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The constancy of ζ in single-clock Inflation at all loops

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ABSTRACT: Studying loop corrections to inflationary perturbations, with particular emphasis on infrared factors, is important to understand the consistency of the inflationary theory, its predictivity and to establish the existence of the slow-roll eternal inflation phenomena and its recently found volume bound. In this paper we show that ζ -correlators are time-independent at large distances at all-loop level in single clock inflation. We write the n -th order correlators of $\dot{\zeta}$ as the time-integral of Green's functions times the correlators of local sources that are function of the lower order fluctuations. The Green's functions are such that only non-vanishing correlators of the sources at late times can lead to non-vanishing correlators for $\dot{\zeta}$ at long distances. When the sources are connected by high wavenumber modes, the correlator is peaked at short distances, and these diagrams cannot lead to a time-dependence by simple diff. invariance arguments. When the sources are connected by long wavenumber modes one can use similar arguments once the constancy of ζ at lower orders was established. Therefore the conservation of ζ at a given order follows from the conservation of ζ at the lower orders. Since at tree-level ζ is constant, this implies constancy at all-loops by induction.

KEYWORDS: Cosmology of Theories beyond the SM, Models of Quantum Gravity, Space-Time Symmetries, Renormalization Regularization and Renormalons

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1 Introduction and summary

This is the sequel to a series of recent papers [1–3] that we wrote studying loop corrections to inflationary observables, with the purpose of understanding possible large infrared effects that might arise when considering the theory of inflation at quantum level. In the latest of these [3], we proved at one-loop level that in single clock inflation, that is inflationary theories where there is only one relevant degree of freedom, the correlation of the curvature perturbation ζ is time-independent on distances much longer than the horizon. Studying loop effect is in general important for the consistency of the theory at quantum level. However, the time-independence of ζ is very important also for the predictivity of inflation and for the existence of slow roll eternal inflation and its recently found universal volume bound [4–6] (see the introduction of [3] where we recently discussed in detail about these points).

The purpose of this paper is to extend the former calculation to all-loop orders. We show the constancy of ζ iteratively in the number of perturbations. At linear level it is well known that ζ becomes time independent after horizon crossing. Then, assuming ζ is constant at a given perturbative order, we show that ζ is constant at the next perturbative order which provides a proof by induction. We achieve this in the following way. The non-linear nature of the equations forces a coupling between different modes. The nature of the Green’s function for the ζ fluctuations forces any interaction that happen in the past not to induce any time-dependence. Time dependence can only come from interactions that happened during the last Hubble time. Further, as it has been already shown in several ways (see for example [7–10]), all interactions involving modes longer than the horizon are vanishingly small. This means that in the latest possible interaction at least two short wavelength fluctuations must be present, plus an additional arbitrary number of long wavelength fluctuations. This reduces the problem to computing correlation functions of local sources made of short-wavelength and long-wavelength fluctuations. The long wavelength fluctuations are lower order in the fluctuations, and so they can be assumed to be constant in time. However a constant-in-time long wavelength fluctuations is locally unobservable and therefore it cannot affect the short wavelength fluctuations. This means that the correlation of sources is dominated either by correlation of short modes at far away distances, that are too small to induce a time-dependence, or by correlations of derivatives

of lower-order long-wavelength fluctuations, that decay in time. This implies that $\dot{\zeta}$ goes to zero at late times, proving the constancy of ζ at all loop order in single clock inflation. In other words, $\dot{\zeta}$ goes to zero in single clock inflation as an operator.¹

2 Local gauge for a long mode

It will turn out to be useful for the proof of the constancy of ζ to introduce the following set of coordinates, which was introduced for the first time in [3, 11] at linear level in the fluctuations. Here we generalize them to the non-linear level. If we consider a set of ζ fluctuations ζ_L that involve, at the time of interest, long fluctuations characterized by the highest wavenumber k_{low} already very outside of the horizon, we aim here to provide a change of coordinates valid in a local patch of volume V much smaller than k_{low}^{-3} and much larger than H^{-3} , that takes us from the metric written in standard ζ -gauge to a form that is locally of the form of a homogeneous anisotropic universe, up to corrections that go as $(k_{\text{low}}^3 V)^{1/3}$. We wish to find a set of coordinates where this fact is manifest. Let us start in standard ζ -gauge, defined by setting the scalar field to be uniform ($\delta\phi = 0$ or the Goldstone boson $\pi = 0$), and the metric to take the form

$$ds^2 = -N^2 dt^2 + \sum_{ij} \hat{h}_{ij} (dx^i + N^i dt) (dx^j + N^j dt) , \tag{2.1}$$

in the ADM parametrization. The spatial metric \hat{h}_{ij} takes the following form

$$\hat{h}_{ij} = \delta_{ij} e^{2\zeta} . \tag{2.2}$$

The variables N and N^i solve some constraint equations. For standard slow roll inflation (but the argument can be straightforwardly generalized to all models described by the EFT

¹As described in [3], the non-linear dependence of ζ on lower order scalar or tensor fluctuations is parametrically different. This implies that the analysis can be done neglecting tensor modes, as we will do, though we expect that very similar arguments apply also when tensor modes are included. For example, working in the context of the Effective Field Theory of Inflation [12, 13], one can add to the slow-roll Lagrangian an operator of the form $(\delta g^{00})^2$. This has the effect of changing the speed of sound of the scalar fluctuations, without changing the one of tensor modes. The speed of sound of tensors modes can be changed by adding terms proportional to the extrinsic curvature K , such as $\delta K^i_j \delta K^j_i$. This means that the power spectrum of ζ depends on two independent parameters, the speeds of sound of the scalar and of the tensors, and therefore, if there is no induced time-dependence at loop level, it must happen so independently for the scalar and the tensor contribution. In fact, by assuming the contrary, that is that the time independence of ζ is due to a cancellation of the scalar and the tensor contribution, then by changing the speed of sound of the scalars without changing the one of the tensor, the cancellation would not happen anymore. This means that the cancellation must be independent.

