Bahadur Zada et al. *Journal of Inequalities and Applications* (2017) 2017:22
DOI 10.1186/s13660-016-1286-7

Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH **Open Access**

Existence of unique common solution to the system of non-linear integral equations via fixed point results in incomplete metric spaces

Mian Bahadur Zada[1], Muhammad Sarwar[1*] and Stojan Radenović[2]

*Correspondence: sarwarswati@gmail.com
[1]Department of Mathematics, University of Malakand, Chakdara, Dir(L), Pakistan
Full list of author information is available at the end of the article

Abstract

In this article, we apply common fixed point results in incomplete metric spaces to examine the existence of a unique common solution for the following systems of Urysohn integral equations and Volterra-Hammerstein integral equations, respectively:

$$u(s) = \phi_i(s) + \int_a^b K_i(s, r, u(r))\, dr,$$

where $s \in (a,b) \subseteq \mathbb{R}$; $u, \phi_i \in C((a,b), \mathbb{R}^n)$ and $K_i : (a,b) \times (a,b) \times \mathbb{R}^n \to \mathbb{R}^n$, $i = 1, 2, \ldots, 6$ and

$$u(s) = p_i(s) + \lambda \int_0^t m(s,r) g_i(r, u(r))\, dr + \mu \int_0^\infty n(s,r) h_i(r, u(r))\, dr,$$

where $s \in (0, \infty)$, $\lambda, \mu \in \mathbb{R}$, u, p_i, $m(s,r)$, $n(s,r)$, $g_i(r, u(r))$ and $h_i(r, u(r))$, $i = 1, 2, \ldots, 6$, are real-valued measurable functions both in s and r on $(0, \infty)$.

MSC: 47H10; 54H25

Keywords: common fixed point; weakly compatible maps; common (CLR)-property; common ($E.A$)-property; Urysohn integral equations; Volterra-Hammerstein integral equations

1 Introduction and preliminaries

Mathematical models are very powerful and important parts of the mathematical analysis with numerous applications to real world problems. Several problems that appear in applied mathematics, physical sciences, geology, mechanics, engineering, economics, and biology generate mathematical models interpreted by functional equations, integral equations, matrix equations, and differential equations *etc.* There are multifarious and advanced methods, focusing on the existence of unique solutions to these models. To handle the existence of unique solution to such equations, one of these methods is the fixed point method; for example, refer to [1–4]. In metric fixed point theory the first remarkable result was given by Banach, usually known as the Banach contraction principle. This principle is a prominent tool for solving problems in non-linear analysis. Several mathematicians improved and extended this principle by modifying the interpretation and pattern of the metric function for instance: cone metric spaces [5], G-metric spaces [6], partial metric spaces [7] and fuzzy metric spaces [8] *etc.* After the proper introduction of cone metric space by Huang and Zhong [5], there was a drawback that fixed point results under rational type contractions are unsubstantial in a cone metric space as it is a vector-valued metric. Azam *et al.* [9] offered the conception of a complex-valued metric space for finding the fixed point results satisfying rational type contractive conditions.

Definition 1.1 ([9]) Let Y be non-empty set and $\mathbb{C}_+ = \{c \in \mathbb{C} : c \succsim 0\}$. Then the mapping $d : Y \times Y \to \mathbb{C}_+$ is a complex-valued metric if it satisfies the following axioms:
(1) $d(c_1, c_2) = 0 \Leftrightarrow c_1 = c_2$;
(2) $d(c_1, c_2) = d(c_2, c_1)$, for all $c_1, c_2 \in Y$;
(3) $d(c_1, c_2) \precsim d(c_1, c_3) + d(c_3, c_2)$, for all $c_1, c_2, c_3 \in Y$.
The set Y together with d is called a complex-valued metric space.

In this setting, Azam *et al.* [9] generalized the Banach contraction principle for two self-maps under rational type contraction. Inspired by the impact of a complex-valued metric space, several authors [4, 9–12] proceeded with the investigation of common fixed point results.

Many mathematicians applied fixed point methods to the existence of unique solutions to non-linear integral equations, for example, refer to [3, 4, 9, 13–16]. Particularly, Sintunavarat *et al.* [4] and Rashwan and Saleh [17] established fixed point results to find the existence of a unique common solution to a system of Urysohn integral equations. On the other hand, Pathak *et al.* [18] and Rashwan and Saleh [17] studied the existence of unique common solution to the system of Volterra-Hammerstein non-linear integral equations.

Throughout this manuscript Y represents a complex-valued metric space, unless otherwise specified. For two self-maps f_1 and f_2 defined on a non-empty set Y, $w \in Y$ is a common fixed point of f_1 and f_2 if $f_1 w = f_2 w = w$. To study common fixed points, Jungck [19] initiated the concept of weak compatibility of maps thus: f_1 and f_2 on Y are weakly compatible maps if $f_1 f_2 w = f_2 f_1 w$ whenever $f_1 w = f_2 w$, for some $w \in Y$. In the study of common fixed point results of weakly compatible mappings we often require the assumption of the continuity of mappings or the completeness of the underlying space. Regarding this Aamri and Moutawakil [20] relaxed these conditions by introducing the notion of the $(E.A)$-property. In 2011, the new notion of Common Limit in the Range property (for short (CLR)-property) was given by Sintunavarat and Kumam [21], which does not enforce the above mention conditions. Liu *et al.* [22] extended the $(E.A)$-property [20] to the common $(E.A)$-property and Imdad *et al.* [23] extended the (CLR)-property [21] to common (CLR)-property. Sarwar and Bahadur Zada [12] defined these views in the complex-valued metric space as follows.

Definition 1.2 Let $f_1, f_2, f_3, f_4 : Y \to Y$ be four maps. If there are two sequences $\{z_n\}$ and $\{w_n\}$ in Y. Then the pairs (f_1, f_3) and (f_2, f_4) satisfy

(1) the common $(E.A)$-property if

$$\lim_{n\to\infty} f_1 z_n = \lim_{n\to\infty} f_3 z_n = \lim_{n\to\infty} f_2 w_n = \lim_{n\to\infty} f_4 w_n = t \in Y;$$

(2) the common $(CLR_{f_3 f_4})$-property if

$$\lim_{n\to\infty} f_1 z_n = \lim_{n\to\infty} f_3 z_n = \lim_{n\to\infty} f_2 w_n = \lim_{n\to\infty} f_4 w_n = t \in f_3(Y) \cap f_4(Y).$$

Note that the $(E.A)$-property tolerates the condition of closeness of the range of subspaces of the involved mappings. However, the significance of the (CLR)-property reveals that closeness of the range of subspaces is not essential.

Sarwar and Bahadur Zada [12] established the following common fixed point results.

