Letter

Light quark masses in multi-quark interactions

A.A. Osipov[a], B. Hiller[b], and A.H. Blin[c]

Centro de Física Computacional, Departamento de Física da Universidade de Coimbra, 3004-516 Coimbra, Portugal

Abstract. We suggest and discuss in detail a multi-quark three-flavor Lagrangian of the Nambu–Jona-Lasinio type, which includes a set of effective interactions proportional to the current quark masses. It is shown that within the dynamical chiral symmetry-breaking regime, the masses of the pseudo-Goldstone bosons and their chiral partners, members of the low-lying scalar nonet, are in perfect agreement with current phenomenological expectations. The role of the new interactions is analyzed.

1 Introduction

A long history of applying the Nambu–Jona-Lasinio (NJL) model in hadron physics shows the importance of the concept of effective multi-quark interactions for modelling QCD at low energies. Originally formulated only in terms of four-fermion chiral-symmetric couplings [1,2], the model has been extended to the realistic three-flavor and color case with $U(1)_A$ breaking six-quark 't Hooft interactions [3–16] and a set of eight-quark interactions [17]. The last ones complete the number of vertices which are important in four dimensions for dynamical chiral symmetry breaking [18,19].

The explicit breaking of chiral symmetry in the model is described by the quark mass term of the QCD Lagrangian, *e.g.*, [20, 21]. As a result, deviations from the exact symmetry predictions are expressed by functions of the light quark masses. The current quark mass dependence is of importance for several reasons, in particular for the phenomenological description of meson spectra and meson-meson interactions, and for the critical point search in hot and dense hadronic matter. In the latter case it has a strong impact on the phase diagram. The mass effects may lead to a different phase structure. For instance, the large mass difference between s and $u(d)$ quarks may disfavor the formation of the color-flavor-locked phase at intermediate density, and the conjecture regarding the two critical points structure finally may not be true [22].

The explicit chiral symmetry breaking (ChSB) by the standard mass term of the free Lagrangian is only a part of the more complicated picture arising in effective models beyond leading order [23]. Chiral perturbation

theory [24–27] gives a well-known example of a self-consistent accounting of the mass terms, order by order, in an expansion in the masses themselves. In fact, NJL-type models should not be an exception from this rule. If one considers multi-quark effective vertices, to the extent that 't Hooft and eight-quark terms are included in the Lagrangian, certain mass-dependent multi-quark interactions must be also taken into account. It is the purpose of this paper to study such higher-order terms in the quark mass expansion. In particular, we show the ability of the model with new quark-mass–dependent interactions to describe the spectrum of the pseudo-Goldstone bosons, including the fine tuning of the $\eta-\eta'$ splitting, and the spectrum of the light scalar mesons: $\sigma(600)$, $\kappa(850)$, $f_0(980)$, and $a_0(980)$.

There are several motivations for this work. In the first place, the quark masses are the only parameters of the QCD Lagrangian which are responsible for an explicit ChSB, and it is important for the effective theory to trace this dependence in full detail. In this paper it will be argued that it is from the point of view of the $1/N_c$ expansion that the new quark-mass–dependent interactions must be included in the NJL-type Lagrangian already when the $U(1)_A$ breaking 't Hooft determinantal interaction is considered. This point is somehow completely ignored in the current literature.

A second reason is that nowadays it is getting clear that the eight-quark interactions, which are almost inessential for the mesonic spectra in the vacuum, can be important for the quark matter in a strong magnetic background [28–32]. We will show that there is a set of the effective quark-mass–dependent interactions which are of importance here and have not been considered yet.

A further motivation comes from the hadronic matter studies in a hot and dense environment. It is known that

[a] e-mail: osipov@nu.jinr.ru
[b] e-mail: brigitte@teor.fis.uc.pt
[c] e-mail: alex@teor.fis.uc.pt

lattice QCD at finite density suffers from the numerical sign problem. Thus, the phase diagram is notoriously difficult to compute *ab initio*, except for an extremely high-density regime where perturbative QCD methods are applicable. In such circumstances effective models designed to shed light on the phase structure of QCD are valuable, especially if such models are known to be successful in the description of the hadronic matter at zero temperature and density. Reasonable modifications of the NJL model are of special interest in this context and our work aims at future applications in that area.

2 Effective multi-quark interactions

The chiral quark Lagrangian has predictive power for the energy range which is of order $\Lambda \simeq 4\pi f_\pi \sim 1\,\mathrm{GeV}$ [33]. Λ characterizes the spontaneous chiral symmetry-breaking scale. Consequently, the effective multi-quark interactions, responsible for this dynamical effect, are suppressed by Λ, which provides a natural expansion parameter in the chiral effective Lagrangian. The scale above which these interactions disappear and QCD becomes perturbative enters the NJL model as an ultraviolet cut-off for the quark loops. Thus, to build the NJL-type Lagrangian we have only three elements: the quark fields q, the scale Λ, and the external sources χ, which generate explicit symmetry-breaking effects —resulting in mass terms and mass-dependent interactions.

The color quark fields possess definite transformation properties with respect to the chiral flavor $U(3)_L \times U(3)_R$ global symmetry of the QCD Lagrangian with three massless quarks (in the large N_c limit). It is convenient to introduce the $U(3)$ Lie-algebra valued field $\Sigma = (s_a - ip_a)\frac{1}{2}\lambda_a$, where $s_a = \bar{q}\lambda_a q$, $p_a = \bar{q}\lambda_a i\gamma_5 q$, and $a = 0, 1, \ldots, 8$, $\lambda_0 = \sqrt{2/3} \times 1$, λ_a being the standard $SU(3)$ Gell-Mann matrices for $1 \leq a \leq 8$. Under chiral transformations, $q' = V_R q_R + V_L q_L$, where $q_R = P_R q$, $q_L = P_L q$, and $P_{R,L} = \frac{1}{2}(1 \pm \gamma_5)$. Hence, $\Sigma' = V_R \Sigma V_L^\dagger$, and $\Sigma^{\dagger'} = V_L \Sigma^\dagger V_R^\dagger$. The transformation property of the source is supposed to be $\chi' = V_R \chi V_L^\dagger$.

