

Y-system, TBA and Quasi-Classical Strings in $AdS_4 \times CP^3$

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ABSTRACT: We study the exact spectrum of the AdS_4/CFT_3 duality put forward by Aharony, Bergman, Jafferis and Maldacena (ABJM). We derive thermodynamic Bethe ansatz (TBA) equations for the planar ABJM theory, starting from “mirror” asymptotic Bethe equations which we conjecture. We also propose generalization of the TBA equations for excited states. The recently proposed Y-system is completely consistent with the TBA equations for a large subsector of the theory, but should be modified in general. We find the general asymptotic infinite length solution of the Y-system, and also several solutions to all wrapping orders in the strong coupling scaling limit. To make a comparison with results obtained from string theory, we assume that the all-loop Bethe ansatz of N.G. and P. Vieira is the valid worldsheet theory description in the asymptotic regime. In this case we find complete agreement, to all orders in wrappings, between the solution of our Y-system and generic quasi-classical string spectrum in $AdS_3 \times S^1$.

KEYWORDS: AdS-CFT Correspondence, Integrable Field Theories

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1 Introduction

The AdS/CFT correspondence [1] continues to be a source of exciting new results in gauge and string theories. The best-studied example of the duality is the correspondence between four-dimensional $\mathcal{N} = 4$ super Yang-Mills (SYM) theory and Type IIB superstring theory on $\text{AdS}_5 \times \text{S}^5$. Another example is the recently found duality between Type IIA string theory on $\text{AdS}_4 \times \text{CP}^3$ and three-dimensional $\mathcal{N} = 6$ super Chern-Simons (SCS) theory [2, 3].

Remarkably, evidence for integrability has been found both in the gauge theory [4–14] and in the string theory [15–23] in the planar limit of large number of colors. In

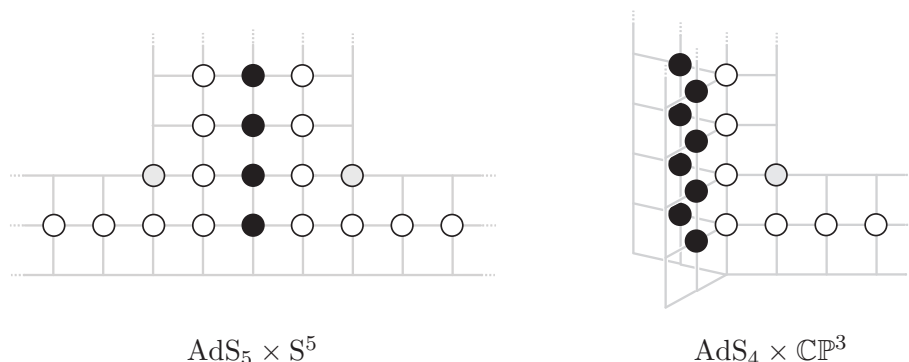


Figure 1. Graphical representation of the Y-systems [54]. Circles correspond to Y-functions. Black nodes are the “massive” nodes which are suppressed for asymptotically large length L . For gray circles in the corners the equation cannot be written “locally” in terms of Y’s.

SYM further intensive development [24–33] has led to complete description of anomalous dimensions of infinitely long operators by means of the Asymptotic Bethe Ansatz (ABA) equations [34, 35]. Similar equations were found in [37, 38] for SCS. Very recently the integrability approach was also extended to $\text{AdS}_3/\text{CFT}_2$ dual pairs [39].

For complete solution of the planar AdS/CFT spectral problem one should be able to solve the integrable two dimensional worldsheet theory in finite volume. The program of applying the methods of relativistic integrable field theories for finite size spectrum of AdS/CFT was started in [40]. In [41, 42] a generalization of the Lüscher type formula was proposed for the first finite volume correction to the asymptotic spectrum generated by ABA. This information, as well as experience with relativistic integrable theories [44–53], led to the Y-system proposed in [54] for exact solution of both $\text{AdS}_5 \times \text{S}^5$ and $\text{AdS}_4 \times \text{CP}^3$ theories. As we show in this work, the proposal of [54] for ABJM theory is only valid in a certain large subsector of the theory, and should be modified to describe the general case.

A graphical representation of the Y-systems of [54] is given in figure 1 where the Y-functions are represented by circles. Each value of the index A , which labels the Y-functions (functions of the spectral parameter u), corresponds to a node of this diagram. For each node A , except the gray ones, the Y-system equation has the form

$$Y_A^+ Y_A^- = \frac{\prod_B (1 + Y_B)}{\prod_C (1 + 1/Y_C)}, \quad (1.1)$$

where $Y_A^\pm = Y_A(u \pm i/2)$ and the index B (resp. C) labels the nodes connected to the A node by horizontal (resp. vertical) lines.¹

¹For the gray node the equations cannot be written as functional equations in terms of Y’s. In many cases it is convenient to parameterize the Y-functions in terms of T-functions, which satisfy the Hirota functional equation. The “non-local” equation for the gray nodes is replaced (see for example [54]) by a “local” one in terms of T-functions.

In this paper we argue that in the AdS₄/CFT₃ case equation (1.1) for black nodes (figure 1, on the right) should be replaced by rather unusual equations

$$Y_{\blacktriangleright_a}^+ Y_{\blacktriangleleft_a}^- = \frac{1 + Y_{\Delta_a}}{(1 + 1/Y_{\blacktriangleleft_{a+1}})(1 + 1/Y_{\blacktriangleright_{a-1}})}, \quad a > 1, \quad (1.2)$$

$$Y_{\blacktriangleleft_a}^+ Y_{\blacktriangleright_a}^- = \frac{1 + Y_{\Delta_a}}{(1 + 1/Y_{\blacktriangleright_{a+1}})(1 + 1/Y_{\blacktriangleleft_{a-1}})}, \quad a > 1, \quad (1.3)$$

$$Y_{\blacktriangleright_1}^+ Y_{\blacktriangleleft_1}^- = \frac{1 + Y_{\otimes}}{(1 + 1/Y_{\blacktriangleleft_2})}, \quad Y_{\blacktriangleleft_1}^+ Y_{\blacktriangleright_1}^- = \frac{1 + Y_{\otimes}}{(1 + 1/Y_{\blacktriangleright_2})}, \quad (1.4)$$

while all the other Y-system equations of [54] need not be changed.² Notice that for the case $Y_{\blacktriangleright_a} = Y_{\blacktriangleleft_a}$ the new equations (1.2)–(1.4) coincide with the ones originally proposed in [54].

Once the Y-functions are found the energy of the state can be computed from

$$E = \sum \int_{-\infty}^{\infty} \frac{du}{2\pi i} \frac{\partial \epsilon_a^{\text{mir}}(u)}{\partial u} \log(1 + Y_{\blacktriangleright_a}^{\text{mir}})(1 + Y_{\blacktriangleleft_a}^{\text{mir}}) + \sum_{j=1}^{K_4} e^{\text{ph}}(u_{4,j}) + \sum_{j=1}^{K_{\bar{4}}} e^{\text{ph}}(u_{\bar{4},j}), \quad (1.5)$$

where u_j are the exact Bethe roots given by

$$Y_{\blacktriangleleft_1}^{\text{ph}}(u_{4,j}) = -1, \quad Y_{\blacktriangleright_1}^{\text{ph}}(u_{\bar{4},j}) = -1, \quad (1.6)$$

and ϵ is the single magnon dispersion introduced in (2.6) (see section 3.3 for more details).

In the AdS₅ case the Y-system passes some nontrivial tests – in [54] the 4-loop perturbative result [55, 56] was reproduced,³ and more recently a comparison was made at 5 loops in [58]. In [59, 60, 62] the Y-system was also shown to be consistent with the thermodynamic Bethe ansatz (TBA) approach.⁴

The TBA equations, describing the ground state energy, do not lead to any nontrivial dependence of that energy on the coupling since the ground state is protected by supersymmetries. In [60] an extension of these equations was proposed to describe the excited states. These equations were solved numerically in [65] for the first non-trivial Konishi operator [66–68], giving for the first time the anomalous dimension of a non-protected operator in a wide range of values of the 't Hooft coupling λ for a 4D gauge theory in the

²With the following identification between Y-functions of [54] and new Y-functions: $Y_{a,0}^4 = Y_{\blacktriangleleft_a}, Y_{a,0}^{\bar{4}} = Y_{\blacktriangleright_a}$

³Technically the derivation of [54] is very similar to [43], where the 4-loop perturbative results were reproduced for the first time.

⁴In [62] the Y-system was only obtained in the interval $-2g < u < 2g$. At the same time the authors of [62] failed to get the Y-system of [54] for $|u| > 2g$ and the discrepancy was stated. The reason of this misunderstanding is that some of the Y-functions have branch points at $u = \pm 2g \pm \frac{i}{2}$ with branch cuts going to $\pm\infty \pm \frac{i}{2}$, parallel to the real axis. Thus for real u the quantity $Y(u + \frac{i}{2})Y(u - \frac{i}{2})$ can be understood for instance as $Y(u + \frac{i}{2} - i0)Y(u - \frac{i}{2} + i0)$ or $Y(u + \frac{i}{2} + i0)Y(u - \frac{i}{2} + i0)$. The first prescription (chosen in [62]) leads to the discrepancy whereas the second does not. Note that the problem is only present for real u i.e. for measure zero subset of the complex plane. Any prescription which preserves continuity leads to agreement with [54]. In [63] (after private communication with P.Vieira) the issue was resolved.

planar limit. The numerical results also indicate agreement with the string prediction [69].⁵ The results of [65] disagree in the sub-leading $1/\lambda^{1/4}$ order with two string computations [72, 73] and [71] which also disagree with each other and are based on rather strong assumptions. In [71] a truncated model is considered whereas [72, 73] assumes the applicability of the quasiclassics in the small charge limit. We hope that a first principles calculation can be done using Berkoviz’s pure spinor formalism [76].

Another very recent test of the Y-system of [54] for $\text{AdS}_5/\text{CFT}_4$ was done at strong coupling [70]. An *analytical* solution of the Y-system was found for generic classical string motion inside $\text{AdS}_3 \times \text{S}$. It was shown to agree with the quasi-classical one-loop spectrum to all orders in wrapping providing thus a deep structural test of the Y-system in the regime where the ABA fails completely.

In this paper we apply the technique of TBA for the $\text{AdS}_4 \times \text{CP}^3$ theory to test the Y-system we propose. We also present the general asymptotic infinite length solution of the Y-system. The asymptotic solution is very important since it allows to establish a correspondence between the exact solution of the Y-system and the physical states of the theory. It can be also used at weak coupling where it is a good approximation to study the leading wrapping effects. In addition, we find strong coupling solutions of the Y-system in two cases and compare results with the quasi-classical string spectrum thus testing deeply the structure of the Y-system to all orders in wrapping.

2 Asymptotic large L solution of Y-system

The asymptotic spectrum of the theory can be found using asymptotic Bethe ansatz (ABA) techniques. In this section we describe the ABA equations of [37] and link them with the Y-system formalism by presenting the general asymptotic solution of the Y-system. That solution extends the one of [54].

In the asymptotic regime the counting of the states is very clear and well established. One can analytically continue the solution of the Y-system from the asymptotic regime, where the solution is explicit, to finite volume. Usually this continuation is unique (see for example [77]) and allows to fix the solution of Y-system. Technically at the moment it is not known how to perform this procedure for the general excited state in AdS/CFT . We show how to apply this general method [77] for the “ $\mathfrak{sl}(2)$ ” subsector and also at strong coupling.

2.1 Asymptotic Bethe ansatz equations for physical $\text{AdS}_4/\text{CFT}_3$

Here we present the asymptotic Bethe equations for the $\text{AdS}_4/\text{CFT}_3$ theory, which were for the first time obtained in [37]. We will also introduce some notation useful for the sequel.

⁵Much later the equations of [65] were rederived by another group [103]. The authors of [103] confirmed the validity of the equations at least in the range of the coupling $0 < \lambda < 700$ where a numerical solution was obtained. The perturbation theory for the world-sheet sigma model in the formulations of [76] is naturally organized in powers of $1/\sqrt{\lambda}$ and thus the value $\lambda \sim 700$ should already give the asymptotic of $E(\lambda)$ with a good precision especially when an appropriate extrapolation procedure is applied. This holds assuming the analyticity of $E(\lambda)$ for real positive values of λ which is however doubted in [103].

First we define the Zhukowski variable $x(u)$:

$$x + \frac{1}{x} = \frac{u}{h(\lambda)}, \tag{2.1}$$

where $h(\lambda)$ is some unknown function of the 't Hooft coupling λ . It should have the following asymptotics at weak coupling and strong coupling:

$$h(\lambda) = \lambda + h_3\lambda^3 + \mathcal{O}(\lambda^5) = \sqrt{\lambda/2} + h^0 + \mathcal{O}\left(\frac{1}{\sqrt{\lambda}}\right). \tag{2.2}$$

Recently the coefficient $h_3 = -8 + 2\zeta_2$ was computed directly from the Super-Chern-Simons perturbation theory [13, 14]. At strong coupling the situation is less clear: in [75] and [78] the coefficient h^0 was argued to be 0 whereas in [109] some evidence was given in favor of a different value $-\frac{\log 2}{2\pi}$ (see also [74]). Hopefully this issue could be analyzed from world-sheet sigma model first principles calculation like in [79, 80].

Equation (2.1) admits two solutions, and we define two branches of the function $x(u)$, which are called “mirror” and “physical”:

$$x^{\text{ph}}(u) = \frac{1}{2} \left(\frac{u}{h} + \sqrt{\frac{u}{h} - 2} \sqrt{\frac{u}{h} + 2} \right), \quad x^{\text{mir}}(u) = \frac{1}{2} \left(\frac{u}{h} + i\sqrt{4 - \frac{u^2}{h^2}} \right). \tag{2.3}$$

Here, by \sqrt{u} we denote the principal branch of the square root. This definition of mirror and physical branches is the same as in the $\text{AdS}_5/\text{CFT}_4$ case [57, 60], with the $\text{AdS}_5/\text{CFT}_4$ coupling g replaced by $h(\lambda)$. Above the real axis, the mirror and physical branches coincide. $x^{\text{ph}}(u)$ is obtained by analytical continuation from the upper half plane to the plane with the cut $(-2h, 2h)$, and $x^{\text{mir}}(u)$ — by continuation to the plane with the cut $(-\infty, -2h) \cup (2h, +\infty)$. The $\text{AdS}_4/\text{CFT}_3$ Bethe equations [37] for the original (physical) theory are written in terms of $x^{\text{ph}}(u)$, while the mirror Bethe equations we conjecture include $x^{\text{mir}}(u)$, in analogy with the $\text{AdS}_5/\text{CFT}_4$ case [57] (see section 3). In sections 4 and 5 we use the mirror branch of x if its argument is a free variable, and the physical branch for $x(u_j)$, with u_j being the Bethe roots.

