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Exact results for 't Hooft loops in Gauge theories on S^4

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Abstract: The path integral of a general $\mathcal{N} = 2$ supersymmetric gauge theory on S^4 is exactly evaluated in the presence of a supersymmetric 't Hooft loop operator. The result we find — obtained using localization techniques — captures all perturbative quantum corrections as well as non-perturbative effects due to instantons and monopoles, which are supported at the north pole, south pole and equator of S^4. As a by-product, our gauge theory calculations successfully confirm the predictions made for 't Hooft loops obtained from the calculation of topological defect correlators in Liouville/Toda conformal field theory.

Keywords: Supersymmetric gauge theory, Wilson, 't Hooft and Polyakov loops, Solitons Monopoles and Instantons, Nonperturbative Effects

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Contents

1 Introduction

Supersymmetry — apart from being phenomenologically appealing for physics beyond the standard model — is a powerful symmetry which constraints the dynamics of gauge theories. Investigations of supersymmetric gauge theories have yielded important physical (and mathematical) insights and serve as calculable models for the rich dynamics of four dimensional gauge theories. For instance, the exact low energy effective action of $\mathcal{N}=2$ super Yang-Mills constructed by Seiberg and Witten [1] provides an elegant physical realization of quark confinement in terms of the dual Meissner effect, via the condensation of magnetic monopoles.

The correlation functions of gauge invariant operators in supersymmetric gauge theories — despite enjoying more controlled dynamics in comparison to QCD — are highly non-trivial to calculate. Even for supersymmetric observables, which preserve some of the symmetries of the theory, generic correlation functions have perturbative corrections to arbitrary loop order as well as non-perturbative instanton corrections. Only in the past few years, exact calculations for the correlation functions of some supersymmetric operators started to be available. An important early step in this recent development was the calculation of the exact partition function of physical $\mathcal{N}=2$ gauge theories on S^4 and of the expectation value of supersymmetric Wilson loop operators in these theories [2]. Likewise, the computation of certain supersymmetric domain walls in $\mathcal{N}=2$ gauge theories on S^4 — such as Janus and duality walls — were presented in [3] (see also [4]).

Some of the most basic observables of four dimensional gauge theories are loop operators. These operators can be classified according to whether the loop operator is electric or magnetic, giving rise to Wilson and 't Hooft operators respectively. Gauge theory loop operators — which are supported on curves in spacetime — are order parameters for the phases that a gauge theory can exhibit, and serve as probes of the quantum dynamics of gauge theories. Loop operators are also the most basic observables on which S-duality is conjectured to act in supersymmetric gauge theories (or certain nonsupersymmetric lattice models), and therefore are ideal probes of this remarkable symmetry exhibited by some supersymmetric gauge theories and M-theory. Calculating these observables exactly allows for a quantitative study of S-duality and serves as a theoretical playground for gaining a deeper understanding of the inner workings of dualities.

In this paper we evaluate the exact path integral which computes the expectation value of supersymmetric 't Hooft loop operators in an arbitrary $\mathcal{N}=2$ supersymmetric gauge theory on S^4 admitting a Lagrangian description. The expectation value of 't Hooft loop operators — originally introduced [5] to probe the phase structure of gauge theories — are calculated by explicit evaluation of the path integral using localization [6]. In the localization framework, the path integral is one-loop exact with respect to an effective $\hbar$-parameter, but nevertheless the computation yields the exact result with respect to the gauge theory coupling constant of the theory. Our analysis of 't Hooft loops together with the results of [2] for Wilson loops, provide a suite of complete, exact calculations of the most elementary loop operators in supersymmetric gauge theories.

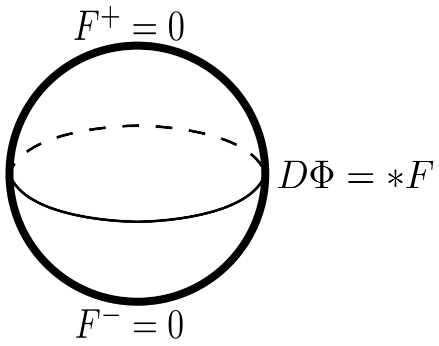

Figure 1. Instanton, monopole and anti-instanton field configurations

We find that for an $\mathcal{N} = 2$ gauge theory in S^4, the expectation value of a supersymmetric 't Hooft operator carrying magnetic charge labeled by a coweight[1] B of the gauge group G takes the form

$$\begin{aligned} \langle T(B) \rangle_{\mathcal{N}=2} &= \int da \sum_v Z_{\text{north}}(v) Z_{\text{south}}(v) Z_{\text{equator}}(B, v) \\ &= \int da \sum_v |Z_{\text{north}}(v)|^2 \, Z_{\text{equator}}(B, v) \,. \end{aligned} \tag{1.1}$$

The integral is over the Cartan subalgebra of the gauge group. The coweight B of G can be identified with the highest weight for a representation of the Langlands (or GNO [7]) dual group ${}^L G$.[2] The sum is then over the coweights v of G such that their corresponding weights of ${}^L G$ appear in the representation specified by B.

The path integral in the localization computation receives contributions which localize to the north and south poles of S^4 as well as to the equator, where the 't Hooft operator is supported. Each factor has an elegant interpretation as arising from specific field configurations in the effective path integral arising in the localization computation. The magic of localization is that it restricts the integral over the space of all field configurations to the submanifold of field configurations invariant under a fermionic symmetry Q, which also preserves the supersymmetric 't Hooft operator. These field configurations are solutions to the localization saddle point equations. Integrating out the fluctuations around each of the saddle points and summing over them in the path integral yield the exact result for the expectation value of the 't Hooft loop operator.

The north pole factor captures the effects of point-like instantons while the south pole one incorporates the contributions of point-like anti-instantons. These configurations are the solutions to the localization saddle point equations at the north and south poles of S^4, given by $F^+ = 0$ and $F^- = 0$ respectively. The result of summing over these saddle points can be written in terms of Nekrasov's instanton partition function [8] of the corresponding

[1] We recall that a coweight, denoted as B here, is an element of the Cartan subalgebra $\mathfrak{t}$ of G such that the product $\alpha \cdot B$ is an integer for all roots $\alpha \in \mathfrak{t}^*$ of G.

[2] The Cartan subalgebra ${}^L\mathfrak{t}$ of ${}^L G$ can be identified with the dual $\mathfrak{t}^*$ of the Cartan subalgebra of G and vice versa: ${}^L\mathfrak{t} \simeq \mathfrak{t}^*$, ${}^L\mathfrak{t}^* \simeq \mathfrak{t}$.

$\mathcal{N} = 2$ theory in $\mathbb{R}^4$ (more precisely in the Ω-background), with arguments depending on the effective magnetic charge v

$$
\begin{aligned}
Z_{\text{north}}(v) =& Z_{\text{cl}}\left(ia - \frac{v}{2r}, q\right) Z_{\text{1-loop,pole}}\left(ia - \frac{v}{2r}, im_f\right) Z_{\text{inst}}\left(ia - \frac{v}{2r}, \frac{1}{r} + im_f, \frac{1}{r}, \frac{1}{r}, q\right) \\
Z_{\text{south}}(v) =& Z_{\text{cl}}\left(ia + \frac{v}{2r}, \overline{q}\right) Z_{\text{1-loop,pole}}\left(ia + \frac{v}{2r}, im_f\right) Z_{\text{inst}}\left(ia + \frac{v}{2r}, \frac{1}{r} + im_f, \frac{1}{r}, \frac{1}{r}, \overline{q}\right),
\end{aligned} \tag{1.2}
$$

with $Z_{\text{cl}}, Z_{\text{1-loop,pole}}$ and Z_{inst} given in (4.10), (6.32), (5.2). The parameters m_f are the masses of the hypermultiplets in the $\mathcal{N} = 2$ gauge theory, r is the radius of S^4 and $q = \exp(2\pi i\tau)$, where τ is the gauge theory coupling constant[3]

$$
\tau = \frac{\theta}{2\pi} + \frac{4\pi i}{g^2}.
$$

A crucial new contribution to the 't Hooft loop expectation value arises from the equator of S^4, where the localization saddle point equations are the Bogomolny equations $D\Phi = *F$. $Z_{\text{equator}}(B, v)$ captures the contribution to the path integral of field configurations which are solutions to the Bogomolny equations in the presence of a singular monopole background labeled by the magnetic charge B, created by the 't Hooft loop operator insertion. The sum over v in (1.1) appears due to the physics of monopole screening, whereby smooth non-abelian monopole field configurations screen the charge B of the singular mononopole down to an effective magnetic charge v. In the path integral we must sum over all possible effective magnetic charges labeled by coweights v, which are attainable given a singular monopole of magnetic charge B.[4] The ${}^LG$-weights corresponding to v precisely span the weights of the representation of ${}^LG$ for which B corresponds to the highest weight.[5] The equatorial contribution is

$$
Z_{\text{equator}}(B, v) = Z_{\text{1-loop,eq}}(ia, im_f, v)\, Z_{\text{mono}}(ia, im_f; B, v), \tag{1.3}
$$

where $Z_{\text{1-loop,eq}}$ is given in (6.61) and Z_{mono} in section 7. Combining all the various contributions produces the exact expectation value for the supersymmetric 't Hooft loop operator in $\mathcal{N} = 2$ gauge theories on S^4.

