$$

$$d(Tx,Ty) \le k \max\left\{d(x,y), \frac{d(x,Tx)+d(y,Ty)}{2}, \frac{d(x,Ty)+d(y,Tx)}{2}\right\}, \quad \textit{where } k \in [0,1); \tag{11}$$

$$d(Tx,Ty) \leq k\max\{d(x,Tx),d(y,Ty)\}, \quad \textit{where } k\in[0,1); \tag{12}$$

$$\begin{aligned} d(Tx,Ty) &\leq a_1 d(x,y) + a_2\bigl[d(x,Tx)+d(y,Ty)\bigr] + a_3\bigl[d(x,Ty)+d(y,Tx)\bigr], \\ &\quad \textit{where } a_1,a_2,a_3\in[0,1) \textit{ and } a_1+2a_2+2a_3<1; \end{aligned} \tag{13}$$

$$\begin{aligned} d(Tx,Ty) &\leq k\max\left\{d(x,y),\frac{d(x,Tx)+d(y,Ty)}{2},d(x,Ty),d(y,Tx)\right\}, \\ &\quad \textit{where } k\in[0,1); \end{aligned} \tag{14}$$

$$\begin{aligned} d(Tx,Ty) &\leq kd(x,y) + L\min\{d(x,Tx),d(y,Ty),d(x,Ty),d(y,Tx)\}, \\ &\quad \textit{where } k\in[0,1) \textit{ and } L\geq 0; \end{aligned} \tag{15}$$

$$\begin{aligned} d(Tx,Ty) &\leq a_1 d(x,y) + a_2 d(x,Tx) + a_3 d(y,Ty) + a_4\bigl[d(x,Ty)+d(y,Tx)\bigr], \\ &\quad \textit{where } a_1,a_2,a_3,a_4\geq 0 \textit{ and } a_1+a_2+a_3+2a_4<1; \end{aligned} \tag{16}$$

$$\begin{aligned} d(Tx,Ty) &\leq k\max\left\{d(x,y),d(x,Tx),d(y,Ty),\frac{d(x,Ty)+d(x,Ty)}{2}\right\} \\ &\quad + L\min\{d(x,Tx),d(y,Ty),d(x,Ty),d(y,Tx)\}, \\ &\quad \textit{where } k\in[0,1) \textit{ and } L\geq 0; \end{aligned} \tag{17}$$

$$\begin{aligned} d(Tx,Ty) &\leq k\max\{d(x,y),d(x,Tx),d(y,Ty),d(x,Ty),d(y,Tx)\}, \\ &\quad \textit{where } k\in[0,1/2); \end{aligned} \tag{18}$$

$$\begin{aligned} d(Tx,Ty) &\leq a_1 d(x,y) + a_2 d(x,Tx) + a_2 d(y,Ty) + a_4 d(x,Ty) + a_5 d(y,Tx), \\ &\quad \textit{where } a_i\textit{'s} > 0 \ (\textit{for } i=1,2,3,4,5) \textit{ and sum of them is strictly} \\ &\quad \textit{less than } 1; \end{aligned} \tag{19}$$

$$\begin{aligned} d(Tx,Ty) &\leq k\max\left\{d(x,y),d(x,Tx),d(y,Ty),\frac{d(x,Ty)}{2},\frac{d(y,Tx)}{2}\right\}, \\ &\quad \textit{where } k\in[0,1); \end{aligned} \tag{20}$$

$$\begin{aligned} d(Tx,Ty) &\leq k\max\{d(x,y),d(x,Tx),d(y,Ty)\} + (1-k)\bigl[ad(x,Ty)+bd(y,Tx)\bigr], \\ &\quad \textit{where } k\in[0,1) \textit{ and } 0\leq a,b<1/2; \end{aligned} \tag{21}$$