In the physical ABA equations of [37] there are five types of Bethe roots: u_1, u_2, u_3, u_4 and $u_{\bar{4}}$. Conserved local charges (the heights Hamiltonians) in $\text{AdS}_4/\text{CFT}_3$ are expressed in terms of the momentum-carrying roots u_4 and $u_{\bar{4}}$:

$$\mathcal{Q}_n = \sum_{j=1}^{K_4} \mathbf{q}_n(u_{4,j}) + \sum_{j=1}^{K_{\bar{4}}} \mathbf{q}_n(u_{\bar{4},j}), \quad \mathbf{q}_n = \frac{i}{n-1} \left(\frac{1}{(x^+)^{n-1}} - \frac{1}{(x^-)^{n-1}} \right), \tag{2.4}$$

where we have used general notation

$$f^\pm(u) \equiv f(u \pm i/2), \quad f^{[+a]} \equiv f(u + ia/2). \tag{2.5}$$

In particular, string state energies in $\text{AdS}_4 \times \mathbb{CP}^3$ or operator anomalous dimensions in the dual gauge theory are obtained from $E = h(\lambda)\mathcal{Q}_2$.

The momentum and energy which correspond to a single Bethe root u_4 or $u_{\bar{4}}$ are given by

$$p = \frac{1}{i} \log \frac{x^+}{x^-}, \epsilon = \frac{1}{2} + h(\lambda) \left(\frac{i}{x^+} - \frac{i}{x^-} \right), \quad (2.6)$$

and the charge \mathcal{Q}_1 is the sum of all momenta:

$$\mathcal{Q}_1 = \sum_{j=1}^{K_4} p(u_{4,j}) + \sum_{j=1}^{K_{\bar{4}}} p(u_{\bar{4},j}). \quad (2.7)$$

To write the Bethe equations in compact form, we introduce the following notation:

$$R_l^{(\pm)} = \prod_{j=1}^{K_l} \frac{x(u) - x_{l,j}^{\mp}}{(x_{l,j}^{\mp})^{1/2}}, \quad R_l = \prod_{j=1}^{K_l} (x(u) - x_{l,j}), \quad (2.8)$$

$$B_l^{(\pm)} = \prod_{j=1}^{K_l} \frac{1/x(u) - x_{l,j}^{\mp}}{(x_{l,j}^{\mp})^{1/2}}, \quad B_l = \prod_{j=1}^{K_l} (1/x(u) - x_{l,j}), \quad (2.9)$$

$$Q_l(u) \equiv \prod_{j=1}^{K_l} (u - u_{l,j}), \quad S_l(u) \equiv \prod_{j=1}^{K_l} \sigma_{\text{BES}}(x(u), x_{l,j}) \quad (2.10)$$

where σ_{BES} is the Beisert-Eden-Staudacher dressing kernel [37]. The Bethe equations of [37] in \mathfrak{sl}_2 favored grading have the form⁶

$$\begin{aligned} +1 &= e^{-\frac{1}{2}i\mathcal{Q}_1} \left. \frac{Q_2^+ B_4^{(-)} B_4^{(-)}}{Q_2^- B_4^{(+)} B_4^{(+)}} \right|_{u_{1,k}}, \\ -1 &= \left. \frac{Q_2^{--} Q_1^+ Q_3^+}{Q_2^{++} Q_1^- Q_3^-} \right|_{u_{2,k}}, \\ +1 &= e^{\frac{1}{2}i\mathcal{Q}_1} \left. \frac{Q_2^+ R_4^{(-)} R_4^{(-)}}{Q_2^- R_4^{(+)} R_4^{(+)}} \right|_{u_{3,k}}, \\ +1 &= e^{\frac{1}{2}i\mathcal{Q}_1} e^{-Lip(u_{4,k})} \left. \frac{B_1^+ R_3^+ Q_4^{++} R_4^{(-)} R_4^{(-)}}{B_1^- R_3^- Q_4^{--} R_4^{(+)} R_4^{(+)}} S_4 S_{\bar{4}} \right|_{u_{4,k}}, \\ +1 &= e^{\frac{1}{2}i\mathcal{Q}_1} e^{-Lip(u_{\bar{4},k})} \left. \frac{B_1^+ R_3^+ Q_4^{++} R_4^{(-)} R_4^{(-)}}{B_1^- R_3^- Q_4^{--} R_4^{(+)} R_4^{(+)}} S_4 S_{\bar{4}} \right|_{u_{\bar{4},k}}, \end{aligned} \quad (2.11)$$

where L is the length of the effective spin chain, and corresponds to the string momentum or length of the operator in the CS theory. The above equations describe the spectrum correctly in the limit $L \rightarrow \infty$. We stress again that in those equations the physical branch x^{ph} of the function x should be used in all places, e.g. inside expressions (2.8), (2.9), (2.10) for B_l , R_l and S_l .

⁶Here, as well as when constructing the asymptotic solution of Y-system, one should be careful with the sign ambiguity in the square root factors inside $e^{\frac{1}{2}i\mathcal{Q}_1}$ and B_l, R_l .

The Bethe roots are additionally constrained by the zero momentum condition

$$1 = \prod_{j=1}^{K_4} \frac{x_{4,j}^+}{x_{4,j}^-} \prod_{j=1}^{K_{\bar{4}}} \frac{x_{\bar{4},j}^+}{x_{\bar{4},j}^-} \Leftrightarrow \mathcal{Q}_1 = 2\pi m. \quad (2.12)$$

2.2 General asymptotic solution

As we mentioned in the beginning of this section the asymptotic (large L) solution of the Y-system plays an important role in the whole Y-system construction. It allows to link a particular solution of the Y-system with an actual state of the theory. The asymptotic solutions are in one-to-one correspondence with the solutions of ABA equations.

In many cases one can analytically continue a solution from asymptotically large volume to finite volume. In [70] another way to inject information about the state of the theory was proposed: demanding that the exact functions Y_{as} approach the formal asymptotic solution for infinite a or s .⁷ Then one can still use the same counting of the states as in the ABA even for finite volumes.⁸ This prescription was shown to work especially successfully in the strong coupling scaling limit [70], which we describe below.

In view of its importance we will review the construction of [54] for the asymptotic large L solution of the Y-system in this section and extend it to the case $Y_{\blacktriangleleft} \neq Y_{\blacktriangleright}$. To distinguish the asymptotic Y functions from the exact ones we use the bold font:

$$\mathbf{Y}_{\Delta_a} = \frac{\mathbf{T}_{a,1}^+ \mathbf{T}_{a,1}^-}{\mathbf{T}_{a+1,1} \mathbf{T}_{a-1,1}} - 1, \quad 1/\mathbf{Y}_{\mathcal{O}_s} = \frac{\mathbf{T}_{1,s}^+ \mathbf{T}_{1,s}^-}{\mathbf{T}_{1,s+1} \mathbf{T}_{1,s-1}} - 1 \quad (2.13)$$

$$\mathbf{Y}_{\blacktriangleleft_a} \simeq \left(\frac{x^{[-a]}}{x^{[+a]}} \right)^L \mathbf{T}_{a,1} \prod_{n=-\frac{a-1}{2}}^{\frac{a-1}{2}} \Phi_4^{\theta_{na}^E}(u+in) \Phi_{\bar{4}}^{\theta_{na}^O}(u+in), \quad (2.14)$$

$$\mathbf{Y}_{\blacktriangleright_a} \simeq \left(\frac{x^{[-a]}}{x^{[+a]}} \right)^L \mathbf{T}_{a,1} \prod_{n=-\frac{a-1}{2}}^{\frac{a-1}{2}} \Phi_4^{\theta_{na}^O}(u+in) \Phi_{\bar{4}}^{\theta_{na}^E}(u+in) \quad (2.15)$$

where θ_{na}^E is 0 for even and 1 for odd terms in the product:

$$\theta_{na}^E \equiv \begin{cases} 1, & n + \frac{a-1}{2} \text{ is even} \\ 0, & n + \frac{a-1}{2} \text{ is odd} \end{cases} \quad (2.16)$$

and $\theta_{na}^O \equiv 1 - \theta_{na}^E$. The factors $\Phi_4(u)$ and $\Phi_{\bar{4}}(u)$ are constructed in such a way that the ABA equations (2.11) for the momentum carrying nodes are given by $\mathbf{Y}_{\blacktriangleleft_1}^{\text{ph}}(u_{4,j}) = -1$ and $\mathbf{Y}_{\blacktriangleright_1}^{\text{ph}}(u_{\bar{4},j}) = -1$. This leads to (using that $\mathbf{T}_{1,1}(u_{4,j}) = -Q_3^+/Q_3^-$)

$$\Phi_4(u) = S_4 S_{\bar{4}} \frac{B_4^{(+)+} R_{\bar{4}}^{(-)-} B_1^+ B_3^-}{B_4^{(-)-} R_{\bar{4}}^{(+)+} B_1^- B_3^+} e^{-i\mathcal{Q}_1/2}, \quad \Phi_{\bar{4}}(u) = S_4 S_{\bar{4}} \frac{B_4^{(+)+} R_{\bar{4}}^{(-)-} B_1^+ B_3^-}{B_4^{(-)-} R_{\bar{4}}^{(+)+} B_1^- B_3^+} e^{+i\mathcal{Q}_1/2}. \quad (2.17)$$

⁷ This should give the same result as analytical continuation in L . Usually, variations of Y 's in L vanish at large a and s . The values of the Bethe roots inside the asymptotic solution should be equal to their exact values e.g. $Y_{\blacktriangleleft_1}^{\text{ph}}(u_{4,j}) = -1$. One should study this point in more detail.

⁸ One cannot exclude completely that this procedure fails for some particular small volumes.

The $\mathbf{T}_{a,s}$ functions which enter the definitions of $\mathbf{Y}_{a,s}$ can be computed from the generating functional [82–84]

$$\mathcal{W} = \left[1 - \frac{Q_1^- B^{(+)+} R^{(+)-}}{Q_1^+ B^{(-)+} R^{(-)-}} D \right] \frac{1}{\left[1 - \frac{Q_3^+ Q_2^- R^{(+)-}}{e^{\frac{1}{2}iQ_1} Q_3^- Q_2 R^{(-)-}} D \right] \left[1 - \frac{Q_1^- Q_2^{++} R^{(+)-}}{e^{\frac{1}{2}iQ_1} Q_1^+ Q_2 R^{(-)-}} D \right]} \left[1 - \frac{Q_3^+}{Q_3^-} D \right] \quad (2.18)$$

where $D = e^{-i\partial_u}$ is the shift operator and $R = R_4 R_{\bar{4}}$, $B = B_4 B_{\bar{4}}$. Expansion of this generating functional yields eigenvalues of the $\mathfrak{su}(2|2)$ transfer matrices:

$$\mathcal{W} = \sum_{s=0}^{\infty} \mathbf{T}_{1,s}(u + i\frac{1-s}{2}) D^s, \quad \mathcal{W}^{-1} = \sum_{a=0}^{\infty} (-1)^a \mathbf{T}_{a,1}(u + i\frac{1-a}{2}) D^a. \quad (2.19)$$

In appendix B we also present the expressions for the asymptotic solution after the duality transformation, which exchanges the $sl(2)$ and $su(2)$ sectors. One can see from those formulas that for $u_{4,j} = u_{\bar{4},j}$ the asymptotic solution exactly coincides with the one proposed in [54].

In the next subsection we expand the asymptotic solution in the scaling strong coupling limit.

2.3 Asymptotic solution in scaling limit

The scaling limit is the strong coupling limit $\lambda \rightarrow \infty$ where the number of Bethe roots M and the operator length L go to infinity as $\sqrt{\lambda}$. The Bethe roots x_j are distributed along cuts \mathcal{C} on the complex plane x in this limit [81]. These cuts can be understood as branch cuts of a 10-sheet Riemann surface which corresponds to a certain function. One can interpret them as the eigenvalues of the classical monodromy matrix, which are usually written as $\lambda_a = e^{-iq_a}$, with q_a being the so called quasi-momenta. Similarly to [37] for the $\eta = -1$ grading we get

$$\begin{aligned} q_2 &= \frac{Lx/h+Q_2x}{x^2-1} + H_1 - \bar{H}_4 - \bar{H}_{\bar{4}} + \bar{H}_3 \\ q_3 &= \frac{Lx/h-Q_1}{x^2-1} - H_2 + H_1 + \bar{H}_3 - \bar{H}_2 \\ q_4 &= \frac{Lx/h-Q_1}{x^2-1} - H_3 + H_2 + \bar{H}_2 - \bar{H}_1 \\ q_1 &= \frac{Lx/h+Q_2x}{x^2-1} + H_4 + H_{\bar{4}} - H_3 - \bar{H}_1 \\ q_5 &= \quad \quad \quad + H_4 - H_{\bar{4}} + \bar{H}_4 - \bar{H}_{\bar{4}} \\ q_a &= -q_{11-a}, \quad a = 6, \dots, 10, \end{aligned} \quad (2.20)$$

where the resolvents H_a have the form

$$H_a(x) = \sum_j \frac{x^2}{x^2-1} \frac{1}{x-x_{a,j}}, \quad \bar{H}_a(x) = H_a(1/x).$$

In these terms the Bethe equations (2.11) are equivalent to the condition that the two eigenvalues of the monodromy matrix are equal along the branch cut

$$q_i(x+i0) - q_j(x-i0) = 2\pi n, \quad x \in \mathcal{C}. \quad (2.21)$$

We can now simplify (2.18) for strong coupling. First of all we notice that the shift operator D becomes a formal expansion parameter. Then we use

$$\begin{aligned} \frac{Q_1^- B^{(+)+} R^{(+)-}}{Q_1^+ B^{(-)+} R^{(-)-}} &\simeq \exp \left[-i \left(H_1 - H_4 - H_{\bar{4}} + \bar{H}_1 + \bar{H}_4 + \bar{H}_{\bar{4}} \right) \right] \\ \frac{Q_3^+ Q_2^{--} R^{(+)-}}{e^{\frac{1}{2}i\mathcal{Q}_1} Q_3^- Q_2 R^{(-)-}} &\simeq \exp \left[-i \left(\frac{\mathcal{Q}_1 + x\mathcal{Q}_2}{x^2 - 1} + H_2 - H_3 - \bar{H}_3 + \bar{H}_2 - H_4 - H_{\bar{4}} \right) \right] \\ \frac{Q_1^+ Q_2^{--} R^{(+)-}}{e^{\frac{1}{2}i\mathcal{Q}_1} Q_1^- Q_2 R^{(-)-}} &\simeq \exp \left[-i \left(\frac{\mathcal{Q}_1 + x\mathcal{Q}_2}{x^2 - 1} + H_2 - H_1 - \bar{H}_1 + \bar{H}_2 - H_4 - H_{\bar{4}} \right) \right] \\ \frac{Q_3^+}{Q_3^-} &\simeq \exp \left[+i \left(H_3 + \bar{H}_3 \right) \right]. \end{aligned} \quad (2.22)$$

The generating functional (2.18) becomes

$$\mathcal{W} = \frac{(1 - \lambda_1 d)(1 - \lambda_2 d)}{(1 - \lambda_3 d)(1 - \lambda_4 d)}, \quad (2.23)$$

where we have redefined the formal expansion parameter in the following way

$$d = \exp \left[i \left(\frac{Lx/h + x\mathcal{Q}_2}{x^2 - 1} + H_4 + H_{\bar{4}} - \bar{H}_1 + \bar{H}_3 \right) \right] D. \quad (2.24)$$

Expanding the generating function (2.23) we get

$$\begin{aligned} T_{1,s} &= \frac{\lambda_4^{s-1}(\lambda_4 - \lambda_1)(\lambda_4 - \lambda_2) - \lambda_3^{s-1}(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)}{\lambda_4 - \lambda_3} \left(\frac{d}{D} \right)^s \\ T_{a,1} &= (-1)^a \frac{\lambda_1^{a-1}(\lambda_1 - \lambda_3)(\lambda_1 - \lambda_4) - \lambda_2^{a-1}(\lambda_2 - \lambda_3)(\lambda_2 - \lambda_4)}{\lambda_1 - \lambda_2} \left(\frac{d}{D} \right)^a. \end{aligned} \quad (2.25)$$

It is now straightforward to compute \mathbf{Y}_{Δ_a} and $\mathbf{Y}_{\mathcal{O}_s}$ from (2.13). Note that the factors $(\frac{d}{D})^s$ and $(\frac{d}{D})^a$ are irrelevant here and thus \mathbf{Y}_{Δ_a} are rational functions of λ_a only!