Theorem 1.3 *Let f_1, f_2, f_3, f_4, f_5, and f_6 be six maps on Y such that*

(1) $f_1(Y) \subseteq f_4(Y), f_1(Y) \subseteq f_5(Y), f_2(Y) \subseteq f_3(Y)$ *and* $f_2(Y) \subseteq f_6(Y)$;

(2) *for all $u, v \in Y$ and $0 < k < 1$,*

$$d(f_1u,f_2v) \precsim k\left\{\frac{d(f_3u,f_1u)d(f_6u,f_1u)d(f_3u,f_2v)d(f_6u,f_2v)}{1+d(f_3u,f_2v)d(f_6u,f_2v)+d(f_4v,f_1u)d(f_5v,f_1u)} + \frac{d(f_4v,f_2v)d(f_5v,f_2v)d(f_4v,f_1u)d(f_5v,f_1u)}{1+d(f_3u,f_2v)d(f_6u,f_2v)+d(f_4v,f_1u)d(f_5v,f_1u)}\right\};$$

(3) *the pairs (f_1,f_3), (f_2,f_4), (f_1,f_6), and (f_2,f_5) are weakly compatible*;

(4) *either both the pairs (f_1,f_3) and (f_1,f_6) satisfies common (CLR_{f_1})-property or both the pairs (f_2,f_4) and (f_2,f_5) satisfies common (CLR_{f_2})-property.*

Then f_1, f_2, f_3, f_4, f_5, and f_6 have a unique common fixed point in Y.

Theorem 1.4 *Let f_1, f_2, f_3, f_4, f_5, and f_6 be six maps on Y such that all the conditions of Theorem* 1.3 *except condition* (4) *holds. In addition if either the pairs (f_1,f_3) and (f_1,f_6) or the pairs (f_2,f_4) and (f_2,f_5) satisfy the common $(E.A)$-property such that either $f_4(Y)$ and $f_5(Y)$ or $f_3(Y)$ and $f_6(Y)$ are closed subspaces of Y, then f_1, f_2, f_3, f_4, f_5, and f_6 have a unique common fixed point in Y.*

The aim of this manuscript is to study the existence of unique common solution for the systems of:

- Urysohn integral equations in complex-valued metric spaces,
- Volterra-Hammerstein integral equations in ordinary metric spaces.

2 Existence of unique common solution to the systems of Urysohn integral equations

Our plan is to apply Theorem 1.3 to the existence of a unique common solution to the following system:

$$u(s) = \phi_i(s) + \int_a^b K_i\big(s, r, u(r)\big)\,dr, \tag{2.1}$$

where $s \in (a,b) \subseteq \mathbb{R}$; $u, \phi_i \in C((a,b),\mathbb{R}^n)$ and $K_i : (a,b) \times (a,b) \times \mathbb{R}^n \to \mathbb{R}^n$, $i = 1, 2, \ldots, 6$.

Let us denote

$$\Omega_i\big(u(s)\big) = \int_a^b K_i\big(s, r, u(r)\big)\,dr,$$

where $i = 1, 2, \ldots, 6$.

Assume that the following conditions hold:

(C_1) for $i = 4, 5$,

$$\Omega_1 u(s) + \phi_1(s) + \phi_i(s) - \Omega_i\big(\Omega_1 u(s) + \phi_1(s) + \phi_i(s)\big) = 0,$$

(C_2) for $j = 3, 6$,

$$\Omega_2 u(s) + \phi_2(s) + \phi_j(s) - \Omega_j\big(\Omega_2 u(s) + \phi_2(s) + \phi_j(s)\big) = 0,$$

(C_3) for $j = 3, 6$,

$$\phi_1(s) + 3\phi_j(s) + 2\Omega_j u(s) + \Omega_1\big(\Omega_1 u(s) + \phi_1(s)\big) + \Omega_j\big(2u(s) - \Omega_j u(s) - \phi_j(s)\big) = 4u(s),$$

(C_4) for $i = 4, 5$,

$$\phi_2(s) + 3\phi_i(s) + 2\Omega_i u(s) + \Omega_2\big(\Omega_2 u(s) + \phi_2(s)\big) + \Omega_i\big(2u(s) - \Omega_i u(s) - \phi_i(s)\big) = 4u(s).$$

Let $Y = C((a, b), \mathbb{R}^n)$, $a > 0$ be an incomplete complex-valued metric space with metric

$$d(u, v) = \max_{s\in(a,b)} \big\|u(s) - v(s)\big\|_\infty \sqrt{1 + a^2}\, e^{i \arctan a}, \quad \text{for all } u, v \in Y.$$

Define six operators $f_1, f_2, f_3, f_4, f_5, f_6 : Y \to Y$ by

$$\left\{\begin{array}{l} f_1 u(s) = \Omega_1 u(s) + \phi_1(s), \\ f_2 u(s) = \Omega_2 u(s) + \phi_2(s), \\ f_3 u(s) = 2u(s) - \Omega_3 u(s) - \phi_3(s), \\ f_4 u(s) = 2u(s) - \Omega_4 u(s) - \phi_4(s), \\ f_5 u(s) = 2u(s) - \Omega_5 u(s) - \phi_5(s), \\ f_6 u(s) = 2u(s) - \Omega_6 u(s) - \phi_6(s). \end{array}\right\} \tag{2.2}$$

Now, we are in a position to formulate the existence results.

Theorem 2.1 *Under the assumptions* (C_1)-(C_4) *if*

(1) *there exist two sequences* $\{z_n\}$ *and* $\{w_n\}$ *in* Y *such that*

$$\lim_{n\to\infty} f_2 z_n = \lim_{n\to\infty} f_4 z_n = \lim_{n\to\infty} f_2 w_n = \lim_{n\to\infty} f_5 w_n = z \in f_2(Y); \tag{2.3}$$