$$\begin{aligned} d^2(Tx,Ty) &\leq d(Tx,Ty)\bigl[a_1 d(x,y) + a_2 d(x,Tx) + a_3 d(y,Ty)\bigr] + a_4 d(x,Ty)d(y,Tx), \\ &\quad \textit{where } a_1>0; a_2,a_3,a_4\geq 0; a_1+a_2+a_3<1 \textit{ and } a_1+a_4<1; \end{aligned} \tag{22}$$

$$\begin{aligned} d(Tx,Ty) &\leq \begin{cases} kd(x,y)\frac{d(x,Ty)+d(y,Tx)}{d(Tx,Ty)+d(x,y)}, & \textit{if } (Tx,Ty)+d(x,y)\neq 0; \\ 0 & \textit{if } (Tx,Ty)+d(x,y)=0, \end{cases} \\ &\quad \textit{where } k\in[0,1); \end{aligned} \tag{23}$$

$$\begin{aligned} d^2(Tx,Ty) &\leq a_1\max\{d^2(x,y),d^2(x,Tx),d^2(y,Ty)\} \\ &\quad + a_2\max\{d(x,Tx)d(x,Ty),d(y,Ty)d(y,Tx)\} + a_3 d(x,Ty)d(y,Tx), \\ &\quad \textit{where } a_1>0, a_2,a_3\geq 0, a_1+2a_2<1 \textit{ and } a_1+a_3<1; \end{aligned} \tag{24}$$

$$d^3(Tx,Ty) \leq k\big(d^3(x,y)+d^3(x,Tx)+d^3(y,Ty)+d^3(x,Ty)+d^3(y,Tx)\big),$$
$$\textit{where } k \in [0,1); \tag{25}$$

$$d(Tx,Ty) \leq \begin{cases} a_1 \frac{d(x,y)d(y,Ty)}{d(x,y)+d(y,Ty)} + a_2 \frac{d(x,Tx)d(y,Tx)}{d(x,Ty)+d(y,Tx)+1}, & \textit{if } d(x,y)+d(y,Ty) \neq 0; \\ 0 & \textit{if } d(x,y)+d(y,Ty)=0, \end{cases}$$
$$\textit{where } a_1, a_2 > 0 \textit{ and } a_1 < 2. \tag{26}$$

Proof The proof of Corollary 2 follows from Theorems 1, 2, and the examples of the implicit function, (1)-(16). □

Remark 3 Theorem 2, corresponding to condition (8), remains true if we replace the condition (f) by the following relatively weaker condition besides retaining the rest of the hypotheses:

$(\tilde{\mathrm{f}}) : \Upsilon(x,y,\mathcal{R}^s)$ is non-empty, for each $x,y \in X$.

Now, we mention some special cases corresponding to some core contraction conditions.

- Corollary 2 corresponding to condition (8) can be viewed a relation-theoretic version of the Banach contraction principle which was established by Alam and Imdad [23].
- Corollary 2 corresponding to condition (9) is a relation-theoretic version of famous Kannan fixed point theorem proved in [2], which remains a new result.
- Corollary 2 corresponding to condition (10) is a relation-theoretic version of a fixed point theorem of Chatterjea [5], which is not reported in the literature till date.
- Corollary 2 corresponding to condition (12) is a relation-theoretic version of a fixed point theorem due to Bianchini [4], which is new to the existing literature.
- Corollary 2 corresponding to condition (16) with $a_4 = 0$ is a relation-theoretic version of a fixed point theorem of Reich [3], which is indeed new.
- Corollary 2 corresponding to condition (18) is merely a partial (due to the fact $k \in [0,1/2)$) relation-theoretic version of Ćirić [7], which has remained unreported in the literature.
- Corollary 2 corresponding to condition (19) is a relation-theoretic version of Hardy and Rogers [6], which is yet another addition to the existing literature.

As specified in Corollary 2, results corresponding to (11), (13)-(15), (17), (20), (21) are relation-theoretic versions of several known fixed point theorems of the existing literature, whereas, the results corresponding to (22)-(26) are new.