Our gauge theory computations are in elegant agreement with the conjectures and calculations in [10–12] for 't Hooft operators in certain $\mathcal{N} = 2$ gauge theories using topological defect operators in two dimensional nonrational conformal field theory. In these papers, gauge theory loop operators in $\mathcal{N} = 2$ gauge theories were identified with loop operators (topological webs more generically) in two dimensional Liouville/Toda conformal field theory, and some correlation functions were explicitly calculated. The Liouville/Toda conformal field theory computations are shown to capture in detail all the features of our gauge theory computation, thereby establishing the proposal put forward in [10–12].

[3]For a semi-simple gauge group there is a coupling constant for each simple factor.

[4]The necessity to sum over such configurations was conjectured in [9], where the perturbative analysis of the expectation value of 't Hooft operators in $\mathcal{N} = 4$ super Yang-Mills was performed.

[5]We recall that regular monopoles are labeled by coroots, which when acting on the singular monopole, labeled by a coweight B, generate all coweights associated to B.

The localization calculation performed in this paper is the first example of an exact computation of a path integral in the presence of a genuine singularity due to a disorder operator — an operator characterized by the singularities induced on the fields — and of which a 't Hooft operator is a prime example.[6] In order to treat precisely the fluctuations around the singular field configuration, we employ the mathematical correspondence between singular monopoles in three dimensions and U(1)-invariant instantons in four dimensions [14]. This turns out to be a particularly clean way to carry out the relevant index calculations.

The plan of the rest of the paper is as follows. Section 2 briefly introduces the key ingredients that will be needed to perform the localization computation of 't Hooft operators in $\mathcal{N}=2$ gauge theories on S^4. In section 3 we derive the localization saddle point equations relevant for the localization computation, demonstrate that these equations interpolate between the anti-self-duality, self-duality and Bogomolny equations at the north pole, south pole and equator respectively, and find the most general non-singular solution to these equations. This section also describes the singular field configuration produced by the supersymmetric 't Hooft operator as well as the symmetries of the theory used to carry out the localization computation. Section 4 contains the calculation of the classical contribution of the 't Hooft loop path integral, which includes a discussion of the relevant boundary terms. In this section we demonstrate that the classical result can be factored into a contribution arising from the north pole and one from the south pole. Section 5 computes the contribution due to the singular solutions to the saddle point equations arising at the north and south poles, described by pointlike instantons and anti-instantons. In section 6 we calculate the localization one-loop determinants arising from the north and south poles of S^4 as well as from the equator. Section 7 describes the effect of monopole screening in the study of the equatorial Bogomolny equations and explains how to calculate the contribution to the 't Hooft loop expectation value due to screening. In section 8 we compare our gauge theory results with the Liouville/Toda computations conjectured to capture 't Hooft operators in certain $\mathcal{N}=2$ gauge theores. We finish with conclusions in section 9. The appendices contain some technical details and computations

2 $\mathcal{N}=2$ Gauge theories in S^4 and localization

In this section we introduce the main ingredients of the localization analysis in [2] that we require to calculate the exact expectation value of supersymmetric 't Hooft operators in an arbitrary four dimensional $\mathcal{N}=2$ gauge theory on S^4 admitting a Lagrangian description.[7] Such a theory is completely characterized by the choice of a gauge group G and of a representation R of G under which the $\mathcal{N}=2$ hypermultiplet transforms, the $\mathcal{N}=2$ vectormultiplet transforming in the adjoint representation of G. This includes gauge theories

[6] A monopole operator in three dimensions is a closely related disorder operator. The work [13] performed localization computations for monopole operators in three dimensions to compute the supersymmetry index via radial quantization, thus removing the singularity by a coordinate change. In this paper we deal with the monopole singularity more directly.

[7] Localization of some $\mathcal{N}=2$ gauge theories was also considered in [15].

with several gauge group factors and multiple matter representations by letting G be the product of several gauge groups and by taking R to be a reducible representation of G. It therefore applies to any gauge theory with a Lagrangian description.

The on-shell field content of the $\mathcal{N} = 2$ multiplets is given by

$$\begin{aligned} \text{vectormultiplet} &: (A_\mu, \Phi_0, \Phi_9, \Psi) \\ \text{hypermultiplet} &: (q, \tilde{q}^\dagger, \chi)\,. \end{aligned}$$

In this notation, the usual complex scalar field of the $\mathcal{N} = 2$ vectormultiplet is constructed out of the real fields Φ_0 and Φ_9. One complication in the construction of the $\mathcal{N} = 2$ Lagrangian in S^4 overcome in [2] was to turn on in a supersymmetric way mass parameters for the flavour symmetries associated to the hypermultiplet. These $\mathcal{N} = 2$ gauge theories on S^4 are invariant under the superalgebra OSp(2|4), where Sp(4) $\simeq$ SO(5) is the isometry group on S^4 and SO$(2)_R$ is a subgroup of the SU(2) R-symmetry of the corresponding $\mathcal{N} = 2$ gauge theory in flat spacetime.

The key idea behind localization [6] exploits that the path integral — possibly enriched with any observables invariant under the action of a supercharge Q — is unchanged upon deforming the supersymmetric Lagrangian of the theory by a Q-exact term

$$\mathcal{L} \to \mathcal{L} + t\,Q \cdot V\,. \tag{2.1}$$

The restriction on the choice of V is such that if Q^2 generates a symmetry and a gauge transformation, as will be the case in our analysis, then V must be gauge invariant and also invariant under the action of the symmetry. Also we require the path integral to be still convergent after the deformation, and that the contribution from the boundary in the space of fields vanishes. In order to localize the gauge fixed path integral, the supersymmetry generated by Q must be realized off-shell, and a gauge fixing procedure must be implemented. This was accomplished in [2] by introducing suitable auxiliary fields and a ghost multiplet, which plays a key role in precisely determining the measure of integration of the fluctuations.

Since the path integral is independent of t, we can study it in the $t \to \infty$ limit. In this limit the saddle points of the path integral are the solutions to the localization equations, which are the saddle points of the deformed action $Q \cdot V$. In this limit, the path integral becomes one-loop exact with respect to the effective $\hbar = 1/t$ parameter and can be evaluated by summing over all saddle points. Therefore, it can be calculated by evaluating the original Lagrangian $\mathcal{L}$ on the saddle points and by integrating out the quadratic fluctuations of all the fields in the Lagrangian deformation $Q \cdot V$ expanded around the solutions to the saddle point equations.[8] Of course, even though the path integral is one-loop exact with respect to t, it yields results to all orders in perturbation theory with respect to the original gauge coupling constant τ of the theory. This underlies the power of localization. In favorable situations, for a judicious choice of V, the deformation freezes out most of the fields that must be integrated over in the path integral, thus yielding a path integral for a reduced model, with fewer degrees of freedom.

[8] The original Lagrangian $\mathcal{L}$ is irrelevant for the localization one-loop analysis.

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In the analysis in [2], as well as in our analysis, it suffices to single out a single supersymmetry generator Q of the OSp(2|4) symmetry algebra present in any $\mathcal{N} = 2$ gauge theory on S^4. This supercharge generates an SU(1|1) subalgebra of OSp(2|4), given explicitly by

$$Q^2 = J + R\,, \qquad [J + R, Q] = 0\,. \tag{2.2}$$

J is the generator of a $\mathrm{U}(1)_J$ subgroup of the SO(5) isometry group of the S^4 while R is the $\mathrm{SO}(2)_R \simeq \mathrm{U}(1)_R$ symmetry generator in OSp(2|4). If we represent the S^4 of radius r by the embedding equation

$$X_1^2 + \ldots + X_5^2 = r^2\,, \tag{2.3}$$

then J acts as follows

$$\begin{aligned} X_1 + iX_2 &\to e^{i\varepsilon}(X_1 + iX_2) \\ X_3 + iX_4 &\to e^{i\varepsilon}(X_3 + iX_4)\,. \end{aligned} \tag{2.4}$$

We note that the action of J has two antipodal fixed points on S^4, which can be used to define the north and south pole of S^4. The U(1) symmetry associated to $J + R$ will be denoted by $\mathrm{U}(1)_{J+R} \equiv (\mathrm{U}(1)_J \times \mathrm{U}(1)_R)_{\mathrm{diag}}$.

