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Strong convergence theorems for nonlinear operator equations with total quasi-φ-asymptotically nonexpansive mappings and applications

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Abstract

The purpose of this article is first to introduce the concept of *total quasi-φ-asymptotically nonexpansive mapping* which contains many kinds of mappings as its special cases, and then by using the hybrid algorithm to introduce a new iterative scheme for finding a common element of set of solutions for a system of generalized mixed equilibrium problems, the set of common fixed points of a countable family of total quasi-φ-asymptotically nonexpansive mappings and null spaces of finite family of γ-inverse strongly monotone mappings in a 2-uniformly convex and uniformly smooth real Banach space. As an application, we shall utilize our results to study the iterative solutions of the nonlinear Hammerstian type equation. The results presented in the article improve and extend the corresponding results announced by some authors.

AMS (MOS) subject classification: 47H06; 47H09; 47J05; 47J25.

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1 Introduction

Throughout this article, we assume that E is a real Banach space with a dual E^*, C is a nonempty closed convex subset of E and $\langle \cdot, \cdot \rangle$ is the duality pairing between members of E and E^*, $\mathscr{R}$ is the set of all real numbers. In the sequel, we denote by $x_n \rightharpoonup x$ and $x_n \to x$ the weak convergence and strong convergence of sequence $\{x_n\}$, respectively. The mapping $J : E \to 2^{E^*}$ defined by

$$J(x) = \left\{ f^* \in E^* : \left\langle x, f^* \right\rangle = \|x\|^2; \left\| f^* \right\| = \|x\| \right\}, \quad x \in E,$$

is called *the normalized duality mapping.*

Let $F : C \times C \to \mathscr{R}$ be a bifunction, B: $C \to E^*$ be a nonlinear mapping and $\Psi : C \to \overline{\mathbb{R}}$ be a proper extended real-valued function. The "so called" *generalized mixed equilibrium problem* (MEP) *for F, B,* Ψ is to find $x^* \in C$ such that

$$F(x^*, y) + \left\langle y - x^*, Bx^* \right\rangle + \Psi(y) - \Psi(x^*) \geq 0, \quad \forall y \in C. \tag{1.1}$$

The set of solutions of (1.1) is denoted by GMEP(F, B, Ψ), i.e.,

$$\begin{aligned}\mathrm{GMEP}(F, B, \Psi) = \{x^* \in C : F(x^*, y) + \langle y - x^*, Bx^* \rangle \\ + \Psi(y) - \Psi(x^*) \geq 0, \quad \forall y \in C\}.\end{aligned}$$

1.1 Special examples

(1) If $B = 0$, $\Psi = 0$, then problem (1.1) is reduced to the *equilibrium problem* (EP), and the set of its solutions is denoted by

$$\mathrm{EP}(F) = \{x^* \in C : F(x^*, y) \geq 0, \quad \forall y \in C\}.$$

(2) If $\Psi \equiv 0$, then the problem (1.1) is reduced to the *generalized equilibrium problem* (GEP), and the set of its solutions is denoted by

$$\mathrm{GEP}(F, B) = \{x^* \in C : F(x^*, y) + \langle y - x^*, Bx^* \rangle \geq 0, \quad \forall y \in C\}.$$

(3) If $B \equiv 0$, then the problem (1.1) is reduced to the MEP, and the set of its solutions is denoted by

$$\mathrm{MEP}(F, \Psi) = \{x^* \in C : F(x^*, y) + \Psi(y) - \Psi(x^*) \geq 0, \quad \forall y \in C\}.$$

These show that the problem (1.1) is very general in the sense that numerous problems in physics, optimization, and economics reduce to finding a solution of (1.1). Recently, some methods have been proposed for the generalized mixed equilibrium problem in Banach space (see, for examples [1-5]).

Let E be a smooth, strictly convex and reflexive Banach space and C be a nonempty closed convex subset of E. Throughout this article we assume that $\phi : E \times E \to \mathscr{R}^+$ is *the Lyapunov function* which is defined by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy \rangle + \|y\|^2, \quad \forall x, y \in E.$$

Following Alber [6], *the generalized projection* Π_C: $E \to C$ is defined by

$$\Pi_C(x) = \arg\min_{y \in C} \phi(y, x), \quad \forall x \in E.$$

Let S: $C \to C$ be a mapping and $F(S)$ be the set of fixed points of S.

Recall that a point $p \in C$ is called an *asymptotic fixed point of S*, if there exists a sequence $\{x_n\} \subset C$ such that $x_n \rightharpoonup p$ and $||x_n - Sx_n|| \to 0$. We denoted the set of all asymptotic fixed points of S by $\tilde{F}(S)$. A point $p \in C$ is called a *strong asymptotic fixed point of S*, if there exists a sequence $\{x_n\} \subset C$ such that $x_n \to p$ and $||x_n - Sx_n|| \to 0$. We denoted the set of all strong asymptotic fixed points of S by $\hat{F}(S)$.

A mapping S: $C \to C$ is said to be *nonexpansive*, if

$$\|Sx - Sy\| \leq \|x - y\|, \quad \forall x, y \in C.$$

A mapping S: $C \to C$ is said to be *relatively nonexpansive* [7] if, $F(S) \neq \emptyset, F(S) = \tilde{F}(S)$, and

$$\phi(p, Sx) \leq \phi(p, x), \quad \forall x \in C, \quad p \in F(S).$$

A mapping S: $C \to C$ is said to be *weak relatively nonexpansive* [8], if $F(S) \neq \emptyset, F(S) = \tilde{F}(S)$, and

$$\phi(p, Sx) \leq \phi(p, x), \quad \forall x \in C, \quad p \in F(S).$$

A mapping S: $C \to C$ is said to be *closed*, if for any sequence $\{x_n\} \subset C$ with $x_n \to x$ and $Sx_n \to y$, then $Sx = y$.

A mapping S: $C \to C$ is said to be *quasi-φ-nonexpansive*, if $F(S) \neq \emptyset$ and

$$\phi(p, Sx) \leq \phi(p, x), \quad \forall x \in C, \quad p \in F(S).$$

A mapping S: $C \to C$ is said to be *quasi-φ-asymptotically nonexpansive*, if $F(S) \neq \emptyset$ and there exists a real sequence $\{k_n\} \subset [1, \infty)$ with $k_n \to 1$ such that

$$\phi(p, S^n x) \leq k_n \phi(p, x), \quad \forall n \geq 1, x \in C, \quad p \in F(S). \tag{1.2}$$

Definition 1.1 (1) A mapping S: $C \to C$ is said to be *total quasi-φ-asymptotically nonexpansive*, if $F(S) \neq \emptyset$ and there exist nonnegative real sequences $\{v_n\}$, $\{\mu_n\}$ with $v_n \to 0$, $\mu_n \to 0$ (as $\to \infty$) and a strictly increasing continuous function $\varsigma : \mathscr{R}^+ \to \mathscr{R}^+$ with $\zeta(0) = 0$ such that for all $x \in C$, $p \in F(S)$

$$\phi(p, S^n x) \leq \phi(p, x) + v_n \varsigma(\phi(p, x)) + \mu_n, \quad \forall n \geq 1. \tag{1.3}$$

