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On finite-times degenerate Cauchy numbers and polynomials

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Abstract

Cauchy polynomials are also called Bernoulli polynomials of the second kind and these polynomials are very important to study mathematical physics. Kim *et al.* have studied some properties of Bernoulli polynomials of the second kind associated with special polynomials arising from umbral calculus. Kim introduced the degenerate Cauchy numbers and polynomials which are derived from the degenerate function e^t. In this paper, we try to degenerate Cauchy numbers and polynomials k-times and investigate some properties of these k-times degenerate Cauchy numbers and polynomials.

MSC: 11B68; 11S40; 11S80

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1 Introduction

In [1], Comtet introduced Cauchy numbers, denoted by C_n, by the integral of the following formula:

$$\begin{aligned} C_n &= \int_0^1 (x)_n\,dx = \int_0^1 x(x-1)\cdots(x-n+1)\,dx \\ &= n!\int_0^1 \binom{x}{n}\,dx. \end{aligned} \tag{1}$$

From (1), we can derive the generating function as follows:

$$\begin{aligned} \sum_{n=0}^{\infty}\frac{C_n}{n!}t^n &= \sum_{n=0}^{\infty}\int_0^1\binom{x}{n}\,dx\,t^n = \int_0^1\sum_{n=0}^{\infty}\binom{x}{n}t^n\,dx \\ &= \int_0^1(1+t)^x\,dx = \frac{t}{\log(1+t)} \quad \text{(see [1–6])}. \end{aligned} \tag{2}$$

Also we have

$$C_n = \int_0^1 (x)_n\,dx = \sum_{n_0=0}^{n} S_1(n,n_0)\frac{1}{n_0+1}. \tag{3}$$

In [7], Kim introduced a new class of numbers and polynomials which are called the *degenerate Cauchy numbers* and *polynomials*, denoted by $C_{n,\lambda}$ and $C_{n,\lambda}(x)$, respectively, as follows:

$$\begin{aligned}\int_0^1 \left(1+\log(1+\lambda t)^{\frac{1}{\lambda}}\right)^{x+y} dy &= \frac{\log(1+\lambda t)^{\frac{1}{\lambda}}}{\log(1+\log(1+\lambda t)^{\frac{1}{\lambda}})}\left(1+\log(1+\lambda t)^{\frac{1}{\lambda}}\right)^x \\ &= \sum_{n=0}^{\infty} C_{n,\lambda}(x)\frac{t^n}{n!}. \end{aligned} \tag{4}$$

When $x=0$, $C_{n,\lambda}=C_{n,\lambda}(0)$ are called the *degenerate Cauchy numbers*.

The degenerate Cauchy numbers above are degenerated 1-time from Cauchy numbers C_n, and we denote these numbers by $C_{n,\lambda}^{(1)}$.

For $r \in \mathbb{N}$, the *Bernoulli polynomials of order r* are defined by the generating function

$$\left(\frac{t}{e^t-1}\right)^r e^{xt} = \sum_{n=0}^{\infty} B_n^{(r)}(x)\frac{t^n}{n!} \quad \text{(see [1–6, 8–13])}. \tag{5}$$

When $x=0$, $B_n^{(r)}=B_n^{(r)}(0)$ are called *Bernoulli numbers of order r*. Thus, by (5), we get

$$B_n^{(r)}(x) = \sum_{l=0}^{n}\binom{n}{l}B_l^{(r)}x^{n-l} \quad \text{(see [1–6, 8–13])}. \tag{6}$$

In particular, for $r=1$, $B_n(x)=B_n^{(1)}(x)$ are called *ordinary Bernoulli polynomials*. As is well known, we have

$$\left(\frac{t}{\log(1+t)}\right)^r (1+t)^{x-1} = \sum_{n=0}^{\infty} B_n^{(n-r+1)}(x)\frac{t^n}{n!} \quad \text{(see [11])}. \tag{7}$$

It is well known that

$$e^t = \lim_{\lambda\to 0}(1+\lambda t)^{\frac{1}{\lambda}} \quad \text{(see [1, 4, 10])}.$$

The function $(1+\lambda t)^{\frac{1}{\lambda}}$ is called the *degenerate function of* e^t. So, for $t=\log e^t$, we have $\log(1+\lambda t)^{\frac{1}{\lambda}}$ as the degenerating function. The first quadrant of the following diagram comes from [7].

We extend Kim's idea (see [7]) to the k-times degenerate of $t=\log e^t$, then we have $\underbrace{\log(1+\log(1+\cdots+\log(1+\lambda t)\cdots))}_{k\text{-logarithms}}{}^{\frac{1}{\lambda}}$ as the k-times degenerating function, see Figure 1.