We conclude this section by mentioning a property of the localization equations that we will exploit in the following section when studying the $\mathcal{N} = 2$ gauge theory path integral on S^4 in the presence of a supersymmetric 't Hooft loop operator. The deformation term $Q{\cdot}V$ that we add to the action naturally splits into two pieces, one giving rise to localization equations for the vectormultiplet and one for the hypermultiplet. In formulas

$$V = V_{\mathrm{vm}} + V_{\mathrm{hm}} = \mathrm{Tr}(\overline{Q\cdot\Psi}\,\Psi) + \mathrm{Tr}(\overline{Q\cdot\chi}\,\chi)\,, \tag{2.5}$$

where Ψ and χ are the fermions in the vectormultiplet and hypermultiplet respectively. We represent the fermion fields in the $\mathcal{N} = 2$ gauge theory by sixteen component, ten dimensional Weyl spinors of Spin(10) subject to the projection conditions (see appendix A for spinor notations and conventions)

$$\begin{aligned} \Gamma^{5678}\Psi &= -\Psi \\ \Gamma^{5678}\chi &= +\chi\,. \end{aligned} \tag{2.6}$$

Since the bosonic part of deformed action $Q \cdot V$ — given by $\mathrm{Tr}(|Q\cdot\Psi|^2) + \mathrm{Tr}(|Q\cdot\chi|^2)$ — is positive definite, the saddle point equations are

$$\begin{aligned} Q\cdot\Psi &= 0 \\ Q\cdot\chi &= 0\,. \end{aligned} \tag{2.7}$$

As shown in [2], the only solution of the saddle point equations

$$Q\cdot\chi = 0 \tag{2.8}$$

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forces all the fields in the hypermultiplet to vanish.[9] Therefore, we are left to analyze the non-trivial saddle point equations for the vectormultiplet fields[10]

$$Q\cdot\Psi=\frac{1}{2}F_{mn}\Gamma^{mn}\epsilon_Q-\frac{1}{2}\Phi_A\Gamma^{A\mu}\nabla_\mu\epsilon_Q+iK_j\Gamma^{8\,j+4}\epsilon_Q=0\,, \tag{2.9}$$

where $A_m\equiv(A_\mu,\Phi_A)=(A_\mu,\Phi_9,\Phi_0)$ and $K_j\equiv(K_1,K_2,K_3)$ are the propagating bosonic fields and three auxiliary fields of the $\mathcal{N}=2$ vectormultiplet respectively. Therefore in our conventions $m=1,2,3,4,9,0$, while $\mu=1,2,3,4$ and $A=9,0$. ϵ_Q is the conformal Killing spinor that parametrizes the supersymmetry transformation generated by the supercharge Q.

The equations (2.9) are Weyl invariant. That is $Q\cdot\Psi=0$ is invariant under the Weyl transformation

$$g_{\mu\nu}\to\Omega^2g_{\mu\nu},\quad A_\mu\to A_\mu,\quad \Phi_A\to\Omega^{-1}\Phi_A,\quad K_j\to\Omega^{-2}K_j,\quad \epsilon_Q\to\Omega^{1/2}\epsilon_Q\,. \tag{2.10}$$

We will use this symmetry to study (2.9) in a Weyl frame where the localization equations take a simpler form.

3 ’t Hooft loop in S^4 and localization equations

In this section we initiate our study of the expectation value of a supersymmetric ’t Hooft loop operator [5] in an arbitrary $\mathcal{N}=2$ gauge theory in S^4. We start by constructing a supersymmetric ’t Hooft loop operator which is annihilated by Q (and therefore by $J+R$). This implies that we can localize the ’t Hooft loop path integral using the supercharge Q. The derivation and interpretation of the localization saddle point equations $Q\cdot\Psi=0$ in (2.9) follow. We will then find the most general non-singular solution to the localization equations in the presence of a supersymmetric ’t Hooft loop operator.

A ’t Hooft loop operator inserts a Dirac monopole into (an arbitrary) spacetime. The operator has support on the loop/curve spanned by the wordline of the monopole. In an arbitrary gauge theory, the operator is characterized by a boundary condition near the support of the loop operator that specifies the magnetic flux created by the monopole. Since the choice of ’t Hooft operator depends on the embedding of the U(1) gauge group of a Dirac monopole into the gauge group G, these operators are labeled by a coweight or magnetic weight vector B, which takes values in the coweight lattice Λ_{cw} of the gauge group G [7].

Locally, near the location of any point on the loop — where the loop is locally a straight line — the ’t Hooft operator creates quantized magnetic flux [16]

$$F=\frac{B}{4}\epsilon_{ijk}\frac{x^i}{|\vec{x}|^3}dx^k\wedge dx^j\,, \tag{3.1}$$

[9] This can be shown by writing $V_{\rm hm}$ as a sum of squares. One of the terms that is generated is a mass term for the scalars in the hypermultiplet — that is $qq^\dagger+\tilde{q}\tilde{q}^\dagger$ — which implies that on the saddle point $q=\tilde{q}=0$.

[10] This formula should be dimensionally reduced to four dimensions using that $F_{mn}=[D_m,D_n]$ and that $D_A\cdot=[\Phi_A,\cdot]$ for $A=9,0$. See appendix B for gauge theory conventions.

where x^i for $i = 1, 2, 3$ denote the three local transverse coordinates to any point in the loop. Since $B \equiv B^a H_a \in \mathfrak{t}$ takes values in the Cartan subalgebra $\mathfrak{t}$ of the Lie algebra $\mathfrak{g}$ of the gauge group G, the magnetic flux (3.1) is abelian. Locally, this operator inserts quantized flux through the S^2 that surrounds any point in the loop

$$\int_{S^2} \frac{F}{2\pi} = -B \,. \tag{3.2}$$

In order to be able to apply localization we must consider supersymmetric 't Hooft loop operators invariant under the action of Q. These operators create a local singularity on the scalar fields of the $\mathcal{N} = 2$ vectormultiplet. The singularity which will be locally compatible with our choice of Q is[11]

$$\Phi_9 = \frac{B}{2|\vec{x}|} \,. \tag{3.3}$$

A 't Hooft loop operator which is globally annihilated by Q and $J + R$, can be constructed by choosing — without loss of generality — the support of the 't Hooft operator to be the maximal circle on S^4

$$X_1^2 + X_2^2 = r^2, \qquad X_3 = X_4 = X_5 = 0 \,, \tag{3.4}$$

which is located at the equator of S^4 and left invariant by the action of J (see (2.4)). We find it convenient to study the localization equations $Q \cdot \Psi = 0$ in (2.9) in the presence of the circular 't Hooft loop by choosing the following coordinates on S^4 (see appendix C for various useful coordinate systems)

$$ds^2 = \frac{\sum_{i=1}^3 dx_i^2}{\left(1 + \frac{|\vec{x}|^2}{4r^2}\right)^2} + r^2 \frac{\left(1 - \frac{|\vec{x}|^2}{4r^2}\right)^2}{\left(1 + \frac{|\vec{x}|^2}{4r^2}\right)^2} d\tau^2 \,. \tag{3.5}$$

The coordinates x_i, where $|\vec{x}|^2 \leq 4r^2$, define a three-ball B_3. In these coordinates, the support of the circular 't Hooft loop (3.4) is the maximal circle parametrized by the coordinate τ located at $x_i = 0$. In these coordinates the action of J, defined in (2.4), is (see equation (C.15))

$$\begin{aligned} x_1 + ix_2 &\to e^{i\varepsilon}(x_1 + ix_2) \\ \tau &\to \tau + \varepsilon \,. \end{aligned} \tag{3.6}$$

Therefore, the north and south poles of the S^4 — the fixed points of the action of J — are located at $\vec{x} = (0, 0, 2r)$ and $\vec{x} = (0, 0, -2r)$ respectively.