We can give some details of the generalization for tensors of the arguments that we give later in the text to prove that scalar loops do not induce a time dependence of ζ . Even in the presence of tensor modes, one could go to a local anisotropic frame. The generalization with respect to eq. (2.12) will be that now more coefficients $\tilde{\lambda}_{L,i}$ will not be zero in the local frame. However, as for the scalar case, these coefficients will shift under a residual change of coordinates proportional to exponentials of the matrixes $S_{1,\dots,5}$ defined in (2.14) that keep ourselves in the local frame. This ensures that an analogous to eq. (3.6) holds also for tensor, from which the proof of the constancy of ζ for tensor modes is straightforward.

of Inflation [12, 13]), they are the following

$$M_{\text{Pl}}^2 \nabla_i \left[N^{-1} \left(E^i_j + \delta^i_j E^l_l \right) \right] = 0, \quad (2.3)$$

$$\frac{M_{\text{Pl}}^2}{2} \left(R^{(3)} - \frac{1}{N^2} \left(E^i_j E^j_i - E^i_i{}^2 \right) \right) + \frac{M_{\text{Pl}}^2}{N^2} \dot{H} - M_{\text{Pl}}^2 \left(3H^2 + \dot{H} \right) = 0,$$

where $R^{(3)}$ and ∇ are taken with respect to the spatial metric \hat{h}_{ij} . What makes the long wavelength limit in ζ -gauge non-trivial is the fact that at linear order in Fourier space the solution for N^i in the long wavelength limit reads

$$N^i \sim \frac{k_{\text{low}}^i}{k_{\text{low}}^2} \dot{\zeta}_L(k), \quad (2.4)$$

which does not have a nice $k_{\text{low}} \rightarrow 0$ limit for a generic time-dependence of ζ_L . The same holds at higher order. If we go back to real space, this shows that at linear level $N^i \propto x^i$: N^i becomes larger and larger at longer distances. The reason why this is not a problem is that N^i is not really a physical observable: as we will show, what matters is rather $\partial_j N^i$.

In order to show this explicitly, we will solve the constraint equations (2.3) around the origin, at leading order in $(k_{\text{low}}^3 V)^{1/3}$. We can Taylor expand $\zeta_L(\vec{x}, t) = \zeta_L(0, t) + \partial_i \zeta|_0 x^i$. Because of rotational invariance, we can choose the gradient of ζ_L to point in the z direction. It can be checked that, non-linearly in ζ_L , but at leading order in $(k_{\text{low}}^3 V)^{1/3}$, we have

$$N = 1 + \frac{\dot{\zeta}_L}{H} + \mathcal{O} \left((k_{\text{low}} V)^{1/3} \right), \quad (2.5)$$

$$N_i = N_{i,0}(t) - \delta_{i3} \frac{\dot{H}}{2H^2} \frac{2H + \dot{\zeta}_L}{H + \dot{\zeta}_L} \dot{\zeta}_L z + \mathcal{O} \left((k_{\text{low}}^3 V)^{1/3} \right),$$

where all the ζ_L are evaluated at the origin and $N_{i,0}$ are three space-independent functions. For later convenience, let us define

$$F(\dot{\zeta}_L) = \frac{\dot{H}}{2H^2} \frac{2H + \dot{\zeta}_L}{H + \dot{\zeta}_L} \dot{\zeta}_L. \quad (2.6)$$

As mentioned, the problematic term is N_i : it does not appear to go to zero as $k_{\text{low}} \rightarrow 0$, even after it has been acted upon by a spatial derivative. Notice however that this term diverges linearly in z , suggesting that the physically sensible quantity is rather $\partial_j N_i$. Indeed, the structure of the constraint equations show that one can restrict oneself to solutions that diverge at most linearly in the spatial coordinates. This means that in order to find a reference frame where the universe looks locally manifestly like an homogeneous anisotropic universe, it is enough to find a coordinate transformation that set to zero locally N_i and $\partial_j N_i$, with the remaining fluctuations going to zero as $k_{\text{low}} \rightarrow 0$, after have been acted upon by a spatial derivative. We will work without assuming any specific time dependence for ζ_L , so that the following change of coordinates holds also when one cannot use the solution to the linear equations for ζ_L , as it happens for example when computing loop corrections to inflationary correlators [3].

We are going to perform the following change of coordinates:

$$\tilde{x}^i = M^i_j(t) x^j + C^i(t), \quad \text{with} \quad \det[M] \neq 0, \quad (2.7)$$

which keeps the inflaton fluctuations zero. Under this change of coordinates, and expanding around the origin, the metric takes the form

$$\begin{aligned} ds^2 = & -N^2 dt^2 + e^{2\zeta(\tilde{x},t)} \delta_{ij} \\ & \times \left[M^i_l d\tilde{x}^l + \left(\dot{M}^i_l \tilde{x}^l + \dot{C}^i + N^i|_0 + \partial_l N^i|_0 \left(M^l_m \tilde{x}^m + C^l \right) \right) dt \right] \\ & \times \left[M^j_n d\tilde{x}^n + \left(\dot{M}^j_n \tilde{x}^n + \dot{C}^j + N^j|_0 + \partial_n N^j|_0 \left(M^n_p \tilde{x}^p + C^n \right) \right) dt \right]. \end{aligned} \quad (2.8)$$

To bring the metric in our desired form, we therefore need to solve the following two equations:

$$\begin{aligned} \dot{C}^i + \partial_l N^i|_0 C^l + N^i|_0 &= 0, \\ \left(\dot{M}^i_l + \partial_n N^i|_0 M^n_l \right) \tilde{x}^l &= 0. \end{aligned} \quad (2.9)$$

The generic solution reads

$$\begin{aligned} C^i(t) &= -[U(t)^{-1}]^i_j \left[\int_{t_2}^t dt' U(t')^j_l N^l|_0(t') \right] + [U(t)^{-1}]^i_j C_1^i, \\ M(t)^i_j &= [U(t)^{-1}]^i_l M_{1j}^l, \end{aligned} \quad (2.10)$$

where

$$U = T e^{\int_{t_1}^t dt' \partial N|_0(t')}, \quad (2.11)$$

with T staying for time-ordering, $t_{1,2}$, C_1 , M_{1j}^i representing integration constants, and $\partial N|_0$ is the matrix such that $[\partial N|_0]^i_j = \partial_j N^i|_0$.

