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The L^1 stability of strong solutions to a generalized *BBM* equation

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Abstract
A nonlinear generalized Benjamin-Bona-Mahony equation is investigated. Using the estimates of strong solutions derived from the equation itself, we establish the $L^1(R)$ stability of the solutions under the assumption that the initial value $u_0(x)$ lies in the space $H^1(R)$.

MSC: 35G25; 35L05

Keywords: generalized *BBM* equation; strong solutions; L^1 stability

1 Introduction

Benjamin, Bona and Mahony [1] established the *BBM* model

$$u_t + au_x - bu_{xxt} + k(u^2)_x = 0, \tag{1}$$

where a, b and k are constants. Equation (1) is often used as an alternative to the *KdV* equation which describes unidirectional propagation of weakly long dispersive waves [2]. As a model that characterizes long waves in nonlinear dispersive media, the *BBM* equation, like the *KdV* equation, was formally derived to describe an approximation for surface water waves in a uniform channel. Equation (1) covers not only the surface waves of long wavelength in liquids, but also hydromagnetic waves in cold plasma, acoustic waves in anharmonic crystals, and acoustic gravity waves in compressible fluids (see [2, 3]). Nonlinear stability of nonlinear periodic solutions of the regularized Benjamin-Ono equation and the Benjamin-Bona-Mahony equation with respect to perturbations of the same wavelength is analytically studied in [4]. Unique continuation property and control for the Benjamin-Bona-Mahony equation on a periodic domain are discussed in [5]. The L^q ($q \geq 2$) asymptotic property of solutions for the Benjamin-Bona-Mahony-Burgers equations is studied in [6] under certain assumptions on the initial data. The tanh technique is employed in [7] to get the compact and noncompact solutions for *KP-BBM* and *ZK-BBM* equations.

Applying the tanh method and the sine-cosine method, Wazwaz [8] obtained compactons, solitons, solitary patterns and periodic solutions for the following generalized Benjamin-Bona-Mahony equation

$$u_t + au_x - bu_{xxt} + k(u^m)_x = 0, \tag{2}$$

where $a \neq 0$, $b > 0$ and $k \neq 0$ are constants, and $m \geq 1$ is an integer.

The objective of this work is to investigate Eq. (2). Using the methods of the Kruzkov's device of doubling the variables presented in Kruzkov's paper [9], we obtain the L^1 stability of strong solutions. Namely, for any solutions $u_1(t,x)$ and $u_2(t,x)$ satisfying Eq. (2), we will derive that

$$\left\|u_1(t,x)-u_2(t,x)\right\|_{L^1(R)} \le ce^{ct}\left\|u_1(0,x)-u_2(0,x)\right\|_{L^1(R)}, \quad t\in[0,T], \tag{3}$$

where T is the maximum existence time of solutions u_1 and u_2 and c depends on $\|u_1(0,x)\|_{H^1(R)}$ and $\|u_2(0,x)\|_{H^1(R)}$. From our knowledge, we state that the L^1 stability of strong solutions for Eq. (2) has never been acquired in the literature.

This paper is organized as follows. Section 2 gives several lemmas and Section 3 establishes the proofs of the main result.

2 Several lemmas

Let $\eta_T = [0,T]\times R$ for an arbitrary $T>0$. We denote the space of all infinitely differentiable functions $f(t,x)$ with compact support in $[0,T]\times R$ by $C_0^\infty(\eta_T)$. We define $\gamma(\sigma)$ to be a function which is infinitely differentiable on $(-\infty,+\infty)$ such that $\gamma(\sigma)\ge 0$, $\gamma(\sigma)=0$ for $|\sigma|\ge 1$ and $\int_{-\infty}^{\infty}\gamma(\sigma)\,d\sigma = 1$. For any number $\varepsilon>0$, we let $\gamma_\varepsilon(\sigma)=\frac{\gamma(\varepsilon^{-1}\sigma)}{\varepsilon}$. Then we have that $\gamma_\varepsilon(\sigma)$ is a function in $C^\infty(-\infty,\infty)$ and

$$\begin{cases}\gamma_\varepsilon(\sigma)\ge 0, & \gamma_\varepsilon(\sigma)=0 \quad \text{if } |\sigma|\ge\varepsilon,\\ |\gamma_\varepsilon(\sigma)|\le\frac{c}{\varepsilon}, & \int_{-\infty}^{\infty}\gamma_\varepsilon(\sigma)\,d\sigma=1.\end{cases} \tag{4}$$