Now by using the invariance of the saddle point equations (2.9) under the Weyl transformation (2.10), the solutions to the saddle point equations on S^4 can be obtained from the solutions of the saddle point equations in $B_3 \times S^1$

$$ds^2_{B_3 \times S^1} = \sum_{i=1}^3 dx_i^2 + r^2 \left(1 - \frac{|\vec{x}|^2}{4r^2}\right)^2 d\tau^2 \,. \tag{3.7}$$

They are related by the transformation (2.10) with $\Omega = \left(1 + \frac{|\vec{x}|^2}{4r^2}\right)$.

[11] This follows by noting that a 't Hooft loop sourcing the scalar field Φ_9 shares common supersymmetries with the Wilson loop considered in [2] — which couples to the scalar field Φ_0 — and which by construction is annihilated by the supercharge Q. We will soon explicitly show that the exact 't Hooft loop singularity is invariant under the action of Q.

One advantage of this choice of Weyl frame is that the exact singularity produced by the circular 't Hooft loop operator in $B_3 \times S^1$ is identical to the one produced by inserting a static point-like monopole in flat spacetime. The exact circular 't Hooft loop background on $B_3 \times S^1$ annihilated by Q when the topological angle vanishes — that is when $\theta = 0$ — is given by[12]

$$\begin{aligned} F &= \frac{B}{4}\epsilon_{ijk}\frac{x^i}{|\vec{x}|^3}dx^k \wedge dx^j \\ \Phi_9 &= \frac{B}{2|\vec{x}|}\,. \end{aligned} \tag{3.8}$$

When the topological angle is non-trivial — that is when $\theta \neq 0$ — then the particle inserted by the 't Hooft operator is a dyon, which acquires electric charge through the Witten effect [17]. If the 't Hooft operator is labeled by a magnetic weight B, the induced electric weight is $g^2\theta B/4\pi$. Moreover, the scalar field Φ_0 also acquires a singularity near the loop. The exact background created by a supersymmetric 't Hooft loop on $B_3 \times S^1$ is given by [12]

$$\begin{aligned} F_{jk} &= -\frac{B}{2}\epsilon_{ijk}\frac{x_i}{|\vec{x}|^3}\,, \quad & F_{i\hat{4}} &= -ig^2\theta\frac{B}{16\pi^2}\frac{x_i}{|\vec{x}|^3}\,, \\ \Phi_9 &= \frac{B}{2|\vec{x}|}\,, & \Phi_0 &= -g^2\theta\frac{B}{16\pi^2}\frac{1}{|\vec{x}|}\,. \end{aligned} \tag{3.9}$$

The corresponding singularity created by the insertion of the circular 't Hooft loop in S^4 can then be simply obtained by performing the Weyl transformation (2.10) with $\Omega = \left(1 + \frac{|\vec{x}|^2}{4r^2}\right)$.

3.1 Symmetries and fields

We now proceed to determining the partial differential equations for the bosonic fields in the $\mathcal{N} = 2$ vectormultiplet on $B_3 \times S^1$ whose solutions yield the saddle points of the localization path integral upon Weyl transforming them back to S^4.[13] We first have to choose the supercharge Q with which to localize the 't Hooft loop path integral.

The supersymmetry transformations of an $\mathcal{N} = 2$ gauge theory on a four manifold with metric $h_{\mu\nu}$ is parametrized by a sixteen component Weyl spinor of Spin(10) which solves the conformal Killing spinor equation[14]

$$\nabla_\mu \epsilon = \tilde{\Gamma}_\mu \tilde{\epsilon} \tag{3.10}$$

subject to the projection

$$\Gamma^{5678}\epsilon = -\epsilon\,. \tag{3.11}$$

$\tilde{\epsilon}$ is determined in terms of ϵ by $\tilde{\epsilon} = \frac{1}{4}\Gamma^\mu\nabla_\mu\epsilon$.[15] It satisfies

$$\tilde{\Gamma}^\mu\nabla_\mu\tilde{\epsilon} = -\frac{R}{12}\epsilon\,, \tag{3.12}$$

where R is the scalar curvature derived from $h_{\mu\nu}$.

[12] In appendix D we show that this background solves the localization saddle point equations $Q \cdot \Psi = 0$ derived in the next subsection.

[13] As we mentioned earlier, the saddle point equations for the hypermultiplets force the fields in the multiplet to vanish.

[14] The theory has maximal number of supersymmetries when the metric is conformally flat.

[15] The spinor ϵ is referred to as a conformal Killing spinor because its defining equation $\nabla_\mu\epsilon = \frac{1}{4}\tilde{\Gamma}_\mu\Gamma^\nu\nabla_\nu\epsilon$ is invariant under the Weyl transformation (2.10).

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The equations on $B_3 \times S^1$ that we need to analyze are[16]

$$\frac{1}{2}F_{mn}\Gamma^{mn}\epsilon - \frac{1}{2}\Phi_A\Gamma^{A\mu}\nabla_\mu\epsilon + iK_j\Gamma^{8\,j+4}\epsilon = 0\,, \tag{3.13}$$

for an specific choice of $\epsilon = \epsilon_Q$. Here ϵ is the (commuting) conformal Killing spinor of the $\mathcal{N} = 2$ gauge theory on $B_3 \times S^1$ which parametrizes the supersymmetry transformation generated by the supercharges of the OSp(2|4) symmetry of the $\mathcal{N} = 2$ gauge theory. The general conformal Killing spinor on $B_3 \times S^1$ — with metric (3.7) — is given by (see appendix A for details and conventions)

$$\epsilon = \cos(\tau/2)\left(\hat{\varepsilon}_s + x^i\tilde{\Gamma}_i\,\hat{\varepsilon}_c\right) + \sin(\tau/2)\,\tilde{\Gamma}^4\left(2r\,\hat{\varepsilon}_c + \frac{x^i}{2r}\Gamma_i\,\hat{\varepsilon}_s\right)\,, \tag{3.14}$$

where $\hat{\varepsilon}_s$ and $\hat{\varepsilon}_c$ are two constant ten dimensional Weyl spinors of opposite ten dimensional chirality obeying $\Gamma^{5678}\hat{\varepsilon}_s = -\varepsilon_s$ and $\Gamma^{5678}\hat{\varepsilon}_c = -\varepsilon_c$.

We now identify the spinor ϵ_Q which parametrizes the supersymmetry transformations of the supercharge Q generating the SU(1|1) subgroup of the OSp(2|4) symmetry of an $\mathcal{N} = 2$ theory in S^4. This is the supercharge used in our localization analysis. We take the spinor $\hat{\varepsilon}_s$ to be

$$\hat{\varepsilon}_s = \tfrac{1}{2}(1, 0, 0, 0, 0^4, 1, 0, 0, 0, 0^4) \tag{3.15}$$

and the spinor $\hat{\varepsilon}_c$

$$\hat{\varepsilon}_c = -\frac{i}{2r}\Gamma^{120}\hat{\varepsilon}_s = \frac{1}{4r}(0, 0, 0, -1, 0^4, 0, 0, 0, -1, 0^4)\,. \tag{3.16}$$

Therefore, the conformal Killing spinor associated to Q is given by

$$\epsilon_Q = \frac{1}{4r}\begin{pmatrix} 2r\cos\left(\frac{\tau}{2}\right) - x_3\cos\left(\frac{\tau}{2}\right) \\ x_1\sin\left(\frac{\tau}{2}\right) + x_2\cos\left(\frac{\tau}{2}\right) \\ x_2\sin\left(\frac{\tau}{2}\right) - x_1\cos\left(\frac{\tau}{2}\right) \\ x_3\sin\left(\frac{\tau}{2}\right) - 2r\sin\left(\frac{\tau}{2}\right) \\ 0^4 \\ 2r\cos\left(\frac{\tau}{2}\right) + x_3\cos\left(\frac{\tau}{2}\right) \\ -x_1\sin\left(\frac{\tau}{2}\right) - x_2\cos\left(\frac{\tau}{2}\right) \\ x_1\cos\left(\frac{\tau}{2}\right) - x_2\sin\left(\frac{\tau}{2}\right) \\ -2r\sin\left(\frac{\tau}{2}\right) - x_3\sin\left(\frac{\tau}{2}\right) \\ 0^4 \end{pmatrix}, \tag{3.17}$$

and has norm $\epsilon_Q\epsilon_Q = \frac{1}{2}\left(1 + \frac{|\vec{x}|^2}{4r^2}\right)$.[17]

We note that the spinor ϵ_Q generates one of the unbroken supersymmetries[18] preserved by the circular Wilson loop coupled to the scalar field Φ_0 in the $\mathcal{N} = 2$ vectormultiplet

$$\mathrm{Tr}\exp\left(\oint_{S^1}\left[A_\mu\frac{dx^\mu}{ds} + i|\dot{x}|\Phi_0\right]ds\right) \tag{3.18}$$

[16] In our conventions Q acts on a field as a fermionic operator and therefore ϵ is a commuting spinor.
[17] Using (2.10) the norm of the spinor on S^4 is therefore 1/2.
[18] Which obey $(1 + i\tilde{\Gamma}^4\Gamma^0)\hat{\varepsilon}_s = 0$ and $(1 + i\Gamma^0\tilde{\Gamma}^4)\hat{\varepsilon}_c = 0$.

supported on the maximal circle at $\vec{x}=0$ in $B_3 \times S^1$. Therefore Q can be used to localize the path integral in the presence of this Wilson loop operator, as in [2].