(2) *for each* $u, v \in Y$ *and* $0 < \lambda < 1$,

$$\Upsilon_1 \sqrt{1+a^2} e^{i \arctan a} \precsim \lambda \left\{ \frac{\Upsilon_2 \times \Upsilon_3 \times \Upsilon_4 \times \Upsilon_5 + \Upsilon_6 \times \Upsilon_7 \times \Upsilon_8 \times \Upsilon_9}{1 + (\max_{s\in(a,b)} \Upsilon_4) \cdot (\max_{s\in(a,b)} \Upsilon_5) + (\max_{s\in(a,b)} \Upsilon_8) \cdot (\max_{s\in(a,b)} \Upsilon_9)} \right\},$$

where

$$\begin{aligned}
\Upsilon_1 &= \left\| \Omega_1 u(s) - \Omega_2 v(s) + \phi_1(s) - \phi_2(s) \right\|_\infty \sqrt{1+a^2} e^{i \arctan a}, \\
\Upsilon_2 &= \left\| 2u(s) - \Omega_3 u(s) - \Omega_1 u(s) - \phi_1(s) - \phi_3(s) \right\|_\infty \sqrt{1+a^2} e^{i \arctan a}, \\
\Upsilon_3 &= \left\| 2u(s) - \Omega_6 u(s) - \Omega_1 u(s) - \phi_1(s) - \phi_6(s) \right\|_\infty \sqrt{1+a^2} e^{i \arctan a}, \\
\Upsilon_4 &= \left\| 2u(s) - \Omega_3 u(s) - \Omega_2 v(s) - \phi_2(s) - \phi_3(s) \right\|_\infty \sqrt{1+a^2} e^{i \arctan a}, \\
\Upsilon_5 &= \left\| 2u(s) - \Omega_6 u(s) - \Omega_2 v(s) - \phi_2(s) - \phi_6(s) \right\|_\infty \sqrt{1+a^2} e^{i \arctan a}, \\
\Upsilon_6 &= \left\| 2v(s) - \Omega_4 v(s) - \Omega_2 v(s) - \phi_2(s) - \phi_4(s) \right\|_\infty \sqrt{1+a^2} e^{i \arctan a}, \\
\Upsilon_7 &= \left\| 2v(s) - \Omega_5 v(s) - \Omega_2 v(s) - \phi_2(s) - \phi_5(s) \right\|_\infty \sqrt{1+a^2} e^{i \arctan a}, \\
\Upsilon_8 &= \left\| 2v(s) - \Omega_4 v(s) - \Omega_1 u(s) - \phi_1(s) - \phi_4(s) \right\|_\infty \sqrt{1+a^2} e^{i \arctan a}, \\
\Upsilon_9 &= \left\| 2v(s) - \Omega_5 v(s) - \Omega_1 u(s) - \phi_1(s) - \phi_5(s) \right\|_\infty \sqrt{1+a^2} e^{i \arctan a};
\end{aligned}$$

(3) $f_1(Y) \subseteq f_4(Y)$, $f_1(Y) \subseteq f_5(Y)$, $f_2(Y) \subseteq f_3(Y)$, *and* $f_2(Y) \subseteq f_6(Y)$ *such that* (f_1,f_3), (f_2,f_4), (f_2,f_5), *and* (f_1,f_6) *are weakly compatible.*

Then the system (2.1) *of Urysohn integral equations has a unique common solution.*

Proof Notice that the system (2.1) of Urysohn integral equations has a unique common solution if and only if the system (2.2) of operators has a unique common fixed point.

Now,

$$\left\{ \begin{aligned}
d(f_1u,f_2v) &= \max_{s\in(a,b)} \left\| \Omega_1 u(s) - \Omega_2 v(s) + \phi_1(s) - \phi_2(s) \right\|_\infty \sqrt{1+a^2} e^{i \arctan a}, \\
d(f_3u,f_1u) &= \max_{s\in(a,b)} \left\| 2u(s) - \Omega_3 u(s) - \Omega_1 u(s) - \phi_1(s) - \phi_3(s) \right\|_\infty \sqrt{1+a^2} e^{i \arctan a}, \\
d(f_6u,f_1u) &= \max_{s\in(a,b)} \left\| 2u(s) - \Omega_6 u(s) - \Omega_1 u(s) - \phi_1(s) - \phi_6(s) \right\|_\infty \sqrt{1+a^2} e^{i \arctan a}, \\
d(f_3u,f_2v) &= \max_{s\in(a,b)} \left\| 2u(s) - \Omega_3 u(s) - \Omega_2 v(s) - \phi_2(s) - \phi_3(s) \right\|_\infty \sqrt{1+a^2} e^{i \arctan a}, \\
d(f_6u,f_2v) &= \max_{s\in(a,b)} \left\| 2u(s) - \Omega_6 u(s) - \Omega_2 v(s) - \phi_2(s) - \phi_6(s) \right\|_\infty \sqrt{1+a^2} e^{i \arctan a}, \\
d(f_4v,f_2v) &= \max_{s\in(a,b)} \left\| 2v(s) - \Omega_4 v(s) - \Omega_2 v(s) - \phi_2(s) - \phi_4(s) \right\|_\infty \sqrt{1+a^2} e^{i \arctan a}, \\
d(f_5v,f_2v) &= \max_{s\in(a,b)} \left\| 2v(s) - \Omega_5 v(s) - \Omega_2 v(s) - \phi_2(s) - \phi_5(s) \right\|_\infty \sqrt{1+a^2} e^{i \arctan a}, \\
d(f_4v,f_1u) &= \max_{s\in(a,b)} \left\| 2v(s) - \Omega_4 v(s) - \Omega_1 u(s) - \phi_1(s) - \phi_4(s) \right\|_\infty \sqrt{1+a^2} e^{i \arctan a}, \\
d(f_5v,f_1u) &= \max_{s\in(a,b)} \left\| 2v(s) - \Omega_5 v(s) - \Omega_1 u(s) - \phi_1(s) - \phi_5(s) \right\|_\infty \sqrt{1+a^2} e^{i \arctan a}.
\end{aligned} \right\} \quad (2.4)$$

From condition (2) of Theorem 2.1, we have

$$\begin{aligned}&\Upsilon_1\sqrt{1+a^2}e^{i\arctan a}\\&\quad\precsim\lambda\left\{\frac{\Upsilon_2\times\Upsilon_3\times\Upsilon_4\times\Upsilon_5+\Upsilon_6\times\Upsilon_7\times\Upsilon_8\times\Upsilon_9}{1+(\max_{s\in(a,b)}\Upsilon_4)\cdot(\max_{s\in(a,b)}\Upsilon_5)+(\max_{s\in(a,b)}\Upsilon_8)\cdot(\max_{s\in(a,b)}\Upsilon_9)}\right\},\end{aligned}$$

which implies that

$$\begin{aligned}&\max_{s\in(a,b)}\Upsilon_1\sqrt{1+a^2}e^{i\arctan a}\\&\quad\precsim\lambda\left\{\frac{(\max_{s\in(a,b)}\Upsilon_2)\cdot(\max_{s\in(a,b)}\Upsilon_3)\cdot(\max_{s\in(a,b)}\Upsilon_4)\cdot(\max_{s\in(a,b)}\Upsilon_5)}{1+(\max_{s\in(a,b)}\Upsilon_4)\cdot(\max_{s\in(a,b)}\Upsilon_5)+(\max_{s\in(a,b)}\Upsilon_8)\cdot(\max_{s\in(a,b)}\Upsilon_9)}\right.\\&\qquad\left.+\frac{(\max_{s\in(a,b)}\Upsilon_6)\cdot(\max_{s\in(a,b)}\Upsilon_7)\cdot(\max_{s\in(a,b)}\Upsilon_8)\cdot(\max_{s\in(a,b)}\Upsilon_9)}{1+(\max_{s\in(a,b)}\Upsilon_4)\cdot(\max_{s\in(a,b)}\Upsilon_5)+(\max_{s\in(a,b)}\Upsilon_8)\cdot(\max_{s\in(a,b)}\Upsilon_9)}\right\},\end{aligned}$$