(2) A countable family of mappings $\{S_n\}$: $C \to C$ is said to be *uniformly total quasi-φ-asymptotically nonexpansive*, if $\cap_{i=1}^{\infty} F(S_i) \neq \emptyset$ and there exist nonnegative real sequences $\{v_n\}$, $\{\mu_n\}$ with $v_n \to 0$, $\mu_n \to 0$ (as $n \to \infty$) and a strictly increasing continuous function $\varsigma : \mathscr{R}^+ \to \mathscr{R}^+$ with $\zeta(0) = 0$ such that for for each $i > 1$, and each $\phi(p, S_i^n x) \leq \phi(p, x) + v_n \varsigma(\phi(p, x)) + \mu_n, \quad \forall n \geq 1.$

$$\phi(p, S_i^n x) \leq \phi(p, x) + v_n \varsigma(\phi(p, x)) + \mu_n, \quad \forall n \geq 1. \tag{1.4}$$

Remark 1.1 From the definitions, it is easy to know that

(1) Each relatively nonexpansive mapping is closed;

(2) Taking $\zeta(t) = t$, $t \geq 0$, $v_n = (k_n - 1)$ and $\mu_n = 0$, then (1.2) can be rewritten as

$$\phi(p, S^n x) \leq \phi(p, x) + v_n \varsigma(\phi(p, x)) + \mu_n, \quad \forall n \geq 1, x \in C, \quad p \in F(S). \tag{1.5}$$

This implies that each quasi-φ-asymptotically nonexpansive mapping must be a total quasi-φ-asymptotically nonexpansive mapping, but the converse is not true.

(3) The class of quasi-φ-asymptotically nonexpansive mappings contains properly the class of quasi-φ-nonexpansive mappings as a subclass, but the converse is not true;

(4) The class of quasi-φ-nonexpansive mappings contains properly the class of weak relatively nonexpansive mappings as a subclass, but the converse may be not true;

(5) The class of weak relatively nonexpansive mappings contains properly the class of relatively nonexpansive mappings as a subclass, but the converse is not true.

A mapping A: $C \to E^*$ is said to be *α-inverse strongly monotone*, if there exists $\alpha > 0$ such that

$$\langle x - y, Ax - Ay \rangle \geq \alpha \left\| Ax - Ay \right\|^2.$$

Remark 1.2 If A is an α-inverse strongly monotone mapping, then it is $\frac{1}{\alpha}$-Lipschitz continuous.

Iterative approximation of fixed points for relatively nonexpansive mappings in the setting of Banach spaces has been studied extensively by many authors. In 2005, Matsushita and Takahashi [7] obtained some weak and strong convergence theorems to approximate a fixed point of a single relatively nonexpansive mapping. Recently, Ofoedu and Malonza [4], Zhang [5], Su et al. [8], Zhang and Su [9], Zegeye and Shahzad [10], Wattanawitoon and Kumam [11], Qin et al. [12], Takahashi and Zembayashi [13] extend the notions from relatively nonexpansive mappings, weakly relatively nonexpansive mappings or quasi-φ-nonexpansive mappings to quasi-φ-asymptotically nonexpansive mappings and also proved some strong convergence theorems to approximate a common fixed point of quasi-φ-nonexpansive mappings or quasi-φ-asymptotically nonexpansive mappings.

The purpose of this article is first to introduce the concept of *total quasi-φ-asymptotically nonexpansive mapping* which contains many kinds of mappings as its special cases, and then by using hybrid algorithm to introduce a new iterative scheme for finding a common element of set of solutions for a system of generalized mixed equilibrium problems, set of common fixed points of a countable family of total quasi-φ-asymptotically nonexpansive mappings and null spaces of finite family of γ-inverse strongly monotone mappings in a 2-uniformly convex and uniformly smooth real Banach space. As an application, we shall utilize our results to study the iterative solutions of the nonlinear Hammerstian type equation. The results presented in the article improve and extend the corresponding results in [1-5,7-16].

2 Preliminaries

For the sake of convenience, we first recall some definitions and conclusions which will be needed in proving our main results.

A Banach space E is said to be *strictly convex*, if $\frac{\|x+y\|}{2} < 1$ for all $x, y \in U = \{z \in E: ||z|| = 1\}$ with $x \neq y$. It is said to be *uniformly convex*, if for each $\epsilon \in (0, 2]$, there exists $\delta > 0$ such that $\frac{\|x+y\|}{2} \leq 1 - \delta$ for all $x, y \in U$ with $||x - y|| \geq \epsilon$. The *convexity modulus of E* is the function δ_E: $(0, 2] \to [0, 1]$ defined by

$$\delta_E(\varepsilon) = \inf\left\{1 - \left\|\frac{1}{2}(x+y)\right\| : x, y \in U, \|x - y\| \geq \varepsilon\right\},$$

for all $\epsilon \in (0, 2]$. It is well known that $\delta_E(\epsilon)$ is a strictly increasing and continuous function with $\delta_E(0) = 0$ and $\frac{\delta_E(\varepsilon)}{\varepsilon}$ is nondecreasing for all $\epsilon \in (0,2]$. Let $p > 1$, then E is said to be *p-uniformly convex*, if there exists a constant $c > 0$ such that $\delta_E(\epsilon) \geq c\epsilon^p$, $\forall \epsilon \in (0, 2]$. The space E is said to be *smooth*, if the limit

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t}$$

exists for all $x, y \in U$. And E is said to be *uniformly smooth*, if the limit exists uniformly in $x, y \in U$.

In the sequel, we shall make use of the following lemmas.

Lemma 2.1 [17] Let E be a 2-uniformly convex real Banach space, then for all $x, y \in E$, the following inequality holds:

$$\|x - y\| \leq \frac{2}{c^2} \|Jx - Jy\|, \tag{2.1}$$

where $0 < c \leq 1$, and c is called the *2-uniformly convex constant of E.*

Lemma 2.2 [18] Let E be a smooth, strict convex and reflexive Banach space and C be a nonempty closed convex subset of E. Then, the following conclusions hold:

(i) $\varphi(x, \Pi_C y) + \varphi(\Pi_C y, y) \le \varphi(x, y), \forall x \in C, y \in E$.

(ii) Let $x \in E$ and $z \in C$, then

$$z = \Pi_C x \Leftrightarrow \langle z - y, Jx - Jz \rangle \ge 0, \quad \forall y \in C.$$

Lemma 2.3 [18] Let E be a uniformly convex and smooth Banach space and $\{x_n\}$, $\{y_n\}$ be sequences of E. If $\varphi(x_n, y_n) \to 0$ $(as\ n \to \infty)$ and either $\{x_n\}$ or $\{y_n\}$ is bounded, then $x_n - y_n \to 0(as\ n \to \infty)$.