Assume that the function $v(x)$ is locally integrable in $(-\infty,\infty)$. We define the approximation of function $v(x)$ as

$$v^\varepsilon(x)=\frac{1}{\varepsilon}\int_{-\infty}^{\infty}\gamma\left(\frac{x-y}{\varepsilon}\right)v(y)\,dy, \quad \varepsilon>0.$$

We call x_0 a Lebesgue point of function $v(x)$ if

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_{|x-x_0|\le\varepsilon}\left|v(x)-v(x_0)\right|dx=0.$$

At any Lebesgue point x_0 of the function $v(x)$, we have $\lim_{\varepsilon\to 0} v^\varepsilon(x_0)=v(x_0)$. Since the set of points which are not Lebesgue points of $v(x)$ has measure zero, we get $v^\varepsilon(x)\to v(x)$ as $\varepsilon\to 0$ almost everywhere.

We introduce notations connected with the concept of a characteristic cone. For any $M>0$, we define $N>\sup_{t\in[0,\infty)}\|u\|_{L^\infty(R)}<\infty$. Let $\mho$ denote the cone $\{(t,x): |x|\le M-Nt, 0<t<T_0=\min(T,MN^{-1})\}$. We let S_τ represent the cross section of the cone $\mho$ by the plane $t=\tau$, $\tau\in[0,T_0]$. Let $H_r=\{x:|x|\le r\}$, where $r>0$.

Lemma 2.1 ([9]) *Let the function $v(t,x)$ be bounded and measurable in cylinder $\Omega=[0,T]\times H_r$. If for any $\delta\in(0,\min[r,T])$ and any number $\varepsilon\in(0,\delta)$, then the function*

$$V_\varepsilon=\frac{1}{\varepsilon^2}\iiiint_{|\frac{t-\tau}{2}|\le\varepsilon,\delta\le\frac{t+\tau}{2}\le T-\delta,|\frac{x-y}{2}|\le\varepsilon,|\frac{x+y}{2}|\le r-\delta}\left|v(t,x)-v(\tau,y)\right|dx\,dt\,dy\,d\tau$$

satisfies $\lim_{\varepsilon\to 0}V_\varepsilon=0$.

In fact, for Eq. (2), we have the conservation law

$$\int_R \left(u^2 + bu_x^2\right) dx = \int_R \left(u^2(0,x) + bu_x^2(0,x)\right) dx, \tag{5}$$

from which we have

$$\|u\|_{L^\infty(R)} \le c\|u_0\|_{H^1(R)}, \tag{6}$$

where c only depends on b.

We write the equivalent form of Eq. (2) in the form

$$u_t + \Lambda^{-2}\left(au + ku^m\right)_x = 0, \tag{7}$$

where the operator $\Lambda^{-2}g = \frac{1}{2\sqrt{b}}\int_{-\infty}^{\infty} e^{-\frac{1}{\sqrt{b}}|x-y|} g(y)\, dy$ for any $g \in L^2(R)$.

Lemma 2.2 *Let $u_0 = u(0,x) \in H^1(R)$, $K_u(t,x) = \Lambda^{-2}(au + ku^m)$ and $P_u(t,x) = \partial_x K_u(t,x)$. For any $t \in [0,\infty)$, it holds that*

$$\left\|K_u(t,x)\right\|_{L^\infty(R)} < C, \qquad \left\|P_u(t,x)\right\|_{L^\infty(R)} < C,$$

where the constant C is independent of time t.