Given our choice of supercharge Q, we can now calculate Q^2, that is the symmetries and gauge transformation that Q^2 generates when acting on the fields of the $\mathcal{N}=2$ gauge theory. Due to the addition of suitable auxiliary fields, this symmetry is realized off-shell, as required for localization.

The spacetime symmetry transformation induced by Q^2 is generated by the Killing vector

$$v^\mu(x,\tau) \equiv e^\mu{}_{\hat{\mu}} v^{\hat{\mu}} = 2\epsilon_Q \Gamma^\mu \epsilon_Q = \left(-\frac{x_2}{r}, \frac{x_1}{r}, 0, \frac{1}{r}\right) , \tag{3.19}$$

where $e^{\hat{i}} = e^i = dx^i$ for $i = 1,2,3$ and $e^{\hat{4}} = r\left(1 - \frac{|\vec{x}|^2}{4r^2}\right) d\tau$ is a vielbein basis for the metric on $B_3 \times S^1$ given in (3.7). Therefore, Q^2 yields the infinitesimal $\mathrm{U}(1)_J$ spacetime transformation (3.6) generated by J.

The operator Q^2 also generates a $\mathrm{U}(1)_R$ R-symmetry transformation. It acts on the fields of the theory as a $\mathrm{U}(1)_R \subset \mathrm{SU}(2)_R$ subgroup of the $\mathrm{SU}(2)_R$ symmetry present when the $\mathcal{N}=2$ theory is in flat spacetime. Therefore, it acts on the gauginos Ψ in the vectormultiplet and the scalars $(q, \tilde{q}^\dagger)$ in the hypermultiplet. The infinitesimal R-symmetry transformation generated by Q^2 is parametrized by the rotation parameter[19]

$$v_R \equiv -4\tilde{\epsilon}_Q \, \Gamma^{56} \epsilon_Q = -4\tilde{\epsilon}_Q \, \Gamma^{78} \epsilon_Q = \frac{1}{r} \, . \tag{3.20}$$

As advertised, our choice of supercharge Q corresponding to the Killing spinor (3.17) generates an $\mathrm{SU}(1|1)$ subalgebra of $\mathrm{OSp}(2|4)$

$$Q^2 = J + R \qquad [J+R, Q] = 0 \, , \tag{3.21}$$

where $J+R$ generates $\mathrm{U}(1)_{J+R} \equiv (\mathrm{U}(1)_J \times \mathrm{U}(1)_R)_{\mathrm{diag}}$.

In the presence of N_{F} hypermultiplets transforming in a representation R of G, the $\mathcal{N}=2$ gauge theory has a flavour symmetry group G_{F}, and the masses m_f with $f = 1, \ldots N_{\mathrm{F}}$ of the hypermultiplets take values in the Cartan subalgebra of the flavour symmetry algebra, which has rank N_{F}. The action of Q^2 on the hypermultiplets fields generates an infinitessimal flavour symmetry transformation with parameters m_f, while the flavour symmetry action on vectomultiplet fields is trivial.

Finally, the operator Q^2 further generates a gauge transformation with gauge group G on all the fields in the theory. The gauge transformation is a function of the scalar fields $\Phi_A = (\Phi_9, \Phi_0)$ of the $\mathcal{N}=2$ vectormultiplet. The associated gauge parameter is given by

$$\Lambda \equiv \Phi_A v^A \, , \tag{3.22}$$

where

$$v^A \equiv 2\epsilon_Q \Gamma^A \epsilon_Q \qquad A = 9, 0 \, . \tag{3.23}$$

[19] Where $\tilde{\epsilon}_Q = \frac{1}{4}\Gamma^\mu \nabla_\mu \epsilon = \frac{1}{4r}\left(-\sin(\frac{\tau}{2}), 0^2, -\cos(\frac{\tau}{2}), 0^4, -\sin(\frac{\tau}{2}), 0^2, -\cos(\frac{\tau}{2}), 0^4\right)$.

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Explicit calculation using (3.17) and Weyl transforming to S^4 using (2.10) gives[20]

$$\Lambda = i\Phi_0 - \frac{x_3/r}{1+\frac{|\vec{x}|^2}{4r^2}}\Phi_9\,. \tag{3.24}$$

This implies that the gauge transformation parameter at the north and south poles of S^4 — which are located at $\vec{x} = (0,0,\pm 2r)$ — are

$$\begin{aligned}\Lambda(N) &= i\Phi_0(N) - \Phi_9(N)\\ \Lambda(S) &= i\Phi_0(S) + \Phi_9(S)\,.\end{aligned} \tag{3.25}$$

Therefore the gauge transformation acts differently at the north and south poles of the S^4, which are the fixed points of the action of the U(1) generator J. This observation will have far reaching consequences in our computation of the expectation value of 't Hooft operators in these theories. At the equator of S^4, the component A_4 of the gauge field also enters in the gauge parameter,

$$\Lambda(E) = v^0\Phi_0 + v^4 A_4\,. \tag{3.26}$$

In summary, Q^2 acting on the bosonic fields of the $\mathcal{N}=2$ vectormultiplet — whose localization equations we are after — generates a $J+R$ and a G-gauge transformation that can be encoded in terms of the vector field

$$v^m = 2\epsilon_Q\Gamma^m\epsilon_Q = \left(-\frac{x_2}{r},\frac{x_1}{r},0,\frac{1}{r},-\frac{x_3}{r},i\left(1+\frac{|\vec{x}|^2}{4r^2}\right)\right) \qquad m=1,2,3,4,9,0\,. \tag{3.27}$$

More explicitly, the action of Q^2 on these fields is [10]

$$\begin{aligned}Q^2\cdot A_\mu &= -[v^m D_m, D_\mu]\\ Q^2\cdot\Phi_A &= -[v^m D_m,\Phi_A]\,.\end{aligned} \tag{3.28}$$

Including the action on hypermultiplets, we conclude that the action of Q^2 on all fields in the $\mathcal{N}=2$ theory defined on $B_3\times S^1$ generates an $\mathrm{U}(1)_{J+R}\times G\times G_{\mathrm{F}}$ transformation.

3.2 Localization equations in $B_3 \times S^1$

Given our choice of supercharge Q, we can now proceed to finding the saddle point equations (2.9) of the localization path integral

$$Q\cdot\Psi = \frac{1}{2}F_{mn}\Gamma^{mn}\epsilon_Q - 2\Phi_A\tilde{\Gamma}^A\tilde{\epsilon}_Q + iK_j\Gamma^{8\,j+4}\epsilon_Q = 0\,, \tag{3.29}$$

where we have used that

$$\nabla_\mu\epsilon_Q = \tilde{\Gamma}_\mu\tilde{\epsilon}_Q\,, \tag{3.30}$$

and ϵ_Q is given in (3.17). The equations can be found by projecting (3.29) on a basis of spinors generated by[21]

$$\Gamma_m\epsilon_Q \qquad m=1,2,3,4,9 \tag{3.31}$$

$$\overline{\Gamma^{8\,j+4}\epsilon_Q} \qquad j=1,2,3\,. \tag{3.32}$$

[20] See below for more details.

[21] Since the $\mathcal{N}=2$ theory has eight supercharges, there are eight independent equations.

We note that the projection equations along $\Gamma_0\epsilon_Q$ can be obtained from a linear combination of the projection equations along (3.31) since the conformal Killing spinor ϵ_Q satisfies the linear constraint

$$v^m\Gamma_m\epsilon_Q = 0\,, \tag{3.33}$$

with v_m given in (3.27).