Lemma 2.4 [19] Let E be a uniformly convex Banach space, r be a positive number and $B_r(\theta)$ be a closed ball of E. For any given points $\{x_1, x_2, \ldots, x_n, \ldots\} \subset B_r(\theta)$ and for any given positive numbers $\{\lambda_1, \lambda_2, \ldots\}$ with $\sum_{n=1}^{\infty} \lambda_n = 1$, then there exists a continuous, strictly increasing and convex function g: $[0, 2r) \to [0, \infty)$ with $g(0) = 0$ such that for any $i, j \in \{1, 2, \ldots, \}$, $i < j$,

$$\left\| \sum_{n=1}^{\infty} \lambda_n x_n \right\|^2 \le \sum_{n=1}^{\infty} \lambda_n \|x_n\|^2 - \lambda_i \lambda_j g(\|x_i - x_i\|). \tag{2.2}$$

Lemma 2.5 [20] Let E be a smooth, strict convex and reflexive Banach space and C be a nonempty closed convex subset of E, $f : C \times C \to \mathscr{R}$ be a bifunction satisfying the following conditions:

(A1) $f(x, x) = 0$ for all $x \in C$;

(A2) f is monotone, i.e., $f(x, y) + f(y, x) \le 0, \forall x, y \in C$;

(A3) $\lim \sup_{t \downarrow 0} f(x + t(z - x), y) \le f(x, y), \forall x, y, z \in C$;

(A4) The function $y \mapsto f(x, y)$ is convex and lower semicontinuous.

Then the following conclusions hold:

(1) For any given $r > 0$ and $x \in E$, there exists a unique $z \in C$ such that

$$f(z, y) + \frac{1}{r}\langle y - z, Jz - Jx \rangle \ge 0, \quad \forall y \in C. \tag{2.3}$$

(2) For given $r > 0$ and $x \in E$, define a mapping $K_r^f : E \to C$ by

$$K_r^f(x) = \left\{ z \in C : f(z, y) + \frac{1}{r}\langle y - z, Jz - Jx \rangle \ge 0, \quad \forall y \in C \right\}. \tag{2.4}$$

Then, the following hold:

(i) K_r^f is single-valued;

(ii) K_r^f is a firmly nonexpansive-type mapping, i.e., for any $x, y \in E$,

$$\langle K_r^f(x) - K_r^f(y), JK_r^f(x) - JK_r^f(y) \rangle \le \langle K_r^f(x) - K_r^f(y), Jx - Jy \rangle;$$

(iii) $F\left(K_r^f\right) = EP(f)$ (the set of solutions of equilibrium problem for function f);

(iv) $EP(f)$ is closed and convex.

(v) $\phi\left(p, K_r^f(x)\right) + \phi\left(K_r^f(x), x\right) \le \phi(p, x), \quad \forall p \in F\left(K_r^f\right)$..

For solving the generalized mixed equilibrium problem (1.1), let us assume that the following conditions are satisfied:

(1) E is a smooth, strictly convex and reflexive Banach space and C is a nonempty closed convex subset of E

(2) B: $C \to E^*$ is a β-inverse strongly monotone mapping;

(3) $F : C \times C \to \mathscr{R}$ is a bifunction satisfying the conditions (A1), (A3), (A4) in Lemma 2.5 and the following condition (A2)':

(A2)' for some $\gamma \geq 0$ with $\gamma \leq \beta$

$$F(x, y) + F(y, x) \leq \gamma \left\|Bx - By\right\|^2, \quad \forall x, y \in C;$$

(4) $\Psi : C \to \mathscr{R}$ is a lower semi-continuous and convex function.

Under the assumptions as above, we have the following results.

Lemma 2.6 Let E, C, B, F, Ψ satisfy the above conditions (1)-(4). Denote by

$$\Gamma(x, y) = F(x, y) + \Psi(y) - \Psi(x) + \langle y - x,\ Bx\rangle, \quad \forall x, y \in C. \tag{2.5}$$

For any given $r > 0$ and $x \in E$, define a mapping $K_r^\Gamma : E \to C$ by

$$K_r^\Gamma(x) = \left\{ z \in C : \Gamma(z, y) + \frac{1}{r}\langle y - z, Jz - Jx\rangle \geq 0, \quad \forall y \in C \right\}. \tag{2.6}$$

Then, the following hold:

(i) K_r^Γ is single-valued;

(ii) K_r^Γ is a firmly nonexpansive-type mapping, i.e., for all $x,y \in E$,

$$\langle K_r^\Gamma(x) - K_r^\Gamma(y), JK_r^\Gamma(x) - JK_r^\Gamma(y)\rangle \leq \langle K_r^\Gamma(x) - K_r^\Gamma(y), Jx - Jy\rangle;$$

(iii) $F(K_r^\Gamma) = EP(\Gamma) = \mathrm{GMEP}(F, \Phi, B)$;

(iv) GMEP(F, Φ, B) is closed and convex.

(v) $\phi(p, K_r^\Gamma(x)) + \phi(K_r^\Gamma(x), x) \leq \phi(p, x), \quad \forall p \in F(K_r^\Gamma)$.

Proof. It follows from Lemma 2.5 that in order to prove the conclusions of Lemma 2.6 it is sufficient to prove that the function $\Gamma : C \times C \to \mathscr{R}$ satisfies the conditions (A1)-(A4) in Lemma 2.5.

In fact, by the similar method as given in the proof of Lemma 2.5 in [1], we can prove that the function Γ satisfies the conditions (A1), (A3), and (A4). Now we prove that Γ also satisfies the condition (A2).

Indeed, for any x, $y \in C$, by condition (A2)' we have

$$\begin{aligned}\Gamma(x, y) + \Gamma(y, x) &= F(x, y) + \Psi(y) - \Psi(x) + \langle y - x,\ Bx\rangle \\ &\quad + F(y, x) + \Psi(x) - \Psi(y) + \langle x - y,\ By\rangle \\ &= F(x, y) + F(y, x) - \langle x - y,\ Bx - By\rangle \\ &\leq (\gamma - \beta)\left\|Bx - By\right\|^2 \leq 0.\end{aligned}$$

This implies that the function Γ satisfies the condition (A2). Therefore the conclusions of Lemma 2.6 can be obtained from Lemma 2.5 immediately.

In the sequel, we make use of the function $V : E \times E^* \to \mathscr{R}$ which is defined by

$$V(x, x^*) = \|x\|^2 - 2\langle x, x^*\rangle + \|x^*\|^2,$$

for all $x \in E$ and $x^* \in E^*$. Observe that $V(x, x^*) = \varphi(x, J^{-1} x^*)$ for all $x \in E$ and $x^* \in E^*$.

The following lemma is well known.