Proof We have

$$K_u(t,x) = \frac{1}{2\sqrt{b}}\int_{-\infty}^{\infty} e^{-\frac{1}{\sqrt{b}}|x-y|}\left[au(t,y) + ku^m(t,y)\right] dy \tag{8}$$

and

$$\begin{aligned}\left|P_u(t,x)\right| &= \left|\partial_x \frac{1}{2\sqrt{b}}\int_{-\infty}^{\infty} e^{-\frac{1}{\sqrt{b}}|x-y|}\left[au(t,y) + ku^m(t,y)\right] dy\right| \\ &= \left|-\frac{1}{2b}e^{-\frac{1}{\sqrt{b}}x}\int_{-\infty}^{x} e^{\frac{1}{\sqrt{b}}y}\left[au(t,y) + ku^m(t,y)\right] dy \right.\\ &\quad \left. + \frac{1}{2b}e^{\frac{1}{\sqrt{b}}x}\int_{x}^{\infty} e^{-\frac{1}{\sqrt{b}}y}\left[au(t,y) + ku^m(t,y)\right] dy\right| \\ &\le \frac{1}{2b}\int_{-\infty}^{\infty} e^{-\frac{1}{\sqrt{b}}|x-y|}\left|au(t,y) + ku^m(t,y)\right| dy. \end{aligned} \tag{9}$$

Using (5)-(6), the integral $\int_{-\infty}^{\infty} e^{-\frac{1}{\sqrt{b}}|x-y|}\, dy = 2\sqrt{b}$ and (8)-(9), we obtain the proof of Lemma 2.2. □

Lemma 2.3 *Let u be the strong solution of Eq. (2), $f(t,x) \in C_0^\infty(\eta_T)$. Then*

$$\iint_{\eta_T} \left\{|u-k|f_t - \operatorname{sign}(u-k)P_u(t,x)f\right\} dx\, dt = 0, \tag{10}$$

where k is an arbitrary constant.

Proof Let $\Phi(u)$ be an arbitrary twice smooth function on the line $-\infty < u < \infty$. We multiply Eq. (7) by the function $\Phi'(u)f(t,x)$, where $f(t,x) \in C_0^\infty(\eta_T)$. Integrating over η_T and transferring the derivatives with respect to t and x to the test function f, we obtain

$$\iint_{\eta_T} \left\{ \Phi(u)f_t - \Phi'(u)P_u(t,x)f \right\} dx\,dt = 0. \tag{11}$$

Let $\Phi^\varepsilon(u)$ be an approximation of the function $|u-k|$ and set $\Phi(u) = \Phi^\varepsilon(u)$. Letting $\varepsilon \to 0$, we complete the proof. □

In fact, the proof of (10) can also be found in [9].

Lemma 2.4 *Assume that $u_1(t,x)$ and $u_2(t,x)$ are two strong solutions of Eq.* (2) *associated with the initial data $u_{10} = u_1(0,x)$ and $u_{20} = u_2(0,x)$. Then, for any $f \in C_0^\infty(\eta_T)$,*

$$\begin{aligned} &\left| \int_{-\infty}^{\infty} \operatorname{sign}(u_1 - u_2)\left[P_{u_1}(t,x) - P_{u_2}(t,x)\right] f\,dx \right| \\ &\quad \le c \int_{-\infty}^{\infty} |u_1 - u_2|\,dx, \end{aligned} \tag{12}$$

where c depends on $\|u_{10}\|_{H^1(R)}$ and $\|u_{20}\|_{H^1(R)}$ and f.

Proof Using (9), we have

$$\begin{aligned} &\left|P_{u_1}(t,x) - P_{u_2}(t,x)\right| \\ &\quad = \left| -\frac{1}{2b} e^{-\frac{1}{\sqrt{b}}x} \int_{-\infty}^{x} e^{\frac{1}{\sqrt{b}}y} \left[au_1(t,y) + ku_1^m(t,y) - au_2(t,y) - ku_2^m(t,y)\right] dy \right. \\ &\qquad \left. + \frac{1}{2b} e^{\frac{1}{\sqrt{b}}x} \int_{x}^{\infty} e^{-\frac{1}{\sqrt{b}}y} \left[au_1(t,y) + ku_1^m(t,y) - au_2(t,y) - ku_2^m(t,y)\right] dy \right| \\ &\quad \le c \int_{-\infty}^{\infty} e^{-\frac{1}{\sqrt{b}}|x-y|} \left|au_1(t,y) + ku_1^m(t,y) - au_2(t,y) - ku_2^m(t,y)\right| dy \\ &\quad \le c \int_{-\infty}^{\infty} e^{-\frac{1}{\sqrt{b}}|x-y|} \left|u_1(t,y) - u_2(t,y)\right| dy, \end{aligned} \tag{13}$$

in which we have used $\|u_1\|_{L^\infty} \le \|u_{10}\|_{H^1(R)}$ and $\|u_2\|_{L^\infty} \le \|u_{20}\|_{H^1(R)}$. Using the Fubini theorem completes the proof. □