Moreover, using the relation

$$\left(\frac{x^-}{x^+} \right)^L \Phi_4(u) \simeq \exp \left[-i \left(\frac{xL/h + x\mathcal{Q}_2}{x^2 - 1} + 2H_{\bar{4}} - \bar{H}_4 + \bar{H}_{\bar{4}} - \bar{H}_1 + \bar{H}_3 \right) \right], \quad (2.26)$$

and the same relation with 4 and $\bar{4}$ exchanged, we obtain expressions for the massive nodes:

$$\begin{aligned} Y_{\blacktriangleleft a} &= (-\lambda_5)^{-\omega_a} \frac{\lambda_1^{a-1}(\lambda_1 - \lambda_3)(\lambda_1 - \lambda_4) - \lambda_2^{a-1}(\lambda_2 - \lambda_3)(\lambda_2 - \lambda_4)}{\lambda_1 - \lambda_2}, \\ Y_{\blacktriangleright a} &= (-\lambda_5)^{+\omega_a} \frac{\lambda_1^{a-1}(\lambda_1 - \lambda_3)(\lambda_1 - \lambda_4) - \lambda_2^{a-1}(\lambda_2 - \lambda_3)(\lambda_2 - \lambda_4)}{\lambda_1 - \lambda_2}, \end{aligned} \quad (2.27)$$

which are again written solely in terms of the eigenvalues of the classical monodromy matrix! Here, we have introduced ω_a , which is defined to be 1 for odd a and zero for even a .

3 TBA equations for AdS₄/CFT₃

In this section we derive the Thermodynamic Bethe ansatz equations for AdS₄/CFT₃. Let us first describe the general form of the TBA method [102] (see a nice introductory paper [57]). We start with an integrable quantum field theory in 1+1 dimensions, on a circle of circumference L . The partition function of this theory at temperature $1/R$ is

$$Z(L, R) = \sum_k e^{-RE_k(L)}, \tag{3.1}$$

and in the limit $R \rightarrow \infty$, $R \gg L$ we have

$$Z(L, R) \sim e^{-RE_0(L)}, \tag{3.2}$$

where $E_0(L)$ is the ground state energy. Denoting by ϕ and ψ the bosonic and fermionic fields, respectively, we can write the partition function as a functional integral

$$Z(L, R) = \int \mathcal{D}\phi \mathcal{D}\psi e^{-S_E} \tag{3.3}$$

where S_E is the theory's Euclidean action. In this integral, fermionic fields are periodic (resp. antiperiodic) in space (resp. time), while bosonic fields are periodic in both space and time:

$$\begin{aligned} \psi(x + L, t) &= \psi(x, t), & \psi(x, t + R) &= -\psi(x, t) \\ \phi(x + L, t) &= \phi(x, t), & \phi(x, t + R) &= \phi(x, t). \end{aligned} \tag{3.4}$$

Using this representation of the partition function, one can relate it to the Witten index of the “mirror” theory in volume R :

$$W(R, L) = \sum_k (-1)^F e^{-LE_k^{\text{mir}}(R)} = \sum_k e^{-RE_k(L)} = Z(L, R). \tag{3.5}$$

The mirror theory is obtained from the original one by a double Wick rotation, and F in (3.5) is 1 for fermionic states and 0 otherwise. Introducing the mirror bulk free energy $\mathcal{F}^{\text{mir}}(L)$, defined by the mirror theory's Witten index at temperature $1/L$,

$$-RL\mathcal{F}^{\text{mir}}(L) = \ln W(R, L), \tag{3.6}$$

we see that the finite volume ground state energy is related to the infinite volume mirror free energy:

$$E_0(L) = L\mathcal{F}^{\text{mir}}(L). \tag{3.7}$$

The mirror theory's infinite volume spectrum is described by the ABA equations, which allow one to find $\mathcal{F}^{\text{mir}}(L)$ and then the original theory's ground state energy.

To compute \mathcal{F}^{mir} it is essential to know the structure of the solutions of infinite volume mirror ABA equations. For numerous theories (see [44–53]), so-called string hypotheses have been formulated, which describe the complexes Bethe roots form in the infinite volume limit (simplest of those complexes are strings of roots). We will use indices A, B, \dots to label the complexes, and denote the energy and momentum of a complex by, respectively, ip_A^* and $i\epsilon_A^*$, to underline that the mirror theory is obtained from the physical one by a double Wick rotation.

Multiplying the Bethe equations for all roots in a complex, one obtains equations for the density $\rho_A(u)$ of complexes, with $u \in \mathbb{R}$ being the center of the complex. Those equations have the form

$$\bar{\rho}_A(u) + \rho_A(u) = \frac{i}{2\pi} \frac{d\epsilon_A^*(u)}{du} - K_{BA}(v, u) * \rho_B(v) \quad (3.8)$$

where $\bar{\rho}$ is the density of holes, $K(v, u) * f(v) \equiv \int_{-\infty}^{+\infty} dv K(v, u) f(v)$ and summation over B is assumed. Also, we use the normalization

$$\int_{-\infty}^{\infty} du \rho_A(u) = \frac{\text{total number of complexes of type } A}{R}. \quad (3.9)$$

The free energy is given by the minimal value of a functional of the densities

$$\mathcal{F}^{\text{mir}}(L) = \min \sum_A \int_{-\infty}^{\infty} du \left((Lip_A^* + h_A)\rho_A - \left[\rho_A \log \left(1 + \frac{\bar{\rho}_A}{\rho_A} \right) + \bar{\rho}_A \log \left(1 + \frac{\rho_A}{\bar{\rho}_A} \right) \right] \right), \quad (3.10)$$

with constraints (3.8) on the densities. Here, $h_A \equiv \log[(-1)^{N_A}]$, where N_A is the number of fermionic Bethe roots in the complex A . Minimization of this functional gives the TBA equations

$$\log \mathcal{Y}_A(u) = K_{AB}(u, v) * \log[1 + 1/\mathcal{Y}_B(v)] + iLp_A^* + h_A, \quad (3.11)$$

where $\mathcal{Y}_A \equiv \frac{\bar{\rho}_A}{\rho_A}$ and $K(u, v) * f(v) \equiv \int dv K(u, v) f(v)$. Lastly, the free energy can be expressed in terms of a solution of TBA equations:

$$\mathcal{F}^{\text{mir}}(L) = \sum_A \int \frac{du}{2\pi i} \frac{d\epsilon_A^*}{du} \log(1 + 1/\mathcal{Y}_A(u)), \quad (3.12)$$

From the free energy, one can compute the ground state energy of the physical theory via (3.7). In addition, the TBA equations can be modified in such a way that their solutions provide also energies of certain excited states in finite volume.

3.1 Ground state TBA equations for AdS₄/CFT₃

We first present, as a conjecture, the ABA equations for the mirror of AdS₄/CFT₃ theory. Like the physical Bethe equations [37], those equations involve Bethe roots $u_1, u_2, u_3, u_4, u_{\bar{4}}$, with all roots except u_2 being fermionic. The only roots which carry energy or momentum are u_4 and $u_{\bar{4}}$. For a single root, we denote energy by ip_1^* , and momentum by $i\epsilon_1^*$, where

$$p_1^* = \frac{1}{i} \log \frac{x^+}{x^-}, \quad \epsilon_1^* = \frac{1}{2} + h(\lambda) \left(\frac{i}{x^+} - \frac{i}{x^-} \right). \quad (3.13)$$

Here, and everywhere in section 3 unless otherwise stated, we use the mirror branch of the function $x(u)$. Note that p_1^* (resp. ϵ_1^*), evaluated in physical instead of mirror kinematics, coincides with the momentum (resp. energy) of a single Bethe root in the physical theory [37]. This is in accordance with the fact that the mirror theory is obtained from the

physical one by a double Wick rotation. The momentum/energy in mirror and physical AdS₅/CFT₄ are related in a similar way.

The mirror Bethe equations we propose are written in terms of the functions B_l, R_l, S_l, Q_l , which were introduced in section 2 (note that x in them should now be understood as x^{mir}). The equations for u_1, u_2 and u_3 are:

$$1 = \frac{Q_2^+ B_4^{(-)} B_4^{(-)}}{Q_2^- B_4^{(+)} B_4^{(+)}} \Big|_{u_{1,k}}, \quad -1 = \frac{Q_2^{--} Q_1^+ Q_3^+}{Q_2^{++} Q_1^- Q_3^-} \Big|_{u_{2,k}}, \quad 1 = \frac{Q_2^+ R_4^{(-)} R_4^{(-)}}{Q_2^- R_4^{(+)} R_4^{(+)}} \Big|_{u_{3,k}} \quad (3.14)$$

The r.h.s. of the equations for u_1 and u_3 is not always unimodular, because $B_l^{(\pm)}(u)$ and $R_l^{(\pm)}(u)$ have cuts on the real axis. However, in the thermodynamic limit (see below) the single fermion roots u_1, u_3 are distributed [62] within the interval $-2h < u_1 < 2h$, $-2h < u_3 < 2h$, and unimodularity of the r.h.s then follows. Note that no conditions have to be imposed on the u_1, u_3 roots which are parts of pyramid complexes Δ_n , as the terms containing cuts cancel during fusion of Bethe equations. This can be seen from the fact that the kernels $\mathcal{K}(u, v)$ in TBA equations (see below) for interactions involving pyramids are real for real u, v and have no cuts on the real axis.

The equations for momentum-carrying roots are:

$$-1 = e^{R\epsilon_1^*(u_{4,k})} \left(\frac{B_1^+ R_3^+ Q_4^{++} R_4^{(-)} R_4^{(-)}}{B_1^- R_3^- Q_4^{--} R_4^{(+)} R_4^{(+)}} \right) S_4 S_{\bar{4}} \left(\frac{x_{4,k}^+}{x_{4,k}^-} \right)^{\frac{K_1 - K_3}{2}} \prod_{j=1}^{K_4} \sqrt{\frac{x_{4,j}^+}{x_{4,j}^-}} \prod_{j=1}^{K_{\bar{4}}} \sqrt{\frac{x_{4,j}^+}{x_{4,j}^-}} \quad (3.15)$$

for $u = u_{4,k}$, and for $u = u_{\bar{4},k}$ we have

$$-1 = e^{R\epsilon_1^*(u_{\bar{4},k})} \left(\frac{B_1^+ R_3^+ Q_4^{++} R_4^{(-)} R_4^{(-)}}{B_1^- R_3^- Q_4^{--} R_4^{(+)} R_4^{(+)}} \right) S_{\bar{4}} S_4 \left(\frac{x_{\bar{4},k}^+}{x_{\bar{4},k}^-} \right)^{\frac{K_1 - K_3}{2}} \prod_{j=1}^{K_4} \sqrt{\frac{x_{4,j}^+}{x_{4,j}^-}} \prod_{j=1}^{K_{\bar{4}}} \sqrt{\frac{x_{\bar{4},j}^+}{x_{\bar{4},j}^-}}. \quad (3.16)$$

Note that the combination

$$S_l^{\text{mir}}(u) \equiv \prod_{j=1}^{K_l} \sigma^{\text{mir}}(x(u), x_{l,j}) = \prod_{j=1}^{K_l} \sqrt{\frac{x_{l,j}^+ x^-}{x_{l,j}^- x^+}} \sigma_{BES}(x(u), x_{l,j}) = S_l(u) \prod_{j=1}^{K_l} \sqrt{\frac{x_{l,j}^+ x^-}{x_{l,j}^- x^+}} \quad (3.17)$$

is a unimodular function (see [60, 61]). By $\sigma_{BES}(x(u), x_{l,j})$ we denote the usual Beisert-Eden-Staudacher dressing kernel analytically continued from $\text{Im } u > i/2$ between the branch points $u = \pm 2h + i/2$.⁹ The above equations are similar to the Bethe equations for physical ABA (2.11). The difference is in the choice of the mirror branch of $x(u)$, interchange of the energy and momentum (with multiplication by i) and various factors of $\sqrt{x^+/x^-}$, tuned in such a way that the right-hand sides are unimodular functions. This prescription is based on the corresponding conjecture in the AdS₅/CFT₄ case [57].

In the thermodynamic limit, solutions of the above ABA are described by complexes of Bethe roots. Among those complexes are $\circ_n, \Delta_n, \oplus, \otimes$, which are the same complexes as

⁹“Physical” choice of the branch corresponds to analytical continuation to the plane with the cut $[-2h + i/2, 2h + i/2]$. In the “mirror” kinematics all cuts should go through infinity.

\circ_n :	string of roots	:	$u_2 = u + ij,$	$j = -\frac{n-2}{2}, \dots, \frac{n-2}{2}$
:		:	$u_3 = u + ij,$	$j = -\frac{n-1}{2}, \dots, \frac{n-1}{2}$
Δ_n :	pyramid	:	$u_2 = u + ij,$	$j = -\frac{n-2}{2}, \dots, \frac{n-2}{2}$
:		:	$u_1 = u + ij,$	$j = -\frac{n-3}{2}, \dots, \frac{n-3}{2}$
\oplus :	single fermion root	:	$u_1 = u$	
\otimes :	single fermion root	:	$u_3 = u$	
\blacktriangleleft_n :	Odd-Even complex	:	$u_4 = u + ij$ when $\theta_{jn}^E = 1,$	$j = -\frac{n-1}{2}, \dots, \frac{n-1}{2}$
:		:	$u_{\bar{4}} = u + ij$ when $\theta_{jn}^O = 1$	
\blacktriangleright_n :	Even-Odd complex	:	$u_4 = u + ij$ when $\theta_{jn}^O = 1,$	$j = -\frac{n-1}{2}, \dots, \frac{n-1}{2}$
:		:	$u_{\bar{4}} = u + ij$ when $\theta_{jn}^E = 1$	

Table 1. Complexes of Bethe roots.