using (2.4), we obtain

$$\begin{aligned}d(f_1u,f_2v)\precsim\lambda&\left\{\frac{d(f_3u,f_1u)d(f_6u,f_1u)d(f_3u,f_2v)d(f_6u,f_2v)}{1+d(f_3u,f_2v)d(f_6u,f_2v)+d(f_4v,f_1u)d(f_5v,f_1u)}\right.\\&\left.+\frac{d(f_4v,f_2v)d(f_5v,f_2v)d(f_4v,f_1u)d(f_5v,f_1u)}{1+d(f_3u,f_2v)d(f_6u,f_2v)+d(f_4v,f_1u)d(f_5v,f_1u)}\right\}.\end{aligned}$$

Now, to show that $f_1(Y)\subseteq f_4(Y)$, we have

$$\begin{aligned}f_4\bigl(f_1u(s)+\phi_4(s)\bigr)&=2\bigl[f_1u(s)+\phi_4(s)\bigr]-\Omega_4\bigl(f_1u(s)+\phi_4(s)\bigr)-\phi_4(s)\\&=f_1u(s)+f_1u(s)+\phi_4(s)-\Omega_4\bigl(f_1u(s)+\phi_4(s)\bigr)\\&=f_1u(s)+\Omega_1u(s)+\phi_1(s)+\phi_4(s)-\Omega_4\bigl(\Omega_1u(s)+\phi_1(s)+\phi_4(s)\bigr).\end{aligned}$$

Using (C_1), we get $f_4(f_1u(s)+\phi_4(s))=f_1u(s)$, which implies that $f_1(Y)\subseteq f_4(Y)$. Similarly, one can prove that $f_1(Y)\subseteq f_5(Y)$, $f_2(Y)\subseteq f_3(Y)$ and $f_2(Y)\subseteq f_6(Y)$.

Next, we need to show the weak compatibility of the pair (f_1,f_3). For this, we have

$$\begin{aligned}\bigl\|f_3f_1u(s)-f_1f_3u(s)\bigr\|&=\bigl\|f_3\bigl(\Omega_1u(s)+\phi_1(s)\bigr)-f_1\bigl(2u(s)-\Omega_3u(s)-\phi_3(s)\bigr)\bigr\|\\&=\bigl\|2\bigl(\Omega_1u(s)+\phi_1(s)\bigr)-\Omega_3\bigl(\Omega_1u(s)+\phi_1(s)\bigr)-\phi_3(s)\\&\quad-\Omega_1\bigl(2u(s)-\Omega_3u(s)-\phi_3(s)\bigr)-\phi_1(s)\bigr\|.\end{aligned}\tag{2.5}$$

If $f_1u(s)=f_3u(s)$, for $u(s)\in Y$. Then $\Omega_1u(s)+\phi_1(s)=2u(s)-\Omega_3u(s)-\phi_3(s)$, thus (2.5) becomes

$$\begin{aligned}\bigl\|f_3f_1u(s)-f_1f_3u(s)\bigr\|&=\bigl\|2\bigl(2u(s)-\Omega_3u(s)-\phi_3(s)\bigr)-\Omega_3\bigl(2u(s)-\Omega_3u(s)-\phi_3(s)\bigr)\\&\quad-\phi_3(s)-\Omega_1\bigl(\Omega_1u(s)+\phi_1(s)\bigr)-\phi_1(s)\bigr\|\\&=\bigl\|4u(s)-2\Omega_3u(s)-3\phi_3(s)-\Omega_3\bigl(2u(s)-\Omega_3u(s)-\phi_3(s)\bigr)\\&\quad-\Omega_1\bigl(\Omega_1u(s)+\phi_1(s)\bigr)-\phi_1(s)\bigr\|,\end{aligned}$$

with the help of (C_3), we get $\|f_3f_1u(s) - f_1f_3u(s)\| = 0$, which implies that $f_3f_1u(s) = f_1f_3u(s)$, whenever $f_1u(s) = f_3u(s)$. Thus (f_1,f_3) is weakly compatible. In a similarly way one can easily show the weakly compatibility of the pairs (f_2,f_4), (f_1,f_6) and (f_2,f_5). Also, from condition (1) of Theorem 2.1, the pairs (f_2,f_4) and (f_2,f_5) satisfy the common (CLR_{f_2})-property.

Thus by Theorem 2.1 we can find a unique common fixed point of f_1, f_2, f_3, f_4, f_5, and f_6 in Y, that is, the system (2.1) of Urysohn integral equations has a unique common solution in Y. □

In the next result we use the common $(E.A)$-property and the proof is simple, so we omit it.

Theorem 2.2 *Under the assumptions* (C_1)-(C_4) *and the conditions* (2), (3) *of Theorem* 2.1, *if there exist two sequences* $\{z_n\}$ *and* $\{w_n\}$ *in* Y *such that*

$$\lim_{n\to\infty} f_2z_n = \lim_{n\to\infty} f_4z_n = \lim_{n\to\infty} f_2w_n = \lim_{n\to\infty} f_5w_n = z, \quad \text{for some } z \in Y, \tag{2.6}$$

and both $f_4(Y)$ *and* $f_5(Y)$ *are closed subspaces of* Y, *then the system* (2.1) *of Urysohn integral equations has a unique common solution.*

3 Existence of unique common solution to the systems of Volterra-Hammerstein integral equations

In this section, we present the real-valued metric version of Theorem 1.3 and Theorem 1.4 and the proof can easily be obtained, so we omit its proof here.