After substitution, the metric takes the form

$$ds^2 = -N^2 dt^2 + e^{2\zeta(\tilde{x},t)} \delta_{ij} M^i_l M^j_m d\tilde{x}^l d\tilde{x}^m. \quad (2.12)$$

Since M contains only ∂N terms, the metric goes to zero as $k_{\text{low}} \rightarrow 0$, if acted upon by spatial derivatives, as we wished to show. The local spatial metric becomes

$$g_{L,ij}^{\text{space}} = a^2 e^{2\tilde{\zeta}_L(t)} \mathbb{1} + \sum_{n=1,\dots,5} \tilde{\lambda}_{L,n}(t) S_{n,ij} \quad (2.13)$$

where $\tilde{\zeta}_L$ is the local ζ fluctuation, that multiplies the identity matrix $\mathbb{1}$, and $\tilde{\lambda}_{L,n}$'s are the local anisotropy coefficients, represented as coefficients of five independent symmetric traceless matrixes S_n .²

²These are given by

$$\begin{aligned} S_{1,ij} &= \delta_{(i2}\delta_{j)3}, & S_{2,ij} &= \delta_{(i1}\delta_{j)3}, & S_{3,ij} &= \delta_{(i1}\delta_{j)2}, \\ S_{4,ij} &= +\frac{1}{3}\delta_{i1}\delta_{j1} - \frac{2}{3}\delta_{i2}\delta_{j2} + \frac{1}{3}\delta_{i3}\delta_{j3}, & S_{5,ij} &= +\frac{1}{3}\delta_{i1}\delta_{j1} + \frac{1}{3}\delta_{i2}\delta_{j2} - \frac{2}{3}\delta_{i3}\delta_{j3}, \end{aligned} \quad (2.14)$$

where (i, j) means symmetrization with respect to the i, j indexes.

Our choice to align the gradient of ζ_L with the z direction simplifies formulae remarkably. We have:

$$[\partial N|_0]_{ij} = \delta_{i3}\delta_{j3}F(\dot{\zeta}_L), \quad U = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{\int_{t_1}^t dt' F(\dot{\zeta}_L(t'))} \end{pmatrix}, \quad (2.15)$$

In the local frame the fluctuations, represented by a tilde, are given in terms of ζ by the following relation

$$\begin{aligned} \tilde{\zeta}_L(\vec{x}, t) &= \zeta_L(\vec{x}(\vec{x}), t) + \frac{1}{3} \int_{t_1}^t dt' \frac{\dot{H}}{2H^2} \frac{2H + \dot{\zeta}_L(\vec{0}, t')}{H + \dot{\zeta}_L(\vec{0}, t')} \dot{\zeta}_L(\vec{0}, t'), \\ \tilde{\lambda}_{L,5} &= - \int_{t_1}^t dt' \frac{\dot{H}}{2H^2} \frac{2H + \dot{\zeta}_L(\vec{0}, t')}{H + \dot{\zeta}_L(\vec{0}, t')} \dot{\zeta}_L(\vec{0}, t'). \end{aligned} \quad (2.16)$$

All the remaining $\lambda_{L,n}$ vanish. For this reason, in the rest of the paper, we will call $\tilde{\lambda}_{L,5}$ as simply $\tilde{\lambda}_L$.³ Notice that under this change of coordinates only the long wavelength ζ_L is transformed. Short wavelength fluctuations $\tilde{\zeta}_S$ transform as a scalar. The long wavelength metric takes locally the following form

$$g_{L,\mu\nu} = -N^2 dt^2 + g_{L,ij}^{\text{space}} dx^i dx^j, \quad (2.17)$$

with the spatial metric defined in (2.13) and given here in matrix form

$$g_L^{\text{space}}(t) = a(t)^2 \begin{pmatrix} e^{2(\tilde{\zeta}_L(t) + \frac{1}{3}\tilde{\lambda}_L(t))} & 0 & 0 \\ 0 & e^{2(\tilde{\zeta}_L(t) + \frac{1}{3}\tilde{\lambda}_L(t))} & 0 \\ 0 & 0 & e^{2(\tilde{\zeta}_L(t) - \frac{2}{3}\tilde{\lambda}_L(t))} \end{pmatrix}, \quad (2.18)$$

and

$$N(t) = 1 + \frac{\dot{\zeta}_L(\dot{\zeta}_L(t))}{H} \quad (2.19)$$

Notice that $\det[g_L] = N^2 a^6 e^{6\tilde{\zeta}_L}$. The long wavelength mode has been locally reabsorbed in an homogenous anisotropic background up to corrections that are manifestly subleading in $(k_{\text{low}}^3 V)^{1/3} \rightarrow 0$ limit. Short wavelength fluctuations live in this very smooth universe.