In order to develop intuition for the saddle point equations, we first study them in the point $\vec{x}=0, \tau=0$ in $B_3\times S^1$. Projecting (3.29) along $\Gamma_{\hat{m}}\epsilon_Q$ yields

$$2\epsilon_Q\Gamma_{\hat{m}}Q\cdot\Psi = (F_{\hat{m}4}+iD_{\hat{m}}\Phi_0) - \frac{1}{r}\Phi_9\,\delta_{\hat{m}3} = 0 \qquad \hat{m}=1\ldots 4,9. \tag{3.34}$$

These equations have a simple interpretation. They describe the Q^2-invariance equations of the bosonic fields in the $\mathcal{N}=2$ vectormultiplet (obtained by setting equations (3.28) to zero), which at $\vec{x}=\tau=0$ are generated by the vector field $v^m=(0,0,0,\frac{1}{r},0,i)$ (see equation (3.27)). This captures the combined action of a J and a G-gauge transformation with vector field $v^\mu=(0,0,0,\frac{1}{r})$ and gauge parameter $\Lambda=i\Phi_0$ respectively. The invariance equation for the scalar field Φ_0 is a linear combination of (3.34), a fact which follows from (3.33).

Projection of (3.29) along $\overline{\Gamma^{8\,j+4}\epsilon_Q}$ gives three dimensional equations. They are the Bogomolny equations

$$2\overline{\epsilon_Q\Gamma_{j+4,8}}Q\cdot\Psi = -D_j\Phi_9 + (*_3F)_j + iK_j + \frac{i}{r}\delta_{j3}\Phi_0 = 0 \qquad j=1,2,3\,. \tag{3.35}$$

We can move to an arbitrary point $\tau\neq 0$ at $\vec{x}=0$ by acting on the equations (3.34) (3.35) by the $\mathrm{U}(1)_J$ transformation generated by J. The invariance equations (3.34) remain the same while the three dimensional equations take the same form (3.35) upon replacing the auxiliary scalar fields K_i by rotated ones

$$\begin{pmatrix} K_1 \\ K_2 \end{pmatrix} \to \begin{pmatrix} \cos\tau & \sin\tau \\ -\sin\tau & \cos\tau \end{pmatrix} \begin{pmatrix} K_1 \\ K_2 \end{pmatrix}\,. \tag{3.36}$$

We can now consider the general equations with $\vec{x}\neq 0$ and $\tau\neq 0$. The Q^2-invariance equations, obtained by projecting (3.29) along $\Gamma_m\epsilon_Q$, are given by[22]

$$\begin{aligned}
\frac{1}{2r}F_{14} + \left[D_1, \frac{i}{2}\left(1+\frac{|\vec{x}|^2}{4r^2}\right)\Phi_0 - \frac{x_3}{2r}\Phi_9\right] + \frac{x_1}{2r}F_{12} = 0 \quad [D_1, v^mD_m]=0\\
\frac{1}{2r}F_{24} + \left[D_2, \frac{i}{2}\left(1+\frac{|\vec{x}|^2}{4r^2}\right)\Phi_0 - \frac{x_3}{2r}\Phi_9\right] - \frac{x_2}{2r}F_{21} = 0 \quad [D_2, v^mD_m]=0\\
\frac{1}{2r}F_{34} + \left[D_3, \frac{i}{2}\left(1+\frac{|\vec{x}|^2}{4r^2}\right)\Phi_0 - \frac{x_3}{2r}\Phi_9\right] + \frac{x_1}{2r}F_{32} - \frac{x_2}{2r}F_{31} = 0 \quad [D_3, v^mD_m]=0\\
\left[D_4, \frac{i}{2}\left(1+\frac{|\vec{x}|^2}{4r^2}\right)\Phi_0 - \frac{x_3}{2r}\Phi_9\right] + \frac{x_1}{2r}F_{42} - \frac{x_2}{2r}F_{41} = 0 \quad [D_4, v^mD_m]=0\\
\frac{1}{2r}[\Phi_9, D_\tau] + \left[\Phi_9, \frac{i}{2}\left(1+\frac{|\vec{x}|^2}{4r^2}\right)\Phi_0\right] + \frac{x_1}{2r}[\Phi_9, D_2] - \frac{x_2}{2r}[\Phi_9, D_1] = 0 \quad [\Phi_9, v^mD_m]=0\,.
\end{aligned} \tag{3.37}$$

[22] As already mentioned, the invariance equation for Φ_0 is a linear combinations of these equations.

The three dimensional equations, which we call deformed monopole equations, are obtained by projecting (3.29) along $\overline{\Gamma^{8j+4}\epsilon_Q}$. They are given by

$$\begin{aligned}-(4r^2+x_1^2-x_2^2-x_3^2)[D_1\Phi_9]-2x_1x_2[D_2\Phi_9]-2x_1x_3[D_3\Phi_9]-4rx_2[D_{\hat4}\Phi_9]+\\ -2x_1\Phi_9-2x_1x_3F_{12}+2x_1x_2F_{13}+(4r^2-x_1^2+x_2^2+x_3^2)F_{23}+\\ -4rx_3F_{1\hat4}+i(4r^2+|\vec{x}|^2)K_1+4rx_1F_{3\hat4}=0\,.\end{aligned} \tag{3.38}$$

$$\begin{aligned}-(4r^2-x_1^2+x_2^2-x_3^2)[D_2\Phi_9]-2x_1x_2[D_1\Phi_9]-2x_2x_3[D_3\Phi_9]+4rx_1[D_{\hat4}\Phi_9]+\\ -2x_2\Phi_9-2x_2x_3F_{12}-2x_1x_2F_{23}-(4r^2+x_1^2-x_2^2+x_3^2)F_{13}+\\ -4rx_3F_{2\hat4}+i(4r^2+|\vec{x}|^2)K_2+4rx_2F_{3\hat4}=0\,,\end{aligned} \tag{3.39}$$

$$\begin{aligned}2x_1x_3[D_1\Phi_9]+2x_2x_3[D_2\Phi_9]-(4r^2+x_1^2+x_2^2-x_3^2)[D_3\Phi_9]+2x_3\Phi_9+4ir\Phi_0+\\ +(4r^2-x_1^2-x_2^2+x_3^2)F_{12}+2x_1x_3F_{23}-2x_2x_3F_{13}+\\ -4rx_2F_{2\hat4}-4rx_3F_{3\hat4}-4rx_1F_{1\hat4}+i\left(4r^2+|\vec{x}|^2\right)K_3=0\,.\end{aligned} \tag{3.40}$$

We note that the equations near the location of the 't Hooft loop — at $\vec{x}=0$ — reduce to the familiar Bogomolny equations in $\mathbb{R}^3$, thus justifying their name. These equations are a supersymmetric extension of well known equations, which interpolate between $F^+=0$ at the north pole, the Bogomolny equations at the equator and $F^-=0$ at the south pole. This concludes our derivation of the saddle point equations of the 't Hooft loop path integral.

In appendix D we explicitly show that the background created by the insertion of a circular 't Hooft loop — given in equations (3.8) (and (3.9) when $\theta\neq 0$) — is a solution of the localization equations derived in this section. This confirms that we can study the expectation value of a supersymmetric circular 't Hooft loop operator in any $\mathcal{N}=2$ gauge theory on S^4 by localizing the path integral with our choice of supercharge Q.

We can now anticipate some key features in the evaluation of the 't Hooft loop path integral of the $\mathcal{N}=2$ theory defined on S^4. As explained earlier, the fields and conformal Killing spinor in $B_3\times S^1$ and S^4 are related by the Weyl transformation (2.10) with $\Omega=\left(1+\frac{|\vec{x}|^2}{4r^2}\right)$. We note that the conformal Killing spinor in S^4 — which we denote by $\epsilon_Q^{\text{sphere}}$ — has negative/positive four dimensional chirality at the fixed points of the U(1) action of J, denoted as north/south poles of S^4 respectively. In formulas[23]

$$\begin{aligned}\Gamma^{\hat4\hat1\hat2\hat3}\epsilon_Q^{\text{sphere}}(N)&=-\epsilon_Q^{\text{sphere}}(N)\\ \Gamma^{\hat4\hat1\hat2\hat3}\epsilon_Q^{\text{sphere}}(S)&=+\epsilon_Q^{\text{sphere}}(S)\,,\end{aligned} \tag{3.41}$$

and therefore instantons and anti-instantons are supersymmetric at the north and south poles of S^4 respectively.