Any term of the effective multi-quark Lagrangian without derivatives can be written as a certain combination of fields which is invariant under chiral $SU(3)_R \times SU(3)_L$ transformations and conserves C, P and T discrete symmetries. These terms have the general form

$$L_i \sim \frac{\bar{g}_i}{\Lambda^\gamma} \chi^\alpha \Sigma^\beta, \qquad (1)$$

where $\bar{g}_i$ are dimensionless coupling constants (starting from eq. (21) the dimensional couplings $g_i = \bar{g}_i/\Lambda^\gamma$ will be also considered). Using dimensional arguments we find $\alpha + 3\beta - \gamma = 4$, with integer values for α, β and γ.

We obtain a second restriction by considering only the vertices which make essential contributions to the gap equations in the regime of dynamical chiral symmetry breaking, *i.e.* we collect only the terms whose contributions to the effective potential survive at $\Lambda \to \infty$. We get

this information by contracting quark lines in L_i, finding that this term contributes to the power counting of Λ in the effective potential as $\sim \Lambda^{2\beta - \gamma}$, *i.e.* we obtain that $2\beta - \gamma \geq 0$ (we used the fact that in four dimensions each quark loop contributes as Λ^2).

Combining both restrictions we come to the conclusion that only vertices with

$$\alpha + \beta \leq 4 \qquad (2)$$

must be taken into account in the approximation considered. On the basis of this inequality one can conclude that i) there are only four classes of vertices which contribute at $\alpha = 0$; those are four- six- and eight-quark interactions, corresponding to $\beta = 2, 3$ and 4, respectively; the $\beta = 1$ class is forbidden by chiral symmetry requirements; ii) there are only six classes of vertices depending on external sources χ, they are $\alpha = 1, \beta = 1, 2, 3$; $\alpha = 2, \beta = 1, 2$; and $\alpha = 3, \beta = 1$.

Let us consider now the structure of multi-quark vertices in detail. The Lagrangian corresponding to the case i) is well known,

$$L_{int} = \frac{\bar{G}}{\Lambda^2} \operatorname{tr}\left(\Sigma^\dagger \Sigma\right) + \frac{\bar{\kappa}}{\Lambda^5} \left(\det \Sigma + \det \Sigma^\dagger\right)$$
$$+ \frac{\bar{g}_1}{\Lambda^8} \left(\operatorname{tr} \Sigma^\dagger \Sigma\right)^2 + \frac{\bar{g}_2}{\Lambda^8} \operatorname{tr}\left(\Sigma^\dagger \Sigma \Sigma^\dagger \Sigma\right). \qquad (3)$$

It contains four dimensionful couplings G, κ, g_1, g_2.

The second group ii) contains eleven terms

$$L_\chi = \sum_{i=0}^{10} L_i, \qquad (4)$$

where

$$L_0 = -\operatorname{tr}\left(\Sigma^\dagger \chi + \chi^\dagger \Sigma\right)$$

$$L_1 = -\frac{\bar{\kappa}_1}{\Lambda} e_{ijk} e_{mnl} \Sigma_{im} \chi_{jn} \chi_{kl} + \text{h.c.}$$

$$L_2 = \frac{\bar{\kappa}_2}{\Lambda^3} e_{ijk} e_{mnl} \chi_{im} \Sigma_{jn} \Sigma_{kl} + \text{h.c.}$$

$$L_3 = \frac{\bar{g}_3}{\Lambda^6} \operatorname{tr}\left(\Sigma^\dagger \Sigma \Sigma^\dagger \chi\right) + \text{h.c.}$$

$$L_4 = \frac{\bar{g}_4}{\Lambda^6} \operatorname{tr}\left(\Sigma^\dagger \Sigma\right) \operatorname{tr}\left(\Sigma^\dagger \chi\right) + \text{h.c.}$$

$$L_5 = \frac{\bar{g}_5}{\Lambda^4} \operatorname{tr}\left(\Sigma^\dagger \chi \Sigma^\dagger \chi\right) + \text{h.c.}$$

$$L_6 = \frac{\bar{g}_6}{\Lambda^4} \operatorname{tr}\left(\Sigma \Sigma^\dagger \chi \chi^\dagger + \Sigma^\dagger \Sigma \chi^\dagger \chi\right)$$

$$L_7 = \frac{\bar{g}_7}{\Lambda^4} \left(\operatorname{tr} \Sigma^\dagger \chi + \text{h.c.}\right)^2$$

$$L_8 = \frac{\bar{g}_8}{\Lambda^4} \left(\operatorname{tr} \Sigma^\dagger \chi - \text{h.c.}\right)^2$$

$$L_9 = -\frac{\bar{g}_9}{\Lambda^2} \operatorname{tr}\left(\Sigma^\dagger \chi \chi^\dagger \chi\right) + \text{h.c.}$$

$$L_{10} = -\frac{\bar{g}_{10}}{\Lambda^2} \operatorname{tr}\left(\chi^\dagger \chi\right) \operatorname{tr}\left(\chi^\dagger \Sigma\right) + \text{h.c.} \qquad (5)$$

Each term in the Lagrangian L_6 is hermitian by itself, but because of the parity symmetry of strong interactions,

which transforms one of them into the other, they have a common coupling $\bar{g}_6$.

Some useful insight into the Lagrangian above can be obtained by considering it from the point of view of the $1/N_c$ expansion. Indeed, the number of color components of the quark field q^i is N_c, hence summing over color indices in Σ gives a factor of N_c, *i.e.* one counts $\Sigma \sim N_c$.

The cut-off Λ that gives the right dimensionality to the multi-quark vertices scales as $\Lambda \sim N_c^0 = 1$. On the other hand, since the leading quark contribution to the vacuum energy is known to be of order N_c, the first term in (3) is estimated as N_c, and we conclude that $G \sim 1/N_c$.

Furthermore, the $U(1)_A$ anomaly contribution (the second term in (3)) is suppressed by one power of $1/N_c$, it yields $\kappa \sim 1/N_c^3$.