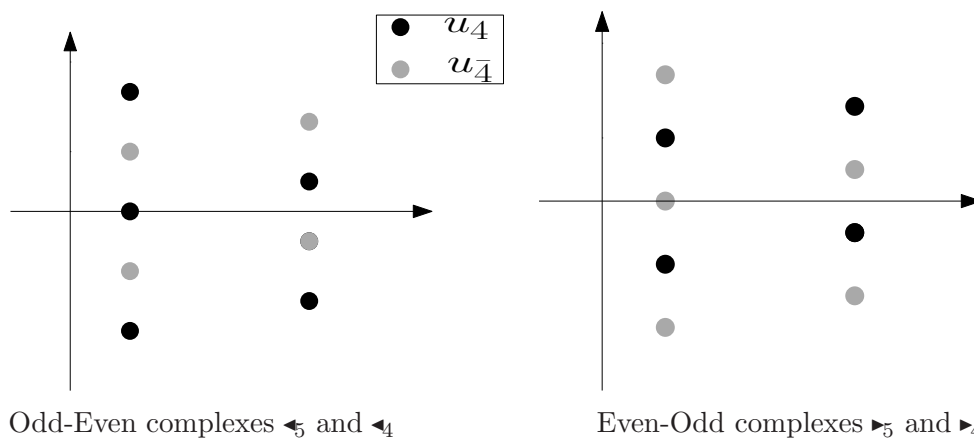


Figure 2. Strings of alternating roots on the $\{\text{Re } u, \text{Im } u\}$ plane. Black circles denote u_4 roots, gray circles denote $u_{\bar{4}}$ roots. Vertical spacing between roots is i .

in the mirror $\text{AdS}_5/\text{CFT}_4$ (see [64]). In addition, the momentum-carrying roots u_4 and $u_{\bar{4}}$ form two new types of complexes, which we call Odd-Even and Even-Odd. They were recently considered in [85]. Those complexes are real-centered strings of alternating u_4 and $u_{\bar{4}}$ roots, adjacent roots being spaced by i . In the Odd-Even complex, the lowest root of the string on the complex plane is u_4 , while in the Even-Odd complex, the lowest root is $u_{\bar{4}}$ (see figure 2). The list of all complexes is given in the table 1.

Here, $u \in \mathbb{R}$ denotes the center of a complex, notation $j = -\frac{n-1}{2}, \dots, \frac{n-1}{2}$ means that j takes the values $-\frac{n-1}{2}, -\frac{n-3}{2}, \dots, \frac{n-3}{2}, \frac{n-1}{2}$, and θ 's were defined in (2.16).

The energy (in our notation ip_A^* , with index A taking the values $\{\circ_n, \oplus, \otimes, \Delta_n, \blacktriangleleft_n, \blacktriangleright_n\}$) which corresponds to a complex is the sum of energies of the roots in a complex, and the same is true for momentum. We have $p_{\blacktriangleleft_n}^* = p_{\blacktriangleright_n}^* = p_n^*, \epsilon_{\blacktriangleleft_n}^* = \epsilon_{\blacktriangleright_n}^* = \epsilon_n^*$, where

$$p_n^*(u) \equiv \frac{1}{i} \log \frac{x^{[+n]}}{x^{[-n]}}, \quad \epsilon_n^*(u) \equiv \frac{n}{2} + h(\lambda) \left(\frac{i}{x^{[+n]}} - \frac{i}{x^{[-n]}} \right), \quad (3.18)$$

while for other complexes, p_A^* and ϵ_A^* are zero. Note also that the only complexes with odd number of fermion roots are those denoted by $\oplus, \otimes, \blacktriangleleft_{2n-1}$ and $\blacktriangleright_{2n-1}$. Hence in our case the

$A \setminus B$	\circ_m	\oplus	\otimes	Δ_m	\blacktriangleleft_m	\blacktriangleright_m
\circ_n	$+K_{n-1,m-1}$	$-K_{n-1}$	$+K_{n-1}$	0	0	0
\oplus	$-K_{m-1}$	0	0	$+K_{m-1}$	$-\mathcal{B}_{1m}^{(01)}$	$-\mathcal{B}_{1m}^{(01)}$
\otimes	$-K_{m-1}$	0	0	$+K_{m-1}$	$-\mathcal{R}_{1m}^{(01)}$	$-\mathcal{R}_{1m}^{(01)}$
Δ_n	0	$-K_{n-1}$	$+K_{n-1}$	$+K_{n-1,m-1}$	$-\mathcal{R}_{nm}^{(01)} - \mathcal{B}_{n-2,m}^{(01)}$	$-\mathcal{R}_{nm}^{(01)} - \mathcal{B}_{n-2,m}^{(01)}$
\blacktriangleleft_n	0	$\mathcal{B}_{n1}^{(10)}$	$-\mathcal{R}_{n1}^{(10)}$	$-\mathcal{R}_{nm}^{(10)} - \mathcal{B}_{n,m-2}^{(10)}$	$-\mathcal{T}_{nm}^{\parallel}$	$-\mathcal{T}_{nm}^{\perp}$
\blacktriangleright_n	0	$\mathcal{B}_{n1}^{(10)}$	$-\mathcal{R}_{n1}^{(10)}$	$-\mathcal{R}_{nm}^{(10)} - \mathcal{B}_{n,m-2}^{(10)}$	$-\mathcal{T}_{nm}^{\perp}$	$-\mathcal{T}_{nm}^{\parallel}$

Table 2. Integration kernels for TBA equations.

quantity h_A in (3.11) has to be $\log(-1)$ for these complexes and $\log(+1)$ otherwise.

Applying the fusion procedure to the mirror ABA equations,¹⁰ we find that in our case, kernels K_{AB} entering (3.11) are given by table 2. Some of those kernels are the same as in the AdS₅/CFT₄ case [60], and we list them in appendix A. The new kernels are

$$\mathcal{T}_{nm}^{\parallel} \equiv \mathcal{B}_{nm}^{(11)} + \tilde{\mathcal{S}}_{nm} - K_{nm}^{\parallel}, \quad (3.19)$$

$$\mathcal{T}_{nm}^{\perp} \equiv \mathcal{B}_{nm}^{(11)} + \tilde{\mathcal{S}}_{nm} - K_{nm}^{\perp}, \quad (3.20)$$

where

$$K_{nm}^{\parallel}(u, v) \equiv \sum_{l=-\frac{n-1}{2}}^{\frac{n-1}{2}} \sum_{k=-\frac{m-1}{2}}^{\frac{m-1}{2}} K_2(u - v + i(l - k)) (\theta_{ln}^E \theta_{km}^O + \theta_{ln}^O \theta_{km}^E), \quad (3.21)$$

$$K_{nm}^{\perp}(u, v) \equiv \sum_{l=-\frac{n-1}{2}}^{\frac{n-1}{2}} \sum_{k=-\frac{m-1}{2}}^{\frac{m-1}{2}} K_2(u - v + i(l - k)) (\theta_{ln}^E \theta_{km}^E + \theta_{ln}^O \theta_{km}^O). \quad (3.22)$$

Let us also introduce the functions Y_A , which will turn out to be the functions which enter the Y-system:

$$\left\{ Y_{\circ_n}, Y_{\oplus}, \frac{1}{Y_{\otimes}}, \frac{1}{Y_{\Delta_n}}, \frac{1}{Y_{\blacktriangleleft_n}}, \frac{1}{Y_{\blacktriangleright_n}} \right\} \equiv \{ \mathcal{Y}_{\circ_n}, \mathcal{Y}_{\oplus}, \mathcal{Y}_{\otimes}, \mathcal{Y}_{\Delta_n}, \mathcal{Y}_{\blacktriangleleft_n}, \mathcal{Y}_{\blacktriangleright_n} \} \quad (3.23)$$

(We recall that $\mathcal{Y}_A \equiv \frac{\bar{\rho}_A}{\rho_A}$.)

¹⁰an important assumption here is monotonicity

We can write the TBA equations for the ground state in the following way:

$$\log Y_{\otimes} = +K_{m-1} * \log \frac{1 + 1/Y_{\circ_m}}{1 + Y_{\Delta_m}} + \mathcal{R}_{1m}^{(01)} * \log(1 + Y_{\blacktriangleleft_m})(1 + Y_{\blacktriangleright_m}) + i\pi \quad (3.24)$$

$$\log Y_{\oplus} = -K_{m-1} * \log \frac{1 + 1/Y_{\circ_m}}{1 + Y_{\Delta_m}} - \mathcal{B}_{1m}^{(01)} * \log(1 + Y_{\blacktriangleleft_m})(1 + Y_{\blacktriangleright_m}) - i\pi \quad (3.25)$$

$$\begin{aligned} \log Y_{\Delta_n} &= -K_{n-1, m-1} * \log(1 + Y_{\Delta_m}) - K_{n-1} \circledast \log(1 + Y_{\otimes}) \\ &\quad + \left(\mathcal{R}_{nm}^{(01)} + \mathcal{B}_{n-2, m}^{(01)} \right) * \log(1 + Y_{\blacktriangleleft_m})(1 + Y_{\blacktriangleright_m}) \end{aligned} \quad (3.26)$$

$$\log Y_{\circ_n} = K_{n-1, m-1} * \log(1 + 1/Y_{\circ_m}) + K_{n-1} \circledast \log(1 + Y_{\otimes}) \quad (3.27)$$

$$\begin{aligned} \log Y_{\blacktriangleleft_n} &= -iLp_n^* + \mathcal{T}_{nm}^{\parallel} * \log(1 + Y_{\blacktriangleleft_m}) + \mathcal{T}_{nm}^{\perp} * \log(1 + Y_{\blacktriangleright_m}) + i\pi n \\ &\quad + \mathcal{R}_{n1}^{(10)} \circledast \log(1 + Y_{\otimes}) + \left(\mathcal{R}_{nm}^{(10)} + \mathcal{B}_{n, m-2}^{(10)} \right) * \log(1 + Y_{\Delta_m}) \end{aligned} \quad (3.28)$$

$$\begin{aligned} \log Y_{\blacktriangleright_n} &= -iLp_n^* + \mathcal{T}_{nm}^{\parallel} * \log(1 + Y_{\blacktriangleright_m}) + \mathcal{T}_{nm}^{\perp} * \log(1 + Y_{\blacktriangleleft_m}) + i\pi n \\ &\quad + \mathcal{R}_{n1}^{(10)} \circledast \log(1 + Y_{\otimes}) + \left(\mathcal{R}_{nm}^{(10)} + \mathcal{B}_{n, m-2}^{(10)} \right) * \log(1 + Y_{\Delta_m}) \end{aligned} \quad (3.29)$$

where $*$ denotes integration over the second variable, as in (3.11). Summation over the repeated index m is assumed with $m \geq 2$ for Δ_m and \circ_m , and $m \geq 1$ for $\blacktriangleleft_m, \blacktriangleright_m$.

Range of integration for fermions is limited to $-2h < u < 2h$. Notice that from (3.24) and (3.25) we can see that $\frac{1}{Y_{\oplus}}$ is the analytical continuation of Y_{\otimes} across the cut $u \in (-\infty, -2h) \cup (2h, +\infty)$. For the convolutions with fermions we introduce the convolutions \circledast which should be understood in the sense of a B-cycle (see [60]), e.g.

$$\begin{aligned} K_{n-1} \circledast \log(1 + Y_{\otimes}) &\equiv \int_{-2h}^{2h} dv K_{n-1} \log \frac{1 + Y_{\otimes}}{1 + 1/Y_{\oplus}}, \\ \mathcal{R}^{(n0)} \circledast \log(1 + Y_{\otimes}) &\equiv \int_{-2h}^{2h} dv \left[\mathcal{R}^{(n0)} \log(1 + Y_{\otimes}) - \mathcal{B}^{(n0)} \log(1 + 1/Y_{\oplus}) \right]. \end{aligned} \quad (3.30)$$

Remarkably, the combination $\mathcal{B}_{nm}^{(11)} + \tilde{\mathcal{S}}_{nm}$, which is part of the kernels \mathcal{T}_{nm}^{\perp} and $\mathcal{T}_{nm}^{\parallel}$, has only two branch cuts for each of the variables u and v . This follows from the integral representation

$$\begin{aligned} \tilde{\mathcal{S}}_{nm}(u, v) + \mathcal{B}_{nm}^{(11)}(u, v) &= \\ &= - \sum_{a=1}^{\infty} \int \left[\mathcal{B}_{n1}^{(10)}(u, w + ia/2) \mathcal{B}_{1m}^{(01)}(w - ia/2, v) + \mathcal{B}_{n1}^{(10)}(u, w - ia/2) \mathcal{B}_{1m}^{(01)}(w + ia/2, v) \right] dw, \end{aligned} \quad (3.31)$$

which can be derived using the results obtained in [60, 61] (see appendix A). As a consequence, the functions $Y_{\blacktriangleleft_n}(u)$ and $Y_{\blacktriangleright_n}(u)$ (see (3.28), (3.29)) should not have branch cuts for $-in/2 < \text{Im } u < in/2$.

In the next section we will establish a relation between the above equations and the AdS₄/CFT₃ Y-system.