Lemma 2.7 [6] Let E be a smooth, strictly convex and reflexive Banach space with E^* as its dual. Then

$$V(x, x^*) + 2\langle J^{-1}x^* - x, y^*\rangle \le V(x, x^* + y^*)$$

for all $x \in E$ and $x^*, y^* \in E^*$.

3 Main results

In this section we shall make use of the following assumptions.

(1) Let E be a 2-uniformly convex and uniformly smooth real Banach space with a dual E^* and C be a nonempty closed and convex subset of E;

(2) Let T_i: $C \to C$, $i = 1, 2,...$ be a countable family of closed and uniformly total quasi-φ-asymptotically nonexpansive mappings with nonnegative real sequences $\{v_n\}$, $\{\mu_n\}$ and a strictly increasing continuous functions $\varsigma : \mathscr{R}^+ \to \mathscr{R}^+$ such that $v_n \to 0$, $\mu_n \to 0$ (as $n \to \infty$) and $\zeta(0) = 0$. Suppose further that for each $i \ge 1$, T_i is a uniformly L_i-Lipschitzian mapping, i.e., there exists a positive number $L_i > 0$ such that

$$\|T_i^n x - T_i^n y\| \le L_i \|x - y\|, \quad \forall x, y \in C, \quad \forall n \ge 1;$$

(3) Let A_i: $C \to E^*$, $i = 1, 2,..., N$ be a δ_i-inverse strongly monotone mapping and denote by $\delta = \min\{\delta_n, i = 1,2,..., N\}$;

(4) *Let* B_i: $C \to E^*$, $i = 1,2,... , M$ be a β_i-inverse strongly monotone mappings;

(5) Let $F_i : C \times C \to \mathscr{R}, i = 1, 2, \ldots, M$ be a finite family of bifunctions satisfying conditions (A1), (A3), (A4) and the following condition (A2)":

(A2)" For each $i = 1, 2,..., M$ there exists $\gamma_i \ge 0$ with $\gamma_i \le \beta_i$ such that

$$F_i(x, y) + F_i(y, x) \le \gamma_i \|B_i x - B_i y\|^2, \quad \forall x, y \in C;$$

(6) Let $\Psi_i : C \to \mathscr{R}, i = 1, 2, \cdot, M$ be a finite family of lower semi-continuous convex functions.

We are now in a position to give the main results of this article.

Theorem 3.1 Let E, C, $\{T_i\}_{i=1}^{\infty}$, $\{A_n\}_{n=1}^{N}$, $\{B_i\}_{i=1}^{M}$, $\{F_i\}_{i=1}^{M}$, $\{\Psi_i\}_{i=1}^{M}$ satisfy the above conditions (1)-(6). Suppose that

$$\mathscr{F} := \bigcap_{i=1}^{\infty} F(T_i) \bigcap \bigcap_{n=1}^{M} A_n^{-1}(0) \bigcap \bigcap_{m=1}^{\infty} \mathrm{GMEP}(F_m, B_m, \Phi_m)$$

is a nonempty and bounded subset of C. For any given $x_0 \in C$, let $\{x_n\}$ be the sequence generated by

$$\begin{cases} x_0 \in C_0 = C, \\ y_n = \Pi_C J^{-1}(Jx_n - \lambda A_{n+1} x_n), \\ z_n = J^{-1}\left(\alpha_{n,0} Jx_n + \sum_{i=1}^{\infty} \alpha_{n,i} J T_i^n y_n\right), \\ u_n = K_{r_{M,n}}^{\Gamma_M} K_{r_{M-1,n}}^{M-1} \ldots K_{r_{2,n}}^{\Gamma_2} K_{r_{1,n}}^{\Gamma_1} z_n, \\ C_{n+1} = \{v \in C_n : \phi(v, u_n) \le \phi(v, x_n) + \eta_n\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0, \quad n \ge 0, \end{cases} \tag{3.1}$$

where

$$\eta_n = \nu_n \sup_{u \in \mathscr{F}} \varsigma(\phi(u, x_n)) + \mu_n, \quad \forall n \geq 1,$$

$K^{\Gamma_k}_{r_{k,n}} : E \to C,\ k = 1, 2, \ldots, M$ is the mapping defined by (2.6) with $\Gamma = \Gamma_k$, $r = r_{k,n}$, and

$$\Gamma_k(x, y) = F_k(x, y) + \langle y - x,\ B_k x\rangle + \Psi_k(y) - \Psi_k(x), \quad \forall x, y \in C, \tag{3.2}$$

$A_n = A_{n(\mathrm{mod} N)}$, $r_{k,n} \in [d, \infty)$, $k = 1,2,..., M$, $n \geq 1$ for some $d > 0$, $0 < \lambda < \frac{c^2\delta}{2}$, where c is the 2-uniformly convex constant of E, and for each $n \geq 1$

(i) for each $n \geq 1$, $\sum_{i=0}^{\infty} \alpha_{n,i} = 1$;
(ii) for each $j \geq 1$, $\liminf_{n\to\infty} \alpha_{n,0}\alpha_{n,j} > 0$.

If $\mathscr{F}$ is a nonempty and bounded subset in C, then $\{x_n\}$ converges strongly to some point $x^* \in \mathscr{F}$.

Proof. We divide the proof of Theorem 3.1 into five steps.

(I) Sequences $\{x_n\}$, $\{y_n\}$ and $\{T_i^n y_n\}$ are all bounded.

In fact, since $x_n = \Pi_{C_n} x_0$, for any $p \in \mathscr{F}$, from Lemma 2.2, we have

$$\phi(x_n, x_0) = \phi(\Pi_{C_n} x_0, x_0) \leq \phi(p, x_0) - \phi(p, x_n) \leq \phi(p, x_0).$$

This implies that the sequence $\{\varphi(x_n, x_0)\}$ is bounded, and so $\{x_n\}$ is bounded.