3 Proof of the conservation of ζ to all loops

Let us consider a long wavelength fluctuations $\zeta_L(\vec{x}, t)$, defined as

$$\zeta_L(\vec{x}, t) = \int d^3k \Theta(k_{\text{low}} - k) e^{i\vec{k}\cdot\vec{x}} \zeta_{\vec{k}}(t), \quad (3.1)$$

³Later in the proof of the constancy of ζ , we will need to consider configurations where there is a generic lower-order long-wavelength configuration sourcing a long wavelength fluctuations. A priori, these might have different gradient directions. However, since we work at linear order in the gradients, rotational invariance implies they must have the same direction. Alternatively, one could imagine to perform a rotation to bring the gradient in the z -direction, obtain the local frame in this simple coordinates, and then perform the inverse rotation. In this way, one would obtain all the $\lambda_{L,n}$. In any event it is straightforward to use the generic $\lambda_{L,n}$ from the get-go.

where k_{low} is some comoving momentum. We are interested in proving the constancy of correlation functions of $\zeta_L(\vec{x}, t)$ once all the comoving modes that are part of its spectrum are outside of the horizon. As we solve iteratively in the interactions the equation of motions for ζ_L , we need to insert interactions at some specific times and spatial locations, and sum over these. In [1, 3], we have shown how solving iteratively in the interactions the equations of motion and then taking correlation functions of the non-linear solutions is equivalent to computing loop corrections. Given a certain solution with the insertion of a given number of interactions, there will be one interaction that has been inserted last. Let us consider it. It is sufficient to work with the tilde variables of the local frame defined in section 2. We can write the equation of motion for $\tilde{\zeta}_L$ and the anisotropy coefficient $\tilde{\lambda}_L$ in the following way

$$\begin{aligned}
 (\partial_t + 3H) \dot{\zeta}_L^{(n)}(\vec{x}, t) &= (\partial_t + 3H) \left(\frac{3H^2}{3H^2 + \dot{H}} \dot{\zeta}_L^{(n)} \right) = S_L^{(\tilde{\zeta}_L, n)}(\vec{x}, t), \\
 (\partial_t + 3H) \dot{\lambda}_L^{(n)}(\vec{x}, t) &= S_L^{(\tilde{\lambda}_L, n)}(\vec{x}, t),
 \end{aligned}
 \tag{3.2}$$

where $S_L^{(\tilde{\zeta}_L, n)}$ and $S_L^{(\tilde{\lambda}_L, n)}$ are some source that depend in a local way on $\tilde{\zeta}_L$, $\tilde{\lambda}_L$, $\delta\tilde{N}_L$, and their derivatives, where $\tilde{\zeta}_L$, $\delta\tilde{N}_L$, $\tilde{\lambda}_L$ are respectively ζ , the time-shift and the anisotropy coefficient of the local space-time described in section 2. S_L additionally depends in a more unconstrained way on the additional short wavelength fluctuations. Let us explain how we obtain the above equation:

1. First of all, notice that the differential operator acting on ζ_L on the left is very similar to the differential operator that acts on linear fluctuations in the standard ζ -gauge, apart for a ∇^2/a^2 term which is missing. The gradient is indeed negligible because we are interested in proving the constancy of ζ_L when k_{low} is already very outside of the horizon. Since, as we will see next, both the ζ -Green's function and $\tilde{\zeta}$ -Green's function of the long mode makes any solution that is sourced by a source in the past approach a constant, it is only the interactions that are happening at the latest possible time that could lead to a time-dependence. This means that in (3.2) we can restrict ourselves to times when k_{low} is outside of the horizon, and so the term $\nabla^2\zeta_L$ is negligible.
2. The right hand side of (3.2) contains the source term S_L . As we said, this term is a local function of $\tilde{\zeta}_L$, $\tilde{\lambda}_L$, $\delta\tilde{N}_L$ and their derivatives, and of the short wavelength fluctuations in a more unconstrained way. In general, each of the fluctuations in S_L will be either a free field, or the results of some interactions. Each interaction is associated to a vertex with some fluctuations, and so it is useful to express the perturbative series in terms of number of fluctuations involved, rather than in the number of vertices included. This is equivalent to the counting of loops.⁴ Let us call $\tilde{\zeta}^{(n)}$ as the solution to the perturbative equations of order n in the fluctuations. If

⁴The same phenomenon happens in standard quantum field theory. For example, in the correction to the propagator at first order in \hbar , diagrams with either one quartic or two cubic vertices contribute. In standard quantum field theory, indeed, fluctuations are of order \hbar .

we wish to source $\tilde{\zeta}_L^{(n)}$, the sum of the orders of all the $\tilde{\zeta}_L$, $\tilde{\lambda}_L$, $\delta\tilde{N}_L$ and the short fluctuations in S_L must be equal to n . In particular, notice that the fluctuations $\tilde{\zeta}_L$, $\tilde{\lambda}_L$ and $\delta\tilde{N}_L$ appearing in S_L are of lower order in the fluctuations. The structure of the equation suites itself therefore to an iterative solution. We will prove that the assumption that the lower order $\tilde{\zeta}_L$ is constant implies that the next higher order $\tilde{\zeta}_L$ is constant as well. Since at tree level $\tilde{\zeta}_L$ is constant, the argument provides a proof to all orders.⁵ Notice immediately that assuming the lower order $\tilde{\zeta}_L$ to be constant simplifies the structure of the equation, as if $\tilde{\zeta}_L$ is constant, $\delta\tilde{N}_L$ must vanish and $\tilde{\lambda}_L$ must go to a constant value.