3.2 Y-system from TBA equations

In this section we show that solutions of ground state TBA equations satisfy the AdS₄/CFT₃ Y-system (1.2)–(1.4) described in the introduction. This derivation of the

Y-system is similar to the AdS₅/CFT₄ case [60]. First, we identify the Y_A functions in TBA equations with the Y-system functions $Y_{a,s}$. We set

$$\{Y_{\ominus_n}, Y_{\oplus}, Y_{\otimes}, Y_{\Delta_n}\} = \{Y_{1,n}, Y_{2,2}, Y_{1,1}, Y_{n,1}\}. \quad (3.32)$$

Let us introduce the discrete Laplacian operator

$$\Delta K_n(u) \equiv K_n(u + i/2 - i0) + K_n(u - i/2 + i0) - K_{n+1}(u) - K_{n-1}(u).$$

Following [60], we apply this operator to the l.h.s. of the TBA equations, acting on the free index n and the free variable u . The action of this Laplacian on some of our kernels has been computed in [60]:

$$\Delta K_n(u) = \delta_{n,1} \delta(u) \quad (3.33)$$

$$\Delta K_{nm}(v - u) = \Delta \mathcal{R}_{nm}^{(11)}(v, u) = \delta_{n,m+1} \delta(v - u) + \delta_{n,m-1} \delta(v - u) \quad (3.34)$$

$$\Delta \mathcal{R}_{nm}^{(01)}(v, u) = \Delta \mathcal{R}_{nm}^{(10)}(v, u) = \delta_{n,m} \delta(v - u) \quad (3.35)$$

$$\Delta \mathcal{B}_{nm} = 0, \quad \Delta \tilde{\mathcal{S}}_{nm} = 0. \quad (3.36)$$

The new kernels $\mathcal{T}_{nm}^{\parallel}$ and \mathcal{T}_{nm}^{\perp} satisfy relations of a new type, which are not written in terms of the Laplacian:

$$\mathcal{T}_{nm}^{\perp}(u + \frac{i-i0}{2}, v) + \mathcal{T}_{nm}^{\parallel}(u - \frac{i-i0}{2}, v) - \mathcal{T}_{n+1,m}^{\parallel}(u, v) - \mathcal{T}_{n-1,m}^{\perp}(u, v) = -\delta_{n,m-1} \delta(u-v) \quad (3.37)$$

$$\mathcal{T}_{nm}^{\parallel}(u + \frac{i-i0}{2}, v) + \mathcal{T}_{nm}^{\perp}(u - \frac{i-i0}{2}, v) - \mathcal{T}_{n+1,m}^{\perp}(u, v) - \mathcal{T}_{n-1,m}^{\parallel}(u, v) = -\delta_{n-1,m} \delta(u-v).$$

Using those identities, we obtain from the TBA equations a set of simpler equations for the functions Y_A . This closely follows [60]. For example, applying the Laplacian to the l.h.s of (3.27), we get

$$\log \frac{Y_{\ominus_n}^+ Y_{\ominus_n}^-}{Y_{\ominus_{n+1}} Y_{\ominus_{n-1}}} = \log(1 + 1/Y_{\ominus_{n+1}})(1 + 1/Y_{\ominus_{n-1}}), \quad n > 2 \quad (3.38)$$

or, equivalently,

$$\log Y_{\ominus_n}^+ Y_{\ominus_n}^- = \log(1 + Y_{\ominus_{n+1}})(1 + Y_{\ominus_{n-1}}), \quad n > 2. \quad (3.39)$$

For $n = 2$ we obtain

$$\log Y_{\ominus_2}^+ Y_{\ominus_2}^- = \log \frac{(1 + Y_{\otimes})(1 + Y_{\ominus_3})}{1 + 1/Y_{\oplus}}. \quad (3.40)$$

Equations (3.24) and (3.26) can be treated in a similar way. We get an equation for Y_{\otimes}

$$\log Y_{\otimes}^+ Y_{\otimes}^- = \log \frac{(1 + Y_{\ominus_2})(1 + Y_{\blacktriangleleft_1})(1 + Y_{\blacktriangleright_1})}{1 + 1/Y_{\Delta_2}}, \quad (3.41)$$

and also equations for Y_{Δ_n} :

$$\log \frac{Y_{\Delta_n}^+ Y_{\Delta_n}^-}{Y_{\Delta_{n+1}} Y_{\Delta_{n-1}}} = \log \frac{(1 + Y_{\blacktriangleleft_n})(1 + Y_{\blacktriangleright_n})}{(1 + Y_{\Delta_{n+1}})(1 + Y_{\Delta_{n-1}})}, \quad n > 2 \quad (3.42)$$

$$\begin{aligned} \log \frac{Y_{\Delta_2}^+ Y_{\Delta_2}^-}{Y_{\Delta_3}} &= \log \frac{(1 + Y_{\oplus})(1 + Y_{\blacktriangleleft_2})(1 + Y_{\blacktriangleright_2}) Y_{\otimes}}{(1 + Y_{\Delta_3})(1 + Y_{\otimes})} \\ &\quad - \log Y_{\otimes} Y_{\oplus} + \sum_m (\mathcal{R}_{1m}^{(01)} - \mathcal{B}_{1m}^{(01)}) * \log(1 + Y_{\blacktriangleleft_m})(1 + Y_{\blacktriangleright_m}) \end{aligned} \quad (3.43)$$

Moreover, adding up eqs. (3.24), (3.25) we find that

$$\log Y_{\otimes} Y_{\oplus} = \sum_m (\mathcal{R}_{1m}^{(01)} - \mathcal{B}_{1m}^{(01)}) * \log(1 + Y_{\blacktriangleleft_n})(1 + Y_{\blacktriangleright_m}). \quad (3.44)$$

Therefore, in eq. (3.43) all summands except the first one cancel, and that equation takes the compact form

$$\log Y_{\Delta_2}^+ Y_{\Delta_2}^- = \log \frac{(1 + Y_{\oplus})(1 + Y_{\blacktriangleleft_2})(1 + Y_{\blacktriangleright_2})}{(1 + 1/Y_{\Delta_3})(1 + 1/Y_{\otimes})}. \quad (3.45)$$

Equations for Y_{\blacktriangleleft_n} , $Y_{\blacktriangleright_n}$ are obtained from (3.28), (3.29) in a similar way with the use of new identities (3.37), and they are precisely equations (1.2)–(1.4) which were given in the introduction:

$$\log Y_{\blacktriangleright_n}^+ Y_{\blacktriangleleft_n}^- = \log \frac{1 + Y_{\Delta_n}}{(1 + 1/Y_{\blacktriangleleft_{n+1}})(1 + 1/Y_{\blacktriangleright_{n-1}})}, \quad n > 1 \quad (3.46)$$

$$\log Y_{\blacktriangleleft_n}^+ Y_{\blacktriangleright_n}^- = \log \frac{1 + Y_{\Delta_n}}{(1 + 1/Y_{\blacktriangleright_{n+1}})(1 + 1/Y_{\blacktriangleleft_{n-1}})}, \quad n > 1, \quad (3.47)$$

while for $n = 1$

$$\log Y_{\blacktriangleright_1}^+ Y_{\blacktriangleleft_1}^- = \log \frac{1 + Y_{\otimes}}{(1 + 1/Y_{\blacktriangleleft_2})}, \quad \log Y_{\blacktriangleleft_1}^+ Y_{\blacktriangleright_1}^- = \log \frac{1 + Y_{\otimes}}{(1 + 1/Y_{\blacktriangleright_2})}. \quad (3.48)$$

3.3 Integral equations for excited states

As we have shown above the equations (3.24)–(3.29) contain important structural information about the Y-system. However, those equations do not make much sense when understood literally since they describe the ground state which is protected by supersymmetry, and the Y-functions are degenerate in this case. There is a way to extend these equations to excited states, with the Y-functions becoming very nontrivial. For the case $Y_{\blacktriangleleft_a} = Y_{\blacktriangleright_a} = Y_{\bullet_a}$, $u_{4,j} = u_{\bar{4},j}$ similarly to [60] we propose, as a conjecture, the following set

of equations:

$$\log Y_{\otimes} = +K_{m-1} * \log \frac{1 + 1/Y_{\circ m}}{1 + Y_{\Delta m}} + 2\mathcal{R}_{1m}^{(01)} * \log(1 + Y_{\bullet m}) + 2 \left[\log \frac{R_4^{(+)}}{R_4^{(-)}} \right] + i\pi \quad (3.49)$$

$$\log Y_{\oplus} = -K_{m-1} * \log \frac{1 + 1/Y_{\circ m}}{1 + Y_{\Delta m}} - 2\mathcal{B}_{1m}^{(01)} * \log(1 + Y_{\bullet m}) - 2 \left[\log \frac{B_4^{(+)}}{B_4^{(-)}} \right] - i\pi \quad (3.50)$$

$$\log Y_{\Delta n} = -K_{n-1, m-1} * \log(1 + Y_{\Delta m}) - K_{n-1} * \log \frac{1 + Y_{\otimes}}{1 + 1/Y_{\oplus}} \quad (3.51)$$

$$+ 2 \left(\mathcal{R}_{nm}^{(01)} + \mathcal{B}_{n-2, m}^{(01)} \right) * \log(1 + Y_{\bullet m})$$

$$+ 2 \left[\sum_{k=-\frac{n-1}{2}}^{\frac{n-1}{2}} \log \frac{R_4^{(+)}(u + ik)}{R_4^{(-)}(u + ik)} \right] + 2 \left[\sum_{k=-\frac{n-3}{2}}^{\frac{n-3}{2}} \log \frac{B_4^{(+)}(u + ik)}{B_4^{(-)}(u + ik)} \right]$$

$$\log Y_{\circ n} = K_{n-1, m-1} * \log(1 + 1/Y_{\circ m}) + K_{n-1} * \log \frac{1 + Y_{\otimes}}{1 + 1/Y_{\oplus}} \quad (3.52)$$

$$\log Y_{\bullet n} = J \log \frac{x^{[-n]}}{x^{[+n]}} - \mathcal{B}_{n1}^{(10)} * \log(1 + 1/Y_{\oplus}) + \mathcal{R}_{n1}^{(10)} * \log(1 + Y_{\otimes}) \quad (3.53)$$

$$+ \left(\mathcal{R}_{nm}^{(10)} + \mathcal{B}_{n, m-2}^{(10)} \right) * \log(1 + Y_{\Delta m})$$

$$+ \left(2\tilde{\mathcal{S}}_{nm} - \mathcal{R}_{nm}^{(11)} + \mathcal{B}_{nm}^{(11)} \right) * \log(1 + Y_{\bullet m}) + \left[\sum_{k=-\frac{n-1}{2}}^{\frac{n-1}{2}} \log \Phi_4(u + ik) \right] + i\pi n$$

where $J = L + K_4$, Φ_4 is given by (2.17) (with B_1^{\pm}, B_3^{\pm} in that expression replaced by unity) and the exact positions of the Bethe roots are determined by

$$Y_{\bullet 1}^{\text{ph}}(u_{4, j}) = -1, \quad j = 1, \dots, K_4. \quad (3.54)$$

The label “ph” here means that one should analytically continue the equation for $Y_{\bullet 1}$ to the physical sheet, like it was done for the first time in [65]. The Bethe roots are additionally constrained by a condition imposed on total momentum (the trace cyclicity condition). We can write this constraint in a form similar to (1.5):

$$\sum_{a=1}^{\infty} \int_{-\infty}^{\infty} \frac{du}{2\pi i} \frac{\partial p_a^{\text{mir}}(u)}{\partial u} \log(1 + Y_{\blacktriangleleft a}^{\text{mir}})(1 + Y_{\blacktriangleright a}^{\text{mir}}) + \sum_{j=1}^{K_4} p^{\text{ph}}(u_{4, j}) + \sum_{j=1}^{K_4} p^{\text{ph}}(u_{\bar{4}, j}) = 2\pi m, \quad m \in \mathbb{Z} \quad (3.55)$$

(recall that the momentum $p(u)$ was introduced in (3.13)). This expression can be simplified in our case, as $Y_{\blacktriangleleft a} = Y_{\blacktriangleright a} = Y_{\bullet a}$ and $u_{4, j} = u_{\bar{4}, j}$.

Note that in equations for excited states, in the terms without convolutions the branch x^{ph} should be used for $x(u_{4, j})$ and x^{mir} should be used for $x(u)$ with u being the free variable.

Strictly speaking these equations are only valid for some particular values of λ and configurations of roots. In other cases the equations may require some modification. This question is usually subjected to case-by-case study (see e.g. [96–98]).

In general the procedure is the following - one can start from a sufficiently large L or small λ where the terms with $\log(1 + Y_{\blacktriangleleft_n})$ are irrelevant and the asymptotic solution of [54] should be a good approximation. The condition (3.54) can be discarded for a while, and one should find such a configuration of the roots $u_{4,j}$ (usually they are sufficiently close to the origin in this case) that the asymptotic solution satisfies the equations for excited states we proposed above. After that the equations should be analytically continued in L, λ and $u_{4,j}$.

This procedure in general is rather complicated, however our experience with the Konishi operator in $\text{AdS}_5/\text{CFT}_4$ [65] tells us that one can probably use the equations above as they are from $\lambda = 0$ to very large λ 's. At the same time the Y-system functional equations are not affected by these modifications and they are more suitable for the strong coupling analysis [70]. Moreover, they are not restricted to the “ $sl(2)$ ” subsector.

The possibility that some singularities could collide with the integration contours and modify the equations when some parameters (such as the coupling) are changed was studied in detail in [96–98]. For AdS/CFT , this issue was mentioned in [65], and following that proposal, such a possibility was explored in [103] for $\text{AdS}_5/\text{CFT}_4$.¹¹

4 Solution of the Y-system in the scaling limit

In this section we obtain a solution of the $\text{AdS}_4/\text{CFT}_3$ Y-system in the strong coupling scaling limit, considering the $sl(2)$ subsector. In this case, the Y-functions which correspond to the momentum-carrying roots are equal. We show that the spectrum obtained from the Y-system is in complete agreement with the results from quasiclassical string theory.

4.1 Y-system equations in the scaling limit

In the scaling limit the Y-system simplifies in several important ways. In this section and section 5 we use rescaled rapidities $z = \frac{u}{2h(\lambda)}$ (similarly to [70]), and since $h(\lambda) \rightarrow \infty$, we can neglect shifts in the arguments in the l.h.s. of the Y-system equations.¹² Hence with $1/h^2$ precision the Y-system becomes a set of algebraic, instead of functional, equations. Moreover, for the $sl(2)$ subsector $Y_{\blacktriangleleft_a} = Y_{\blacktriangleright_a}$. Also, only $z_{4,k}$ and $z_{\bar{4},k}$ Bethe roots are introduced (see (1.6)), and they coincide pairwise: $z_{4,k} = z_{\bar{4},k}$. Denoting $Y_{\bullet_a} \equiv Y_{\blacktriangleleft_a}$ we get

¹¹In [103] an attempt was also made to estimate the “critical” values of the 't Hooft coupling - values for which the equations for excited states should be modified by extra terms. The result from [103] is $\lambda_{\text{critical}} \simeq 774$. The method used in that work is based on the asymptotic solution [54] of the Y-system. The asymptotic solution works perfectly for very small and very large values of the coupling, however it very badly approximates the exact Y-functions for $\lambda \sim 700$. Thus the only reasonable estimate at the moment for the critical value is $\lambda > 700$, from the results of [65], where no singularity was found in numerical studies of the TBA equations in the range $0 < \lambda < 700$.