On the other hand, by Lemmas 2.2, 2.7, and 2.1, we have that

$$\begin{aligned}
\phi(p, y_n) &= \phi(p, \Pi_C J^{-1}(Jx_n - \lambda A_{n+1}x_n)) \\
&\leq \phi(p, J^{-1}(Jx_n - \lambda A_{n+1}x_n)) \\
&= V(p, Jx_n - \lambda A_{n+1}x_n) \\
&\leq V(p, (Jx_n - \lambda A_{n+1}x_n) + \lambda A_{n+1}x_n) \\
&\quad - 2\langle J^{-1}(Jx_n - \lambda A_{n+1}x_n) - p, \lambda A_{n+1}x_n\rangle \\
&= V(p, Jx_n) - 2\lambda\langle J^{-1}(Jx_n - \lambda A_{n+1}x_n) - p, A_{n+1}x_n\rangle \\
&= \phi(p, x_n) - 2\lambda\langle x_n - p, A_{n+1}x_n\rangle \\
&\quad - 2\lambda\langle J^{-1}(Jx_n - \lambda A_{n+1}x_n) - x_n A_{n+1}x_n\rangle \\
&= \phi(p, x_n) - 2\lambda(x_n - p, A_{n+1}x_n - A_{n+1}p) \text{ (since } A_{n+1}p = 0) \\
&\quad - 2\lambda\langle J^{-1}(Jx_n - \lambda A_{n+1}x_n) - x_n, A_{n+1}x_n\rangle \\
&\leq \phi(p, x_n) - 2\lambda\delta\|A_{n+1}x_n\|^2 \text{(by condition (3))} \\
&\quad + 2\lambda\left\|J^{-1}(Jx_n - \lambda A_{n+1}x_n) - J^{-1}Jx_n\right\| \times \|A_{n+1}x_n\| \\
&\leq \phi(p, x_n) - 2\lambda\delta\|A_{n+1}x_n\|^2 + \frac{4\lambda^2}{c^2}\|A_{n+1}x_n\|^2 \text{(by } \mathit{Lemma}\ 2.1) \\
&= \phi(p, x_n) + 2\lambda\left(\frac{2\lambda}{c^2} - \delta\right)\|A_{n+1}x_n\|^2
\end{aligned} \tag{3.3}$$

Thus, using the fact that $\lambda \leq \frac{c^2}{2}\delta$, we have that

$$\phi(p, y_n) \leq \phi(p, x_n). \tag{3.4}$$

This shows that $\{y_n\}$ is also bounded. Moreover, by condition (2), $\{T_i : C \to C\}_{i=1}^{\infty}$ is a countable family of uniformly total quasi-φ-asymptotically nonexpansive mappings with nonnegative real sequences $\{v_n\}$, $\{\mu_n\}$ and a strictly increasing continuous functions $\varsigma : \mathscr{R}^+ \to \mathscr{R}^+$ such that $v_n \to 0$, $\mu_n \to 0$ (as $n \to \infty$) and $\zeta(0) = 0$. Therefore for each $i \geq 1$ and for given $p \in \mathscr{F}$ we have

$$\phi(p, T_i^n y_n) \leq \phi(p, y_n) + v_n \varsigma(\phi(p, y_n)) + \mu_n, \quad \forall n \geq 1.$$

Since $\{y_n\}$ is bounded, this shows that, $\{T_i^n y_n\}$ is uniformly bounded. Denote by

$$\tilde{K} = \sup_{n\geq 0, i\geq 1} \{\|x_n\|, \|y_n\|, \|T_i^n y_n\|\} < \infty.$$

By the way, from the definition of $\{\eta_n\}$, it is easy to see that

$$\eta_n = v_n \sup_{u\in\mathscr{F}} \varsigma(\phi(u, x_n)) + \mu_n \to 0 \ (as\ n \to \infty). \tag{3.5}$$

(II) For each $n \geq 0$, C_n is a closed and convex subset of C and $\Omega \subset C_n$.

It is obvious that $C_0 = C$ is closed and convex. Suppose that C_n is closed and convex for some $n \geq 1$. Since the inequality $\varphi(v,u_n) \leq \varphi(v, x_n) + \eta_n$ is equivalent to

$$2\langle v, Jx_n - Ju_n\rangle \leq \|x_n\|^2 - \|u_n\|^2 + \eta_n,$$

therefore, we have

$$C_{n+1} = \{v \in C_n : 2\langle v, Jx_n - Ju_n\rangle \leq \|x_n\|^2 - \|u_n\|^2 + \eta_n\}. \tag{3.6}$$

This implies that C_{n+1} is closed and convex. Thus for each $n \geq 0$, C_n is a closed and convex subset of C

Next, we prove that $\mathscr{F} \subset C_n$ for all $n \geq 0$. Indeed, it is obvious that, $\mathscr{F} \subset C_0 = C$. Suppose $\mathscr{F} \subset C_n$ for some $n \geq 1$. Since E is uniformly smooth, E^* is uniformly convex. For any given $p \in \mathscr{F} \subset C_n$ and for any positive integers $j \geq 1$ and any positive integer $m = 1, 2,..., M - 1$, from Lemmas 2.6(v) and 2.4, we have

$$\begin{aligned}
\phi(p, u_n) &= \phi\left(p, K_{r_{M,n}}^{\Gamma_M} K_{r_{M-1,n}}^{\Gamma_M} \cdots K_{r_{2,n}}^{\Gamma_2} K_{r_{1,n}}^{\Gamma_1} z_n\right) \\
&\leq \phi\left(p, K_{r_{m,n}}^{m} K_{r_{m-1,n}}^{\Gamma_{m-1}} \cdots K_{r_{2,n}}^{\Gamma_2} K_{r_{1,n}}^{\Gamma_1} z_n\right) \\
&\leq \phi(p, z_n) \ (\text{by } Lemma\ 2.6(\text{v})) \\
&= \phi\left(p, J^{-1}\left(\alpha_{n,0} Jx_n + \sum_{i=1}^{\infty} \alpha_{n,i} JT_i^n y_n\right)\right) \\
&= \|p\|^2 - 2\left\langle p,\ \alpha_{n,0} Jx_n + \sum_{i=1}^{\infty} \alpha_{n,i} JT_i^n y_n\right\rangle \\
&\quad + \left\|\alpha_{n,0} Jx_n + \sum_{i=1}^{\infty} \alpha_{n,i} JT_i^n y_n\right\|^2 \\
&\leq \|p\|^2 - 2\alpha_{n,0}\langle p,\ Jx_n\rangle - 2\sum_{i=1}^{\infty} \alpha_{n,i}\langle p, JT_i^n y_n\rangle \\
&\quad + \alpha_{n,0}\|x_n\|^2 + \sum_{i=1}^{\infty} \alpha_{n,i}\|T_i^n y_n\|^2 - \alpha_{n,0}\alpha_{n,j} g\left(\left\|Jx_n - JT_j^n y_n\right\|\right) \\
&= \alpha_{n,0}\phi(p, x_n) + \sum_{i=1}^{\infty} \alpha_{n,i}\phi(p, T_i^n y_n) - \alpha_{n,0}\alpha_{n,j} g\left(\left\|Jx_n - JT_j^n y_n\right\|\right) \\
&\leq \alpha_{n,0}\phi(p, x_n) + \sum_{i=1}^{\infty} \alpha_{n,i}\{\phi(p, y_n) + v_n \varsigma(\phi(p, y_n)) + \mu_n\} \\
&\quad - \alpha_{n,0}\alpha_{n,j} g\left(\left\|Jx_n - JT_j^n y_n\right\|\right)
\end{aligned}$$

Combining (3.4), we have

$$\begin{aligned}
\phi(p, u_n) &\le \phi(p, z_n) \\
&\le \alpha_{n,0}\phi(p, x_n) + \sum_{i=1}^{\infty} \alpha_{n,i}\{\phi(p, x_n) + \nu_n \varsigma(\phi(p, x_n)) + \mu_n\} \\
&\quad - \alpha_{n,0}\alpha_{n,j} g\left(\|Jx_n - JT_i^n y_n\|\right) \\
&\le \phi(p, x_n) + \nu_n \sup_{u\in\Omega} \varsigma(\phi(u, x_n)) + \mu_n - \alpha_{n,0}\alpha_{n,j} g\left(\left\|Jx_n - JT_j^n y_n\right\|\right) \\
&= \phi(p, x_n) + \eta_n - \alpha_{n,0}\alpha_{n,j} g\left(\left\|Jx_n - JT_j^n y_n\right\|\right) \\
&\le \phi(p, x_n) + \eta_n.
\end{aligned} \tag{3.8}$$

Hence $p \in C_{n+1}$ and $\mathscr{F} \subset C_n$ for all $n \ge 0$.