Corollary 3.1 *Let* $f_1, f_2, f_3, f_4, f_5, f_6$ *be six maps on a metric space* (Z, d) *such that*

(1) $f_1(Z) \subseteq f_4(Z), f_1(Z) \subseteq f_5(Z), f_2(Z) \subseteq f_3(Z)$, *and* $f_2(Z) \subseteq f_6(Z)$;

(2) *for all* $u, v \in Z$ *and* $0 < k < 1$,

$$\begin{aligned} d(f_1u,f_2v) \leq \lambda\Bigg\{ &\frac{d(f_3u,f_1u)d(f_6u,f_1u)d(f_3u,f_2v)d(f_6u,f_2v)}{1 + d(f_3u,f_2v)d(f_6u,f_2v) + d(f_4v,f_1u)d(f_5v,f_1u)} \\ &+ \frac{d(f_4v,f_2v)d(f_5v,f_2v)d(f_4v,f_1u)d(f_5v,f_1u)}{1 + d(f_3u,f_2v)d(f_6u,f_2v) + d(f_4v,f_1u)d(f_5v,f_1u)}\Bigg\}; \end{aligned}$$

(3) *the pairs* (f_1,f_3), (f_2,f_4), (f_1,f_6) *and* (f_2,f_5) *are weakly compatible*;

(4) *either both the pairs* (f_1,f_3) *and* (f_1,f_6) *satisfies common* (CLR_{f_1})*-property or both the pairs* (f_2,f_4) *and* (f_2,f_5) *satisfies common* (CLR_{f_2})*-property.*

Then f_1, f_2, f_3, f_4, f_5, *and* f_6 *have a unique common fixed point in* Z.

Corollary 3.2 *Let* $f_1, f_2, f_3, f_4, f_5, f_6$ *be six maps on a metric space* (Z, d) *such that all the conditions of corollary* 3.1 *except condition* (4) *holds. In addition if either the pairs* (f_1,f_3) *and* (f_1,f_6) *or* (f_2,f_4) *and* (f_2,f_5) *satisfy the common* $(E.A)$*-property such that either* $f_4(Z)$ *and* $f_5(Z)$ *or* $f_3(Z)$ *and* $f_6(Z)$ *are closed subspaces of* Z, *then* f_1, f_2, f_3, f_4, f_5, *and* f_6 *have a unique common fixed point in* Z.

We apply the above results to study the existence of unique common solution to the following system (3.1) of non-linear Volterra-Hammerstein integral equations.

Let $Z = (L(0,\infty),\mathbb{R})$ be the space of real-valued measurable functions on $(0,\infty)$:

$$u(s) = p_i(s) + \lambda \int_0^t m(s,r)g_i\bigl(r,u(r)\bigr)\,dr + \mu \int_0^\infty n(s,r)h_i\bigl(r,u(r)\bigr)\,dr, \tag{3.1}$$

for all $s \in (0,\infty)$, where $\lambda, \mu \in \mathbb{R}$, u, p_i, $m(s,r)$, $n(s,r)$, $g_i(r,u(r))$ and $h_i(r,u(r))$, $i = 1,2,\dots,6$, are real-valued measurable functions in s and r on $(0,\infty)$.

Let us denote

$$\Delta_i u(s) = \int_0^t m(s,r)g_i\bigl(r,u(r)\bigr)\,dr$$

and

$$\nabla_i u(s) = \int_0^\infty n(s,r)h_i\bigl(r,u(r)\bigr)\,dr,$$

where $i = 1,2,\dots,6$.

Assume that

(C_1^*) for $i = 4,5$,

$$\begin{aligned} &\Delta_1 u(s) + \nabla_1 u(s) + p_1(s) + p_i(s) - \Delta_i\bigl(\Delta_1 u(s) + \nabla_1 u(s) + p_1(s) + p_i(s)\bigr) \\ &\quad - \nabla_i\bigl(\Delta_1 u(s) + \nabla_1 u(s) + p_1(s) + p_i(s)\bigr) = 0, \end{aligned}$$

(C_2^*) for $j = 3,6$,

$$\begin{aligned} &\Delta_2 u(s) + \nabla_2 u(s) + p_2(s) + p_j(s) - \Delta_j\bigl(\Delta_2 u(s) + \nabla_2 u(s) + p_2(s) + p_j(s)\bigr) \\ &\quad - \nabla_j\bigl(\Delta_2 u(s) + \nabla_2 u(s) + p_2(s) + p_j(s)\bigr) = 0, \end{aligned}$$

(C_3^*) for $j = 3,6$,

$$\begin{aligned} &p_1(s) + 3p_j(s) + 2\Delta_j u(s) + 2\nabla_j u(s) + \Delta_1\bigl(\Delta_1 u(s) + \nabla_1 u(s) + p_1(s)\bigr) \\ &\quad + \Delta_j\bigl(2u(s) - \Delta_j u(s) - \nabla_j u(s) - p_j(s)\bigr) + \nabla_1\bigl(\Delta_1 u(s) + \nabla_1 u(s) + p_1(s)\bigr) \\ &\quad + \nabla_j\bigl(2u(s) - \Delta_j u(s) - \nabla_j u(s) - p_j(s)\bigr) = 4u(s), \end{aligned}$$

(C_4^*) for $i = 4,5$,

$$\begin{aligned} &p_2(s) + 3p_i(s) + 2\Delta_i u(s) + 2\nabla_i u(s) + \Delta_2\bigl(\Delta_2 u(s) + \nabla_2 u(s) + p_2(s)\bigr) \\ &\quad + \Delta_i\bigl(2u(s) - \Delta_i u(s) - \nabla_i u(s) - p_i(s)\bigr) + \nabla_2\bigl(\Delta_2 u(s) + \nabla_2 u(s) + p_2(s)\bigr) \\ &\quad + \nabla_i\bigl(2u(s) - \Delta_i u(s) - \nabla_i u(s) - p_i(s)\bigr) = 4u(s). \end{aligned}$$

Let $Z = (L(0,\infty),\mathbb{R})$ be an incomplete metric space with metric

$$d(u,v) = \max_{s\in(0,\infty)} \bigl\|u(s) - v(s)\bigr\|, \quad \text{for all } u,v \in Z.$$

Define the six operators f_1, f_2, f_3, f_4, f_5, and f_6 on Z by

$$\begin{aligned} f_1u(s) &= \Delta_1 u(s) + \nabla_1 u(s) + p_1(s), \\ f_2u(s) &= \Delta_2 u(s) + \nabla_2 u(s) + p_2(s), \\ f_3u(s) &= 2u(s) - \Delta_3 u(s) - \nabla_3 u(s) - p_3(s), \\ f_4u(s) &= 2u(s) - \Delta_4 u(s) - \nabla_4 u(s) - p_4(s), \\ f_5u(s) &= 2u(s) - \Delta_5 u(s) - \nabla_5 u(s) - p_5(s), \\ f_6u(s) &= 2u(s) - \Delta_6 u(s) - \nabla_6 u(s) - p_6(s). \end{aligned} \tag{3.2}$$

Now, we are in a position to formulate the existence results.