In [14], Carlitz introduced the degenerate Bernoulli polynomials and explained a degenerate Staudt-Clausen theorem. Later Ustinov studied the same concept using the name of Korobov polynomials of the second kind.

Recently, Kim and Kim introduced Daehee polynomials with λ-parameter for $\lambda \in \mathbb{C}$. When we approach λ to zero, both the degenerate Bernoulli polynomials and the Daehee polynomials with λ-parameter approach the well-known Bernoulli polynomials. We find that the differences are on the numerators of the degenerate Bernoulli and Daehee polynomials with λ-parameter.

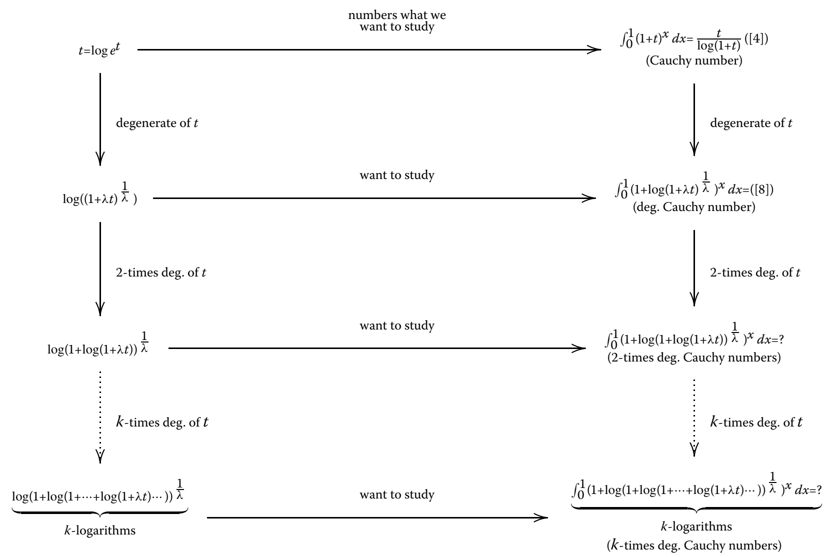

Figure 1 ***k*-times degenerate Cauchy numbers via consecutive degenerating variable *t*.**

By the use of p-adic invariant integral on $\mathbb{Z}_p$, the numerator of the Daehee polynomials with λ-parameter becomes much more natural (see [15]).

On the other hand, if we take $\lambda = 1$ on the Daehee polynomials with λ-parameter, we have the generating function of the Daehee polynomials. If we take $x = 0$, then we have the generating function of the Daehee numbers, which is the multiplicative inverse of the generating function of the Bernoulli numbers of the second kind (see [16]). The higher order Daehee numbers and polynomials are investigated by Kim *et al.* in [17].

Cauchy polynomials are also called Bernoulli polynomials of the second kind and these polynomials are very important to study mathematical physics. (See [18, 19].) In [18, 19], Kim *et al.* have studied some properties of Bernoulli polynomials of the second kind associated with special polynomials arising from umbral calculus.

Kim introduced the degenerate Cauchy numbers and polynomials which are derived from the degenerate function e^t (see [7]). In this paper, we try to degenerate Cauchy numbers and polynomials k-times and investigate some properties of these k-times degenerate Cauchy numbers and polynomials.

For the application, we consider bosonic p-adic integration on $\mathbb{Z}_p$ for the same integrand function $f(x) = (1+t)^x$, then we have Daehee numbers which are defined in [16]. Also, by considering the degenerate of t as above figure, we have degenerate Daehee numbers and polynomials. Thus we can apply our idea in this paper to k-times degenerate Daehee number and polynomials. Then we can get some new identities on these numbers and polynomials.

In [20], Kim *et al.* introduced the q-analog of the Daehee numbers and polynomials which are called the q-Daehee numbers and polynomials. By using the p-adic q-integration, the q-Daehee polynomials $D_{n,q}(x)$ are defined and studied by Kim and Kim (see [16]).

We also apply our idea to k-times degenerate q-Daehee numbers and polynomials. We can get interesting identities on these numbers and polynomials.

Meanwhile, the degenerate Bernoulli polynomials diverged several ways for further study. One way is the q-analog of degenerate Bernoulli polynomials by using the p-adic q-integral on $\mathbb{Z}_p$ by Kim *et al.* Another way is considered and studied the partially degenerate Bernoulli numbers and polynomials of the first kind and second kind by Kim and Seo in [21]. The partially degenerate one uses generating function concept on complex plane. The hidden idea of this lies on the p-adic invariant integral on $\mathbb{Z}_p$. From this idea, Lim studied partially degenerate Daehee numbers and polynomials (see [15]).

We can apply k-times partially degenerate idea to such partially degenerate numbers and polynomials, and we can get interesting identities.