(III) $\{x_n\}$ is a Cauchy sequence.

Since $x_n = \Pi_{C_n} x_0$ and $x_{n+1} = \Pi_{C_{n+1}} x_0 \in C_{n+1} \subset C_n$, we have that

$$\phi(x_n, x_0) \le \phi(x_{n+1}, x_0)$$

which implies that the sequence $\{\varphi(x_n, x_0)\}$ is nondecreasing and bounded, and so $\lim_{n\to\infty} \phi(x_n, x_0)$ exists. Hence for any positive integer m, using Lemma 2.2(i) we have

$$\phi(x_{n+m}, x_n) = \phi(x_{n+m}, \Pi_{C_n} x_0) \le \phi(x_{n+m}, x_0) - \phi(x_n, x_0)$$

for all $n \ge 0$. Since $\lim_{n\to\infty} \phi(x_n, x_0)$ exists, we obtain that

$$\phi(x_{n+m}, x_n) \to 0 (n \to \infty), \quad \forall m \ge 1. \tag{3.9}$$

Thus, by Lemma 2.3 we have that $||x_{n+m} - x_n|| \to 0$ as $n \to \infty$. This implies that the sequence $\{x_n\}$ is a Cauchy sequence in C. Since C is a nonempty closed subset of Banach space E, it is complete. Hence there exists an x^* in C such that

$$x_n \to x^* (n \to \infty). \tag{3.10}$$

(IV) We show that $x^* \in \bigcap_{i=1}^{\infty} F(T_i)$.

Since $x_{n+1} \in C_{n+1}$ by the structure of C_{n+1}, we have that

$$\phi(x_{n+1}, u_n) \le \phi(x_{n+1}, x_n) + \eta_n.$$

Again by (3.5), (3.9), and Lemma 2.3, we get that $\lim_{n\to\infty} \|x_{n+1} - u_n\| = 0$. But

$$\|x_n - u_n\| \le \|x_n - x_{n+1}\| + \|x_{n+1} - u_n\|.$$

Thus

$$\lim_{n\to\infty} \|x_n - u_n\| = 0. \tag{3.11}$$

This implies that $u_n \to x^*$ as $n \to \infty$. Since J is norm-to-norm uniformly continuous on bounded subsets of E we have that

$$\lim_{n\to\infty} \|Jx_n - Ju_n\| = 0. \tag{3.12}$$

From (3.8), (3.11), and (3.12), we have that

$$\begin{aligned}\alpha_{n,0}\alpha_{n,j}g\left(\left\|Jx_n - JT_j^n y_n\right\|\right) &\le \phi(p, x_n) - \phi(p, u_n) + \eta_n \\ &\le \|x_n - u_n\|\left(\|u_n\| + \|x_n\|\right) + 2\langle p, Ju_n - Jx_n\rangle + \eta_n \\ &\to 0\ (n \to \infty).\end{aligned} \tag{3.13}$$

In view of condition $\liminf_{n\to\infty} \alpha_{n,0}\alpha_{n,j} > 0$, we have that

$$g\left(\left\|Jx_n - JT_j^n y_n\right\|\right) \to 0(n \to \infty). \tag{3.14}$$

It follows from the property of g that

$$\left\|Jx_n - JT_j^n y_n\right\| \to 0(n \to \infty). \tag{3.15}$$

Since $x_n \to x^*$ and J is uniformly continuous, it yields $Jx_n \to Jx^*$. Hence from (3.15) we have

$$JT_j^n y_n \to Jx^*(n \to \infty), \quad \forall j \ge 1. \tag{3.16}$$

Since E^* is uniformly smooth, J^{-1} is uniformly continuous, it follows that

$$T_j^n y_n \to x^*(n \to \infty), \quad \forall j \ge 1. \tag{3.17}$$

Moreover, using inequalities (3.7) and (3.3), we obtain that

$$\begin{aligned}\phi(p, u_n) &\le \alpha_{n,0}\phi(p, x_n) + \sum_{i=1}^{\infty} \alpha_{n,i}\{\phi(p, y_n) + \nu_n\varsigma(\phi(p, y_n)) + \mu_n\} \\ &\le \alpha_{n,0}\phi(p, x_n) + \sum_{i=1}^{\infty} \alpha_{n,i}\left\{\phi(p, x_n) + 2\lambda\left(\frac{2\lambda}{c^2} - \delta\right)\|A_{n+1}x_n\|^2\right. \\ &\quad \left. + \nu_n\varsigma(\phi(p, y_n)) + \mu_n\right\} \\ &\le \phi(p, x_n) + \alpha_{n,j}2\lambda\left(\frac{2\lambda}{c^2} - \delta\right)\|A_{n+1}x_n\|^2 + \nu_n\varsigma(\phi(p, x_n)) + \mu_n \\ &\le \phi(p, x_n) + \alpha_{n,j}2\lambda\left(\frac{2\lambda}{c^2} - \delta\right)\|A_{n+1}x_n\|^2 + \eta_n.\end{aligned}$$

This implies that

$$2\alpha_{n,j}\lambda\left(\delta - \frac{2\lambda}{c^2}\right)\|A_{n+1}x_n\|^2 \le \phi(p, x_n) - \phi(p, u_n) + \eta_n \to 0 \text{ (as } n \to \infty).$$

By the assumption that for each $j \ge 1$, $\liminf_{n\to\infty} \alpha_{n,j} \ge \liminf_{n\to\infty} \alpha_{n,0}\alpha_{n,j} > 0$, and $\lambda(\delta - \frac{2\lambda}{c^2}) > 0$, hence we have

$$\lim_{n\to\infty} \|A_{n+1}x_n\|^2 = 0. \tag{3.19}$$

This together with (3.1) shows that

$$\begin{aligned}\lim_{n\to\infty} \left\|y_n - x^*\right\| &= \lim_{n\to\infty} \left\|\Pi_C J^{-1}(Jx_n - \lambda A_{n+1}x_n) - \Pi_C x^*\right\| \\ &\le \lim_{n\to\infty} \left\|J^{-1}(Jx_n - \lambda A_{n+1}x_n) - x^*\right\| = 0\end{aligned} \tag{3.20}$$