3. How can we guarantee that S_L is spatially local? Once a mode is much longer than the Horizon, the Green's function associated to the differential operator in (3.2) is local in space, as we will show next in (3.4). This means that a long wavelength mode at one location is sourced only by a source at the same location. This shows that the interactions of the long mode with the other modes can be effectively described in the local anisotropic frame introduced in section 2, and we can use directly the variables with tildes. Notice that the constancy of $\tilde{\zeta}_L, \tilde{\lambda}_{L,n}$ implies the constancy of ζ_L and viceversa. In this frame it is manifest that the dependence of S_L on the long-wavelength fields is manifestly local. In more detail it follows from the fact that in the constraint equations for the short-wavelength $\delta\tilde{N}_S$ and \tilde{N}_S^i there cannot be spatial derivatives acting on $\tilde{\zeta}_L$ and $\tilde{\lambda}_L$ as they can be taken as homogeneous. Corrections to this are explicitly suppressed by powers of $k_{\text{low}}/(aH)$. Notice that this is not the case for standard ζ -gauge, where operators such as ∂_i/∂^2 acting on long-wavelength fields, appear in the source.

Since we are interested in $\dot{\tilde{\zeta}}_L$, the solution to (3.2) can be written directly:

$$\dot{\tilde{\zeta}}_L^{(n)}(x, t) = \int_V d^4x' \mathcal{D}_{x'} \tilde{G}_L(x, x'); S_L^{(\tilde{\zeta}_L, n)}(x'), \quad (3.3)$$

where $\tilde{G}_L(x, x')$ is the $\tilde{\zeta}_L$ -Green's function for the long mode

$$\tilde{G}_L(x, x') = \delta^{(3)}(x - x') G_L(t, t'), \quad G_L(t, t') = \frac{3H^2 + \dot{H}}{H^2} \frac{e^{-3H(t-t')}}{3H}, \quad (3.4)$$

in the slow roll approximation, and $\mathcal{D}_{x'}$ is the x' -differential operator that acts on ζ in the vertex that gives rise to S_L . Since we can neglect gradients of the long mode, the Green's function is local in real space. We have

$$\dot{\tilde{\zeta}}_L^{(n)}(t) = \int dt' \mathcal{D}_{t'} G_L(t, t') S_L^{(\tilde{\zeta}_L, n)}(t'). \quad (3.5)$$

As promised, for $t \gg t'$, G_L tends to a constant, which means that in order for a time-dependence to be induced, we need to concentrate on $t' \sim t$. All of the above applies

⁵When ζ is not constant at tree-level or alternatively when one cannot go to standard ζ -gauge as defined in (2.2), as for example in [14], our proof does not apply.

identically to the sourcing of $\tilde{\lambda}_L$ after replacing $\tilde{\zeta}_L$ with $\tilde{\lambda}_L$ apart from an irrelevant multiplicative constant in the definition of the Green's function, that is time-independent at leading order in slow-roll. Slow-roll corrections are irrelevant for the purpose of proving the constancy of ζ_L .

Since generic n -point functions are formed by decomposing them as product of connected correlation functions involving a smaller number of fluctuations, we can concentrate on connected correlation functions, that are therefore simply related to connected correlation functions of S_L . Since the case of correlation functions involving also $\tilde{\lambda}_L$ follows virtually unaltered from the case of the correlation function of $\tilde{\zeta}_L$, we quote explicit results only for $\tilde{\zeta}_L$. We concentrate first on the two point function. We summarize the following points:

1. Some of the operators contributing to at least one of the $\tilde{\zeta}_L$ must involve short modes, as otherwise all interactions vanish.
2. We are interested in contributions at late times because the form of the Green's function makes $\tilde{\zeta}_L$ go to a constant if the final interaction is in the distant past.
3. Any vertex connected to the x' by a short mode must be within a space-time distance of order Hubble at most. This is because short modes need to be oscillating and they become uncorrelated on larger distances or longer times.⁶

Let us consider the correlator of $S_L^{(\tilde{\zeta}_L, n)}$, and first consider taking expectation value only over the short modes. This expectation value should be thought of as taken on a diff-invariant regularized theory, for example in dimensional regularization, with the counter terms provided naturally by the operators present in the effective theory of single clock inflation [12, 13].⁷ The general structure takes the form:

$$\begin{aligned} \langle S_L^{(\tilde{\zeta}_L, n)}(x_1) S_L^{(\tilde{\zeta}_L, n)}(x_2) \rangle_{\text{short}} & \quad (3.6) \\ & = \frac{\delta^{(4)}(x_1 - x_2)}{a^3 e^{3\tilde{\zeta}_L}} F_A \left(t, \dot{\tilde{\zeta}}_L(x_1), \dots \right) + F_B \left(t, \dot{\tilde{\zeta}}_L(x_1), \dots \right) F_B \left(t, \dot{\tilde{\zeta}}_L(x_2), \dots \right) . \end{aligned}$$

where \dots represents $\dot{\tilde{\lambda}}_L$, $\delta\tilde{N}_L$ and higher time derivatives of all these functions evaluated at the same location. The first term accounts for cases where the vertices around x_1 and x_2 are connected by high momentum lines (which there must be at least two by

⁶This is quite intuitive, but let us give a detailed explanation. The fact that they become uncorrelated in time can be quickly understood by realizing that one can rotate the contour of integration in such a way that all modes inside the horizon are exponentially damped as $e^{k\eta}$. Given that the last vertex has to be at about horizon crossing, the earlier one cannot be further in the past than about one Hubble time. The fact that short modes k_S with physical wavenumber greater or equal than Hubble do not contribute to the correlation at long physical distances $r_L(t) \gg H^{-1}$ can be seen by a simple Fourier space argument: the contribution of an e -folding of modes to the correlation at a physical distance $r_L(t)$ scales as $\sin(r_L k_S/a)/(r_L k_S/a)$. Since $k_S/a \gtrsim H$, this contribution vanishes as we take $r_L(t) \gg H^{-1}$.