2 *k*-Times degenerate Cauchy numbers

In [7], Kim considered the degenerate Cauchy numbers which are defined by the generating function

$$\begin{aligned}\sum_{n=0}^{\infty} C_{n,\lambda}^{(1)} \frac{t^n}{n!} &= \frac{\log(1+\lambda t)^{\frac{1}{\lambda}}}{\log(1+\log(1+\lambda t)^{\frac{1}{\lambda}})} \\ &= \int_0^1 \bigl(1+\log(1+\lambda t)^{\frac{1}{\lambda}}\bigr)^x \, dx \quad \text{(see [7])}. \end{aligned} \tag{8}$$

The following is shown in [7], Corollary 2.2.

Lemma 1 *For $n_0, n_1, n_2 \geq 0$, we have*

$$\text{(i)} \quad C_{n_2,\lambda}^{(1)} = \sum_{n_1=0}^{n_2} \lambda^{n_2-n_1} S_1(n_2,n_1) C_{n_1},$$

$$\text{(ii)} \quad C_{n_2,\lambda}^{(1)} = \sum_{n_1=0}^{n_2} \sum_{n_0=0}^{n_1} \lambda^{n_2-n_1} S_1(n_2,n_1) S_1(n_1,n_0) \frac{1}{n_0+1}.$$

We degenerate $C_{n_2,\lambda}^{(1)}$ one more time, *i.e.*, we get 2-times degenerate Cauchy numbers. Then the left-hand side of (8) becomes as follows:

$$\begin{aligned}&\sum_{n_2=0}^{\infty} C_{n_2,\lambda}^{(1)} \frac{1}{n_2!} \lambda^{-n_2} \bigl(\log(1+\lambda t)\bigr)^{n_2} \\ &= \sum_{n_2=0}^{\infty} C_{n_2,\lambda}^{(1)} \lambda^{-n_2} \sum_{n_3=n_2}^{\infty} S_1(n_3,n_2) \frac{\lambda^{n_3} t^{n_3}}{n_3!} \\ &= \sum_{n_3=0}^{\infty} \left(\sum_{n_2=0}^{n_3} C_{n_2,\lambda}^{(1)} \lambda^{n_3-n_2} S_1(n_3,n_2) \right) \frac{t^{n_3}}{n_3!} \\ &= \sum_{n_3=0}^{\infty} \sum_{n_2=0}^{n_3} \left(\sum_{n_1=0}^{n_2} \lambda^{n_2-n_1} S_1(n_2,n_1) C_{n_1} \right) \lambda^{n_3-n_2} S_1(n_3,n_2) \frac{t^{n_3}}{n_3!} \\ &= \sum_{n_3=0}^{\infty} \sum_{n_2=0}^{n_3} \sum_{n_1=0}^{n_2} \lambda^{n_3-n_1} S_1(n_3,n_2) S_1(n_2,n_1) C_{n_1} \frac{t^{n_3}}{n_3!}. \end{aligned} \tag{9}$$

On the other hand, the right-hand side of (8) becomes as follows:

$$\begin{aligned}\frac{\log(1+\log(1+\lambda t))^{1/\lambda}}{\log(1+(\log(1+\log(1+\lambda t)))^{1/\lambda})} &= \int_0^1 \bigl(1+\log\bigl(1+\log(1+\lambda t)\bigr)^{1/\lambda}\bigr)^x\,dx \\ &= \sum_{n_3=0}^{\infty} C_{n_3,\lambda}^{(2)} \frac{t^{n_3}}{n_3!}. \end{aligned} \tag{10}$$

Thus, comparing the coefficients of (9) and (10), we have

$$\begin{aligned} C_{n_3,\lambda}^{(2)} &= \sum_{n_2=0}^{n_3} C_{n_2,\lambda}^{(1)} \lambda^{n_3-n_2} S_1(n_3,n_2) \\ &= \sum_{n_2=0}^{n_3}\sum_{n_1=0}^{n_2} \lambda^{n_3-n_1} S_1(n_3,n_2) S_1(n_2,n_1) C_{n_1}. \end{aligned} \tag{11}$$

Now, replacing C_{n_1} in (11) by $\sum_{n_0=0}^{n_1} S_1(n_1,n_0)\frac{1}{n_0+1}$ in (3), we get

$$C_{n_3,\lambda}^{(2)} = \sum_{n_2=0}^{n_3}\sum_{n_1=0}^{n_2}\sum_{n_0=0}^{n_1} \lambda^{n_3-n_1} S_1(n_3,n_2)S_1(n_2,n_1)S_1(n_1,n_0)\frac{1}{n_0+1}. \tag{12}$$

Inductively we get the k-times degenerate Cauchy numbers $C_{n_{k+1},\lambda}^{(k)}$ as follows.