The last two terms in (3) have the same N_c counting as the 't Hooft term. They are of order 1. Indeed, Zweig's rule violating effects are always of order $1/N_c$ with respect to the leading-order contribution $\sim N_c$. This reasoning helps us to find $g_1 \sim 1/N_c^4$. The term with $g_2 \sim 1/N_c^4$ is also $1/N_c$ suppressed. It represents the next to the leading-order contribution with one internal quark loop in N_c counting. Such vertex contains the admixture of the four-quark component $\bar{q}q\bar{q}q$ to the leading quark-antiquark structure at $N_c \to \infty$.

Next, all terms in eq. (5), except L_0, are of order 1. The argument is just the same as before: this part of the Lagrangian is obtained by succesive insertions of the χ-field (χ counts as $\chi \sim 1$) in place of Σ fields in the already known $1/N_c$ suppressed vertices. It means that $\kappa_1, g_9, g_{10} \sim 1/N_c$, $\kappa_2, g_5, g_6, g_7, g_8 \sim 1/N_c^2$, and $g_3, g_4 \sim 1/N_c^3$.

There are two important conclusions here. The first is that at leading order in $1/N_c$ only two terms contribute: the first term of eq. (3), and the first term of eq. (5). This corresponds exactly to the standard NJL model picture, where mesons are pure $\bar{q}q$ states. At the next to leading order we have thirteen terms additionally. They trace the Zweig's rule violating effects ($\kappa, \kappa_1, \kappa_2, g_1, g_4, g_7, g_8, g_{10}$), and an admixture of the four-quark component to the $\bar{q}q$ one (g_2, g_3, g_5, g_6, g_9). Only the phenomenology of the last three terms from eq. (3) has been studied until now. We must still understand the role of the other ten terms to be consistent with the generic $1/N_c$ expansion of QCD.

The second conclusion is that the N_c counting justifies the classification of the vertices made above on the basis of the inequality (2). This is seen as follows: the equivalent inequality $\lceil (\alpha + \beta)/2 \rceil \le 2$ is obtained by restricting the multi-quark Lagrangian to terms that do not vanish at $N_c \to \infty$ (it follows from (1) that $\beta - \lceil \gamma/2 \rceil \ge 0$ by noting that $\bar{g}_i \sim 1/N_c^{\lceil \gamma/2 \rceil}$, where $\lceil \gamma/2 \rceil$ is the nearest integer greater than or equal to $\gamma/2$).

The total Lagrangian is the sum

$$L = \bar{q} i \gamma^\mu \partial_\mu q + L_{int} + L_\chi. \qquad (6)$$

In this $SU(3)_L \times SU(3)_R$ symmetric chiral Lagrangian we neglect terms with derivatives in the multi-quark interactions, as usually assumed in the NJL model. We follow this approximation, because the specific questions for

which these terms might be important, *e.g.*, the radial meson excitations, or the existence of some inhomogeneous phases, characterized by a spatially varying order parameter, are not the goal of this work.

Finally, having all the building blocks conform with the symmetry content of the model, one is now free to choose the external source χ. Putting $\chi = \mathcal{M}/2$, where

$$\mathcal{M} = \mathrm{diag}(\mu_u, \mu_d, \mu_s),$$

we obtain a consistent set of explicitly breaking chiral symmetry terms. This leads to the following mass-dependent part of the NJL Lagrangian:

$$L_\chi \to L_\mu = -\bar{q} m q + \sum_{i=2}^{8} L_i', \qquad (7)$$

where the current quark mass matrix m is equal to

$$
m = \mathcal{M} + \frac{\bar{\kappa}_1}{\Lambda} (\det \mathcal{M}) \mathcal{M}^{-1} + \frac{\bar{g}_9}{4\Lambda^2} \mathcal{M}^3
$$
$$
+ \frac{\bar{g}_{10}}{4\Lambda^2} (\mathrm{tr} \, \mathcal{M}^2) \mathcal{M}, \qquad (8)
$$

and

$$
\begin{aligned}
L_2' &= \frac{\bar{\kappa}_2}{2\Lambda^3} e_{ijk} e_{mnl} \mathcal{M}_{im} \Sigma_{jn} \Sigma_{kl} + \mathrm{h.c.} \\[4pt]
L_3' &= \frac{\bar{g}_3}{2\Lambda^6} \mathrm{tr} \left(\Sigma^\dagger \Sigma \Sigma^\dagger \mathcal{M} \right) + \mathrm{h.c.} \\[4pt]
L_4' &= \frac{\bar{g}_4}{2\Lambda^6} \mathrm{tr} \left(\Sigma^\dagger \Sigma \right) \mathrm{tr} \left(\Sigma^\dagger \mathcal{M} \right) + \mathrm{h.c.} \\[4pt]
L_5' &= \frac{\bar{g}_5}{4\Lambda^4} \mathrm{tr} \left(\Sigma^\dagger \mathcal{M} \Sigma^\dagger \mathcal{M} \right) + \mathrm{h.c.} \\[4pt]
L_6' &= \frac{\bar{g}_6}{4\Lambda^4} \mathrm{tr} \left[\mathcal{M}^2 \left(\Sigma \Sigma^\dagger + \Sigma^\dagger \Sigma \right) \right] \\[4pt]
L_7' &= \frac{\bar{g}_7}{4\Lambda^4} \left(\mathrm{tr} \, \Sigma^\dagger \mathcal{M} + \mathrm{h.c.} \right)^2 \\[4pt]
L_8' &= \frac{\bar{g}_8}{4\Lambda^4} \left(\mathrm{tr} \, \Sigma^\dagger \mathcal{M} - \mathrm{h.c.} \right)^2 .
\end{aligned}
\qquad (9)
$$