⁷An explicit regularization and renormalization using the operator structure of effective field theory of single clock inflation was made in [1] and [3], where the symmetry constraints provided by the effective theory turned out to be extremely useful to simplify the calculations.

momentum conservation). Note that if there are short momentum lines connecting the vertices they have to be close both in space and in time. We call these Short-Connected (*SC*) diagrams and they reduce in the two point function at 1-loop to the cut-in-the-middle diagrams discussed in [1, 3]. Since at large distances these modes become uncorrelated, the contribution is peaked at the same spatial location by a simple $1/N$ argument (see figure 1). We approximated the spatial dependence as a δ -function, though it should be really thought of as a function that has support within a distance of order Hubble. For the purposes of proving the constancy of ζ at long wavelength, this difference is irrelevant.

We made the δ -function a scalar function of the local coordinates. This is why there is a factor of the determinant of the local metric $g_{L,\mu\nu}$, which is given by (2.17). The reason why we made the δ -function a scalar can be understood by noticing that there is a residual gauge-transformation that keeps us in the local homogeneous anisotropic coordinates. This is given by a time-independent rescaling of the coordinates

$$x^i \rightarrow M^i_j x^j . \tag{3.7}$$

This change of coordinates does not affect $\dot{\zeta}_L$ and $\dot{\lambda}_L$, but does change $\tilde{\zeta}_L$ and $\tilde{\lambda}_L$, and it can be checked that all these variables can be changed independently. In particular, for the choice of coordinates in which the gradient of ζ_L is on the z axis, so that only $\tilde{\lambda}_{L,5}$ is non-vanishing, we have the two independent transformations:

$$\begin{aligned}
 x^i &\rightarrow e^\alpha x^i, & \tilde{\zeta}_L &\rightarrow \tilde{\zeta}_L - \alpha, & (3.8) \\
 x^i &= \left[e^{S_5\beta} \right]^i_j x^j = \begin{pmatrix} e^{\beta/3} & 0 & 0 \\ 0 & e^{\beta/3} & 0 \\ 0 & 0 & e^{-2\beta/3} \end{pmatrix}_{ij} x^j, & \tilde{\lambda}_{L,5} &\rightarrow \tilde{\lambda}_{L,5} + \beta .
 \end{aligned}$$

The right hand side of (3.6) must therefore be invariant under the residual gauge-transformation. This implies that every x -dependence must be accompanied with a contraction with the local metric. The factor of a^3 in the local metric must be there because we can choose to re-absorb a isotropic rescaling of the coordinates with a rescaling of a . Additionally, this implies that F_A cannot depend on $\tilde{\zeta}_L$ and $\tilde{\lambda}_L$, but only on $\delta\tilde{N}_L, \tilde{\zeta}_L, \dots$, all of these vanishing as some power of $k_{\text{low}}/(aH)$. We stress that at diagrammatic level the absence of $\tilde{\zeta}_L$ in the expectation value of the short modes appears in general as the result of a cancellation among many different diagrams. The complex structure and detailed physical reasons of how this cancellation occurs was studied in [3].

The second term represents the contribution from diagrams where x_1 and x_2 are connected by low momentum lines (see figure 2). We call these Long-Connected (*LC*) diagrams. At 1-loop in the 2-point function calculation, they reduce to the a cut-in-the-side diagrams discussed in [1, 3]. F_B must therefore vanish if any of the long wavelength fluctuations vanish. The same gauge transformation as in (3.7) implies that F_B cannot depend on $\tilde{\zeta}_L$ and $\tilde{\lambda}_L$ but only on its derivatives, which vanish as powers of $k_{\text{low}}/(aH)$.

We have taken the dependence of the functions F_A and F_B on $\dot{\zeta}_L, \delta\tilde{N}_L, \dots$ as local in time. This is only an approximation. The iterative argument implies that they decay in time with a time scale of order Hubble. However, it should be noticed that, because

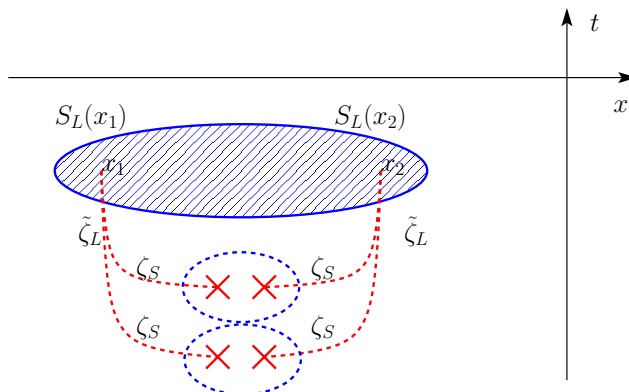


Figure 1. Example of a *SC* (short-connected) diagram. Red-dashed lines ended by a red-cross represent free modes, and two crosses encircled by a blue-dashed ellipse represent a correlation function between those two free modes. The dashed-filled ellipse represents possible additional correlation functions. This is a *SC* diagram as, among the various correlation functions present in the diagram, there are two involving two short modes at different locations x_1 and x_2 that are contracted among each other. If the distance between x_1 and x_2 is much longer than the wavelength and frequency of the two modes, the contribution becomes vanishingly small. The points x_1 and x_2 have therefore to be very close to each other. This diagram contributes to the first term of (3.6).

of the $i\epsilon$ prescription, the time integrals have support of order one Hubble time, and, for the purposes of proving the constancy of $\tilde{\zeta}_L$, we can approximate the dependence as local in time. F_A and F_B can also depend on time, but in a way that is proportional to the slow-roll parameters for example through the slow changes in H . Also note that these terms include connections through long modes that ‘are generated’ in other vertices that might be outside the volumes around x_1 and x_2 and are connected to both via long modes. The $\tilde{\zeta}_L$ appearing inside the F s account for the entire $\tilde{\zeta}_L$ which is constant over each V region (see figure 3).