Theorem 1 *For $n_i \geq 0$, for each $i = 0,1,\ldots,k+1$, we have*

$$\begin{aligned} C_{n_{k+1},\lambda}^{(k)} &= \sum_{n_k=0}^{n_{k+1}} \cdots \sum_{n_1=0}^{n_2} \lambda^{n_{k+1}-n_1} \left(\prod_{j=1}^{k} S_1(n_{j+1},n_j)\right) C_{n_1} \\ &= \sum_{n_k=0}^{n_{k+1}} \cdots \sum_{n_0=0}^{n_1} \lambda^{n_{k+1}-n_1} \left(\prod_{j=0}^{k} S_1(n_{j+1},n_j)\right) \frac{1}{n_0+1}. \end{aligned} \tag{13}$$

By replacing t by $\frac{1}{\lambda}(e^{\lambda t}-1)$ in (10), we get

$$\begin{aligned} \frac{\log(1+\lambda t)^{\frac{1}{\lambda}}}{\log(1+\log(1+\lambda t)^{\frac{1}{\lambda}})} &= \int_0^1 \bigl(1+\log(1+\lambda t)^{\frac{1}{\lambda}}\bigr)^x\,dx \\ &= \sum_{n=0}^{\infty} C_{n,\lambda}^{(2)} \frac{1}{n!}\left(\frac{1}{\lambda}\bigl(e^{\lambda t}-1\bigr)\right)^n. \end{aligned} \tag{14}$$

By (8), we have

$$\begin{aligned} \sum_{n=0}^{\infty} C_{n,\lambda}^{(1)} \frac{t^n}{n!} &= \sum_{n=0}^{\infty} C_{n,\lambda}^{(2)} \lambda^{-n} \frac{(e^{\lambda t}-1)^n}{n!} \\ &= \sum_{n=0}^{\infty} C_{n,\lambda}^{(2)} \lambda^{-n} \sum_{m=n}^{\infty} S_2(m,n)\frac{\lambda^m t^m}{m!} \\ &= \sum_{m=0}^{\infty} \left(\sum_{n=0}^{m} C_{n,\lambda}^{(2)} \lambda^{m-n} S_2(m,n)\right)\frac{t^m}{m!}, \end{aligned} \tag{15}$$

where $S_2(m,n)$ is the Stirling number of the second kind. Thus comparing the coefficients of (15), we have the inversion of (11) as follows:

$$C^{(1)}_{m,\lambda} = \sum_{n=0}^{m} C^{(2)}_{n,\lambda}\lambda^{m-n} S_2(m,n). \tag{16}$$

Now applying one more time the above process, we obtain the following:

$$C^{(1)}_{n_2,\lambda} = \sum_{n_3=0}^{n_2}\sum_{n_4=0}^{n_3} C^{(3)}_{n_4,\lambda}\lambda^{n_2-n_4} S_2(n_2,n_3)S_2(n_3,n_4).$$

Inductively, we have the following identity.

Theorem 2 *For $n_i \geq 0$, where $i = 1,\dots,k+1$, we have*

$$C^{(1)}_{n_2,\lambda} = \sum_{n_3=0}^{n_2}\sum_{n_4=0}^{n_3}\cdots\sum_{n_{k+1}=0}^{n_k} C^{(k)}_{n_{k+1},\lambda}\lambda^{n_2-n_{k+1}} \prod_{i=2}^{k} S_2(n_i,n_{i+1}).$$

3 *k*-Times degenerate Cauchy polynomials

We recall that Cauchy polynomials $C_n(x)$ are defined by the generating function as follows:

$$\sum_{n=0}^{\infty} C_n(x)\frac{t^n}{n!} = \frac{t}{\log(1+t)}(1+t)^x = \int_0^1 (1+t)^{x+y}\,dy \quad \text{(see [1–6])}.$$

In [7], Kim defined the degenerate Cauchy polynomials by the generating function as the form in (4), and in this paper we shall denote these polynomials by $C^{(1)}_{n,\lambda}(x)$.

The following observation has been made in [7], Theorem 2.1 and Theorem 2.3.

Theorem 3 *For $n \geq 0$, we have*

$$\begin{aligned} C^{(1)}_{n,\lambda}(x) &= \sum_{l=0}^{n} \lambda^{n-l} S_1(n,l)\int_0^1 (x+y)_l\,dy \\ &= \sum_{l=0}^{n} \lambda^{n-l} S_1(n,l) C_l(x) \\ &= \sum_{l=0}^{n}\sum_{m=0}^{l} \lambda^{n-l} S_1(n,l)\binom{l}{m}(x)_m C_{l-m}. \end{aligned} \tag{17}$$