Furthermore, by the assumption that for each $j \geq 1$, T_j is uniformly L_j-Lipschitz continuous, hence we have

$$\begin{aligned} \left\| T_j^{n+1}y_n - T_j^n y_n \right\| \leq & \left\| T_j^{n+1}y_n - T_j^{n+1}y_{n+1} \right\| + \left\| T_j^{n+1}y_{n+1} - y_{n+1} \right\| \\ & + \left\| y_{n+1} - y_n \right\| + \left\| y_n - T_j^n y_n \right\| \\ \leq & \left(L_j + 1 \right) \left\| y_{n+1} - y_n \right\| + \left\| T_j^{n+1}y_{n+1} - y_{n+1} \right\| \\ & + \left\| y_n - T_j^n y_n \right\| . \end{aligned} \tag{3.21}$$

This together with (3.17) and (3.20), yields

$$\lim_{n\to\infty} \left\| T_j^{n+1}y_n - T_j^n y_n \right\| = 0.$$

Hence from (3.17) we have

$$\lim_{n\to\infty} T_j^{n+1}y_n = x^*,$$

i.e.,

$$\lim_{n\to\infty} T_j T_j^n y_n = x^*.$$

In view of (3.17) and the closeness of T_j it yields that $T_j x^* = x^*$ for all $j \geq 1$. This implies that $x^* \in \cap_{j=1}^{\infty} F(T_j)$.

(IV) Now, we prove that $x^* \in \cap_{n=1}^{N} A_n^{-1}(0)$.

It follows from (3.19) that

$$\lim_{n\to\infty} \|A_{n+1}x_n\| = 0. \tag{3.22}$$

Since $\lim_{n\to\infty} x_n = x^*$, we have that for every subsequence $\{x_{n_j}\}_{j\geq 1} \subset \{x_n\}_{n\geq 0}$, $\lim_{j\to\infty} x_{n_j} = x^*$ and

$$\lim_{j\to\infty} A_{n_j+1}x_{n_j} = 0. \tag{3.23}$$

Let $\{n_q\}_{q\ \geq 1} \subset \mathbb{N}$ be an increasing sequence of natural numbers such that $A_{n_q+1} = A_1, \forall_q \in \mathbb{N}$. Then $\lim_{p\to\infty} \left\| x_{n_q} - x^* \right\| = 0$ and

$$0 = \lim_{q\to\infty} A_{n_q+1}x_{n_q} = \lim_{q\to\infty} A_1 x_{n_q}.$$

Since A_1 is γ-inverse strongly monotone, it is $\frac{1}{\gamma}$-Lipschitz continuous and thus

$$A_1 x^* = A_1 \left(\lim_{q\to\infty} x_{n_q} \right) = \lim_{q\to\infty} A_1 x_{n_q} = 0.$$

Hence

$$x^* \in A_1^{-1}(0).$$

Continuing this process, we obtain that $x^* \in A_i^{-1}(0), \forall i = 1, 2, \ldots, N$, $\forall i = 1, 2, \ldots, N$. Hence

$$x^* \in \bigcap_{n=1}^{N} A_n^{-1}(0).$$

(V) Next, we prove that $x^* \in \bigcap_{m=1}^{M} \text{GMEP}(F_m, B_m, \Psi_m)$.

Denote

$$\mathscr{S}_n^m = K_{r_{m,n}}^{\Gamma_m} K_{r_{m-1,n}}^{\Gamma_{m-1}} \dots K_{r_{2,n}}^{\Gamma_2} K_{r_{1,n}}^{\Gamma_1}, \quad m = 1, 2, \dots, M$$

and $\mathscr{S}_n^0 = I$ for all $n \geq 1$. By Lemma 2.6, for any $p \in \mathscr{F}$, we have

$$\begin{aligned}\phi(\mathscr{S}_n^m z_n, \mathscr{S}_n^{m-1} z_n) &\leq \phi(p, \mathscr{S}_n^{m-1} z_n) - \phi(p, \mathscr{S}_n^m z_n) \\ &\leq \phi(p, z_n) - \phi(p, \mathscr{S}_n^m z_n) \\ &\leq \phi(p, x_n) + \eta_n - \phi(p, \mathscr{S}_n^m z_n) (\text{by (3.8) and (3.7)}) \\ &\leq \phi(p, x_n) + \eta_n - \phi(p, u_n) (\text{by (3.7)}) \to 0 \ (\text{as } n \to \infty).\end{aligned} \tag{3.24}$$

From (3.13) we have that $\lim_{n\to\infty} \phi(\mathscr{S}_n^m z_n, \mathscr{S}_n^{m-1} z_n) = 0$. Since E is 2-uniformly convex and uniformly smooth Banach space and $\{z_n\}$ is bounded, we have that

$$\lim_{n\to\infty} \left\| \mathscr{S}_n^m z_n - \mathscr{S}_n^{m-1} z_n \right\| = 0, \quad m = 1, 2, \dots, M. \tag{3.25}$$

Next we prove that for each $m = 1, 2,..., M$

$$\mathscr{S}_n^m z_n \to x^* (\text{as } n \to \infty).$$

In fact, since $x_n \to x^*$ and $u_n \to x^*$ (as $n \to \infty$), if $m = M$ then we have

$$\lim_{n\to\infty} \left\| \mathscr{S}_n^M z_n - \mathscr{S}_n^{M-1} z_n \right\| = \lim_{n\to\infty} \left\| u_n - \mathscr{S}_n^{M-1} z_n \right\| = 0.$$

This implies that $\mathscr{S}_n^{M-1} z_n \to x^*$. By induction, the conclusion can be obtained.

Since J is norm-to-norm uniformly continuous on bounded subsets of E, from (3.25) we have

$$\lim_{n\to\infty} \left\| J\mathscr{S}_n^m z_n - J\mathscr{S}_n^{m-1} z_n \right\| = 0, \quad \forall m = 1, 2, \dots, M. \tag{3.26}$$

Again since $\{r_{m,n}\}_{m=1}^M \subset [d, \infty)$ for some $d > 0$, we have that

$$\lim_{n\to\infty} \frac{\left\| J\mathscr{S}_n^m z_n - J\mathscr{S}_n^{m-1} z_n \right\|}{r_{m,n}} = 0, \quad \forall m = 1, 2, \dots, M. \tag{3.27}$$

In the proof of Lemma 2.6 we have proved that the function Γ_m, $m = 1, 2,... M$ defined by (3.2) satisfies the condition (A1)-(A4) and

$$\Gamma_m(\mathscr{S}_n^m z_n, y) + \frac{1}{r_{m,n}} \langle y - \mathscr{S}_n^m z_n, J\mathscr{S}_n^m z_n - J\mathscr{S}_n^{m-1} z_n \rangle \geq 0, \quad \forall y \in C,$$