Let us now take the average over the long modes:

$$\begin{aligned} \langle S_L^{(\tilde{\zeta}_L, n)}(x_1) S_L^{(\tilde{\zeta}_L, n)}(x_2) \rangle_{\text{short \& long}} &= \frac{\delta^{(4)}(x_1 - x_2)}{a(t_1)^3} \langle e^{-3\tilde{\zeta}_L} F_A(t, \dot{\tilde{\zeta}}_L, \dots) \rangle \\ &+ \left\{ F_{B1}(t_1) F_{B1}(t_2) \langle \dot{\tilde{\zeta}}_L(x_1) \dot{\tilde{\zeta}}_L(x_2) \rangle + \dots \right\}, \end{aligned} \tag{3.9}$$

where $F_{B1} = \partial F_B / \partial \dot{\tilde{\zeta}}_L|_0$. Let us consider the expectation value in the first line. Since the lower order $\dot{\tilde{\zeta}}_L$ vanish, this is dominated by the expectation value of the determinant of the metric. We have

$$\langle e^{-3\tilde{\zeta}_L} \rangle \simeq 1 + \frac{9}{2} \langle \tilde{\zeta}_L^2 \rangle + \dots, \tag{3.10}$$

where we have expanded the argument of the expectation value to the leading order. Since $\tilde{\zeta}_L$ is made of lower order modes with wavenumber up to k_{low} , all of which are constant, this expectation value is non-perturbatively time- (and space-) independent. The neglected terms decay in time as powers of $k_{\text{low}}/(aH)$. Analogously the expectation value on the second line decays in time as it is made up of lower order $\dot{\tilde{\zeta}}_L$ that are assumed to be decaying

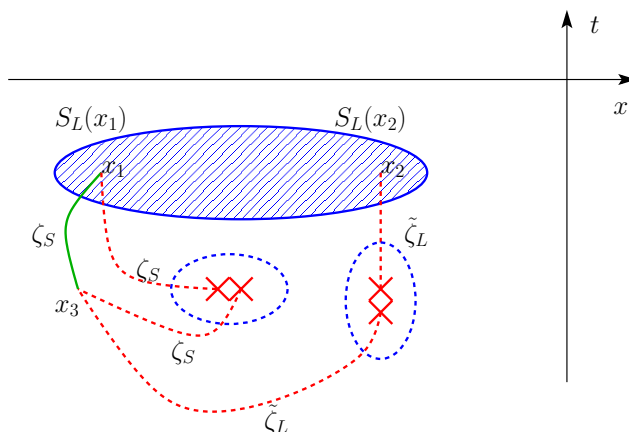


Figure 2. Example of a *LC* (long-connected) diagram. Green line represent Green’s function. The dashed-filled ellipse contains possible additional correlations that connect x_1 and x_2 only through long modes. This is a *LC* diagram as x_1 and x_2 are connected only by low momentum lines. Notice that x_3 must be very close to x_1 because it is connected to x_1 by a correlation of short modes. This diagram contributes to the second term of (3.6).

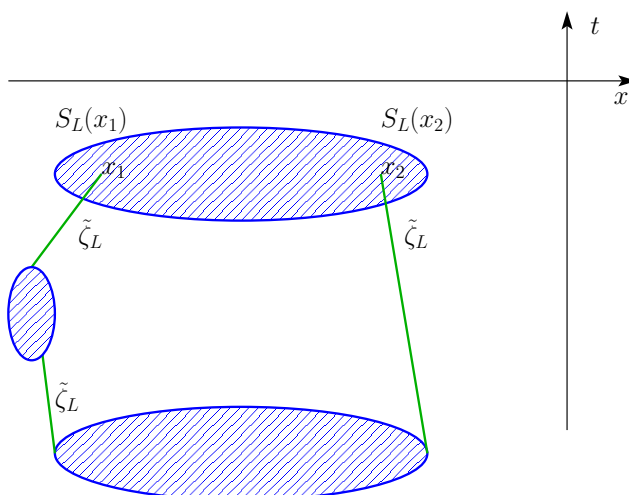


Figure 3. Example of a diagram where the ζ_L dependence of the functions F_A and F_B appearing in the average over short modes of $\langle S_L^{(\zeta_L, n)}(x_1) S_L^{(\zeta_L, n)}(x_2) \rangle_{\text{short}}$ (top shaded region) is generated by some interaction in the past (shaded regions). If the ellipses enclosing x_1 and x_2 contains a correlation function made of short modes that connect x_1 and x_2 we obtain a *SC* diagram, and it contributes to the first term of (3.6). If instead x_1 and x_2 are connected only by long modes, we have a *LC* diagram, and a contribution to the second term of (3.6). After taking expectation value over the long modes, we obtain a contribution respectively to the first and the second line of (3.9).

as some power of $k_{\text{low}}/(aH)$ by the iterative argument. The same is true for the higher derivative terms. The details of how the decay happens might be dependent on which specific vertex we are considering. In particular, we expect generically the suppression to go as $(k_{\text{low}}/(aH))^2$, but note that one factor of $(k_{\text{low}}/(aH))$ suffices to make the $\tilde{\zeta}_L$ correlation time independent. The $\tilde{\zeta}_L$ dependence in the first line originates for example

from short-connected diagrams that are connected additionally by long modes (see figure 3). Additionally, notice that if there are short vertices around both x_1 and x_2 which are being affected by the $\tilde{\zeta}_L$ in the correlation (as opposed of the the line being connected directly to the external $\tilde{\zeta}_L$) one should be paying a price in *each* vertex.