We utilize the following example to demonstrate the genuineness of our extension.

Example 2 Let $X = [0,2]$ under the natural metric d. Then (X,d) is a complete metric space. Now, we define a mapping $T : X \to X$ by

$$T(x) = \begin{cases} 0, & x \in [0,1]; \\ 1, & x \in (1,2], \end{cases}$$

and a binary relation $\mathcal{R} = \{(0,0),(0,1),(0,2),(1,1),(1,2),(2,2)\}$ on X. Clearly, $\mathcal{R}$ is T-closed but T is not continuous. Choose any $\mathcal{R}$-preserving sequence $\{x_n\}$ with

$$x_n \stackrel{d}{\longrightarrow} x \text{ such that } (x_n, x_{n+1}) \in \mathcal{R}, \quad \text{for all } n \in \mathbb{N}_0.$$

Here, one may notice that $(x_n, x_{n+1}) \in \mathcal{R}$, for all $n \in \mathbb{N}_0$, and there exists an integer $N \in \mathbb{N}_0$ such that $x_n = x \in \{0,1,2\}$ for all $n \geq N$. So, we can take a subsequence $\{x_{n_k}\}$ of the sequence $\{x_n\}$ such that $x_{n_k} = x$ for all $k \in \mathbb{N}_0$, which amounts to saying that $[x_{n_k}, x] \in \mathcal{R}$, for all $k \in \mathbb{N}_0$. Therefore, $\mathcal{R}$ is d-self-closed.

Define a continuous function $F : \mathbb{R}_+^6 \to \mathbb{R}$ by

$$F(s_1, s_2, s_3, s_4, s_5, s_6) = s_1 - \frac{9}{100} s_5 - \frac{9}{10} s_6,$$

which meets the requirements of our implicit function. By straightforward calculation, one can verify assumption (e) of Theorem 1. In all, the requirements (a)-(e) of Theorem 1 are met. Observe that the point $x = 0$ is fixed under T.

With a view to establish the genuineness of our extension, notice that in Example 2,

$$(1,2) \in \mathcal{R} \text{ but } d(T1, T2) \leq kd(1,2), \quad \textit{i.e.}, 1 \leq k,$$

which shows that the contractive condition of Theorem 1 due to Alam and Imdad [23] is not satisfied. Thus, in all, our Theorem 1 is applicable to the present example while Theorem 1 of Alam and Imdad is not, which substantiates the utility of Theorem 1.

5 An application

As an application of Theorem 2, we prove an existence and uniqueness theorem on the solution of a Fredholm integral equation described by

$$x(t) = v(t) + \mu \int_a^b K(t,s) x(s)\, ds, \tag{27}$$

where x is an unknown function on $I = [a,b]$ $(b > a)$, μ a parameter, v a known continuous function on I, and K a kernel defined on $G = I \times I$.

Now, we give the following definitions.

Definition 12 A lower solution for (27) is a function $\alpha \in C(I, \mathbb{R})$ such that

$$\alpha(t) \leq v(t) + \mu \int_a^b K(t,s) \alpha(s)\, ds.$$

Definition 13 An upper solution for (27) is a function $\beta \in C(I, \mathbb{R})$ such that

$$\beta(t) \geq v(t) + \mu \int_a^b K(t,s) \beta(s)\, ds.$$

Theorem 3 *Consider the problem described by* (27), *where* $K: I \times I \to \mathbb{R}$ *is continuous and there exist* $\mu > 0$, $c > 0$ *such that*

$$0 \leq K(t,s) \leq c \quad \text{for all } t,s \in I, \text{ with } \mu c(b-a) < 1.$$

Then the existence of a lower solution of (27) *ensures the existence of a unique solution of* (27).