We degenerate $C^{(1)}_{n,\lambda}(x)$ one more time, *i.e.*, we get 2-times degenerate Cauchy polynomials $C_n(x)$. We have the following identity:

$$\begin{aligned} &\int_0^1 \bigl(\bigl(1+\log\bigl(1+\log(1+\lambda t)\bigr)\bigr)^{\frac{1}{\lambda}}\bigr)^{x+y}\,dy \\ &\quad = \frac{\log(1+\log(1+\lambda t))^{\frac{1}{\lambda}}}{\log(1+\log(1+\log(1+\lambda t))^{\frac{1}{\lambda}})}\bigl(1+\log\bigl(1+\log(1+\lambda t)\bigr)^{\frac{1}{\lambda}}\bigr)^x \\ &\quad = \sum_{n=0}^{\infty} C^{(2)}_{n,\lambda}(x)\frac{t^n}{n!}. \end{aligned} \tag{18}$$

Then the middle term of (18) becomes

$$\begin{aligned}
&\frac{\log(1+\log(1+\lambda t))^{1/\lambda}}{\log(1+(\log(1+\log(1+\lambda t)))^{1/\lambda})}\bigl(1+\log\bigl(1+\log(1+\lambda t)\bigr)^{1/\lambda}\bigr)^{x}\\
&=\left(\sum_{l=0}^{\infty}C_{l,\lambda}^{(2)}\frac{t^l}{l!}\right)\left(\sum_{m=0}^{\infty}(x)_m\bigl[\log\bigl(1+\log(1+\lambda t)\bigr)^{1/\lambda}\bigr]^m\frac{1}{m!}\right)\\
&=\left(\sum_{l=0}^{\infty}C_{l,\lambda}^{(2)}\frac{t^l}{l!}\right)\left(\sum_{m=0}^{\infty}(x)_m\sum_{j=m}^{\infty}\lambda^{-m}S_1(j,m)\frac{(\log(1+\lambda t))^j}{j!}\right)\\
&=\left(\sum_{l=0}^{\infty}C_{l,\lambda}^{(2)}\frac{t^l}{l!}\right)\left(\sum_{m=0}^{\infty}\sum_{j=0}^{m}(x)_m\lambda^{-m}S_1(j,m)S_1(i,j)\sum_{i=j}^{\infty}\frac{\lambda^i t^i}{i!}\right)\\
&=\left(\sum_{l=0}^{\infty}C_{l,\lambda}^{(2)}\frac{t^l}{l!}\right)\left(\sum_{i=0}^{\infty}\sum_{j=0}^{i}\sum_{m=0}^{j}(x)_m\lambda^{i-m}S_1(j,m)S_1(i,j)\frac{t^i}{i!}\right)\\
&=\sum_{n=0}^{\infty}\left(\sum_{i=0}^{n}\binom{n}{i}\sum_{j=0}^{i}\sum_{m=0}^{j}(x)_m\lambda^{i-m}S_1(j,m)S_1(i,j)C_{n-i,\lambda}^{(2)}\right)\frac{t^n}{n!}.
\end{aligned}\tag{19}$$

Hence, from (19), we can rewrite 2-times degenerate Cauchy polynomials as follows.

For each $n_i\in\mathbb{Z}_+$, where $i=0,1,2,\ldots,$

$$\begin{aligned}
C_{n_3,\lambda}^{(2)}(x)={}&\sum_{n_2=0}^{n_3}\sum_{n_1=0}^{n_2}\sum_{n_0=0}^{n_1}\binom{n_3}{n_2}(x)_{n_0}\lambda^{n_2-n_1}\\
&\times S_1(n_3,n_2)S_1(n_2,n_1)S_1(n_1,n_0)C_{n_3-n_2,\lambda}^{(2)}.
\end{aligned}\tag{20}$$

We combine (20) and Theorem 1, and we have

$$\begin{aligned}
C_{n_3,\lambda}^{(2)}(x)={}&\sum_{n_2=0}^{n_3}\sum_{n_1=0}^{n_2}\sum_{n_0=0}^{n_1}\binom{n_3}{n_2}(x)_{n_0}\lambda^{n_2-n_1}S_1(n_3,n_2)S_1(n_2,n_1)S_1(n_1,n_0)\\
&\times\sum_{m_2=0}^{n_3-n_2}\sum_{m_1=0}^{m_2}\Biggl(\lambda^{n_3-n_2-m_2}S_1(n_3-n_2,m_2)S_1(m_2,m_1)\\
&\times\sum_{m_0=0}^{m_1}S_1(m_1,m_0)\frac{1}{m_0+1}\Biggr).
\end{aligned}\tag{21}$$

Inductively, we have k-times degenerate Cauchy polynomials as follows.