¹²This simplification of the Y-system involves certain subtleties, as the shifts in the argument of the Y-functions cannot be neglected close to the branch cuts. This issue can be treated in our case in the same way as in [70].

three infinite series of equations

$$Y_{\mathbb{O}_s}^2 = (1 + Y_{\mathbb{O}_{s+1}})(1 + Y_{\mathbb{O}_{s-1}}) \ , \quad s = 3, 4, \dots \ , \quad (4.1)$$

$$Y_{\Delta_a}^2 = \frac{(1 + Y_{\bullet_a})^2}{(1 + 1/Y_{\Delta_{a+1}})(1 + 1/Y_{\Delta_{a-1}})} \ , \quad a = 3, 4, \dots \ , \quad (4.2)$$

$$Y_{\bullet_a}^2 = \frac{(1 + Y_{\Delta_a})}{(1 + 1/Y_{\bullet_{a+1}})(1 + 1/Y_{\bullet_{a-1}})} \ , \quad a = 2, 3, \dots \ , \quad (4.3)$$

plus four more equations

$$Y_{\Delta_2}^2 = \frac{(1 + Y_{\oplus})(1 + Y_{\bullet_2})^2}{(1 + 1/Y_{\Delta_3})(1 + 1/Y_{\otimes})} \ , \quad (4.4)$$

$$Y_{\mathbb{O}_2}^2 = \frac{(1 + Y_{\mathbb{O}_3})(1 + Y_{\otimes})}{(1 + 1/Y_{\oplus})} \ , \quad (4.5)$$

$$Y_{\otimes}^2 = \frac{(1 + Y_{\mathbb{O}_2})(1 + Y_{\bullet_1})^2}{(1 + 1/Y_{\Delta_2})} \ , \quad (4.6)$$

$$Y_{\bullet_1}^2 = \frac{(1 + Y_{\otimes})}{(1 + 1/Y_{\bullet_2})} \ . \quad (4.7)$$

Together with the Y-system, we have to solve the non-local equation

$$\log Y_{\otimes} Y_{\oplus} = 2 \sum_{m=1}^{\infty} (\mathcal{R}_{1m}^{(01)} - \mathcal{B}_{1m}^{(01)}) * \log(1 + Y_{\bullet_m}) + 2 \log \frac{R_4^{(+)} B_4^{(-)}}{R_4^{(-)} B_4^{(+)}} \ , \quad (4.8)$$

which can be obtained by adding up (3.50) and (3.49) (it corresponds to the gray node in figure 1, right). Introducing the following notation

$$G_k(x) = \frac{1}{h} \sum_j^{M_k} \frac{1}{x - x_{k,j}} \frac{x_{k,j}^2}{x_{k,j}^2 - 1} \ , \quad k = 4, \bar{4} \quad (4.9)$$

$$f_k(z) = \exp \left(-i G_k(x(z)) \right) \ , \quad \bar{f}_k(z) = \exp \left(+i G_k(1/x(z)) \right) \ , \quad (4.10)$$

where the mirror branch of x is used for $x(z)$ and the physical branch for $x_{k,j}$ (this choice of branches is used by default in sections 4 and 5), following [70] we can write the non-local equation in the form

$$F = \frac{1}{f\bar{f}} \prod_{n=1}^{\infty} (1 + Y_{\bullet_n})^2 \quad (4.11)$$

where

$$F \equiv Y_{\oplus} Y_{\otimes} \ , \quad f(z) = f_4^2(z) \ , \quad \bar{f}(z) = \bar{f}_4^2(z) \ . \quad (4.12)$$

Note that, similarly to [70], the Bethe roots have to satisfy the constraint

$$\mathcal{Q}_1 = 2\pi m + \mathcal{O}(1/h) \ , \quad m \in \mathbb{Z} \ , \quad (4.13)$$

where

$$\mathcal{Q}_1 = \sum_{j=1}^{M_4} \frac{1}{i} \log \frac{x_{4,j}^+}{x_{4,j}^-} + \sum_{j=1}^{M_{\bar{4}}} \frac{1}{i} \log \frac{x_{\bar{4},j}^+}{x_{\bar{4},j}^-} \simeq \sum_{j=1}^{M_4} \frac{x_{4,j}}{h(x_{4,j}^2 - 1)} + \sum_{j=1}^{M_{\bar{4}}} \frac{x_{\bar{4},j}}{h(x_{\bar{4},j}^2 - 1)} \ . \quad (4.14)$$

This condition reflects the cyclicity symmetry of single trace operators. Its consistency with the other equations remains to be checked, and we assume that equation to be satisfied.

4.2 Asymptotics of Y-functions

The Y-system equations should be supplemented by boundary conditions on the functions $Y_{a,s}$, i.e. by their large a, s asymptotics. In our case, similarly to $\text{AdS}_5/\text{CFT}_4$ (see [70]), we demand that Y_{Δ_a} and Y_{\bullet_a} have the same asymptotics as the $L \rightarrow \infty$ solution, which was constructed in section 2. As for the functions Y_{\circ_s} , we demand that their large s asymptotics is polynomial in s , which is true for the $L \rightarrow \infty$ solution as well.

It is straightforward to show that the expressions for Y-functions from section 2 can be recast in the following form:

$$\mathbf{Y}_{\bullet_a} = (-1)^a \Delta^a \frac{f(\bar{f}-1)^2 \bar{f}^a - \bar{f}(f-1)^2 f^a}{f\bar{f}(\bar{f}-f)} \quad (4.15)$$

$$\mathbf{Y}_{\circ_s}(z) = (s-A)^2 - 1, \quad \mathbf{Y}_{\Delta_a} = \frac{(T-1)^2 ST^{a-1}}{(ST^{a+1}-1)(ST^{a-1}-1)}, \quad (4.16)$$

where

$$A = \frac{1}{f-1} + \frac{f}{f-1}, \quad S = \frac{\bar{f}(f-1)^2}{f(f-1)^2}, \quad T = \frac{f}{\bar{f}}, \quad (4.17)$$

$$\Delta = \exp\left(-i \frac{Lx/h - \mathcal{Q}_1}{x^2 - 1}\right) \quad (4.18)$$

while f, \bar{f} are given by (4.12). As for real z the quantity $f(z)/\bar{f}(z)$ is a pure phase, to investigate the large a, s limit we consider Y-functions of shifted argument $z - i0$. We have then $|f(z - i0)/\bar{f}(z - i0)| > 1$ and we get

$$\lim_{a \rightarrow \infty} \frac{\log \mathbf{Y}_{\Delta_a}(z - i0)}{a} = \log \frac{\bar{f}}{f}. \quad (4.19)$$

Similarly,

$$\lim_{a \rightarrow \infty} \frac{\log \mathbf{Y}_{\bullet_a}(z - i0)}{a} = \log(-\Delta f). \quad (4.20)$$

Conditions (4.19), (4.20) are the boundary conditions which we impose on the functions $Y_{\Delta_a}, Y_{\bullet_a}$ for finite L at strong coupling.

4.3 Solution in upper and right wings

In this section, we solve the Y-system partially, expressing the upper wing functions $Y_{\Delta_a}, Y_{\bullet_a}$ and the right wing functions Y_{\circ_s} in terms of only three yet unknown functions.

Using the analogy between our Y-system and the one considered in [70], it can be shown that the functions $Y_{\Delta_a}, Y_{\bullet_a}$ can be constructed in the following way:

$$1 + Y_{\Delta_a} = \frac{T_{a,0}^2}{T_{a+1,0}T_{a-1,0}}, \quad 1 + Y_{\bullet_a} = \frac{T_{a,1}^2}{T_{a+1,1}T_{a-1,1}}, \quad (4.21)$$

where the set of functions $T_{a,s}$, which is the general solution of the Hirota equation in the vertical strip, was found in [70]. Those functions are:

$$T_{a,2} = 1 \quad (4.22)$$

$$T_{a,1} = \frac{y_1 y_2}{(y_1 - y_2)(y_1 y_2 - 1)} \left(\frac{y_1}{y_1^2 - 1} \left(S_1 y_1^a + \frac{1}{S_1 y_1^a} \right) - \frac{y_2}{y_2^2 - 1} \left(S_2 y_2^a + \frac{1}{S_2 y_2^a} \right) \right) \quad (4.23)$$

$$T_{a,0} = (T_{a,1}^2 - T_{a+1,1}T_{a-1,1})/T_{a,2}. \quad (4.24)$$

with y_1, y_2, S_1 and S_2 being arbitrary parameters. The functions $Y_{\bullet_a}, Y_{\Delta_a}$, given by (4.21), satisfy the Y-system equations (4.2) and (4.3) for arbitrary $y_1(z), y_2(z), S_1(z)$ and $S_2(z)$. The asymptotic conditions (4.19), (4.20) fix y_1 and y_2 :

$$y_1 = -\frac{f_4}{\bar{f}_4}, \quad y_2 = \Delta f_4 \bar{f}_4. \quad (4.25)$$

The general solution of (4.1) with polynomial large s asymptotics is (see [70])

$$Y_{\circ_s}(z) = (s - A(z))^2 - 1, \quad (4.26)$$

with arbitrary $A(z)$. Thus, the solution of our Y-system in the upper and right wings is expressed in terms of three unknown functions $S_1(z), S_2(z)$ and $A(z)$.

4.4 Matching wings

By now, we have constructed $Y_{\circ_n}, Y_{\Delta_n}$ and Y_{\bullet_n} for all n in terms of $A(z), S_1(z)$ and $S_2(z)$. To find those three functions, as well as Y_{\oplus} and Y_{\otimes} , we have to solve the five remaining equations (4.4), (4.5), (4.6), (4.7), (4.11). Excluding Y_{\oplus} and Y_{\otimes} , we get:

$$\frac{Y_{\Delta_2}^2(1 + 1/Y_{\Delta_3})}{(1 + Y_{\bullet_2})^2} = \frac{F}{(A - 1)^2}, \quad (4.27)$$

$$\frac{1 + 1/Y_{\Delta_2}}{(1 + Y_{\bullet_1})^2} = \frac{1}{A^2} \left(\frac{(A - 1)^2}{F} - 1 \right)^2, \quad (4.28)$$

$$Y_{\bullet_1}^2(1 + 1/Y_{\bullet_2}) = -\frac{(A - 1)^2(F - 1)}{F - (A - 1)^2}, \quad (4.29)$$

$$\frac{1}{f_4^2 \bar{f}_4^2} \prod_{n=1}^{\infty} (1 + Y_{\bullet_n})^2 = F \quad (4.30)$$

The r.h.s. of the four equations above depends only on $F(z)$ and $A(z)$, and they can be solved perturbatively in Δ , like analogous equations in [70]. Namely, we find several terms in the expansion of unknown functions in powers of Δ , notice a simple relation between consecutive terms, and sum up the series in Δ assuming this relation to hold for all terms.¹³ It is then easy to check that the functions obtained in this way are indeed solutions of (4.27), (4.28), (4.29). The result is:

$$A = \frac{(1 + \Delta)((1 - f_4 \bar{f}_4)(1 - \Delta^2 f_4 \bar{f}_4) - \Delta(f_4 - \bar{f}_4)^2)((1 + f_4 \bar{f}_4)(1 + \Delta^2 f_4 \bar{f}_4) - \Delta(f_4 + \bar{f}_4)^2)}{(-1 + \Delta)(f - 1)(\bar{f} - 1)(\Delta^2 f - 1)(\Delta^2 \bar{f} - 1)} \quad (4.31)$$

$$S_1 = \frac{(f - 1)\bar{f}(\Delta^2 \bar{f} - 1)}{f(\Delta^2 f - 1)(\bar{f} - 1)} \quad (4.32)$$

$$S_2 = \frac{(f - 1)(\bar{f} - 1)}{f\bar{f}(\Delta^2 f - 1)(\Delta^2 \bar{f} - 1)} \quad (4.33)$$

$$F = \frac{(\Delta f - 1)^2(\Delta \bar{f} - 1)^2}{(\Delta - 1)^4 f \bar{f}} \quad (4.34)$$

¹³For $\Delta = 0$, there are several solutions of eqs. (4.27)–(4.29). We choose the one consistent with the asymptotic solution of Y-system.

Putting those functions into the expressions for the Y-functions (4.21), (4.26), we obtain all the $Y_{a,s}$ in terms of f, \bar{f} and Δ (using (4.7) to find Y_{\otimes} and then getting Y_{\oplus} from $F = Y_{\oplus}Y_{\otimes}$).

Recalling the definitions (4.12), (4.18) of f, \bar{f} and Δ , we see that we have found all the Y-functions in terms of the Bethe roots. We present our solution of the Y-system in *Mathematica* form in appendix C.

4.5 The spectrum from Y-system

Here we repeat the arguments of [70] to find the equation for the displacement of Bethe roots due to the finite size effects at strong coupling. In this section we assume $Y_{\blacktriangleright n} = Y_{\blacktriangleleft n} \equiv Y_{\bullet n}$ (the situation where this is not the case is considered in the next section). We again start from the TBA equation for the momentum-carrying node

$$\log Y_{\bullet 1} = \mathcal{T}_{1m} * \log(1 + Y_{\bullet m}) + \mathcal{R}^{(10)} \otimes \log(1 + Y_{\otimes}) + \mathcal{R}^{(10)} \otimes K_{m-1} * \log(1 + Y_{\Delta m}) + i\Phi, \quad (4.35)$$

where $\mathcal{T}_{1m} \equiv 2\tilde{\mathcal{S}}_{nm} - \mathcal{R}_{nm}^{(11)} + \mathcal{B}_{nm}^{(11)}$ and Φ represents extra potentials in the TBA equations for the excited states. The only difference with [70] is absence of 2's in front of the second and third terms. Thus we can use the same trick as in [70] to get the expression for $Y_{\bullet 1}$ in physical kinematics:

$$\begin{aligned} \log \frac{Y_{\bullet 1}^{\text{ph}}}{Y_{\bullet 1}^{\text{ph0}}} &= \mathcal{T}_{1m}^{\text{ph,mir}} * \log(1 + Y_{\bullet m}) + \mathcal{R}^{(10)\text{ph,mir}} \otimes \log\left(\frac{1 + Y_{\otimes}}{1 + Y_{\otimes}^0}\right) \\ &\quad + \mathcal{R}^{(10)\text{ph,mir}} \otimes K_{m-1} * \log\left(\frac{1 + Y_{\Delta m}}{1 + Y_{\Delta m}^0}\right) + K_{m-1}(z_k - \frac{i}{4h}) * \log\left(\frac{1 + Y_{\Delta m}}{1 + Y_{\Delta m}^0}\right). \end{aligned} \quad (4.36)$$

Now we simply have to expand the kernels at large h and substitute Y's. Let us denote

$$r(x, z) = \frac{x^2}{x^2 - 1} \frac{\partial_z}{2\pi h} \frac{1}{x - x(z)}, \quad u(x, z) = \frac{x}{x^2 - 1} \frac{\partial_z}{2\pi h} \frac{1}{x^2(z) - 1}.$$

We rearrange the terms in (4.36) to evaluate the following “magic” products¹⁴

$$\begin{aligned} e^{+\mathcal{M}_0} &\equiv \prod_{m=1}^{\infty} (1 + Y_{\bullet m})^{2m} = \frac{(\Delta^2 f - 1)^4 (\Delta^2 \bar{f} - 1)^4}{(\Delta - 1)^6 (\Delta + 1)^2 (\Delta f + 1)^2 (\Delta \bar{f} + 1)^2 (\Delta^2 f \bar{f} - 1)^2} \\ e^{-\mathcal{M}_+} &\equiv \frac{1 + Y_{\otimes}}{1 + Y_{\otimes}^0} \prod_{m=2}^{\infty} \left(\frac{1 + Y_{\Delta m}}{1 + Y_{\Delta m}^0}\right)^m \prod_{m=1}^{\infty} \frac{1}{(1 + Y_{\bullet m})^{2m}} = -\frac{(f\Delta + 1)^2 (f\bar{f}\Delta^2 - 1)}{(f\Delta^2 - 1)^2} \\ e^{+\mathcal{M}_-} &\equiv \frac{1 + 1/Y_{\oplus}^0}{1 + 1/Y_{\oplus}} \prod_{m=2}^{\infty} \left(\frac{1 + Y_{\Delta m}}{1 + Y_{\Delta m}^0}\right)^{m-2} = -\frac{(\bar{f}\Delta^2 - 1)^2}{(\bar{f}\Delta + 1)^2 (f\bar{f}\Delta^2 - 1)}. \end{aligned} \quad (4.37)$$

We get the following corrected Bethe equation for the \mathfrak{sl}_2 sector

$$\begin{aligned} 1 &= \left(\frac{x_k^-}{x_k^+}\right)^L \prod_{j=1}^M \frac{x_k^- - x_j^+}{x_k^+ - x_j^-} \frac{1 - 1/(x_k^+ x_j^-)}{1 - 1/(x_k^- x_j^+)} \sigma^2(z_k, z_j) \\ &\quad \times \exp\left[-\int_{-1}^1 \left(r(x_k, z)\mathcal{M}_+ - r(1/x_k, z)\mathcal{M}_- + u(x_k, z)\mathcal{M}_0\right) dz\right], \end{aligned} \quad (4.38)$$

¹⁴To compute these products we again use the $z - i0$ prescription to ensure their convergence. This prescription is inherited from the TBA equation for excited states where the integration should go slightly below the real axis.

and the equation for the energy at strong coupling is

$$E = \sum_{i=1}^M \frac{x_i^2 + 1}{x_i^2 - 1} + \int_{-1}^1 \frac{dz}{4\pi} \frac{z}{\sqrt{1-z^2}} \partial_z \mathcal{M}_0 . \quad (4.39)$$

The extra factor of 2 in the denominator under the integral is due to the single magnon dispersion relation, which includes an extra 1/2 compared to AdS₅/CFT₄.