Theorem 3.3 *Under the assumptions* (C_1^*)-(C_4^*), *if*

(1) *there exist two sequences* $\{z_n\}$ *and* $\{w_n\}$ *in* Z *such that*

$$\lim_{n\to\infty} f_2 z_n = \lim_{n\to\infty} f_4 z_n = \lim_{n\to\infty} f_2 w_n = \lim_{n\to\infty} f_5 w_n = z \in f_2(Z); \tag{3.3}$$

(2) *for each* $u, v \in Z$ *and* $0 < \lambda < 1$,

$$\begin{aligned} &\left\| \Delta_1 u(s) + \nabla_1 u(s) + p_1(s) - \Delta_2 u(s) - \nabla_2 u(s) - p_2(s) \right\| \\ &\quad \le \lambda \left\{ \frac{\Upsilon_2 \times \Upsilon_3 \times \Upsilon_4 \times \Upsilon_5 + \Upsilon_6 \times \Upsilon_7 \times \Upsilon_8 \times \Upsilon_9}{1 + (\max_{s\in(a,b)} \Upsilon_4) \cdot (\max_{s\in(a,b)} \Upsilon_5) + (\max_{s\in(a,b)} \Upsilon_8) \cdot (\max_{s\in(a,b)} \Upsilon_9)} \right\}, \end{aligned}$$

where

$$\begin{aligned} \Upsilon_2 &= \left\| 2u(s) - \Delta_3 u(s) - \nabla_3 u(s) - p_3(s) - \Delta_1 u(s) - \nabla_1 u(s) - p_1(s) \right\|, \\ \Upsilon_3 &= \left\| 2u(s) - \Delta_6 u(s) - \nabla_6 u(s) - p_6(s) - \Delta_1 u(s) - \nabla_1 u(s) - p_1(s) \right\|, \\ \Upsilon_4 &= \left\| 2u(s) - \Delta_3 u(s) - \nabla_3 u(s) - p_3(s) - \Delta_2 v(s) - \nabla_2 v(s) - p_2(s) \right\|, \\ \Upsilon_5 &= \left\| 2u(s) - \Delta_6 u(s) - \nabla_6 u(s) - p_6(s) - \Delta_2 v(s) - \nabla_2 v(s) - p_2(s) \right\|, \\ \Upsilon_6 &= \left\| 2v(s) - \Delta_4 v(s) - \nabla_4 v(s) - p_4(s) - \Delta_2 v(s) - \nabla_2 v(s) - p_2(s) \right\|, \\ \Upsilon_7 &= \left\| 2v(s) - \Delta_5 v(s) - \nabla_5 v(s) - p_5(s) - \Delta_2 v(s) - \nabla_2 v(s) - p_2(s) \right\|, \\ \Upsilon_8 &= \left\| 2v(s) - \Delta_4 v(s) - \nabla_4 v(s) - p_4(s) - \Delta_1 u(s) - \nabla_1 u(s) - p_1(s) \right\|, \\ \Upsilon_9 &= \left\| 2v(s) - \Delta_5 v(s) - \nabla_5 v(s) - p_5(s) - \Delta_1 u(s) - \nabla_1 u(s) - p_1(s) \right\|; \end{aligned}$$

(3) $f_1(Z) \subseteq f_4(Z), f_1(Z) \subseteq f_5(Z), f_2(Z) \subseteq f_3(Z)$ *and* $f_2(Z) \subseteq f_6(Z)$ *such that the pairs* (f_1, f_3), (f_2, f_4), (f_1, f_6) *and* (f_2, f_5) *are weakly compatible,*

then the system (3.1) *of Volterra-Hammerstein equations has a unique common solution.*

Proof Notice that the system of Volterra-Hammerstein non-linear integral equations (3.1) has a unique common solution if and only if the system of operators (3.2) has a unique common fixed point.

Now,

$$\begin{cases} d(f_1u,f_2v) = \max_{s\in(0,\infty)} \|\Delta_1 u(s) + \nabla_1 u(s) + p_1(s) - \Delta_2 u(s) - \nabla_2 u(s) - p_2(s)\|, \\ d(f_3u,f_1u) = \max_{s\in(0,\infty)} \|2u(s) - \Delta_3 u(s) - \nabla_3 u(s) - p_3(s) - \Delta_1 u(s) - \nabla_1 u(s) - p_1(s)\|, \\ d(f_6u,f_1u) = \max_{s\in(0,\infty)} \|2u(s) - \Delta_6 u(s) - \nabla_6 u(s) - p_6(s) - \Delta_1 u(s) - \nabla_1 u(s) - p_1(s)\|, \\ d(f_3u,f_2v) = \max_{s\in(0,\infty)} \|2u(s) - \Delta_3 u(s) - \nabla_3 u(s) - p_3(s) - \Delta_2 v(s) - \nabla_2 v(s) - p_2(s)\|, \\ d(f_6u,f_2v) = \max_{s\in(0,\infty)} \|2u(s) - \Delta_6 u(s) - \nabla_6 u(s) - p_6(s) - \Delta_2 v(s) - \nabla_2 v(s) - p_2(s)\|, \\ d(f_4v,f_2v) = \max_{s\in(0,\infty)} \|2v(s) - \Delta_4 v(s) - \nabla_4 v(s) - p_4(s) - \Delta_2 v(s) - \nabla_2 v(s) - p_2(s)\|, \\ d(f_5v,f_2v) = \max_{s\in(0,\infty)} \|2v(s) - \Delta_5 v(s) - \nabla_5 v(s) - p_5(s) - \Delta_2 v(s) - \nabla_2 v(s) - p_2(s)\|, \\ d(f_4v,f_1u) = \max_{s\in(0,\infty)} \|2v(s) - \Delta_4 v(s) - \nabla_4 v(s) - p_4(s) - \Delta_1 u(s) - \nabla_1 u(s) - p_1(s)\|, \\ d(f_5v,f_1u) = \max_{s\in(0,\infty)} \|2v(s) - \Delta_5 v(s) - \nabla_5 v(s) - p_5(s) - \Delta_1 u(s) - \nabla_1 u(s) - p_1(s)\|. \end{cases} \tag{3.4}$$