Theorem 4 *For each $n_i\in\mathbb{Z}_+$, where $i=0,1,\ldots,k+1$, we have*

$$\begin{aligned}
C_{n_{k+1},\lambda}^{(k)}(x)&=\sum_{n_k=0}^{n_{k+1}}\cdots\sum_{n_0=0}^{n_1}\binom{n_{k+1}}{n_k}(x)_{n_0}\lambda^{n_k-n_1}\prod_{j=0}^{k}S_1(n_{j+1},n_j)C_{n_{k+1}-n_k,\lambda}^{(k)}\\
&=\sum_{n_k=0}^{n_{k+1}}\cdots\sum_{n_0=0}^{n_1}\binom{n_{k+1}}{n_k}(x)_{n_0}\lambda^{n_k-n_1}\prod_{j=0}^{k}S_1(n_{j+1},n_j)\times\alpha,
\end{aligned}$$

where

$$\alpha = \sum_{m_k=0}^{n_{k+1}-n_k} \sum_{m_{k-1}=0}^{m_k} \cdots \sum_{m_0=0}^{m_1} \lambda^{n_{k+1}-n_k-m_k} S_1(n_{k+1}-n_k, m_k) \prod_{i=0}^{k-1} S_1(m_{i+1}, m_i) \frac{1}{m_0+1}.$$

Observing the left-hand side of (18), we have the following:

$$\begin{aligned}
&\int_0^1 \bigl(1+\log\bigl(1+\log(1+\lambda t)\bigr)^{\frac{1}{\lambda}}\bigr)^{x+y}\,dy \\
&= \sum_{l=0}^{\infty} \int_0^1 (x+y)_l\,dy \frac{1}{l!}\bigl(\log\bigl(1+\log(1+\lambda t)\bigr)^{\frac{1}{\lambda}}\bigr)^l \\
&= \sum_{l=0}^{\infty} C_l(x) \sum_{m=l}^{\infty} \lambda^{-l} S_1(m,l) \frac{\log(1+\lambda t)^m}{m!} \\
&= \sum_{l=0}^{\infty} C_l(x) \sum_{m=l}^{\infty} \lambda^{-l} S_1(m,l) \sum_{j=m}^{\infty} S_1(j,m) \frac{\lambda^j t^j}{j!} \\
&= \sum_{l=0}^{\infty} \left(\sum_{m=0}^{j} \sum_{l=0}^{m} S_1(j,m) S_1(m,l) \lambda^{j-l} C_l(x) \right) \frac{t^j}{j!}.
\end{aligned} \tag{22}$$

By comparing the coefficients of the ends of (22), we have the following identity:

$$\begin{aligned}
C_{n,\lambda}^{(2)}(x) &= \sum_{m=0}^{n} \sum_{l=0}^{m} S_1(n,m) S_1(m,l) \lambda^{n-l} C_l(x) \\
&= \sum_{m=0}^{n} \sum_{l=0}^{m} S_1(n,m) S_1(m,l) \lambda^{n-l} \sum_{j=0}^{l} \binom{l}{j} (x)_j C_{l-j}.
\end{aligned} \tag{23}$$

By inductively k-times degenerating Cauchy polynomials, we obtain the following.

Theorem 5 *For each $n_i \geq 0$ where $i = 0, 1, \ldots, k+1$, we have*

$$\begin{aligned}
C_{n_{k+1},\lambda}^{(k)}(x) &= \sum_{n_k=0}^{n_{k+1}} \cdots \sum_{n_1=0}^{n_2} \lambda^{n_{k+1}-n_1} \left(\prod_{i=1}^{k} S_1(n_{i+1}, n_i) \right) C_{n_1}(x) \\
&= \sum_{n_k=0}^{n_{k+1}} \cdots \sum_{n_1=0}^{n_2} \sum_{n_0=0}^{n_1} \lambda^{n_{k+1}-n_0} \left(\prod_{i=1}^{k} S_1(n_{i+1}, n_i) \right) (x)_{n_0} C_{n_1-n_0}.
\end{aligned}$$

By using (3), the middle term of (18) can be expressed by higher order Bernoulli polynomials,

$$\begin{aligned}
&\sum_{m=0}^{\infty} B_m^{(m)}(x+1) \frac{1}{m!} \bigl(\bigl(\log\bigl(1+\log(1+\lambda t)\bigr)\bigr)^{\frac{1}{\lambda}}\bigr)^m \\
&= \sum_{m=0}^{\infty} B_m^{(m)}(x+1) \sum_{j=m}^{\infty} \lambda^{-j} S_1(j,m) \frac{(\log(1+\lambda t))^j}{j!}
\end{aligned}$$

$$= \sum_{m=0}^{\infty} B_m^{(m)}(x+1) \sum_{j=m}^{\infty} \lambda^{-j} S_1(j,m) \sum_{n=j}^{\infty} S_1(n,j) \frac{\lambda^n t^n}{n!}$$

$$= \sum_{n=0}^{\infty} \sum_{j=0}^{n} \sum_{m=0}^{j} B_m^{(m)}(x+1) \lambda^{-j} S_1(n,j) S_1(j,m) \frac{t^n}{n!}.$$

Thus

$$C_{n,\lambda}^{(2)}(x) = \sum_{j=0}^{n} \sum_{m=0}^{j} B_m^{(m)}(x+1) \lambda^{-j} S_1(n,j) S_1(j,m).$$

Inductively, we have the representation of k-times degenerate Cauchy polynomials via higher order Bernoulli polynomials.

