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On the Densest Packing of Polycylinders in Any Dimension

Wöden Kusner[1]

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Abstract Using transversality and a dimension reduction argument, a result of Bezdek and Kuperberg is applied to polycylinders, showing that the optimal packing density of $\mathbb{D}^2 \times \mathbb{R}^n$ equals $\pi/\sqrt{12}$ for all natural numbers n.

Keywords Polycylinders · Packing · Density · Slicing

1 Introduction

Open and closed Euclidean unit n-balls will be denoted by $\mathbb{B}^n$ and $\mathbb{D}^n$ respectively. The closed unit interval is denoted by $\mathbb{I}$. A general polycylinder C is a set congruent to $\prod_{i=1}^{i=m} \lambda_i \mathbb{D}^{k_i}$ in $\mathbb{R}^{k_1+\cdots+k_m}$, where λ_i is in $[0, \infty]$. For this article, the term polycylinder refers to the special case of an infinite polycylinder over a two-dimensional disk of unit radius. A *polycylinder* is a set congruent to $\mathbb{D}^2 \times \mathbb{R}^n$ in $\mathbb{R}^{n+2}$. A *polycylinder packing of* $\mathbb{R}^{n+2}$ is a family $\mathscr{C} = \{C_i\}_{i \in I}$ of polycylinders $C_i \subset \mathbb{R}^{n+2}$ with mutually disjoint interiors. The *upper density* $\delta^+(\mathscr{C})$ of a packing $\mathscr{C}$ of $\mathbb{R}^n$ is defined to be

$$\delta^+(\mathscr{C}) = \limsup_{r \to \infty} \frac{\mathrm{Vol}(\mathscr{C} \cap r\mathbb{B}^n)}{\mathrm{Vol}(r\mathbb{B}^n)}.$$

The *upper packing density* $\delta^+(C)$ of an object C is the supremum of $\delta^+(\mathscr{C})$ over all packings $\mathscr{C}$ of $\mathbb{R}^n$ by C.

Editor in Charge: János Pach

Wöden Kusner
wkusner@gmail.com

[1] Institute of Analysis and Number Theory, Graz University of Technology, Steyrergasse 30/II, 8010 Graz, Austria

This article proves the following sharp bound for the packing density of infinite polycylinders:

Theorem 1 $\delta^+(\mathbb{D}^2 \times \mathbb{R}^n) = \pi/\sqrt{12}$ *for all natural numbers n.*

Theorem 1 generalizes a result of Bezdek and Kuperberg [1] and improves on results that may be computed using a method of Fejes Tóth and Kuperberg [3], cf. [2,5]; it gives some of the first sharp upper bounds for packing density in high dimensions.

2 Transversality

This section introduces the required transversality arguments from affine geometry. A *d-flat* is a d-dimensional affine subspace of $\mathbb{R}^n$. The *parallel dimension* $\dim_\|\{F, \dots, G\}$ of a collection of flats $\{F, \dots, G\}$ is the dimension of their maximal parallel sub-flats. The notion of parallel dimension can be interpreted in several ways, allowing a modest abuse of notation.

- For a collection of flats $\{F, \dots, G\}$, consider their tangent cones at infinity $\{F_\infty, \dots, G_\infty\}$. The parallel dimension of $\{F, \dots, G\}$ is the dimension of the intersection of these tangent cones. This may be viewed as the limit of a rescaling process $\mathbb{R}^n \to r\mathbb{R}^n$ as r tends to 0, leaving only the scale-invariant information.
- For a collection of flats $\{F, \dots, G\}$, consider each flat as a system of linear equations. The corresponding homogeneous equations determine a collection of linear subspaces $\{F_\infty, \dots, G_\infty\}$. The parallel dimension is the dimension of their intersection $F_\infty \cap \cdots \cap G_\infty$.

Two disjoint d-flats are *parallel* if their parallel dimension is d, that is, if every line in one is parallel to a line in the other.

Lemma 1 *A pair of disjoint n-flats in $\mathbb{R}^{n+k}$ with $n \geq k$, has parallel dimension strictly greater than $n - k$.*

Proof Let F and G be such a pair. By homogeneity of $\mathbb{R}^{n+k}$, let $F = F_\infty$. As F_∞ and G are disjoint, G contains a non-trivial vector $\mathbf{v}$ such that $G = G_\infty + \mathbf{v}$ and $\mathbf{v}$ is not in $F_\infty + G_\infty$. It follows that

$$\begin{aligned}\dim(\mathbb{R}^{n+k}) &\geq \dim\big(F_\infty + G_\infty + \operatorname{span}(\mathbf{v})\big) > \dim(F_\infty + G_\infty)\\ &= \dim(F_\infty) + \dim(G_\infty) - \dim(F_\infty \cap G_\infty).\end{aligned}$$

Count dimensions to find $n + k > n + n - \dim_\|(F_\infty, G_\infty)$. □

Corollary 1 *A pair of disjoint n-flats in $\mathbb{R}^{n+2}$ has parallel dimension at least $n - 1$.*