From condition (2) of Theorem 2.1, we have

$$\begin{aligned} &\max\|\Delta_1 u(s) + \nabla_1 u(s) + p_1(s) - \Delta_2 u(s) - \nabla_2 u(s) - p_2(s)\| \\ &\quad\leq \lambda \max\left\{\frac{\Upsilon_2 \times \Upsilon_3 \times \Upsilon_4 \times \Upsilon_5 + \Upsilon_6 \times \Upsilon_7 \times \Upsilon_8 \times \Upsilon_9}{1 + (\max_{s\in(a,b)} \Upsilon_4) \cdot (\max_{s\in(a,b)} \Upsilon_5) + (\max_{s\in(a,b)} \Upsilon_8) \cdot (\max_{s\in(a,b)} \Upsilon_9)}\right\}, \end{aligned}$$

which implies that

$$\begin{aligned} &\max_{s\in(0,\infty)} \|\Delta_1 u(s) + \nabla_1 u(s) + p_1(s) - \Delta_2 u(s) - \nabla_2 u(s) - p_2(s)\| \\ &\quad\leq \lambda \left\{\frac{(\max_{s\in(0,\infty)} \Upsilon_2) \cdot (\max_{s\in(0,\infty)} \Upsilon_3) \cdot (\max_{s\in(0,\infty)} \Upsilon_4) \cdot (\max_{s\in(0,\infty)} \Upsilon_5)}{1 + (\max_{s\in(0,\infty)} \Upsilon_4) \cdot (\max_{s\in(0,\infty)} \Upsilon_5) + (\max_{s\in(0,\infty)} \Upsilon_8) \cdot (\max_{s\in(0,\infty)} \Upsilon_9)}\right. \\ &\qquad \left. + \frac{(\max_{s\in(0,\infty)} \Upsilon_6) \cdot (\max_{s\in(0,\infty)} \Upsilon_7) \cdot (\max_{s\in(0,\infty)} \Upsilon_8) \cdot (\max_{s\in(0,\infty)} \Upsilon_9)}{1 + (\max_{s\in(0,\infty)} \Upsilon_4) \cdot (\max_{s\in(0,\infty)} \Upsilon_5) + (\max_{s\in(0,\infty)} \Upsilon_8) \cdot (\max_{s\in(0,\infty)} \Upsilon_9)}\right\}, \end{aligned}$$

using (3.4), we get

$$\begin{aligned} d(f_1u,f_2v) &\leq \lambda \left\{\frac{d(f_3u,f_1u)d(f_6u,f_1u)d(f_3u,f_2v)d(f_6u,f_2v)}{1 + d(f_3u,f_2v)d(f_6u,f_2v) + d(f_4v,f_1u)d(f_5v,f_1u)}\right. \\ &\quad \left. + \frac{d(f_4v,f_2v)d(f_5v,f_2v)d(f_4v,f_1u)d(f_5v,f_1u)}{1 + d(f_3u,f_2v)d(f_6u,f_2v) + d(f_4v,f_1u)d(f_5v,f_1u)}\right\}. \end{aligned}$$

Now, to show that $f_1(Z) \subseteq f_4(Z)$, we have

$$\begin{aligned} &f_4\bigl(f_1u(s) + p_4(s)\bigr) \\ &\quad= 2\bigl[f_1u(s) + p_4(s)\bigr] - \Delta_4\bigl(f_1u(s) + p_4(s)\bigr) - \nabla_4\bigl(f_1u(s) + p_4(s)\bigr) - p_4(s) \\ &\quad= f_1u(s) + f_1u(s) + p_4(s) - \Delta_4\bigl(f_1u(s) + p_4(s)\bigr) - \nabla_4\bigl(f_1u(s) + p_4(s)\bigr) \\ &\quad= f_1u(s) + \Delta_1 u(s) + \nabla_1 u(s) + p_1(s) + p_4(s) \end{aligned}$$

$$- \Delta_4\big(\Delta_1 u(s) + \nabla_1 u(s) + p_1(s) + p_4(s)\big)$$
$$- \nabla_4\big(\Delta_1 u(s) + \nabla_1 u(s) + p_1(s) + p_4(s)\big).$$

Using (C_1^*), we get $f_4(f_1u(s) + p_4(s)) = f_1u(s)$, which implies that $f_1(Z) \subseteq f_4(Z)$. Similarly, one can prove that $f_1(Z) \subseteq f_5(Z)$, $f_2(Z) \subseteq f_3(Z)$ and $f_2(Z) \subseteq f_6(Z)$.

Next, we need to show the weak compatibility of the pair (f_1,f_3). For this purpose,

$$\begin{aligned}
&\big\|f_3f_1u(s) - f_1f_3u(s)\big\| \\
&\quad = \big\|f_3\big(\Delta_1 u(s) + \nabla_1 u(s) + p_1(s)\big) - f_1\big(2u(s) - \Delta_3 u(s) - \nabla_3 u(s) - p_3(s)\big)\big\| \\
&\quad = \big\|2\big(\Delta_1 u(s) + \nabla_1 u(s) + p_1(s)\big) - \Delta_3\big(\Delta_1 u(s) + \nabla_1 u(s) + p_1(s)\big) \\
&\qquad - \nabla_3\big(\Delta_1 u(s) + \nabla_1 u(s) + p_1(s)\big) - p_3(s) - \Delta_1\big(2u(s) - \Delta_3 u(s) - \nabla_3 u(s) - p_3(s)\big) \\
&\qquad - \nabla_1\big(2u(s) - \Delta_3 u(s) - \nabla_3 u(s) - p_3(s)\big) - p_1(s)\big\|.
\end{aligned}$$

If $f_1u(s) = f_3u(s)$, for $u(s) \in Z$. Then $\Delta_1 u(s) + \nabla_1 u(s) + p_1(s) = 2u(s) - \Delta_3 u(s) - \nabla_3 u(s) - p_3(s)$, thus the above equation becomes

$$\begin{aligned}
&\big\|f_3f_1u(s) - f_1f_3u(s)\big\| \\
&\quad = \big\|2\big(2u(s) - \Delta_3 u(s) - \nabla_3 u(s) - p_3(s)\big) - \Delta_3\big(2u(s) - \Delta_3 u(s) - \nabla_3 u(s) - p_3(s)\big) \\
&\qquad - \nabla_3\big(2u(s) - \Delta_3 u(s) - \nabla_3 u(s) - p_3(s)\big) - p_3(s) - \Delta_1\big(\Delta_1 u(s) + \nabla_1 u(s) + p_1(s)\big) \\
&\qquad - \nabla_1\big(\Delta_1 u(s) + \nabla_1 u(s) + p_1(s)\big) - p_1(s)\big\| \\
&\quad = \big\|4u(s) - 2\Delta_3 u(s) - 2\nabla_3 u(s) - p_1(s) - 3p_3(s) \\
&\qquad - \Delta_1\big(\Delta_1 u(s) + \nabla_1 u(s) + p_1(s)\big) - \nabla_1\big(\Delta_1 u(s) + \nabla_1 u(s) + p_1(s)\big) \\
&\qquad - \Delta_3\big(2u(s) - \Delta_3 u(s) - \nabla_3 u(s) - p_3(s)\big) \\
&\qquad - \nabla_3\big(2u(s) - \Delta_3 u(s) - \nabla_3 u(s) - p_3(s)\big)\big\|,
\end{aligned}$$

with the help of (C_3^*), we get $\|f_3f_1z(s) - f_1f_3z(s)\| = 0$, which implies that $f_3f_1z(s) = f_1f_3z(s)$, whenever $f_1z(s) = f_3z(s)$. Thus the pair (f_1,f_3) is weakly compatible. In a similar way one can easily show the weakly compatibility of the pairs (f_2,f_4), (f_1,f_6), and (f_2,f_5). Also, from condition (1) of Theorem 3.3 the pairs (f_2,f_4) and (f_2,f_5) satisfy the common (CLR_{f_2})-property. Thus by Corollary 3.1, we can find a unique common fixed point of f_1, f_2, f_3, f_4, f_5, and f_6 in Z, that is, the system (3.1) of Volterra-Hammerstein non-linear integral equations has a unique common solution in Z. □

In the next theorem we use the common $(E.A)$-property.