Theorem 6 *For each $n_i \geq 0$ where $i = 0, 1, \dots, k+1$, we have*

$$C_{n_{k+1},\lambda}^{(k)}(x) = \sum_{n_k=0}^{n_{k+1}} \cdots \sum_{n_1=0}^{n_2} B_{n_1}^{(n_1)}(x+1) \lambda^{n_{k+1}-n_1} \times \prod_{i=1}^{k} S_1(n_{i+1}, n_i).$$

In particular,

$$C_{n_{k+1}}^{(k)} = \sum_{n_k=0}^{n_{k+1}} \cdots \sum_{n_1=0}^{n_2} B_{n_1}^{(n_1)}(1) \lambda^{n_{k+1}-n_1} \times \prod_{i=1}^{k} S_1(n_{i+1}, n_i).$$

As is well known, the Cauchy polynomials of the second kind are given by the generating function

$$\int_0^1 (1+t)^{-y+x}\, dy = \frac{t}{(1+t)\log(1+t)} (1+t)^x = \sum_{n=0}^{\infty} \widetilde{C}_n(x) \frac{t^n}{n!} \quad \text{(see [7])}. \tag{24}$$

Kim defined the degenerate Cauchy polynomials of the second kind as follows:

$$\begin{aligned} \int_0^1 \bigl(1+\log(1+\lambda t)^{\frac{1}{\lambda}}\bigr)^{-y+x}\, dy &= \frac{\log(1+\lambda t)^{\frac{1}{\lambda}} (1+\log(1+\lambda t)^{\frac{1}{\lambda}})^x}{\log(1+\log(1+\lambda t)^{\frac{1}{\lambda}})(1+\log(1+\lambda t)^{\frac{1}{\lambda}})} \\ &= \sum_{n=0}^{\infty} \widetilde{C}_{n,\lambda}(x) \frac{t^n}{n!} \quad \text{(see [7])}. \end{aligned} \tag{25}$$

The following is well known for describing degenerate Cauchy polynomials of the second kind. We record some results as a theorem for convenience.

Theorem 7 (Kim [7]) *For $l \geq 0$, we have*

$$\begin{aligned} \widetilde{C}_{l,\lambda}(x) &= \sum_{j=0}^{l} \lambda^{l-j} S_1(l,j) C_j(x) \\ &= \sum_{j=0}^{l} \lambda^{l-j} S_1(l,j) B_j^{(j)}(x) = \int_0^1 (x-y)_l\, dy. \end{aligned}$$

Now we consider degenerating one more time the degenerate Cauchy polynomials of the second kind as follows:

$$\begin{aligned}
&\int_0^1 \bigl(1+\log\bigl(1+\log(1+\lambda t)\bigr)^{\frac{1}{\lambda}}\bigr)^{x-y}\,dy \\
&\quad= \frac{\log(1+\log(1+\lambda t))^{\frac{1}{\lambda}}(1+\log(1+\log(1+\lambda t))^{\frac{1}{\lambda}})^{x}}{\log(1+\log(1+\log(1+\lambda t))^{\frac{1}{\lambda}})(1+\log(1+\log(1+\lambda t))^{\frac{1}{\lambda}})} = \sum_{n=0}^{\infty} \widetilde{C}_{n,\lambda}^{(2)}(x)\frac{t^n}{n!}. \qquad (26)
\end{aligned}$$

Observing the left-hand side of (26), we have

$$\begin{aligned}
&\int_0^1 \bigl(1+\log\bigl(1+\log(1+\lambda t)\bigr)^{\frac{1}{\lambda}}\bigr)^{x-y}\,dy \\
&\quad= \sum_{l=0}^{\infty}\int_0^1 (x-y)_l\,dy\,\bigl(\log\bigl(1+\log(1+\lambda t)\bigr)^{\frac{1}{\lambda}}\bigr)^{l}\frac{1}{l!} \\
&\quad= \sum_{l=0}^{\infty}\lambda^{-l}\widetilde{C}_{l,\lambda}(x)\sum_{n=l}^{\infty}S_1(n,l)\frac{(\log(1+\lambda t))^n}{n!} \\
&\quad= \sum_{n=0}^{\infty}\sum_{l=0}^{n}\lambda^{-l}\widetilde{C}_{l,\lambda}(x)S_1(n,l)\frac{\log(1+\lambda t)^n}{n!} \\
&\quad= \sum_{j=0}^{\infty}\sum_{n=0}^{j}\sum_{l=0}^{n}\lambda^{j-l}S_1(j,n)S_1(n,l)\widetilde{C}_{l,\lambda}(x)\frac{t^j}{j!}. \qquad (27)
\end{aligned}$$