Let us note that there is a definite freedom in the definition of the external source χ. In fact, the sources

$$
\begin{aligned}
\chi^{(c_i)} &= \chi + \frac{c_1}{\Lambda} \left(\det \chi^\dagger \right) \chi \left(\chi^\dagger \chi \right)^{-1} + \frac{c_2}{\Lambda^2} \chi \chi^\dagger \chi \\
&\quad + \frac{c_3}{\Lambda^2} \mathrm{tr} \left(\chi^\dagger \chi \right) \chi,
\end{aligned}
\qquad (10)
$$

with three independent constants c_i have the same symmetry transformation property as χ. Therefore, we could have used $\chi^{(c_i)}$ everywhere that we used χ. As a result, we would come to the same Lagrangian with the following redefinitions of couplings:

$$
\begin{aligned}
\bar{\kappa}_1 &\to \bar{\kappa}_1' = \bar{\kappa}_1 + \frac{c_1}{2}, \quad \bar{g}_5 \to \bar{g}_5' = \bar{g}_5 - \bar{\kappa}_2 c_1, \\[4pt]
\bar{g}_7 &\to \bar{g}_7' = \bar{g}_7 + \frac{\bar{\kappa}_2}{2} c_1, \quad \bar{g}_8 \to \bar{g}_8' = \bar{g}_8 + \frac{\bar{\kappa}_2}{2} c_1, \\[4pt]
\bar{g}_9 &\to \bar{g}_9' = \bar{g}_9 + c_2 - 2\bar{\kappa}_1 c_1, \\[4pt]
\bar{g}_{10} &\to \bar{g}_{10}' = \bar{g}_{10} + c_3 + 2\bar{\kappa}_1 c_1.
\end{aligned}
\qquad (11)
$$

Since c_i are arbitrary parameters, this corresponds to a continuous family of equivalent Lagrangians. This family

reflects the known Kaplan-Manohar ambiguity [34–37] in the definition of the quark mass, and means that several different parameter sets (11) may be used to represent the data. In particular, without loss of generality we can use the reparametrization freedom to obtain the set with $\bar{\kappa}'_1 = \bar{g}'_9 = \bar{g}'_{10} = 0$.

The effective multi-quark Lagrangian can be written now as

$$L = \bar{q}(i\gamma^\mu \partial_\mu - m)q + L_{int} + \sum_{i=2}^{8} L'_i. \qquad (12)$$

It contains eighteen parameters: the scale Λ, three parameters which are responsible for explicit chiral symmetry breaking μ_u, μ_d, μ_s, and fourteen interaction couplings $\bar{G}, \bar{\kappa}, \bar{\kappa}_1, \bar{\kappa}_2, \bar{g}_1, \ldots, \bar{g}_{10}$. Three of them, $\bar{\kappa}_1, \bar{g}_9, \bar{g}_{10}$, contribute to the current quark masses m. Seven more describe the strength of multi-quark interactions with explicit symmetry-breaking effects. These vertices contain new details of the quark dynamics which have not been studied yet in any NJL-type models.

3 From quarks to mesons: stationary phase calculations

The model can be solved by path integral bosonization of this quark Lagrangian. Indeed, following [7] we may equivalently introduce auxiliary fields $s_a = \bar{q}\lambda_a q$, $p_a = \bar{q}i\gamma_5\lambda_a q$, and physical scalar and pseudoscalar fields $\sigma = \sigma_a\lambda_a$, $\phi = \phi_a\lambda_a$. In these variables the Lagrangian is a bilinear form in quark fields (once the replacement has been done the quarks can be integrated out giving us the kinetic terms for the physical fields ϕ and σ),

$$L = \bar{q}\left[i\gamma^\mu\partial_\mu - (\sigma + i\gamma_5\phi)\right]q + L_{aux},$$
$$L_{aux} = s_a\sigma_a + p_a\phi_a - s_a m_a + L_{int}(s,p)$$
$$+ \sum_{i=2}^{8} L'_i(s,p,\mu). \qquad (13)$$

It is clear, that after the elimination of the fields σ, ϕ by means of their classical equations of motion, one can rewrite this Lagrangian in its original form (12). On the other hand, written in terms of auxiliary bosonic variables, the Lagrangian becomes

$$L_{int}(s,p) = L_{4q} + L_{6q} + L_{8q}^{(1)} + L_{8q}^{(2)},$$
$$L_{4q}(s,p) = \frac{\bar{G}}{2\Lambda^2}\left(s_a^2 + p_a^2\right),$$
$$L_{6q}(s,p) = \frac{\bar{\kappa}}{4\Lambda^5}A_{abc}s_a(s_b s_c - 3p_b p_c), \qquad (14)$$
$$L_{8q}^{(1)}(s,p) = \frac{\bar{g}_1}{4\Lambda^8}\left(s_a^2 + p_a^2\right)^2,$$
$$L_{8q}^{(2)}(s,p) = \frac{\bar{g}_2}{8\Lambda^8}\left[d_{abe}d_{cde}\left(s_a s_b + p_a p_b\right)\left(s_c s_d + p_c p_d\right)\right.$$
$$\left. + 4f_{abe}f_{cde}s_a s_c p_b p_d\right],$$

and the quark-mass–dependent part is as follows:

$$L'_2 = \frac{3\bar{\kappa}_2}{2\Lambda^3}A_{abc}\mu_a\left(s_b s_c - p_b p_c\right),$$
$$L'_3 = \frac{\bar{g}_3}{4\Lambda^6}\mu_a\left[d_{abe}d_{cde}s_b\left(s_c s_d + p_c p_d\right) - 2f_{abe}f_{cde}p_b p_c s_d\right],$$
$$L'_4 = \frac{\bar{g}_4}{2\Lambda^6}\mu_b s_b\left(s_a^2 + p_a^2\right),$$
$$L'_5 = \frac{\bar{g}_5}{4\Lambda^4}\mu_b\mu_d\left(d_{abe}d_{cde} - f_{abe}f_{cde}\right)\left(s_a s_c - p_a p_c\right),$$
$$L'_6 = \frac{\bar{g}_6}{4\Lambda^4}\mu_a\mu_b d_{abe}d_{cde}\left(s_c s_d + p_c p_d\right),$$
$$L'_7 = \frac{\bar{g}_7}{\Lambda^4}\left(\mu_a s_a\right)^2,$$
$$L'_8 = -\frac{\bar{g}_8}{\Lambda^4}\left(\mu_a p_a\right)^2, \qquad (15)$$

where

$$A_{abc} = \frac{1}{3!}e_{ijk}e_{mnl}(\lambda_a)_{im}(\lambda_b)_{jn}(\lambda_c)_{kl}, \qquad (16)$$

and the $U(3)$ antisymmetric f_{abc} and symmetric d_{abc} constants are standard.