3 Main results

Theorem 3.1 *Let u_1 and u_2 be two local or global strong solutions of Eq.* (2) *with initial data $u_1(0,x) = u_{10} \in H^1(R)$ and $u_2(0,x) = u_{20} \in H^1(R)$, respectively. Let T_0 be the maximum existence time of solutions u_1 and u_2. For any $t \in [0, T_0)$, it holds that*

$$\left\| u_1(t,\cdot) - u_2(t,\cdot) \right\|_{L^1(R)} \le ce^{ct} \|u_{10} - u_{20}\|_{L^1(R)}, \tag{14}$$

where c depends on $\|u_{10}\|_{H^1(R)}$ and $\|u_{20}\|_{H^1(R)}$.

Proof For an arbitrary $T > 0$, set $\eta_T = [0, T] \times R$. Let $f(t,x) \in C_0^\infty(\eta_T)$. We assume that $f(t,x) = 0$ outside the cylinder

$$\biguplus = \{(t,x)\} = [\delta, T - 2\delta] \times H_{r-2\delta}, \quad 0 < 2\delta \leq \min(T, r). \tag{15}$$

We define

$$g = f\left(\frac{t+\tau}{2}, \frac{x+y}{2}\right)\gamma_\varepsilon\left(\frac{t-\tau}{2}\right)\gamma_\varepsilon\left(\frac{x-y}{2}\right) = f(\cdots)\lambda_\varepsilon(*), \tag{16}$$

where $(\cdots) = (\frac{t+\tau}{2}, \frac{x+y}{2})$ and $(*) = (\frac{t-\tau}{2}, \frac{x-y}{2})$. The function $\gamma_\varepsilon(\sigma)$ is defined in (4). Note that

$$g_t + g_\tau = f_t(\cdots)\lambda_\varepsilon(*), \qquad g_x + g_y = f_x(\cdots)\lambda_\varepsilon(*). \tag{17}$$

Taking $u = u_1(t,x)$ and $k = u_2(\tau,y)$ and assuming $f(t,x) = 0$ outside the cylinder $\biguplus$, from Lemma 2.3, we have

$$\begin{aligned} &\iiiint_{\eta_T\times\eta_T} \{|u_1(t,x) - u_2(\tau,y)|g_t \\ &\quad + \operatorname{sign}(u_1(t,x) - u_2(\tau,y))P_{u_1}(t,x)g\}\,dx\,dt\,dy\,d\tau = 0. \end{aligned} \tag{18}$$

Similarly, it holds

$$\begin{aligned} &\iiiint_{\eta_T\times\eta_T} \{|u_2(\tau,y) - u_1(t,x)|g_\tau \\ &\quad + \operatorname{sign}(u_2(\tau,y) - u_1(t,x))P_{u_2}(\tau,y)g\}\,dx\,dt\,dy\,d\tau = 0, \end{aligned} \tag{19}$$

from which we obtain

$$\begin{aligned} 0 &\leq \iiiint_{\eta_T\times\eta_T} |u_1(t,x) - u_2(\tau,y)|(g_t + g_\tau) \\ &\quad + \left|\iiiint_{\eta_T\times\eta_T} \operatorname{sign}(u_1(t,x) - u_2(\tau,y))(P_{u_1}(t,x) - P_{u_2}(\tau,y))g\,dx\,dt\,dy\,d\tau\right| \\ &= \iiiint_{\eta_T\times\eta_T} I_1\,dx\,dt\,dy\,d\tau + \left|\iiiint_{\eta_T\times\eta_T} I_2\,dx\,dt\,dy\,d\tau\right|. \end{aligned} \tag{20}$$

We will show that

$$\begin{aligned} 0 &\leq \iint_{\eta_T} |u_1(t,x) - u_2(t,x)|f_t \\ &\quad + \left|\iint_{\eta_T} \operatorname{sign}(u_1(t,x) - u_2(t,x))[P_{u_1}(t,x) - P_{u_2}(t,x)]f\,dx\,dt\right|. \end{aligned} \tag{21}$$

We note that the first term in the integrand of (20) can be represented in the form

$$Y_\varepsilon = Y(t,x,\tau,y,u_1(t,x),u_2(\tau,y))\lambda_\varepsilon(*). \tag{22}$$