Moreover, in the neighborhood of the north pole the Q-complex of the $\mathcal{N}=2$ theory on S^4 generated by $\epsilon_Q^{\text{sphere}}$ reduces to the complex of the equivariant Donaldson-Witten twist[24]

[23] The volume form is given by $\epsilon^{\hat4\hat1\hat2\hat3}=1$.
[24] Also known as $\mathcal{N}=2$ gauge theory in the Ω-background [8].

in $\mathbb{R}^4$ [8], described by the instanton equations $F^+ = 0$. Likewise, in the neighborhood of the south pole the Q-complex of the $\mathcal{N} = 2$ theory on S^4 reduces to that of the equivariant conjugate Donaldson-Witten twist in $\mathbb{R}^4$, described by the anti-instanton equations $F^- = 0$.

This implies that the path integral for a 't Hooft loop receives contributions from equivariant instantons at the north pole and equivariant anti-instantons at the south pole. These are singular solutions to the localization equations which must be included in the computation of the 't Hooft loop expectation value. The equivariant instanton/anti-instanton partition function in $\mathbb{R}^4$ is captured by the so-called Nekrasov partition function [8], which will play a prominent role in our analysis.

From our expression for the action of Q^2 on the fields (see (3.27)), we find that the $\mathrm{U}(1)_{\varepsilon_1} \times \mathrm{U}(1)_{\varepsilon_2}$ equivariant rotation parameters $(\varepsilon_1, \varepsilon_2)$ in Nekrasov's partition function [8] at the north and south poles in S^4 are fixed to

$$\varepsilon_1 = \varepsilon_2 = \varepsilon = \frac{1}{r}\,, \tag{3.42}$$

since $\mathrm{U}(1)_{\varepsilon_1} \times \mathrm{U}(1)_{\varepsilon_2}$ acts on $\mathbb{R}^4$ as

$$X_1 + iX_2 \to e^{i\varepsilon_1}(X_1 + iX_2) \qquad\qquad X_3 + iX_4 \to e^{i\varepsilon_2}(X_3 + iX_4)\,. \tag{3.43}$$

Here $(X_1, \ldots, X_4)$ are the S^4 embedding coordinates (2.3) which parametrize the local $\mathbb{R}^4$ near the north and south poles. As Nekrasov's partition function is for the $\mathcal{N} = 2$ topologically twisted theory in $\mathbb{R}^4$ — which mixes the SU(2) Lorentz with SU(2) R-symmetry generators — the $\mathrm{U}(1)_{J+R}$ symmetry generated by Q^2 in the physical theory on S^4 gets identified at the north and south poles with the $(\mathrm{U}(1)_{\varepsilon_1} \times \mathrm{U}(1)_{\varepsilon_2})_{\mathrm{diag}}$ symmetry in Nekrasov's partition function.

Moreover, it follows from equation (3.27) that the equivariant parameter $\hat{a} \in \mathfrak{t}$ for the action of constant G-gauge transformations in $\mathbb{R}^4$ in Nekrasov's partition function is fixed at the north and and south poles of the S^4 to[25]

$$\hat{a}(N) = i\Phi_0(N) - \Phi_9(N) \qquad \hat{a}(S) = i\Phi_0(S) + \Phi_9(S) \tag{3.44}$$

respectively. Since the 't Hooft loop induces a non-trivial background for the scalar field Φ_9 (3.9), which is non-vanishing at the north and south poles, the instanton/anti-instanton partition function contributions arising from the fixed points of J explicitly depend on the magnetic weight B labeling the 't Hooft operator. We will return to the instanton and anti-instanton contributions to the 't Hooft loop path integral in section 5.

Likewise, there are singular solutions to the localization equations arising from the equator in S^4, where the 't Hooft loop is inserted. As we have shown, near the equator we must consider solutions to the Bogomolny equations in the presence of the singular monopole configuration created by the 't Hooft loop operator. We will consider the contribution of these singular solutions to the saddle point equations in sections 6.3 and 7.

Our next task is to study the non-singular solutions of the localization equations.

[25] Here we note that the value of scalar fields at the north and south poles of $B_3 \times S^1$ and S^4 are related by $\Phi_{S^4} = 2\Phi_{B_3 \times S^1}$ through Weyl rescaling, while at the equator $\Phi_{S^4} = \Phi_{B_3 \times S^1}$.

3.3 Vanishing theorem

In the evaluation of the 't Hooft loop path integral using localization we must sum over all the saddle points of the localization action $Q\cdot V$ which have a prescribed singularity, induced by insertion of the 't Hooft operator. Therefore, we wish to obtain the most general solution of the localization equations (3.37)–(3.40) satisfying the appropriate boundary conditions imposed by the presence of the circular 't Hooft loop operator. The boundary condition requires that the solutions to the localization equations approach the background (3.9) near the location of the 't Hooft loop, supported at the equator of S^4.

In this section we obtain the most general non-singular solution to these equations (besides the singularity due to the ' t Hooft operator). Singular solutions to the localization equations, however, will play a central role in our computations. We will discuss singular solutions supported at the north and south poles of the S^4 and their contribution to the expectation value of the 't Hooft loop operator in section 5, while the contribution of the singular solutions supported at the equator will be analyzed in section 7.

In appendix D we show that the field configuration

$$\begin{aligned} &F_{jk} = -\frac{B}{2}\epsilon_{ijk}\frac{x_i}{|\vec{x}|^3}\,, && F_{i\hat{4}} = -ig^2\theta\frac{B}{16\pi^2}\frac{x_i}{|\vec{x}|^3}\,, \qquad \Phi_9 = \frac{B}{2|\vec{x}|}\,, \\ &\Phi_0 = -g^2\theta\frac{B}{16\pi^2}\frac{1}{|\vec{x}|} + \frac{a}{1+\frac{|\vec{x}|^2}{4r^2}}\,, && K_3 = -\frac{a/r}{\left(1+\frac{|\vec{x}|^2}{4r^2}\right)^2}\,, \end{aligned} \tag{3.45}$$

solves the saddle point equations $Q\cdot\Psi = 0$. This field configuration is the 't Hooft loop background (3.9) deformed by a "zeromode"[26] of Φ_0, which is labeled by a. The auxiliary field K_3 in the $\mathcal{N}=2$ vectormultiplet is also turned on. Therefore, evaluation of the path integral requires integrating over the "zeromode" $a\in\mathfrak{t}$, which takes values in the Cartan subalgebra $\mathfrak{t}$ of the gauge group G.

We will now show that the only solutions to $Q\cdot\Psi=0$ which are smooth away from the loop are given by (3.45). For this it suffices to consider the deformed monopole equations, the differential equations (3.38)–(3.40). We find it more transparent, however, to take instead a projection of the localization equations $Q\cdot\Psi = 0$ along $\overline{\Gamma_{9\mu}\epsilon_Q}$. This gives

$$0 = \overline{\epsilon_Q\Gamma_{\mu 9}}Q\cdot\Psi = -(*F)_{\mu\nu}v^\nu + \frac{i}{2r}D_\mu(x_3\Phi_0) - D_\mu\left[\frac{1}{2}\left(1+\frac{|\vec{x}|^2}{4r^2}\right)\Phi_9\right] + i\sum_{j=1}^{3} w_\mu^{(j)}K_j\,, \tag{3.46}$$

where we have used $[\Phi_9,\Phi_0]=0$, which follows from the imaginary part of the last equation in (3.37). We have also defined three real 1-forms $w_\mu^{(j)} = \overline{\epsilon_Q\Gamma_{\mu 9}}\Gamma^{8\,j+4}\epsilon_Q$.

The field strength $F = F^{(r)} + iF^{(i)}$ has real and imaginary parts. The imaginary part is due to the presence of the 't Hooft operator background with $\theta\neq 0$, while the fluctuating part of the field that we integrate over in the path integral must be real. The imaginary part of equations (3.38) and (3.39) imply that

$$K_1 = K_2 = 0\,, \tag{3.47}$$

[26] The corresponding field configuration is annihilated by $D_\mu\left[\left(1+\frac{|\vec{x}|^2}{4r^2}\right)\Phi_0\right] = 0$ since the background gauge field is abelian.

while the imaginary part of (3.37) requires that

$$\begin{gathered}\frac{1}{2r}F^{(i)}_{j4}+\left[D_j,\frac{i}{2}\left(1+\frac{|\vec{x}|^2}{4r^2}\right)\Phi_0\right]=0 \qquad j=1,2,3\\ x_1F^{(i)}_{42}-x_2F^{(i)}_{41}=0\,.\end{gathered} \tag{3.48}$$

Therefore, these equations completely determine Φ_0 in terms of the electric field produced by the 't Hooft operator when $\theta\neq 0$ up to a zeromode, which we parametrize by a in (3.45). Moreover, the imaginary part of (3.40) locks in the value of the auxiliary field K_3 in terms of the zeromode part of Φ_0. Therefore, the most general solution to the localization equations for the electric field F_{j4} and the scalar fields Φ_0, K_1, K_2 and K_3 is given in (3.45). Now it remains to show that the most general solution to the localization equations for the magnetic field F_{ij} and the scalar field Φ_9 is also given by (3.45).