4.6 Non-symmetric strong coupling solution

In this section we present a simple strong coupling solution with $Y_{\blacktriangleleft} \neq Y_{\blacktriangleright}$. It can be used as a test of the new structure of the Y-system which we proposed in the introduction.

We consider the limit where the massive nodes are completely decoupled from the rest of the system. We solve the following infinite set of equations

$$Y_{\blacktriangleright n} Y_{\blacktriangleleft n} = \frac{1}{(1 + 1/Y_{\blacktriangleleft n+1})(1 + 1/Y_{\blacktriangleright n-1})} , \quad n > 1 \quad (4.40)$$

$$Y_{\blacktriangleleft n} Y_{\blacktriangleright n} = \frac{1}{(1 + 1/Y_{\blacktriangleright n+1})(1 + 1/Y_{\blacktriangleleft n-1})} , \quad n > 1. \quad (4.41)$$

The explicit general solution of this system with two parameters α and β is

$$Y_{\blacktriangleleft n} = \begin{cases} \frac{(\alpha^2-1)^2 \alpha^n}{\beta \left(\alpha^n - \frac{\alpha+\beta}{\alpha\beta+1}\right) \left(\alpha + \frac{1}{\beta}\right) \left(\alpha^{n+2} - \frac{\alpha\beta+1}{\alpha+\beta}\right) (\alpha+\beta)} , & n \text{ is odd} \\ \frac{(\alpha^2-1)^2 \alpha^n}{(\alpha^n-1) \left(\alpha + \frac{1}{\beta}\right) (\alpha^{n+2}-1) (\alpha+\beta)} , & n \text{ is even} \end{cases} \quad (4.42)$$

and $Y_{\blacktriangleright n}$ is obtained from $Y_{\blacktriangleleft n}$ by replacing $\beta \rightarrow 1/\beta$. We can easily compute \mathcal{M}_0 for this solution,

$$e^{\mathcal{M}_0} = \frac{(\alpha\beta + 1)(\alpha + \beta)}{(\alpha^2 - 1)^2 \beta} \quad (4.43)$$

and by matching with the asymptotic solution we identify

$$\lambda_1 = \alpha , \quad \lambda_2 = \lambda_3 = \lambda_4 = 0 , \quad \lambda_5 = -\beta . \quad (4.44)$$

so that we get

$$e^{\mathcal{M}_0} = \frac{(\lambda_5 - \lambda_1)(1 - \lambda_1 \lambda_5)}{(\lambda_1^2 - 1)^2 \lambda_5} . \quad (4.45)$$

5 One-loop strong coupling quasi-classical string spectrum

In this section we briefly describe the construction of [70], applied for the ABJM model. The algebraic curve described in [36] can be used to compute the one-loop correction for a generic finite gap classical string state by computing the spectrum of fluctuations around a given solution. We assume that the one-loop shift computed from the algebraic curve agrees with the strong coupling expansion of ABA in the limit $L/h \gg 1$. This assumption was explicitly verified for the folded string in [104]. The general proof like in [105] is still missing.

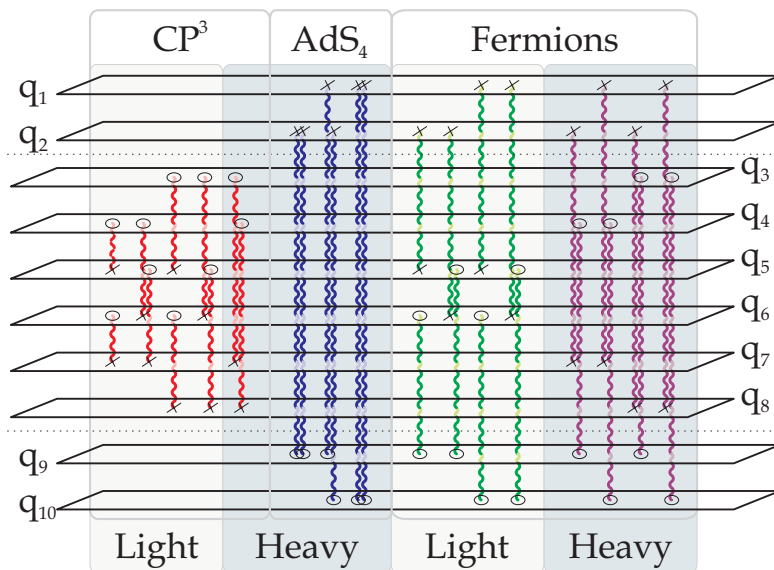


Figure 3. Elementary excitations of the string in $AdS_4 \times CP^3$.

There is yet another way to compute the one-loop shift directly from the world-sheet action which is similar to the algebraic curve computation. Whereas for the folded string both computations give the same excitation frequencies¹⁵, for the circular string a negative result was obtained in [109]. Recently it was shown that one should be more careful with the periodicity of the fermionic fields in the world-sheet approach and the corrected derivation leads to agreement with the algebraic curve frequencies [110] so we assume all approaches to be consistent with each other.

The pattern of excitations in the ABJM theory is quite different from that of $AdS_5 \times S^5$. The string in $AdS_4 \times CP^3$ has 8 bosonic (3 modes of AdS_4 and 5 of CP^3) and 8 fermionic excitations. They are divided into heavy and light modes (see figure 3). The dispersion relations for the heavy and light modes differ by a factor of two. We will see that this complicated structure of heavy and light fluctuations is captured by the Y-system. As usual the one-loop shift is given by a sum over the fluctuation energies [111]. In the algebraic curve language the fluctuations are the small cuts (i.e. poles) connecting different sheets of the algebraic curve q_i . The poles could be placed only in certain special positions x_n^{ij} given by

$$q_i(x_n^{ij}) - q_j(x_n^{ij}) = 2\pi n . \tag{5.1}$$

The quasiclassical Bohr-Sommerfeld quantization condition constrains the minimal residue of the pole. Insertion of the pole results in displacement of the other singularities. Moreover the pole by itself carries the energy $\omega(x_n^{ij})$ for the light mode and $2\omega(x_n^{ij})$ for the heavy mode where

$$\omega(x) = \frac{1}{x^2 - 1} . \tag{5.2}$$

¹⁵similar analysis for the giant magnon was done in [74, 75], McLoughlin:2008ms, Gromov:2008fy

Following [70] we first compute this second part of the one-loop shift which does not take into account the back-reaction of the fluctuation on the large cuts. Then we have

$$\delta E_{1-loop}^0 = \frac{1}{2} \sum_n \sum_{\text{light}} (-1)^{F_{ij}} \omega(x_n^{ij}) + \sum_n \sum_{\text{heavy}} (-1)^{F_{ij}} \omega(x_n^{ij}) \quad (5.3)$$

where $F_{ij} = 1$ for bosonic modes and -1 for fermionic. The modes are

$$(i, j) = (4, 5), (4, 6), (3, 5), (3, 6) \quad , \quad \text{light bosonic modes} \quad (5.4)$$

$$(i, j) = (3, 7), (2, 9), (1, 9), (1, 10) \quad , \quad \text{heavy bosonic modes} \quad (5.5)$$

$$(i, j) = (2, 5), (2, 6), (1, 5), (1, 6) \quad , \quad \text{light fermionic modes} \quad (5.6)$$

$$(i, j) = (2, 7), (1, 7), (2, 8), (1, 8) \quad , \quad \text{heavy fermionic modes} \quad (5.7)$$

Rewriting the sum (5.3) as an integral we get

$$\delta E_{1-loop}^0 = \sum_{(ij)} \oint_{\mathbb{U}^+} \frac{dx}{2\pi i} \omega(x) \partial_x \mathcal{N}_0 \quad (5.8)$$

where the integration goes over the upper half of the unit circle $|x| = 1$ and we denote

$$e^{\mathcal{N}_0} = \prod_{\text{light}} (1 - e^{-ip_i + ip_j})^{F_{ij}} \prod_{\text{heavy}} (1 - e^{-ip_i + ip_j})^{2F_{ij}} \quad (5.9)$$

Note that $\partial_x \mathcal{N}_0$ is constructed to have the residue ± 1 (± 2) exactly at the position of the light (heavy) mode x_n^{ij} . More explicitly we can write

$$e^{\mathcal{N}_0} = \frac{(\lambda_1^2 - 1)^2 (\lambda_2^2 - 1)^2 (\lambda_1 \lambda_2 - 1)^2 (\lambda_3 \lambda_4 - 1)^2 (\lambda_3 - \lambda_5) (\lambda_4 - \lambda_5) (\lambda_3 \lambda_5 - 1) (\lambda_4 \lambda_5 - 1)}{(\lambda_1 \lambda_3 - 1)^2 (\lambda_2 \lambda_3 - 1)^2 (\lambda_1 \lambda_4 - 1)^2 (\lambda_2 \lambda_4 - 1)^2 (\lambda_1 - \lambda_5) (\lambda_2 - \lambda_5) (\lambda_1 \lambda_5 - 1) (\lambda_2 \lambda_5 - 1)} \quad (5.10)$$

where

$$\lambda_a = e^{-iq_a} \quad . \quad (5.11)$$

For sl_2 sector [37] there are only cuts connecting 2nd and 9th sheets so that

$$\lambda_5 = 1 \quad , \quad \lambda_3 = \lambda_4 = \Delta \quad , \quad \lambda_2 = \Delta f \quad , \quad \lambda_2 = \Delta \bar{f} \quad (5.12)$$

and (5.10) simplifies to

$$e^{\mathcal{N}_0} = \frac{(\Delta - 1)^6 (\Delta + 1)^2 (\Delta f + 1)^2 (\Delta \bar{f} + 1)^2 (\Delta^2 f \bar{f} - 1)^2}{(\Delta^2 f - 1)^4 (\Delta^2 \bar{f} - 1)^4} \quad , \quad (5.13)$$

where we recognize $\mathcal{N}_0 = -\mathcal{M}_0$ and (5.8) coincides precisely with the second term in (4.39)!

Then one should also take into account the back-reaction of the fluctuations. As it is explained in detail in [70] for that one should work with the modified Bethe equations. We need to consider only the fluctuations touching those sheets where the macroscopic cuts are located. Hence, one of those sheets has to be the 2nd or the 9th sheet. Computing the r.h.s of (5.9) with that restriction imposed, we get, similarly to [70],

$$e^{\mathcal{N}_+} = - \frac{(\lambda_1 \lambda_2 - 1) (\lambda_2^2 - 1)^2}{(\lambda_2 \lambda_3 - 1) (\lambda_2 \lambda_4 - 1) (\lambda_2 - \lambda_5) (\lambda_2 - 1/\lambda_5)} \quad (5.14)$$

$$e^{\mathcal{N}_-} = - \frac{(\lambda_1^2 - 1)^2 (\lambda_1 \lambda_2 - 1)}{(\lambda_1 \lambda_3 - 1) (\lambda_1 \lambda_4 - 1) (\lambda_1 - \lambda_5) (\lambda_1 - 1/\lambda_5)} \quad (5.15)$$

and from (5.12) we get

$$e^{\mathcal{N}_+} = -\frac{(\Delta f + 1)^2(\Delta^2 f \bar{f} - 1)}{(\Delta^2 f - 1)^2}, \quad e^{\mathcal{N}_-} = -\frac{(\Delta \bar{f} + 1)^2(\Delta^2 f \bar{f} - 1)}{(\Delta^2 f - 1)^2} \quad (5.16)$$

and we recognize exactly the same structures (4.37) we got from solving the Y-system!

Finally by putting $\lambda_2 = \lambda_3 = \lambda_4 = 0$ we obtain

$$e^{\mathcal{N}_0} = \frac{(\lambda_1^2 - 1)^2 \lambda_5}{(\lambda_1 - \lambda_5)(\lambda_1 \lambda_5 - 1)} \quad (5.17)$$

which is again precisely the quantity $e^{-\mathcal{M}_0}$ obtained for this sector in (4.45)!

We see that the nontrivial pattern of the fluctuations is reflected in the Y-system thus providing a direct link with the worldsheet theory. This is also a deep test of the structure of the Y-system equations we proposed.

6 Conclusion

In this paper we refined the Y-system for the ABJM theory which was conjectured in [54]. We derived it directly through the TBA approach and then made several highly nontrivial tests at strong coupling. In particular we constructed the general \mathfrak{sl}_2 solution for the new Y-system in the scaling limit, and also made a test for a subsector where the difference between the old and the new Y-systems is crucial.

We also constructed the general asymptotic solution of the Y-system for arbitrary excited states. It can be used, in particular, for the weak coupling tests of the conjecture and as an initial configuration for numerical iterative solutions.

Note added. While we were working on the strong coupling solution, the paper [112] appeared, with a similar Y-system and vacuum TBA equations.