By the choice of g, we have $Y_\varepsilon = 0$ outside the region

$$\{(t,x;\tau,y)\} = \left\{\delta \le \frac{t+\tau}{2} \le T-2\delta, \frac{|t-\tau|}{2} \le \varepsilon, \frac{|x+y|}{2} \le r-2\delta, \frac{|x-y|}{2} \le \varepsilon\right\} \tag{23}$$

and

$$\begin{aligned}
&\iiiint_{\eta_T\times\eta_T} Y_\varepsilon \,dx\,dt\,dy\,d\tau \\
&\quad= \iiiint_{\eta_T\times\eta_T} \bigl[Y\bigl(t,x,\tau,y,u_1(t,x),u_2(\tau,y)\bigr) \\
&\qquad - Y\bigl(t,x,t,x,u_1(t,x),u_2(t,x)\bigr)\bigr]\lambda_\varepsilon(*)\,dx\,dt\,dy\,d\tau \\
&\qquad + \iiiint_{\eta_T\times\eta_T} Y\bigl(t,x,t,x,u_1(t,x),u_2(t,x)\bigr)\lambda_\varepsilon(*)\,dx\,dt\,dy\,d\tau = J_1(\varepsilon)+J_2.
\end{aligned} \tag{24}$$

Considering the estimate $|\lambda(*)| \le \frac{c}{\varepsilon^2}$ and the expression of function Y_ε, we have

$$\begin{aligned}
\bigl|J_1(\varepsilon)\bigr| \le c\biggl[\frac{1}{\varepsilon^2}\iiiint_{|\frac{t-\tau}{2}|\le\varepsilon,\delta\le\frac{t+\tau}{2}\le T-\delta,|\frac{x-y}{2}|\le\varepsilon,|\frac{x+y}{2}|\le r-\delta} &\bigl|u_2(t,x) \\
- u_2(\tau,y)\bigr|\,dx\,dt\,dy\,d\tau\biggr]&,
\end{aligned} \tag{25}$$

where the constant c does not depend on ε. Using Lemma 2.1, we obtain $J_1(\varepsilon) \to 0$ as $\varepsilon \to 0$. The integral J_2 does not depend on ε. In fact, substituting $t = \alpha$, $\frac{t-\tau}{2} = \beta$, $x = \eta$, $\frac{x-y}{2} = \xi$ and noting that

$$\int_{-\varepsilon}^{\varepsilon}\int_{-\infty}^{\infty} \lambda_\varepsilon(\beta,\xi)\,d\xi\,d\beta = 1, \tag{26}$$

we have

$$\begin{aligned}
J_2 &= 2^2 \iint_{\eta_T} Y_\varepsilon\bigl(\alpha,\eta,\alpha,\eta,u_1(\alpha,\eta),u_2(\alpha,\eta)\bigr)\left\{\int_{-\varepsilon}^{\varepsilon}\int_{-\infty}^{\infty}\lambda_\varepsilon(\beta,\xi)\,d\xi\,d\beta\right\}d\eta\,d\alpha \\
&= 4\iint_{\eta_T} Y\bigl(t,x,t,x,u_1(t,x),u_2(t,x)\bigr)\,dx\,dt.
\end{aligned} \tag{27}$$

Hence

$$\lim_{\varepsilon\to 0}\iiiint_{\eta_T\times\eta_T} Y_\varepsilon\,dx\,dt\,dy\,d\tau = 4\iint_{\eta_T} Y\bigl(t,x,t,x,u_1(t,x),u_2(t,x)\bigr)\,dx\,dt. \tag{28}$$