Theorem 3.4 *Under the assumptions* (C_1^*)-(C_4^*) *and the conditions* (2), (3) *of Theorem* 3.3, *if there exist two sequences* $\{z_n\}$ *and* $\{w_n\}$ *in* Z *such that*

$$\lim_{n\to\infty} f_2 z_n = \lim_{n\to\infty} f_4 z_n = \lim_{n\to\infty} f_2 w_n = \lim_{n\to\infty} f_5 w_n = z, \quad \textit{for some } z \in Z, \tag{3.5}$$

and both $f_4(Z)$ *and* $f_5(Z)$ *are closed subspaces of* Z, *then the system* (3.1) *of Volterra-Hammerstein equations has a unique common solution*.

4 Conclusions

In the current work, we studied the existence of unique common solution for the systems of Urysohn and Volterra-Hammerstein integral equations in incomplete spaces. Several problems that appear in applied mathematics, physical sciences, geology, mechanics, engineering, economics, and biology generate mathematical models described by integral equations.

Competing interests
The authors declare that they have no competing interests regarding this manuscript.

Authors' contributions
All authors contributed equally to the writing of this manuscript. All authors read and approved the final version.

Author details
[1]Department of Mathematics, University of Malakand, Chakdara, Dir(L), Pakistan. [2]Faculty of Mechanical Engineering, Universitry of Belgrade, Kraljice Marije 16, Beograd, 11 120, Serbia.

Acknowledgements
The authors wish to thank the editor and anonymous referees for their comments and suggestions, which helped to improve this paper. The authors are also grateful to Springer International Publishing for granting full fee waiver.

Received: 15 August 2016 Accepted: 22 December 2016 Published online: 18 January 2017

References

1. Asgari, MS, Badehian, Z: Fixed point theorems for α-β-ψ-contractive mappings in partially ordered sets. J. Nonlinear Sci. Appl. **8**, 518-528 (2015)
2. Bose, S, Hossein, SM, Paul, K: Positive definite solution of a nonlinear matrix equation. J. Fixed Point Theory Appl. (2016). doi:10.1007/s11784-016-0291-2
3. Radenović, S, Došenović, T, Lampert, TA, Golubović, Z: A note on some recent fixed point results for cyclic contractions in *b*-metric spaces and an application to integral equations. Appl. Math. Comput. **273**, 155-164 (2016)
4. Sintunavarat, W, Cho, YJ, Kumam, P: Urysohn integral equations approach by common fixed points in complex-valued metric spaces. Adv. Differ. Equ. **2013**, 49 (2013)
5. Huang, LG, Zhang, X: Cone metric spaces and fixed point theorems of contractive mappings. J. Math. Anal. Appl. **332**, 1468-1476 (2007)
6. Mustafa, Z, Sims, B: Fixed point theorems for contractive mappings in complete *G*-metric spaces. Fixed Point Theory Appl. **2009**, Article ID 917175 (2009)
7. Aydi, H: Some coupled fixed point results on partial metric spaces. Int. J. Math. Math. Sci. **2011**, Article ID 647091 (2011)
8. Fang, JX: On fixed point theorem in fuzzy metric spaces. Fuzzy Sets Syst. **46**, 107-113 (1992)
9. Azam, A, Fisher, B, Khan, M: Common fixed point theorems in complex valued metric spaces. Numer. Funct. Anal. Optim. **32**(3), 243-253 (2011)
10. Abbas, M, Rajić, VĆ, Nazir, T, Radenović, S: Common fixed point of mappings satisfying rational inequalities in ordered complex valued generalized metric spaces. Afr. Math. **26**, 17-30 (2015)
11. Rouzkard, F, Imdad, M: Some common fixed point theorems on complex valued metric spaces. Comput. Math. Appl. **64**, 1866-1874 (2012)
12. Sarwar, M, Bahadur Zada, M: Common fixed point theorems for six self-maps satisfying common (*E.A*) and common (*CLR*) properties in complex valued metric space. Electron. J. Math. Anal. Appl. **3**(1), 215-231 (2015)
13. Gholizadeh, L: A fixed point theorem in generalized ordered metric spaces with application. J. Nonlinear Sci. Appl. **6**, 244-251 (2013)
14. Pathak, HK, Khan, MS, Liu, Z, Ume, JS: Fixed point theorems in metrically convex spaces and applications. J. Nonlinear Convex Anal. **4**(2), 231-244 (2003)
15. Salimi, P, Hussain, N, Shukla, S, Fathollahi, S, Radenović, S: Fixed point results for cyclic α-$\psi\varphi$-contractions with applications to integral equations. J. Comput. Appl. Math. **290**, 445-458 (2015)
16. Shatanawi, W: Some fixed point theorems in ordered *G*-metric spaces and applications. Abstr. Appl. Anal. **2011**, Article ID 126205 (2011)
17. Rashwan, RA, Saleh, SM: Solution of nonlinear integral equations via fixed point theorems in *G*-metric spaces. Int. J. Appl. Math. Res. **3**(4), 561-571 (2014)
18. Pathak, HK, Khan, MS, Tiwari, R: A common fixed point theorem and its application to nonlinear integral equations. Comput. Math. Appl. **53**, 961-971 (2007)
19. Jungck, G: Common fixed points for non-continuous non-self mappings on a non-numeric spaces. Far East J. Math. Sci. **4**(2), 199-212 (1996)
20. Aamri, M, El Moutawakil, D: Some new common fixed point theorems under strict contractive conditions. J. Math. Anal. Appl. **270**(1), 181-188 (2002)
21. Sintunavarat, W, Kumam, P: Common fixed point theorem for a pair of weakly compatible mappings in fuzzy metric space. J. Appl. Math. **2011**, Article ID 637958 (2011)
22. Liu, W, Wu, J, Li, Z: Common fixed points of single-valued and multi-valued maps. Int. J. Math. Math. Sci. **19**, 3045-3055 (2005)
23. Imdad, M, Pant, BD, Chauhan, S: Fixed point theorems in Menger spaces using the (CLR_{ST})-property and applications. J. Nonlinear Anal. Optim., Theory Appl. **3**(2), 225-237 (2012)