Proof Define a mapping $T: C(I,\mathbb{R}) \to C(I,\mathbb{R})$ by

$$(Tx)(t) = v(t) + \mu \int_a^b K(t,s)x(s)\,ds, \quad t \in I,$$

and a binary relation

$$\mathcal{R} = \{(x,y) \in C(I,\mathbb{R}) \times C(I,\mathbb{R}) \mid x(t) \leq y(t), \forall t \in I\}.$$

(i) Notice that $C(I,\mathbb{R})$ equipped with the sup-metric *i.e.*,

$$d(x,y) = \sup_{t\in I}|x(t) - y(t)|, \quad \text{for } x,y \in C(I,\mathbb{R})$$

is a complete metric space and hence $(C(I,\mathbb{R}),d)$ is $\mathcal{R}$-complete.

(ii) Choose an $\mathcal{R}$-preserving sequence $\{x_n\}$ such that $x_n \xrightarrow{d} x$. Then for all $t \in I$, we get

$$x_0(t) \leq x_1(t) \leq x_2(t) \leq \cdots \leq x_n(t) \leq x_{n+1}(t) \leq \cdots$$

and convergence to $x(t)$ implies that $x_n(t) \leq x(t)$ for all $t \in I$, $n \in \mathbb{N}_0$, which amounts to saying that $[x_n,x] \in \mathcal{R}$, for all $n \in \mathbb{N}_0$. Hence, $\mathcal{R}$ is d-self-closed.

(iii) For any $(x,y) \in \mathcal{R}$, *i.e.*, $x(t) \leq y(t)$ for all $t \in I$, $\mu > 0$ and $K(t,s) \geq 0$, we obtain

$$\begin{aligned}(Tx)(t) &= v(t) + \mu \int_a^b K(t,s)x(s)\,ds \\ &\leq v(t) + \mu \int_a^b K(t,s)y(s)\,ds \\ &= (Ty)(t),\end{aligned}$$

which shows that $(Tx,Ty) \in \mathcal{R}$, *i.e.*, $\mathcal{R}$ is T-closed.

(iv) For all $(x,y) \in \mathcal{R}$,

$$\begin{aligned}d(Tx,Ty) &= \sup_{t\in I}|Tx(t) - Ty(t)| \\ &\leq \sup_{t\in I} \mu \int_a^b |K(t,s)||x(s) - y(s)|\,ds \\ &\leq kd(x,y),\end{aligned}$$

where $k = \mu c(b-a) < 1$ (by assumption). This proves that T satisfies hypothesis (e) of Theorem 1 with $k < 1$.

(v) Now, let $\alpha \in C(I,\mathbb{R})$ be a lower solution of (27), therefore, for all $t \in I$,

$$\begin{aligned} \alpha(t) &\leq v(t) + \mu \int_a^b K(t,s)\alpha(s)\,ds, \\ &= (T\alpha)(t). \end{aligned}$$

This implies that $(\alpha, T\alpha) \in \mathcal{R}$, *i.e.*, $X(T,\mathcal{R})$ is non-empty.

(vi) Finally, let x and y be arbitrary elements of $C(I,\mathbb{R})$ and $z := \max\{x,y\}$. Then $x(t) \leq z(t)$ and $y(t) \leq z(t)$ for all $t \in I$. This implies that $(x,z) \in \mathcal{R}$ and $(y,z) \in \mathcal{R}$. Therefore, the finite sequence $\{x,z,y\}$ describes a path which joins x to y in $\mathcal{R}$.

Now, on using Corollary 2 corresponding to (8) (see Remark 3), the mapping T admits a unique fixed point, which also remains a unique solution of the problem described by (27). □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors read and approved the final manuscript.

Acknowledgements
All the authors thank the referees for their valuable suggestions and comments bringing about several improvements. The second author is thankful to UGC-India for a Start-up Grant (No. F.30-62/2014(BSR)).

Received: 4 November 2015 Accepted: 9 March 2016 Published online: 25 March 2016

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