Since

$$I_2 = \operatorname{sign}\bigl(u_1(t,x)-u_2(\tau,y)\bigr)\bigl(P_{u_1}(t,x)-P_{u_2}(\tau,y)\bigr)f\lambda_\varepsilon(*) \tag{29}$$

and

$$\begin{aligned}
\iiiint_{\eta_T\times\eta_T} I_2\,dx\,dt\,dy\,d\tau &= \iiiint_{\eta_T\times\eta_T}\bigl[I_2(t,x,\tau,y)-I_2(t,x,t,x)\bigr]\,dx\,dt\,dy\,d\tau \\
&\quad + \iiiint_{\eta_T\times\eta_T} I_2(t,x,t,x)\,dx\,dt\,dy\,d\tau = K_1(\varepsilon)+K_2,
\end{aligned} \tag{30}$$

we obtain

$$\big|K_1(\varepsilon)\big| \le c\biggl(\varepsilon + \frac{1}{\varepsilon^2}\iiiint_{|\frac{t-\tau}{2}|\le\varepsilon,\delta\le\frac{t+\tau}{2}\le T-\delta,|\frac{x-y}{2}|\le\varepsilon,|\frac{x+y}{2}|\le r-\delta}\big|P_{u_2}(t,x) - P_{u_2}(\tau,y)\big|\,dx\,dt\,dy\,d\tau\biggr). \tag{31}$$

By Lemmas 2.1 and 2.2, we have $K_1(\varepsilon) \to 0$ as $\varepsilon \to 0$. Using (26), we have

$$\begin{aligned} K_2 &= 2^2\iint_{\eta T} I_2\big(\alpha,\eta,\alpha,\eta,u_1(\alpha,\eta),u_2(\alpha,\eta)\big)\biggl\{\int_{-\varepsilon}^{\varepsilon}\int_{-\infty}^{\infty}\lambda_\varepsilon(\beta,\xi)\,d\xi\,d\beta\biggr\}\,d\eta\,d\alpha \\ &= 4\iint_{\eta T} I_2\big(t,x,t,x,u_1(t,x),u_2(t,x)\big)\,dx\,dt \\ &= 4\iint_{\eta T}\operatorname{sign}\big(u_1(t,x)-u_2(t,x)\big)\big(P_{u_1}(t,x)-P_{u_2}(t,x)\big)f(t,x)\,dx\,dt. \end{aligned} \tag{32}$$

From (28) and (32), we prove that inequality (21) holds.

Let

$$w(t) = \int_{-\infty}^{\infty}\big|u_1(t,x)-u_2(t,x)\big|\,dx. \tag{33}$$

We define the following increasing function

$$\theta_\varepsilon(\rho) = \int_{-\infty}^{\rho}\gamma_\varepsilon(\sigma)\,d\sigma \quad \big(\theta'_\varepsilon(\rho) = \gamma_\varepsilon(\rho) \ge 0\big) \tag{34}$$

and choose two numbers τ_1 and $\tau_2 \in (0,T_0)$, $\tau_1 < \tau_2$. In (21), we choose

$$f = \big[\theta_\varepsilon(t-\tau_1) - \theta_\varepsilon(t-\tau_2)\big]\chi(t,x), \quad \varepsilon < \min(\tau_1, T_0-\tau_2), \tag{35}$$

where

$$\chi(t,x) = \chi_h(t,x) = 1 - \theta_h\big(|x| + Nt - M + h\big), \quad h > 0. \tag{36}$$

When h is sufficiently small, we note that function $\chi(t,x) = 0$ outside the cone $\mho$ and $f(t,x) = 0$ outside the set $\uplus$. For $(t,x) \in \mho$, we have the relation

$$\chi_t + N|\chi_x| = 0,$$

which derives

$$\chi_t \le 0. \tag{37}$$

Applying (21), (34)-(37) and the increasing properties of θ_ε, we have the inequality

$$\begin{aligned} 0 \le{}& \iint_{\eta T_0}\big\{\big[\gamma_\varepsilon(t-\tau_1) - \gamma_\varepsilon(t-\tau_2)\big]\chi_h\big|u_1(t,x)-u_2(t,x)\big|\big\}\,dx\,dt \\ &+ \biggl|\iint_{\eta T_0}\big[\theta_\varepsilon(t-\tau_1)-\theta_\varepsilon(t-\tau_2)\big]\big[P_{u_1}(t,x)-P_{u_2}(t,x)\big]B(t,x)\chi_h(t,x)\,dx\,dt\biggr|, \end{aligned} \tag{38}$$

where $B(t,x) = \operatorname{sign}[u_1(t,x)-u_2(t,x)]$.