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A Notation and kernels

We use the following notation for kernels in TBA equations:

$$K_n(u, v) \equiv \frac{1}{2\pi i} \frac{\partial}{\partial v} \ln \frac{u - v + in/2}{u - v - in/2}, \quad (\text{A.1})$$

$$K_{n,m}(u, v) \equiv \sum_{j=-\frac{m-1}{2}}^{\frac{m-1}{2}} \sum_{k=-\frac{n-1}{2}}^{\frac{n-1}{2}} K_{2j+2k+2}(u, v), \quad (\text{A.2})$$

$$\mathcal{S}_{nm}(u, v) \equiv \frac{1}{2\pi i} \frac{\partial}{\partial v} \log \sigma_{BES}(x^{[+n]}(u), x^{[-n]}(u), x^{[+m]}(v), x^{[-m]}(v)) \quad (\text{A.3})$$

$$\tilde{\mathcal{S}}_{nm}(u, v) \equiv \mathcal{S}_{nm}(u, v) + \frac{ni}{2} \mathcal{P}^{(m)}(v) \quad (\text{A.4})$$

$$\mathcal{B}_{nm}^{(ab)}(u, v) \equiv \sum_{j=-\frac{n-1}{2}}^{\frac{n-1}{2}} \sum_{k=-\frac{m-1}{2}}^{\frac{m-1}{2}} \frac{1}{2\pi i} \frac{\partial}{\partial v} \log \frac{b(u + ia/2 + ij, v - ib/2 + ik)}{b(u - ia/2 + ij, v + ib/2 + ik)} \quad (\text{A.5})$$

$$\mathcal{R}_{nm}^{(ab)}(u, v) \equiv \sum_{j=-\frac{n-1}{2}}^{\frac{n-1}{2}} \sum_{k=-\frac{m-1}{2}}^{\frac{m-1}{2}} \frac{1}{2\pi i} \frac{\partial}{\partial v} \log \frac{r(u + ia/2 + ij, v - ib/2 + ik)}{r(u - ia/2 + ij, v + ib/2 + ik)}, \quad (\text{A.6})$$

where

$$b(u, v) = \frac{1/x^{\text{mir}}(u) - x^{\text{mir}}(v)}{\sqrt{x^{\text{mir}}(v)}}, \quad r(u, v) = \frac{x^{\text{mir}}(u) - x^{\text{mir}}(v)}{\sqrt{x^{\text{mir}}(v)}}, \quad (\text{A.7})$$

and

$$\mathcal{P}^{(a)}(v) = -\frac{1}{2\pi} \partial_v \log \frac{x^{\text{mir}}(v + ia/2)}{x^{\text{mir}}(v - ia/2)}. \quad (\text{A.8})$$

To derive (3.31) we use the following integral representation [60, 61]:

$$2\tilde{\mathcal{S}}_{nm}(u, v) - \mathcal{R}_{nm}^{(11)}(u, v) + \mathcal{B}_{nm}^{(11)}(u, v) = -K_{n,m}(u - v) \quad (\text{A.9})$$

$$- 2 \sum_{a=1}^{\infty} \int \left[\mathcal{B}_{n1}^{(10)}(u, w + ia/2) \mathcal{B}_{1m}^{(01)}(w - ia/2, v) + \mathcal{B}_{n1}^{(10)}(u, w - ia/2) \mathcal{B}_{1m}^{(01)}(w + ia/2, v) \right] dw.$$

Equation (3.31) follows due to the identity $\mathcal{R}_{nm}^{(11)}(u, v) + \mathcal{B}_{nm}^{(11)}(u, v) = K_{n,m}(u - v)$.

B Fermionic duality transformation and $\mathfrak{su}(2)$

We can transform a set of Bethe equations into an equivalent one by application of the fermionic duality. This follows [34] closely. We construct the polynomial

$$P(x) = \prod_{j=1}^{K_4} (x - x_{4,j}^+) \prod_{j=1}^{K_{\bar{4}}} (x - x_{4,j}^+) \prod_{j=1}^{K_2} (x - x_{2,j}^-) (x - 1/x_{2,j}^-)$$

$$- \prod_{j=1}^{K_4} (x - x_{4,j}^-) \prod_{j=1}^{K_{\bar{4}}} (x - x_{4,j}^-) \prod_{j=1}^{K_2} (x - x_{2,j}^+) (x - 1/x_{2,j}^+) \quad (\text{B.1})$$

from the Bethe equations of [37] (given in section 2.1 of the present work) for the fermionic roots u_1 and u_3 . We see that this polynomial has zeros for $x = 1/x_{1,j}$ and $x = x_{3,j}$. Denoting the remaining zeros of this polynomial by \tilde{x} , we get

$$P(x) = C \prod_{j=1}^{K_3} (x - x_{3,j}) \prod_{j=1}^{K_1} (x - 1/x_{1,j}) \prod_{j=1}^{\tilde{K}_3} (x - \tilde{x}_{3,j}) \prod_{j=1}^{\tilde{K}_1} (x - 1/\tilde{x}_{1,j}) \quad (\text{B.2})$$

(where C is a constant) or in our notation (with $R \equiv R_4 R_{\bar{4}}$, $B \equiv B_4 B_{\bar{4}}$)

$$P(x) = C R_3 B_1 R_{\bar{3}} B_{\bar{1}} = \left[R^{(-)} Q_2^+ - R^{(+)} Q_2^- \right] \left(\frac{x}{h} \right)^{K_2} \prod_{j=1}^{K_4} \sqrt{x_{4,j}^+} \quad (\text{B.3})$$

$$\frac{P(x^+)}{P(x^-)} = \frac{R_3^+ B_1^+ R_{\bar{3}}^+ B_{\bar{1}}^+}{R_3^- B_1^- R_{\bar{3}}^- B_{\bar{1}}^-} = \left(\frac{x^+}{x^-} \right)^{K_2} \frac{R^{(-)+} Q_2^{++} - R^{(+)+} Q_2^-}{R^{(-)-} Q_2^- - R^{(+)-} Q_2^{--}} \quad (\text{B.4})$$

and

$$\frac{P(1/x^-)}{P(1/x^+)} = \frac{B_3^- R_1^- B_{\bar{3}}^- R_{\bar{1}}^-}{B_3^+ R_1^+ B_{\bar{3}}^+ R_{\bar{1}}^+} = \left(\frac{x^+}{x^-} \right)^{K_2} \frac{B^{(-)-} Q_2^- - B^{(+)-} Q_2^{--}}{B^{(-)+} Q_2^{++} - B^{(+)+} Q_2^-} \quad (\text{B.5})$$

then

$$f(u) = - \left(\frac{x^+}{x^-} \right)^{K_2} \frac{R^{(+)+} B_1^- \tilde{B}_1^- B_3^+ \tilde{B}_3^+}{R^{(-)-} B_1^+ \tilde{B}_1^+ B_3^- \tilde{B}_3^-}, \quad f_a(u) \equiv \prod_{n=-\frac{a-1}{2}}^{\frac{a-1}{2}} f(u + in) \quad (\text{B.6})$$

$$T_{a,1}(u | \{u_{1,j}\}, \{u_{3,j}\}) = f_a(u) \overline{T_{1,a}(u | \{\tilde{u}_{1,j}\}, \{\tilde{u}_{3,j}\})} \quad (\text{B.7})$$

$$T_{1,s}(u | \{u_{1,j}\}, \{u_{3,j}\}) = f_s(u) \overline{T_{s,1}(u | \{\tilde{u}_{1,j}\}, \{\tilde{u}_{3,j}\})} \quad (\text{B.8})$$

Here, the bar means complex conjugation in the physical sense, i.e. the replacement

$$R^{(\pm)\pm} \rightarrow R^{(\mp)\mp}, \quad B^{(\pm)\pm} \rightarrow B^{(\mp)\mp}. \quad (\text{B.9})$$

Notice that x is not inverting under this conjugation.

$$\mathbf{Y}_{\blacktriangleleft a} \simeq \left(\frac{x^{[-a]}}{x^{[+a]}} \right)^{L-K_2} \overline{\mathbf{T}_{a,1}} \prod_{n=-\frac{a-1}{2}}^{\frac{a-1}{2}} \tilde{\Phi}_4^{\theta_{na}^E}(u+in) \tilde{\Phi}_4^{\theta_{na}^O}(u+in), \quad (\text{B.10})$$

$$\mathbf{Y}_{\blacktriangleright a} \simeq \left(\frac{x^{[-a]}}{x^{[+a]}} \right)^{L-K_2} \overline{\mathbf{T}_{a,1}} \prod_{n=-\frac{a-1}{2}}^{\frac{a-1}{2}} \tilde{\Phi}_4^{\theta_{na}^O}(u+in) \tilde{\Phi}_4^{\theta_{na}^E}(u+in) \quad (\text{B.11})$$

As in section 2, the factors $\tilde{\Phi}_4(u)$ and $\tilde{\Phi}_{\bar{4}}(u)$ are constructed in such a way that the ABA equations for the momentum carrying nodes are given by $\mathbf{Y}_{\blacktriangleleft 1}^{\text{ph}}(u_{4,j}) = -1$ and $\mathbf{Y}_{\blacktriangleright 1}^{\text{ph}}(u_{\bar{4},j}) = -1$. We have

$$\tilde{\Phi}_4(u) = -S_4 S_{\bar{4}} \frac{B_4^{(+)+} R_4^{(-)-} \tilde{B}_1^- \tilde{B}_3^+}{B_4^{(-)-} R_4^{(+)+} \tilde{B}_1^+ \tilde{B}_3^-} e^{i\mathcal{Q}_1/2}, \quad \tilde{\Phi}_{\bar{4}}(u) = -S_4 S_4 \frac{B_{\bar{4}}^{(+)+} R_{\bar{4}}^{(-)-} \tilde{B}_1^- \tilde{B}_3^+}{B_{\bar{4}}^{(-)-} R_{\bar{4}}^{(+)+} \tilde{B}_1^+ \tilde{B}_3^-} e^{i\mathcal{Q}_1/2}. \quad (\text{B.12})$$

where \tilde{B}_l, \tilde{R}_l are defined similarly to (2.8), (2.9), with $x_{l,j}$ replaced by $\tilde{x}_{l,j}$.

C Explicit expressions for Y-functions

Here we present the solution of the Y-system in the scaling limit for the $sl(2)$ sector. This solution was obtained in section 4, and below we give its explicit form, which can be used in the *Mathematica* system. We denote $d = \Delta$, $g = f_4$, $gb = \bar{f}_4$, $Ym[a_] = Y_{\bullet a}$, $Yp[a_] = Y_{\Delta a}$, $Yb[s_] = Y_{\circ s}$, and the Y-functions are given by the following code:

```

sb={
A-> ((1+d)(1-d g^-2-g gb+2 d g gb-d^2 g gb-d gb^2+d^2 g^2 gb^2)
      (1-d g^2+g gb-2 d g gb+d^2 g gb-d gb^2+d^2 g^2 gb^2))/
      ((-1+d)(-1+g)(1+g)(-1+d g)(1+d g)(-1+gb)(1+gb)(-1+d gb)(1+d gb)),
S1->((-1+g)(1+g)gb^2(-1+d gb)(1+d gb))/(g^2(-1+d g)(1+d g)(-1+gb)(1+gb)),
S2->((-1+g)(1+g)(-1+gb)(1+gb))/(g^2(-1+d g)(1+d g)gb^2(-1+d gb)(1+d gb)),
P-> ((-1+d g^2)^2(-1+d gb^2)^2)/((-1+d)^4g^2gb^2),
T2->d g gb, T1->-(g/gb)};

Ym[a_]=-1+(S2 T2^(1+a)(-1+T2^2)-S1^2 S2 T1^(4+2a)T2^(1+a)(-1+T2^2)+
S1 T1^(1+a)(-1+T1^2)(-1+S2^2 T2^(4+2a)))^2/
((-S2 T2^a (-1+T2^2)+S1^2 S2 T1^(2+2 a)T2^a (-1+T2^2)-
S1 T1^a (-1+T1^2)(-1+S2^2 T2^(2+2 a)))(-S2 T2^(2+a)(-1+T2^2)+
S1^2 S2 T1^(6+2 a) T2^(2+a)(-1+T2^2)-
S1 T1^(2+a)(-1+T1^2)(-1+S2^2 T2^(6+2 a))))/.sb;

Yp[a_]=((S1 T1^(4+a) T2)/(-1+T1^2)+(T1^-a T2)/(S1-S1 T1^2)-
(T1(T2^-a-S2^2 T2^(4+a)))/(S2 - S2 T2^2))^2/(-((S1 T1^(4+a)T2)/
(-1+T1^2)+(T1^-a T2)/(S1 - S1 T1^2)-(T1(T2^-a-S2^2 T2^(4+a)))/
(S2-S2 T2^2))^2+(T1^(-2 a)T2^(-2 a)(T2^2-S2^2 T2^(4+2 a)-
2 S1 S2 T1^(2+a)T2^(2+a)(-1+T2^2)+2 S1 S2 T1^(4+a)T2^(2+a)(-1+T2^2)+
S1^2 T1^(6+2 a)T2^2(-1+S2^2 T2^(2+2 a))+2 T1 T2 (-1+S2^2 T2^(4+2 a))-
2 S1^2 T1^(5+2 a)T2(-1+S2^2 T2^(4+2 a))+T1^2 (1-S2^2 T2^(6+2 a))+
S1^2 T1^(4+2 a)(-1+S2^2 T2^(6+2 a)))^2)/(S1^2 S2^2 (-1+T1^2)^2
(-1+T2^2)^2(T2+T1^2 T2-T1(1+T2^2))^2))/.sb;

Yb[s_]=(s-A)^2-1/.sb;

Y11=(-1+(S1 S2(-1+T1^2)(-1+T2^2)(T2^2-S2^2 T2^6-2 S1 S2 T1^3 T2^3 (-1+T2^2)+
2 S1 S2 T1^5 T2^3(-1+T2^2)+S1^2 T1^8 T2^2(-1+S2^2 T2^4)+
2 T1 T2(-1+S2^2 T2^6)-2 S1^2 T1^7 T2(-1+S2^2 T2^6)+T1^2 (1-S2^2 T2^8)+
S1^2 T1^6 (-1+S2^2 T2^8))^2)/((S1^2 S2 T1^4 T2 (-1+T2^2)+S2 (T2-T2^3)-
S1 T1 (-1+T1^2)(-1+S2^2 T2^4))^2 (T2^2-S2^2 T2^8-2 S1 S2 T1^4 T2^4 (-1+T2^2)+
2 S1 S2 T1^6 T2^4 (-1+T2^2)+S1^2 T1^10 T2^2 (-1+S2^2 T2^6)+2 T1 T2 (-1+S2^2 T2^8)-
2 S1^2 T1^9 T2 (-1+S2^2 T2^8)+T1^2 (1-S2^2 T2^10)+S1^2 T1^8 (-1+S2^2 T2^10))))/.sb;

Y22=(P/Y11)/.sb;

```

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