Thus for $j \geq 0$,

$$\widetilde{C}_{j,\lambda}^{(2)}(x) = \sum_{n=0}^{j}\sum_{l=0}^{n}\lambda^{j-l}S_1(j,n)S_1(n,l)\widetilde{C}_{l,\lambda}(x).$$

Now we apply the result of (27) and Theorem 7, and we can rewrite $\widetilde{C}_{j,\lambda}^{(2)}(x)$ as follows.

For $j, n, l \geq 0$, we have

$$\begin{aligned}
\widetilde{C}_{j,\lambda}^{(2)}(x) &= \sum_{n=0}^{j}\sum_{l=0}^{n}\sum_{i=0}^{l}\lambda^{j-i}S_1(j,n)S_1(n,l)S_1(l,i)\widetilde{C}_i(x) \\
&= \sum_{n=0}^{j}\sum_{l=0}^{n}\sum_{i=0}^{l}\lambda^{j-i}S_1(j,n)S_1(n,l)S_1(l,i)B_i^{(i)}(x).
\end{aligned}$$

Inductively, we get the following identity for k-times degenerate Cauchy polynomials of the second kind.

Theorem 8 *For $n_i \geq 0$, where $i = 0, 1, \ldots, k+1$, we have*

$$\begin{aligned}
\widetilde{C}_{n_{k+1},\lambda}^{(k)}(x) &= \sum_{n_k=0}^{n_{k+1}}\sum_{n_{k-1}=0}^{n_k}\cdots\sum_{n_1=0}^{n_2}\lambda^{n_{k+1}-n_1}\left(\prod_{i=1}^{k}S_1(n_{i+1},n_i)\right)\widetilde{C}_{n_1,\lambda}(x) \\
&= \sum_{n_k=0}^{n_{k+1}}\sum_{n_{k-1}=0}^{n_k}\cdots\sum_{n_1=0}^{n_2}\lambda^{n_{k+1}-n_1}\left(\prod_{i=1}^{k}S_1(n_{i+1},n_i)\right)B_{n_1}^{(n_1)}(x).
\end{aligned}$$

Moreover, we have the following:

$$\begin{aligned}\widetilde{C}^{(k)}_{n_{k+1},\lambda}(x) &= \sum_{n_k=0}^{n_{k+1}}\cdots\sum_{n_0=0}^{n_1}\lambda^{n_{k+1}-n_0}\prod_{i=0}^{k}S_1(n_{i+1},n_i)\widetilde{C}_{n_0}(x)\\ &= \sum_{n_k=0}^{n_{k+1}}\cdots\sum_{n_0=0}^{n_1}\lambda^{n_{k+1}-n_0}\prod_{i=0}^{k}S_1(n_{i+1},n_i)B^{(n_0)}_{n_0}(x).\end{aligned}$$

From the left-hand side of (26), we have

$$\begin{aligned}\widetilde{C}^{(2)}_{j,\lambda}(x) &= \sum_{n=0}^{j}\sum_{l=0}^{n}\lambda^{j-l}S_1(j,n)S_1(n,l)l!\int_0^1\binom{x-y}{l}\,dy\\ &= \sum_{n=0}^{j}\sum_{l=0}^{n}\lambda^{j-l}l!S_1(j,n)S_1(n,l)(-1)^n\int_0^1\binom{y+n-1-x}{n}\,dy\\ &= \sum_{n=1}^{j}\sum_{l=1}^{n}\sum_{m=1}^{l}\frac{l!}{m!}\lambda^{j-l}S_1(j,n)S_1(n,l)(-1)^m\binom{n-1}{m-1}C_m(-x).\end{aligned}\tag{28}$$

Therefore, from (28), we inductively obtain the following theorem.

Theorem 9 *For $n_i \geq 0$, where $i = 0,1,\ldots,k+1$, we have*

$$\widetilde{C}^{(k)}_{n_{k+1},\lambda}(x) = \sum_{n_k=1}^{n_{k+1}}\cdots\sum_{n_0=1}^{n_1}\frac{n_1!}{n_0!}\lambda^{n_{k+1}-n_1}\left(\prod_{i=1}^{k}S_1(n_{i+1},n_i)\right)(-1)^{n_0}\binom{n_k-1}{n_0-1}C_{n_0}(-x).$$

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to this work. All authors read and approved the final manuscript.

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