Let us now compute the two point function of $\tilde{\zeta}_L$. We have

$$\langle \dot{\tilde{\zeta}}_L(x_1) \dot{\tilde{\zeta}}_L(x_2) \rangle = \int^t dt_1 \int^t dt_2 e^{-3H(t-t_1)} e^{-3H(t-t_2)} \quad (3.11)$$

$$\left\{ \frac{\delta^{(4)}(x_1 - x_2)}{a(t_1)^3} \left[\tilde{F}_{A0}(t_1) + \left(\frac{k_{\text{low}}}{a(t_1)H} \right) \tilde{F}_{A1}(t_1) + \dots \right] \right.$$

$$\left. + \left[\tilde{F}_{B1}(t_1) \tilde{F}_{B1}(t_2) \left(\frac{\partial_{x_1-x_2}}{aH} \right)^2 \Big|_{\text{Min}[t_1, t_2]} C(x_1 - x_2) + \dots \right] \right\} .$$

where $C(x_1 - x_2)$ is the lower-order correlation function of $\tilde{\zeta}_L$, and we have rearranged the terms according to their $\partial_{x_1-x_2}/(aH)$ suppression, naming the coefficients of each term as $\tilde{F}_{A\dots}, \tilde{F}_{B\dots}, \dots$. We have used the slow roll approximation for the Green's function of $\dot{\zeta}$: $G_L(t, t') \sim \text{Exp}(-3H(t - t'))$, the subleading corrections being irrelevant. Notice that since $C(x_1 - x_2)$ is approximately scale invariant, its spatial derivative is peaked at short distances.

The time integrals clearly go to zero at late times.⁸ Notice that the spatial dependence of the correlation function consists of the sum of a δ -function and a term that is peaked at short distances approximately as $1/x^2$.

The same behavior is true when we consider the contribution of n -point correlations involving $\dot{\tilde{\zeta}}_L$ and/or $\dot{\tilde{\lambda}}_{L,n}$. This shows that the correlation functions of $\dot{\tilde{\zeta}}_L$ and $\dot{\tilde{\lambda}}_{L,n}$ at a given order vanish once we take the lower order correlation functions to be time-independent. This proves by iteration that $\dot{\tilde{\zeta}}_L$ and $\dot{\tilde{\lambda}}_{L,n}$ have vanishing correlation functions at all orders at late times. They vanish as operators.

Finally, let us consider the long wavelength $\dot{\tilde{\zeta}}_L$ defined in (3.1). As given by (2.12) and (2.16) in section 2, the relationship between $\dot{\tilde{\zeta}}_L$ and $\dot{\tilde{\lambda}}_L$ is such that once $\dot{\tilde{\zeta}}_L$ and $\dot{\tilde{\lambda}}_L$ vanish so does $\dot{\zeta}_L$, implying that the correlation functions of $\dot{\zeta}_L$ vanish to all orders for $k_{\text{low}}/(aH) \ll 1$. In particular, since for $k \leq k_{\text{low}}$, $\zeta_L(k) = \zeta(k)$, with $\zeta(k)$ being the Fourier transform of the globally defined ζ fluctuation, the correlation functions of $\dot{\zeta}_k$ vanish at all orders for $k/(aH) \ll 1$, as we ultimately wished to show.

We finally make a comment on the IR divergencies present in the above formula (3.11). Indeed notice that if we try to evaluate (3.10) perturbatively, we obtain, at leading order,

$$\langle e^{-3\tilde{\zeta}_L} \rangle \simeq 1 + \frac{9}{2} \langle \tilde{\zeta}_L^2 \rangle + \dots \simeq 1 + \frac{9}{2} \Delta_\zeta \log(k_{\text{low}} L) + \dots , \quad (3.12)$$

⁸It might be useful to notice that

$$\int_{t_{\text{in}}}^t dt_1 e^{-6H(t-t_1)} e^{-3Ht_1} = \frac{e^{-3Ht}}{3H} \left(1 - e^{-3H(t-t_{\text{in}})} \right) ,$$

$$\int_{t_{\text{in}}}^t dt_1 \int_{t_{\text{in}}}^{t_1} dt_2 e^{-3H(t-t_1)} e^{3H(t-t_2)} e^{-2Ht_2} = \frac{e^{-2Ht}}{12H^2} \left(3 - 4e^{-H(t-t_{\text{in}})} + e^{-4H(t-t_{\text{in}})} \right) .$$

where L is the infrared comoving size of the box and where Δ_ζ is the tree-level dimensionless power spectrum of the ζ fluctuations

$$\langle \zeta(\vec{k}, t \rightarrow \infty) \zeta(\vec{k}', t \rightarrow \infty) \rangle_{\text{tree-level}} = (2\pi)^3 \delta^{(3)}(\vec{k} + \vec{k}') \frac{\Delta_\zeta}{k^3}, \quad (3.13)$$

where we have neglected the tilt. We see that there is an IR divergence as $L \rightarrow \infty$, hidden in (3.11) in the coefficients $\tilde{F}_{A\dots}, \tilde{F}_{B\dots}, \dots$. Such IR divergencies were found first in [15, 16] in the context of the long wavelength ζ 1-loop power spectrum, but they are present in every ζ -correlation function that is expressed in comoving coordinates. It was later realized in [2, 17–19] that they cancel completely out once the calculation is performed in local physical coordinates where the long-wavelength fluctuations are re-absorbed within the coordinates: for example $V_{\text{physical}} = \text{Exp}(3\zeta_L) a(t)^3 V_{\text{comoving}}$. Since expressions (3.9) and (3.11) are in comoving coordinates, they are also affected by these IR divergencies, which are not harmful. More in detail, starting at 2-loops, the IR divergence (3.12) comes from the IR divergence of the short-mode correlation functions at horizon crossing. Notice that while for modes longer than the horizon the IR divergence is slow-roll suppressed, this is not the case for modes evaluated at horizon crossing. Indeed the three-point function in the squeezed limit is not slow-roll suppressed in this same kinematical regime [3, 11]. The second line of (3.9) is also IR divergent starting at 2-loops as it also involves short mode correlations in the background of long wavelength modes. These IR divergencies need to be present in correlation functions in comoving coordinates in order to ensure that once the correlation functions are expressed in terms of local physical coordinates, every IR dependence on modes much longer than the ones of interest is absent [2].

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