3 Dimension Reduction

3.1 Pairwise Foliations

The *core* a_i of a polycylinder C_i congruent to $\mathbb{D}^2 \times \mathbb{R}^n$ in $\mathbb{R}^{n+2}$ is the distinguished n-flat defining C_i as the set of points at most distance 1 from a_i. In a packing $\mathscr{C}$ of

$\mathbb{R}^{n+2}$ by polycylinders, Corollary 1 shows that, for every pair of polycylinders C_i and C_j, one can choose parallel $(n-1)$-dimensional subflats $b_i \subset a_i$ and $b_j \subset a_j$ and define a product foliation

$$\mathscr{F}^{b_i,b_j} : \mathbb{R}^{n+2} \to \mathbb{R}^{n-1} \times \mathbb{R}^3$$

with $\mathbb{R}^3$ leaves that are orthogonal to b_i and to b_j. Given a point x in a_i, there is a distinguished $\mathbb{R}^3$ leaf $F_x^{b_i,b_j}$ that contains the point x. The foliation $\mathscr{F}^{b_i,b_j}$ restricts to foliations of C_i and C_j with right-circular-cylinder leaves.

3.2 The Dirichlet Slice

In a packing $\mathscr{C}$ of $\mathbb{R}^{n+2}$ by polycylinders, the *Dirichlet cell* D_i associated with a polycylinder C_i is the set of points in $\mathbb{R}^{n+2}$ which lie no further from C_i than from any other polycylinder in $\mathscr{C}$. The Dirichlet cells of a packing partition $\mathbb{R}^{n+2}$, as $C_i \subset D_i$ for all polycylinders C_i. To bound the density $\delta^+(\mathscr{C})$, it is enough to fix an i in I and consider the density of C_i in D_i.

Consider the following slicing of the Dirichlet cell D_i. Given a fixed polycylinder C_i in a packing $\mathscr{C}$ of $\mathbb{R}^{n+2}$ by polycylinders and a point x on the core a_i, the plane p_x is the 2-flat orthogonal to a_i and containing the point x. The *Dirichlet slice* d_x is the intersection of D_i and p_x.

Note that p_x is a sub-flat of $F_x^{b_i,b_j}$ for all j in I.

3.3 Bezdek–Kuperberg Bound

For any point x on the core a_i of a polycylinder C_i, the results of Bezdek and Kuperberg [1] apply to the Dirichlet slice d_x.

Lemma 2 *A Dirichlet slice is convex and, if bounded, a parabola-sided polygon.*

Proof Construct the Dirichlet slice d_x as an intersection. Define d^j to be the set of points in p_x which lie no further from C_i than from C_j. Then the Dirichlet slice d_x is realized as

$$d_x = \{\bigcap_{j\in I} d^j\}.$$

Each arc of the boundary of d_x in p_x is given by an arc of the boundary of some d^j in p_x. The boundary of d^j in p_x is the set of points in p_x equidistant from C_i and C_j. Since the foliation $\mathscr{F}^{b_i,b_j}$ is a product foliation, the arc of the boundary of d^j in p_x is also the set of points in p_x equidistant from the leaf $C_i \cap F_x^{b_i,b_j}$ of $\mathscr{F}^{b_i,b_j}|_{C_i}$ and the leaf $C_j \cap F_x^{b_i,b_j}$ of $\mathscr{F}^{b_i,b_j}|_{C_j}$. This reduces the analysis to the case of a pair of cylinders in $\mathbb{R}^3$. From [1], it follows that d^j is convex and the boundary of d_j in p_x is a parabola; the intersection of such sets d^j in p_x is convex, and a parabola-sided polygon if bounded. □

Let $S_x(r)$ be the circle of radius r in p_x centered at x.

Lemma 3 *The vertices of d_x are not closer to $S_x(1)$ than the vertices of a regular hexagon circumscribed about $S_x(1)$.*

Proof A vertex of d_x occurs where three or more polycylinders are equidistant, so the vertex is the center of a $(n+2)$-ball B tangent to three polycylinders. Thus B is tangent to three disjoint unit $(n+2)$-balls B_1, B_2, B_3. By projecting into the affine hull of the centers of B_1, B_2, B_3, it is immediate that the radius of B is no less than $2/\sqrt{3}-1$. □

Lemma 4 *Let y and z be points on the circle $S_x(2/\sqrt{3})$. If each of y and z is equidistant from C_i and C_j, then the angle yxz is smaller than or equal to $2\arccos(\sqrt{3}-1) = 85.8828\ldots°$.*

Proof Following [1,4], the existence of a supporting hyperplane of C_i that separates $\mathrm{int}(C_i)$ from $\mathrm{int}(C_j)$ suffices. □

In [1], it is shown that planar objects satisfying Lemmas 2, 3 and 4 have area no less than $\sqrt{12}$. As the bound holds for all Dirichlet slices, it follows that $\delta^+(\mathbb{D}^2 \times \mathbb{R}^n) \leq \pi/\sqrt{12}$ in $\mathbb{R}^{n+2}$. The product of the dense disk packing in the plane with $\mathbb{R}^n$ gives a polycylinder packing in $\mathbb{R}^{n+2}$ that achieves this density. Combining this with the result of Thue [6] for $n = 0$ and the result of Bezdek and Kuperberg [1] for $n = 1$, Theorem 1 follows.

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