Our final goal is to clarify the role of the mass-dependent terms described by the Lagrangian densites of eq. (15). We can gain some understanding of this by considering the low-energy meson dynamics which follows from our Lagrangian. For that we must exclude quark degrees of freedom in (13), e.g., by integrating them out from the corresponding generating functional. The standard Gaussian path integral leads us to the fermion determinant, which we expand by using a heat-kernel technique [38–41]. The remaining part of the Lagrangian, L_{aux}, depends on auxiliary fields which do not have kinetic terms. The equations of motion of such a static system are the extremum conditions,

$$\frac{\partial L}{\partial s_a} = 0, \quad \frac{\partial L}{\partial p_a} = 0, \qquad (17)$$

which must be fulfilled in the neighbourhood of the uniform vacuum state of the theory. To take this into account one should shift the scalar field $\sigma \to \sigma + M$. The new σ-field has a vanishing vacuum expectation value $\langle\sigma\rangle = 0$, describing small amplitude fluctuations about the vacuum, with M being the mass of constituent quarks. We seek solutions of eq. (17) in the form

$$s_a^{st} = h_a + h_{ab}^{(1)}\sigma_b + h_{abc}^{(1)}\sigma_b\sigma_c + h_{abc}^{(2)}\phi_b\phi_c + \ldots$$
$$p_a^{st} = h_{ab}^{(2)}\phi_b + h_{abc}^{(3)}\phi_b\sigma_c + \ldots \qquad (18)$$

Equations (17) determine all coefficients of this expansion giving rise to a system of cubic equations to obtain h_a, and the full set of recurrence relations to find higher-order coefficients in (18). We can gain some insight into the physical meaning of these parameters if we calculate the Lagrangian density L_{aux} on the stationary trajectory. In fact, using the recurrence relations, we are led to the

result

$$
L_{aux} = h_a \sigma_a + \frac{1}{2} h_{ab}^{(1)} \sigma_a \sigma_b + \frac{1}{2} h_{ab}^{(2)} \phi_a \phi_b \tag{19}
$$
$$
+ \frac{1}{3} \sigma_a \left[h_{abc}^{(1)} \sigma_b \sigma_c + \left(h_{abc}^{(2)} + h_{bca}^{(3)} \right) \phi_b \phi_c \right] + \dots
$$

From this one can see that h_a define the quark condensates, $h_{ab}^{(1)}$, $h_{ab}^{(2)}$ contribute to the masses of scalar and pseudoscalar states, and higher order h's are the couplings that measure the strength of the meson-meson interactions.

We proceed now to explain the details of determining h. In the following only the first coefficients h_a, $h_{ab}^{(1)}$, and $h_{ab}^{(2)}$ will be of interest to us. In particular, eq. (17) states that $h_a = 0$, if $a \neq 0, 3, 8$, while h_α ($\alpha = 0, 3, 8$), after the convenient redefinition to the flavor indices $i = u, d, s$,

$$
h_\alpha = e_{\alpha i} h_i, \quad e_{\alpha i} = \frac{1}{2\sqrt{3}} \begin{pmatrix} \sqrt{2} & \sqrt{2} & \sqrt{2} \\ \sqrt{3} & -\sqrt{3} & 0 \\ 1 & 1 & -2 \end{pmatrix}, \tag{20}
$$

satisfy the following system of cubic equations:

$$
\Delta_i + \frac{\kappa}{4} t_{ijk} h_j h_k + \frac{h_i}{2} \left(2G + g_1 h^2 + g_4 \mu h \right) + \frac{g_2}{2} h_i^3
$$
$$
+ \frac{\mu_i}{4} \left[3g_3 h_i^2 + g_4 h^2 + 2(g_5 + g_6) \mu_i h_i + 4g_7 \mu h \right]
$$
$$
+ \kappa_2 t_{ijk} \mu_j h_k = 0. \tag{21}
$$

Here $\Delta_i = M_i - m_i$; t_{ijk} is a totally symmetric quantity, whose nonzero components are $t_{uds} = 1$; there is no summation over the open index i but we sum over the dummy indices, e.g., $h^2 = h_u^2 + h_d^2 + h_s^2, \mu h = \mu_u h_u + \mu_d h_d + \mu_s h_s$.

In particular, eq. (8) reads in this basis

$$
m_i = \mu_i \left(1 + \frac{g_9}{4} \mu_i^2 + \frac{g_{10}}{4} \mu^2 \right) + \frac{\kappa_1}{2} t_{ijk} \mu_j \mu_k. \tag{22}
$$

For the set $g_9 = g_{10} = \kappa_1 = 0$ the current quark mass m_i coincides precisely with the explicit symmetry-breaking parameter μ_i.

Note that the factor multiplying h_i in the third term of eq. (21) is the same for each flavor. This quantity also appears in all meson mass expressions, and there is no further dependence on the couplings G, g_1, g_4 involved for meson states with $a = 1, \dots, 7$. Thus there is a freedom of choice which allows to vary these couplings, condensates and quark masses μ_i, without altering this part of the meson mass spectrum.

It is now straightforward to obtain the inverse matrices to $h_{ab}^{(1)}$ and $h_{ab}^{(2)}$, namely,

$$
-2 \left(h_{ab}^{(1)} \right)^{-1} = \left(2G + g_1 h^2 + g_4 \mu h \right) \delta_{ab} + 4g_1 h_a h_b
$$
$$
+3A_{abc} \left(\kappa h_c + 2\kappa_2 \mu_c \right) + g_2 h_r h_c \left(d_{abe} d_{cre} + 2d_{ace} d_{bre} \right)
$$
$$
+g_3 \mu_r h_c \left(d_{abe} d_{cre} + d_{ace} d_{bre} + d_{are} d_{bce} \right)
$$
$$
+2g_4 \left(\mu_a h_b + \mu_b h_a \right) + g_5 \mu_r \mu_c \left(d_{are} d_{bce} - f_{are} f_{bce} \right)
$$
$$
+g_6 \mu_r \mu_c d_{abe} d_{cre} + 4g_7 \mu_a \mu_b, \tag{23}
$$