Therefore for any $y \in C$ we have

$$\frac{1}{r_{m,n}} \langle y - \mathscr{S}_n^m z_n, J\mathscr{S}_n^m z_n - J\mathscr{S}_n^{m-1} z_n \rangle \geq -\Gamma_m(\mathscr{S}_n^m z_n, y) \geq \Gamma_m(y, \mathscr{S}_n^m z_n). \tag{3.28}$$

This implies that

$$\begin{aligned}\Gamma_m(y, \mathscr{S}_n^m z_n) &\leq \frac{1}{r_{m,n}} \langle y - \mathscr{S}_n^m z_n, J\mathscr{S}_n^m z_n - J\mathscr{S}_n^{m-1} z_n \rangle \\ &\leq (M_1 + \|y\|) \frac{\|J\mathscr{S}_n^m z_n - J\mathscr{S}_n^{m-1} z_n\|}{r_{m,n}},\end{aligned} \tag{3.29}$$

for some constant $M_1 > 0$. Since the function $y \mapsto \Gamma_m(x, y)$ is a convex and lower semi-continuous, from (3.27) and (3.29) we have

$$\Gamma_m(y, x^*) \leq \liminf_{n \to \infty} \Gamma_m(y, \mathscr{S}_n^m z_n) \leq 0, \quad \forall y \in C. \tag{3.30}$$

For any $t \in (0, 1]$ and $y \in C$, then $y_t = ty + (1 - t)x^* \in C$. Since Γ_m satisfies conditions (A1), (A4),
from (3.30) we have

$$\begin{aligned}0 = \Gamma_m(y_t, y_t) &\leq t\Gamma_m(y_t, y) + (1-t)\Gamma_m(y_t, x^*) \\ &\leq t\Gamma_m(y_t, y), \quad \forall m = 1, 2, \ldots M.\end{aligned}$$

Deleting t and then letting $t \to 0$, by condition (A3) we have

$$0 \leq \Gamma_m(x^*, y), \quad \forall y \in C, \quad \forall m = 1, 2, \ldots M,$$

i.e., for each $m = 1, 2, \ldots, M$ we have

$$F_m(x^*, y) + \langle y - x^*, B_m x^* \rangle + \Psi_m(y) - \Psi_m(x^*) \geq 0, \quad \forall y \in C.$$

This implies that $x^* \in \mathrm{GMEP}(F_m, B_m, \Psi_m)$, for each $m = 1, 2, \ldots, M$. Therefore, we have that

$$x^* \in \bigcap_{m=1}^{M} \mathrm{GMEP}(F_m, B_m, \Psi_m).$$

This completes the proof of Theorem 3.1.

4 Application

It is well known that the following *Hammerstian type equation*

$$u + Kfu = 0, \tag{4.1}$$

where K is a linear operator and f is a nonlinear Nemytskii operator, plays a crucial role in the theory of optimal control systems (see, example, [21]). Several existence and uniqueness theorems have been proved for equation (4.1) (see, for examples, [22-25].

We are now ready to give an application of Theorem 3.1 to an iterative solution of the nonlinear Hammerstein type Equation (4.1).

Theorem 4.1 Let E be a real Banach space with a dual E^* such that $X = E \times E^*$ is a 2-uniformly convex and uniformly smooth real Banach space with norm $\|z\|_X^2 = \|u\|_E^2 + \|v\|_{E^*}^2$, $z = (u, v) \in X$. Let C be a nonempty closed convex subset of X. Let f: $E \to E^*$ and K: $E^* \to E$ with $D(K) = f(E) = E^*$ be continuous monotone type operators such that Equation (4.1) has a solution in E and such that the mapping A: $X \to X^*$ defined by

$$Az := A(u, v) = (fu - v, u + Kv) \tag{4.2}$$

is δ-inverse strongly monotone. Let B: $C \to X^*$ be a β-inverse strongly monotone mappings. Let $F : C \times C \to \mathscr{R}$ be a bifunction satisfying condition (A1), (A2)", (A3), (A4) in Theorem 3.1 and $\Psi : C \to \mathscr{R}$ be a lower semi-continuous and convex function. Let T: $C \to C$ be a closed and total quasi-φ-asymptotically nonexpansive mapping with nonnegative real sequences $\{v_n\}$, $\{\mu_n\}$ and a strictly increasing continuous functions $\varsigma : \mathscr{R}^+ \to \mathscr{R}^+$ such that $v_n \to 0$, $\mu_n \to 0$ (as $n \to \infty$) and $\zeta(0) = 0$, and T is uniformly L-Lipschitzian. If $\mathscr{F} := F(T) \cap A^{-1}(0) \cap \mathrm{GMEP}(F, B, \Psi)$ is a nonempty bounded subset of C, then the sequence $\{x_n\}$ defined by

$$\begin{cases} x_0 \in C_0 = C, \\ y_n = \Pi_C J^{-1}(Jx_n - \lambda A x_n), \\ z_n = J^{-1}(\alpha_n J x_n + (1 - \alpha_n J T^n y_n), \\ u_n = K_{r_n}^{\Gamma} z_n, \\ C_{n+1} = \{v \in C_n : \phi(v, u_n) \le \phi(v, x_n) + \eta_n\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0, \quad n \ge 0, \end{cases} \tag{4.3}$$

converges strongly to some point $x^* \in \mathscr{F}$ where

$$\eta_n = v_n \sup_{u \in \mathscr{F}} \varsigma(\phi(u, x_n)) + \mu_n, \quad \forall n \ge 1, \tag{4.4}$$

$K_{r_n}^{\Gamma} : E \to C$ is the mapping defined by (2.6) and

$$\Gamma(x, y) = F(x, y) + \Psi(y) - \Psi(x) + \langle y - x,\ Bx \rangle, \quad \forall x, y \in C,$$

$r_n \in [d, \infty)$, $n \ge 1$ for some $d > 0$, $0 < \lambda < \frac{c^2 \delta}{2}$, where c is the 2-uniformly convex constant of E and $\{\alpha_n\}$ is the sequence in (0, 1) with $\liminf_{n\to\infty} \alpha_n(1 - \alpha_n) > 0$.

Remark 4.1 Since $x^* \in \mathscr{F}$, therefore we have $x^* \in A^{-1}(0)$, i.e., $Ax^* = 0$. Since $x^* = (u^*, v^*)$ for some $u^* \in E$, $v^* \in E^*$, we have $Ax^* = A(u^*, v^*) = (fu^* - v^*, u^* + Kv^*) = (0, 0)$. This implies that $fu^* - v^* = 0$ and $u^* + Kv^* = 0$, i.e., $u^* + Kfu^* = 0$. Hence u^* is a solution of the nonlinear Hammerstein type equation (4.1) and $x_n \to x^*$ (as $n \to \infty$).

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Competing interests
The authors declare that they have no competing interests.

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