From (38), we obtain

$$0 \le \iint_{\eta_{T_0}} \{[\gamma_\varepsilon(t-\tau_1) - \gamma_\varepsilon(t-\tau_2)]\chi_h |u_1(t,x) - u_2(t,x)|\}\, dx\, dt$$
$$+ \int_0^{T_0} (\theta_\varepsilon(t-\tau_1) - \theta_\varepsilon(t-\tau_2)) \left| \int_{-\infty}^{\infty} [P_{u_1}(t,x) - P_{u_2}(t,x)] B(t,x)\chi_h(t,x)\, dx \right| dt. \quad (39)$$

Using Lemma 2.4, we have

$$0 \le \iint_{\eta_{T_0}} \{[\gamma_\varepsilon(t-\tau_1) - \gamma_\varepsilon(t-\tau_2)]\chi_h |u_1(t,x) - u_2(t,x)|\}\, dx\, dt$$
$$+ c\int_0^{T_0} (\theta_\varepsilon(t-\tau_1) - \theta_\varepsilon(t-\tau_2)) \int_{-\infty}^{\infty} |u_1 - u_2|\, dx\, dt, \quad (40)$$

where c is defined in Lemma 2.4.

Letting $h \to 0$ in (40) and letting $M \to \infty$, we have

$$0 \le \int_0^{T_0} \left\{ [\gamma_\varepsilon(t-\tau_1) - \gamma_\varepsilon(t-\tau_2)] \int_{-\infty}^{\infty} |u_1(t,x) - u_2(t,x)|\, dx \right\} dt$$
$$+ c\int_0^{T_0} (\theta_\varepsilon(t-\tau_1) - \theta_\varepsilon(t-\tau_2)) \left(\int_{-\infty}^{\infty} |u_1 - u_2|\, dx \right) dt. \quad (41)$$

By the properties of the function $\gamma_\varepsilon(\sigma)$ for $\varepsilon \le \min(\tau_1, T_0 - \tau_1)$, we have

$$\left| \int_0^{T_0} \gamma_\varepsilon(t-\tau_1) w(t)\, dt - w(\tau_1) \right| = \left| \int_0^{T_0} \gamma_\varepsilon(t-\tau_1)[w(t) - w(\tau_1)]\, dt \right|$$
$$\le c\frac{1}{\varepsilon} \int_{\tau_1-\varepsilon}^{\tau_1+\varepsilon} |w(t) - w(\tau_1)|\, dt \to 0 \quad \text{as } \varepsilon \to 0, \quad (42)$$

where c is independent of ε.

Set

$$L(\tau_1) = \int_0^{T_0} \theta_\varepsilon(t-\tau_1) w(t)\, dt = \int_0^{T_0} \int_{-\infty}^{t-\tau_1} \gamma_\varepsilon(\sigma)\, d\sigma\, w(t)\, dt. \quad (43)$$

Using the similar proof of (42), we get

$$L'(\tau_1) = -\int_0^{T_0} \gamma_\varepsilon(t-\tau_1) w(t)\, dt \to -w(\tau_1) \quad \text{as } \varepsilon \to 0, \quad (44)$$

from which we obtain

$$L(\tau_1) \to L(0) - \int_0^{\tau_1} w(\sigma)\, d\sigma \quad \text{as } \varepsilon \to 0. \quad (45)$$

Similarly, we have

$$L(\tau_2) \to L(0) - \int_0^{\tau_2} w(\sigma)\, d\sigma \quad \text{as } \varepsilon \to 0. \quad (46)$$

Then we get

$$L(\tau_1) - L(\tau_2) \to \int_{\tau_1}^{\tau_2} w(\sigma)\,d\sigma \quad \text{as } \varepsilon \to 0. \tag{47}$$

Letting $\varepsilon \to 0$, $\tau_1 \to 0$ and $\tau_2 \to t$, from (41), (42) and (47), for any $t \in [0, T_0]$, we have

$$\begin{aligned} \int_{-\infty}^{\infty} \left|u_1(t,x) - u_2(t,x)\right| dx \leq{}& \int_{-\infty}^{\infty} \left|u_1(0,x) - u_2(0,x)\right| dx \\ &+ c\int_0^t \int_{-\infty}^{\infty} \left|u_1(t,x) - u_2(t,x)\right| dx\,dt, \end{aligned} \tag{48}$$

from which we complete the proof of Theorem 3.1 by using the Gronwall inequality. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The article is a joint work of two authors who contributed equally to the final version of the paper. All authors read and approved the final manuscript.

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