$$
-2 \left(h_{ab}^{(2)} \right)^{-1} = \left(2G + g_1 h^2 + g_4 \mu h \right) \delta_{ab}
$$
$$
-3A_{abc} \left(\kappa h_c + 2\kappa_2 \mu_c \right) + g_2 h_r h_c \left(d_{abe} d_{cre} + 2f_{are} f_{bce} \right)
$$
$$
+g_3 \mu_r h_c \left(d_{abe} d_{cre} + f_{are} f_{bce} + f_{ace} f_{bre} \right)
$$
$$
-g_5 \mu_r \mu_c \left(d_{are} d_{bce} - f_{are} f_{bce} \right)
$$
$$
+g_6 \mu_r \mu_c d_{abe} d_{cre} - 4g_8 \mu_a \mu_b. \tag{24}
$$

These coefficients are totally defined in terms of h_a and the parameters of the model.

4 From quarks to mesons: heat kernel calculations

We now turn our attention to the total Lagrangian of the bosonized theory. To write down this Lagrangian we should add the terms coming from integrating out the quark degrees of freedom in (13) to our result (19). Fortunately, the result is known. One can find all necessary details of such calculations, for instance, in [38], where we used the modified heat kernel technique [39–41] developed for the case of explicit chiral symmetry breaking. Here we quote the main outcome. The σ tadpole term must be excluded from the total Lagrangian. This gives us a system of gap equations,

$$
h_i + \frac{N_c}{6\pi^2} M_i \left[3I_0 - \left(3M_i^2 - M^2 \right) I_1 \right] = 0. \tag{25}
$$

Here $N_c = 3$ is the number of colors, and $M^2 = M_u^2 + M_d^2 + M_s^2$. The factors I_i ($i = 0, 1, \dots$) are the arithmetic average values $I_i = \frac{1}{3}[J_i(M_u^2) + J_i(M_d^2) + J_i(M_s^2)]$, constructed from the one-quark-loop integrals

$$
J_i(m^2) = \int_0^\infty \frac{dt}{t^{2-i}} \rho(t\Lambda^2) e^{-tm^2}, \tag{26}
$$

with the Pauli-Villars regularization kernel [42, 43]

$$
\rho(t\Lambda^2) = 1 - (1 + t\Lambda^2) \exp(-t\Lambda^2). \tag{27}
$$

In the following we need only to know two of them:

$$
J_0(m^2) = \Lambda^2 - m^2 \ln \left(1 + \frac{\Lambda^2}{m^2} \right) \tag{28}
$$

and

$$
J_1(m^2) = \ln \left(1 + \frac{\Lambda^2}{m^2} \right) - \frac{\Lambda^2}{\Lambda^2 + m^2}. \tag{29}
$$

From now on we will consider the case with an exact $SU(2)$ isospin symmetry, i.e. $\mu_u = \mu_d = \hat{\mu} \neq \mu_s$, and $M_u = M_d = \hat{M} \neq M_s$. We also restrict ourselves to small perturbations, so we retain terms in the bosonized Lagrangian which are quadratic in the perturbations ϕ

and σ. To this order we obtain

$$
\begin{aligned}
L = {} & \frac{N_c I_1}{16\pi^2}\, \mathrm{tr}\left[(\partial_\mu \sigma)^2 + (\partial_\mu \phi)^2\right] + \frac{N_c I_0}{4\pi^2}\left(\sigma_a^2 + \phi_a^2\right) \\
& - \frac{N_c I_1}{12\pi^2}\Big\{ \Delta_{ns}\left[2\sqrt{2}(3\sigma_0\sigma_8 + \phi_0\phi_8) - \phi_8^2 + \phi_i^2\right] \\
& + 2(2\hat{M}^2 + M_s^2)\sigma_0^2 + (\hat{M}^2 + 5M_s^2)\sigma_8^2 \\
& + (7\hat{M}^2 - M_s^2)\sigma_i^2 + (\hat{M} + M_s)(\hat{M} + 2M_s)\sigma_f^2 \\
& + (M_s - \hat{M})(2M_s - \hat{M})\phi_f^2 \Big\} \\
& + \frac{1}{2}\, h_{ab}^{(1)}\sigma_a\sigma_b + \frac{1}{2}\, h_{ab}^{(2)}\phi_a\phi_b + \dots,
\end{aligned}
\tag{30}
$$

where $\Delta_{ns} = \hat{M}^2 - M_s^2$, $\phi_i^2 = \sum_{i=1}^{3}\phi_i^2$, $\phi_f^2 = \sum_{f=4}^{7}\phi_f^2$. The kinetic term requires a redefinition of meson fields,

$$
\sigma_a = g\sigma_a^R, \quad \phi_a = g\phi_a^R, \quad g^2 = \frac{4\pi^2}{N_c I_1},
\tag{31}
$$

to obtain the standard factor $1/4$. The Lagrangian (30) in the chiral limit, $m = 0$, leads to the conserved vector, $\mathcal{V}_\mu^a$, and axial-vector, $\mathcal{A}_\mu^a$, currents. The matrix elements of axial-vector currents,

$$
\langle 0|\mathcal{A}_\mu^a(0)|\phi_R^b(p)\rangle = ip_\mu f^{ab},
\tag{32}
$$

define the weak and electromagnetic decay constants of physical pseudoscalar states (see details in [38]). In fact, we obtain that all new information about the mass-dependent interactions is explicitly absorbed in the last two terms of the Lagrangian, where the matrices $h_{ab}^{(1,2)}$ are block diagonal and mix only in the $(0,8)$ sector, see eqs. (23) and (24). There is also an implicit dependence through the gap and stationary phase equations.

5 Fixing parameters

Now let us fix the values of the various quantities introduced. After choosing the set $\kappa_1 = g_9 = g_{10} = 0$ we still have to fix fourteen parameters: $\Lambda, \hat{m}, m_s, G, \kappa, \kappa_2$ and $g_1, \dots, g_8$. Note that there are two intrinsic restrictions of the model, namely, the stationary phase (21) and the gap (25) equations, which must be solved self-consistently. This is how the explicit symmetry breaking is intertwined with the dynamical symmetry breaking and vice versa. We use (25) to determine $\hat{h}, h_s$ through Λ, M_s and $\hat{M}$. The ratio $M_s/\hat{M}$ is related to the ratio of the weak decay constants of the pion, $f_\pi = 92\,\mathrm{MeV}$, and the kaon, $f_K = 113\,\mathrm{MeV}$. Here we obtain

$$
\frac{M_s}{\hat{M}} = 2\frac{f_K}{f_\pi} - 1 = 1.46.
\tag{33}
$$

Furthermore, the two eqs. (21) can be used to find the values of Λ and $\hat{M}$ if the parameters $\hat{m}, m_s, G, \kappa, \kappa_2, g_1, \dots, g_7$ are known. Thus, together with g_8 we have at this stage thirteen couplings to be fixed. Let us consider the current quark masses $\hat{m}$ and m_s to be an input. Their

values are known, from various analyses of the chiral treatment of the light pseudoscalars, to be around $\hat{m} = 4\,\mathrm{MeV}$ and $m_s = 100\,\mathrm{MeV}$ [44]. Then the remaining eleven couplings can be found by comparing with empirical data. One should stress the possibility (which did not exist before the inclusion of mass-dependent interactions) to fit the low-lying pseudoscalar spectrum, $m_\pi = 138\,\mathrm{MeV}$, $m_K = 494\,\mathrm{MeV}$, $m_\eta = 547\,\mathrm{MeV}$, $m_{\eta'} = 958\,\mathrm{MeV}$, the weak pion and kaon decay constants, $f_\pi = 92\,\mathrm{MeV}$, $f_K = 113\,\mathrm{MeV}$, and the singlet-octet mixing angle $\theta_p = -15°$ to perfect accuracy. One can deduce that the couplings κ_2 and g_8 are essential to improve the description in the pseudoscalar sector; in particular, g_8 is responsible for fine tuning the $\eta - \eta'$ mass splitting.

The remaining five conditions are taken from the scalar sector of the model. Unfortunately, the scalar channel in the region about $1\,\mathrm{GeV}$ became a long-standing problem of QCD. The abundance of meson resonances with 0^{++} quantum numbers shows that one can expect the presence of non-$q\bar{q}$ scalar objects, like glueballs, hybrids, multiquark states and so forth [45]. This creates known difficulties in the interpretation and classification of scalars. For instance, the numerical attempts to organize the $U(3)$ quark-antiquark nonet based on the light scalar mesons, σ or $f_0(600)$, $a_0(980)$, $\kappa(850)$, $f_0(980)$, in the framework of NJL-type models have failed (see, e.g., [8–10, 43, 46–48]). The reason is the ordering of the calculated spectrum which typically is $m_\sigma < m_{a_0} < m_\kappa < m_{f_0}$, as opposed to the empirical evidence: $m_\kappa < m_{a_0} \simeq m_{f_0}$.

On the other hand, it is known that a unitarized non-relativistic meson model can successfully describe the light scalar meson nonet as $\bar{q}q$ states with a meson-meson admixture [49]. Another model which assumes the mixing of $q\bar{q}$ states with others, consisting of two quarks and two antiquarks, $q^2\bar{q}^2$ [50], yields a possible description of the 0^{++} meson spectra as well [51, 52]. The well-known model of Close and Törnqvist [53] is also designed to describe two scalar nonets (above and below $1\,\mathrm{GeV}$). The light scalar nonet below $1\,\mathrm{GeV}$ has a core made of $q^2\bar{q}^2$ states with a small admixture of a $\bar{q}q$ component, rearranged asymptotically as meson-meson states. These successful solutions seemingly indicate on the importance of certain admixtures for the correct description of the light scalars. Our model contains such admixtures in the form of the appropriate effective multi-quark vertices with the asymptotic meson states described by the bosonized $\bar{q}q$ fields. We have found, that the quark-mass–dependent interactions can solve the problem of the light scalar spectrum and these masses can be understood in terms of spontaneous and explicit chiral symmetry breaking only. Indeed, one can easily fit the data: $m_\sigma = 600\,\mathrm{MeV}$, $m_{a_0} = 980\,\mathrm{MeV}$, $m_\kappa = 850\,\mathrm{MeV}$, $m_{f_0} = 980\,\mathrm{MeV}$ with the input value $g_2 = 0$. In this case we obtain for the singlet-octet mixing angle θ_s roughly $\theta_s = 19°$.

To many readers our success with scalars may seem trivial: we have five parameters to fit five numbers. What is not trivial, however, is that the overall result of the fit is also in an agreement with phenomenological expectations. To compare, if we try instead to fit the second

Table 1. Parameter sets of the model: $\hat{m}, m_s$, and Λ are given in MeV. The couplings have the following units: $[G] = \text{GeV}^{-2}$, $[\kappa] = \text{GeV}^{-5}$, $[g_1] = [g_2] = \text{GeV}^{-8}$. We also show here the values of constituent quark masses $\hat{M}$ and M_s in MeV.

Sets	$\hat{m}$	m_s	$\hat{M}$	M_s	Λ	G	$-\kappa$	g_1	g_2
(a)	4.0*	100*	361	526	837	8.96	93.0	1534	0*
(b)	4.0*	100*	361	526	837	7.06	93.3	3420	0*

Table 2. Explicit symmetry-breaking interaction couplings. The couplings have the following units: $[\kappa_1] = \text{GeV}^{-1}$, $[\kappa_2] = \text{GeV}^{-3}$, $[g_3] = [g_4] = \text{GeV}^{-6}$, $[g_5] = [g_6] = [g_7] = [g_8] = \text{GeV}^{-4}$, $[g_9] = [g_{10}] = \text{GeV}^{-2}$.

Sets	κ_1	κ_2	$-g_3$	$-g_4$	g_5	$-g_6$	$-g_7$	g_8	g_9	g_{10}
(a)	0*	9.05	4967	661	192.2	1236	293	52.2	0*	0*
(b)	0*	9.01	4990	653	192.5	1242	293	51.3	0*	0*

scalar nonet $f_0(1370), a_0(1450), K_0^*(1430), f_0(1500)$ with the same input, our attempt fails. The best that we can do is the values $m_{f_0} = 1220 \,\text{MeV}$, $m_{a_0} = 1406 \,\text{MeV}$, $m_{K_0^*} = 1506 \,\text{MeV}$, $m_{f_0'} = 1786 \,\text{MeV}$. However, even these unreasonable masses come out only together with the very large ratio $m_s/\hat{m} = 36$ and phenomenologically unacceptable values for constituent quark masses $\hat{M} = 631 \,\text{MeV}$ and $M_s = 919 \,\text{MeV}$.

We obtain and understand the empirical mass assignment inside the light scalar nonet as a consequence of the quark-mass–dependent interactions, *i.e.* as the result of some predominance of the explicit chiral symmetry-breaking terms over the dynamical chiral symmetry-breaking ones for these states. Indeed, let us consider the difference

$$m_{a_0}^2 - m_\kappa^2 = 2g^2 \left(\frac{1}{H_{a_0}} - \frac{1}{H_\kappa} \right)$$
$$- 2(M_s + 2\hat{M})(M_s - \hat{M}). \tag{34}$$

The sign of this expression is a result of the competition of two terms. In the chiral limit both of them are zero, since at $\hat{\mu}, \mu_s = 0$ we obtain $\hat{M} = M_s$ and $H_{a_0} = H_\kappa$, for H_{a_0} and H_κ being positive. The splitting $H_\kappa > H_{a_0}$ is a necessary condition to get $m_{a_0} > m_\kappa$. The following terms contribute to the difference:

$$H_\kappa - H_{a_0} = \kappa(h_s - \hat{h}) + 2\kappa_2(\mu_s - \hat{\mu}) - g_2(h_s^2 + \hat{h}h_s - 2\hat{h}^2)$$
$$+ \frac{g_3}{2}\left(2\mu_s h_s + \mu_s \hat{h} + \hat{\mu} h_s - 4\hat{\mu}\hat{h} \right)$$
$$+ g_5\hat{\mu}(\mu_s - \hat{\mu}) + \frac{g_6}{2}\left(\mu_s^2 - \hat{\mu}^2 \right). \tag{35}$$

Accordingly, from this formula we deduce the "anatomy" of the successful numerical fit,

$$m_{a_0}^2 - m_\kappa^2 = ([0.007]_\kappa + [0.076]_{\kappa_2} + [0]_{g_2}$$
$$+ [0.832]_{g_3} + [0.003]_{g_5} + [-0.269]_{g_6}$$
$$- [0.41]_M = 0.24) \,\text{GeV}^2, \tag{36}$$

where the contributions of terms with corresponding coupling (see eq. (35)) are indicated in square brackets. The last number, marked by M, is the value of the last term from (34). It is a contribution due to the dynamical chiral symmetry breaking (in the presence of an explicit chiral

symmetry breaking). One can see that the g_3 interaction is the main reason for the reverse ordering $m_{a_0} > m_\kappa$, the coupling g_6 being responsible for the fine tuning of the result.

Let us now show the result of our global fitting of the model parameters. We collect them in two tables. Two sets, (a) and (b), are shown. The difference is the fitted value of the σ mass: in (a) $m_\sigma = 600 \,\text{MeV}$, in (b) $m_\sigma = 500 \,\text{MeV}$. Table 1 contains the standard set of parameters, which are known from previous considerations. Their values are not much affected by the quark mass effects. Table 2 contains the couplings which are responsible for the explicit chiral symmetry-breaking effects in the interactions. Note that these couplings almost do not change from set (a) to (b). We have already learned (as can be seen, again, in table 1) that higher values of g_1 lead to the lower σ mass [38]. This eight-quark interaction violates Zweig's rule, since it involves $q\bar{q}$ annihilation. The mixing angle θ_s is stable with respect to such changes, we obtain $\theta_s = 19.4°$ in case (a), and $\theta_s = 18.9°$ in case (b). The calculated values of quark condensates are the same for both sets: $-\langle \bar{u}u \rangle^{\frac{1}{3}} = 232 \,\text{MeV}$, and $-\langle \bar{s}s \rangle^{\frac{1}{3}} = 206 \,\text{MeV}$. Our calculated values for constituent quark masses agree with the ones found in [8–10,33], showing their insensitivity to the new mass-dependent corrections.

6 Concluding remarks

The purpose of this paper has been to take into account the quark masses at next-to-leading order in the expansion of the effective multi-quark Lagrangian of the NJL type. As a result a picture with some attractive new features has emerged. Let us summarize the details of such a picture.

The main qualitative difference between our result and previous calculations is the possibility to fit the low-lying pseudoscalar spectrum (the pseudo-Goldstone 0^{-+} nonet) and weak decay constants of the pion and the kaon to perfect accuracy. The fitting of the $\eta - \eta'$ mass splitting together with the overall successful description of the whole set of low-energy characteristics is actually a solution for a long-standing problem of NJL-type models. We expect that with such modifications the model is getting more appropriate not only for studying low-energy meson physics, but also in studies of the ground state of hadronic matter

in an environment, which is known to be very sensitive to quark mass effects.

With a set of new quark-mass–dependent interactions we are also capable to describe the spectrum of the light scalar nonet. From that one can conclude that both spectra can be understood on the basis of the dynamical and explicit chiral symmetry breaking only. The splitting inside the scalar nonet is determined by two competing contributions: first it is due to the explicit symmetry breaking (embodied in the stationary phase part of the bosonized Lagrangian), second, it is due to the dynamical symmetry breaking (see the heat kernel part of the bosonized Lagrangian). It is the first type of contribution that changes the ordering inside the light scalar nonet, as compared to the standard approach.

Our result for the scalar sector, being promising by itself, must be considered with some reservation. To report about a real success here, one should explain not only the mass spectrum of scalars, particularly the mass degeneracy of the $f_0(980)$ and $a_0(980)$ states (as we have done here), but answer some known challenges related with radiative decays of these states. Work in this direction is in progress.

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