From the real part of (3.46) we obtain

$$i_v*F^{(r)}-D\left[\left(1+\frac{|\vec{x}|^2}{4r^2}\right)\Phi_9\right]=0\,. \tag{3.49}$$

We also note that the real part of the Q^2-invariance equations (3.37) implies that

$$-i_vF+D(v^9\Phi_9)=0\,, \qquad\qquad i_vD\Phi_9=0\,. \tag{3.50}$$

Let us define a 1-form $\tilde{v}=dx^\mu v_\mu/(v_\nu v^\nu)$ dual to the four-vector v, so that $i_v\tilde{v}=1$.[27] Now, in terms of the redefined gauge field

$$\hat{A}=A+v^9\Phi_9\,\tilde{v}\,, \tag{3.51}$$

the Q^2-invariance equations (3.50) imply that

$$i_v\hat{F}=0\,, \tag{3.52}$$

where $\hat{F}=d\hat{A}+\hat{A}\wedge\hat{A}$. Indeed,

$$\begin{aligned}i_v\hat{F}&=i_vF+i_vD(v^9\Phi_9\,\tilde{v})=i_vF+i_v(D(v^9\Phi_9)\wedge\tilde{v})+i_v(v^9\Phi_9D\tilde{v})=\\&=i_vF+(i_vD(v^9\Phi_9))\wedge\tilde{v}-D(v^9\Phi_9)\wedge(i_v\tilde{v})+v^9\Phi_9\,i_v(D\tilde{v})\,,\end{aligned} \tag{3.53}$$

which using $i_v\tilde{v}=1$ and the equations in (3.50) makes the first three terms vanish. The last term vanishes for any Riemannian metric invariant under the action generated by the vector field v. In this situation $\mathcal{L}_v\tilde{v}=0$ and since $\mathcal{L}_v=Di_v+i_vD$ we have that indeed $i_vD\tilde{v}=-D(i_v\tilde{v})=-D(1)=0$. Therefore, the Q^2-invariance equations (3.52) reduce the whole system of equations in S^4 to equations in the three-dimensional space $M_3=S^4/\mathrm{U}(1)$, since v generates the $\mathrm{U}(1)_J$ spacetime transformation corresponding to J.

The scalar field in the S^4 conformal frame is $\Phi=\Phi_9\left(1+\frac{|\vec{x}|^2}{4r^2}\right)$. In the S^4 metric[28] (see appendix C)

$$ds^2_{S^4}=r^2d\vartheta^2+\frac{r^2}{4}\sin^2\vartheta d\Omega_2+r^2\sin^2\vartheta(d\psi+\omega)^2 \tag{3.54}$$

[27]The definition of $\tilde{v}$ is invariant under Weyl rescaling of the metric.

[28]The orientation is such that the volume form is proportional to $d\tau dx^1dx^2dx^3\propto d\vartheta d\psi\mathrm{vol}(S^2)$.

equation (3.49) reads

$$i_v * F^{(r)} = D\Phi\,. \tag{3.55}$$

In this metric, the 1-form $\tilde{v}$ is given by $\tilde{v} = r(d\psi+\omega)$, and the redefined gauge field (3.51) is

$$\hat{A} = A^{(r)} - \Phi r\cos\vartheta(d\psi+\omega)\,. \tag{3.56}$$

Let us also redefine the scalar as

$$\hat{\Phi} = \Phi\sin^2\vartheta\,. \tag{3.57}$$

In terms of the redefined fields, equation (3.55) becomes

$$i_v * \hat{F} - \frac{1}{\sin^2\vartheta}D\hat{\Phi} = 0\,. \tag{3.58}$$

We can obtain quantities with nice properties by considering the background values of $\hat{A}$ and $\hat{\Phi}$ specified by (3.9). We define 1-forms λ and ρ, as well as a function h as quantities that appear in (3.9).[29]

$$\hat{A} = -B\lambda\,, \qquad A = -B\rho\,, \qquad \hat{\Phi} = \frac{B}{2}h \quad \text{(in the background)}\,. \tag{3.60}$$

Since the background solves the equation (3.58), λ and h satisfy the relation

$$0 = i_v * d\lambda + \frac{dh}{2\sin^2\vartheta}\,. \tag{3.61}$$

In order to derive useful identities, let us square the left-hand side of the equation (3.58) and integrate it with an appropriate measure:

$$\begin{aligned} 0 &= \int_{S^4}\frac{1}{2h}\left|\left|i_v * \hat{F} - \frac{1}{\sin^2\vartheta}D\hat{\Phi}\right|\right|^2 \\ &= \int_{S^4}\frac{1}{2h}\left(\left|\left|i_v * \hat{F} - \frac{\Phi}{h}dh\right|\right|^2 + \left|\left|\frac{h}{\sin^2\vartheta}D\left(\frac{\hat{\Phi}}{h}\right)\right|\right|^2\right) \\ &\quad + \int_{S^4}\frac{1}{\sin^2\vartheta}\operatorname{Tr}\left[D\left(\frac{\hat{\Phi}}{h}\right)\wedge *\left(i_v * \hat{F} - \frac{\Phi}{h}dh\right)\right]\,. \end{aligned} \tag{3.62}$$

We are using both Φ and $\hat{\Phi}$ to simplify the equations. Now we will show that the integrand in the cross term is the total derivative of a suitable 3-form. First we can write it as

$$\tilde{v}\wedge d\operatorname{Tr}\left[\frac{\hat{\Phi}}{h}\left(\hat{F} + \frac{\hat{\Phi}}{h}d\lambda\right)\right]\,. \tag{3.63}$$

[29] In the gauge where the Dirac singularity is at $x_3<0, x_1=x_2=0$, we have explicitly

$$\begin{aligned} \lambda &= \frac{1}{2}d\psi + \frac{1}{2}\left(1 - \frac{1+\cos\eta}{2}\frac{\cos\vartheta}{\left(\cos^2\vartheta + \sin^2\vartheta\sin^2\frac{\eta}{2}\right)^{1/2}}\right)d\varphi\,, \\ \rho &= \frac{1}{2}\left(1 - \frac{\cos\vartheta}{\left(\cos^2\vartheta + \sin^2\vartheta\sin^2\frac{\eta}{2}\right)^{1/2}}\right)(d\psi + d\varphi)\,, \qquad h = \frac{\sin^2\vartheta}{r}\left(1 - \sin^2\vartheta\cos^2\frac{\eta}{2}\right)^{-1/2}\,. \end{aligned} \tag{3.59}$$

We denote by $\hat{D}$ the covariant derivative with respect to the gauge field $\hat{A}$. Note that $\hat{A}+B\lambda$, which is the difference of two connections, is a globally defined smooth 1-form. Using that $d\,\mathrm{Tr}[(\hat{A}+B\lambda)\hat{\Phi}/h] = \mathrm{Tr}\left[(\hat{F}+Bd\lambda+(\hat{A}+B\lambda)\wedge(\hat{A}+B\lambda))\hat{\Phi}/h - (\hat{A}+B\lambda)\wedge\hat{D}(\hat{\Phi}/h)\right]$, one can check that the expression (3.63) equals[30]

$$\frac{\tilde{v}}{4}\wedge d\,\mathrm{Tr}\left[\left(\hat{F}+\frac{2}{h}\hat{\Phi}d\lambda-(\hat{A}+B\lambda)\wedge(\hat{A}+B\lambda)\right)\left(\frac{2}{h}\hat{\Phi}-B\right)+2(\hat{A}+B\lambda)\wedge\hat{D}\left(\frac{\hat{\Phi}}{h}\right)\right].$$

We try to write this as a total derivative:

$$\begin{aligned}
&-d\left(\frac{\tilde{v}}{4}\wedge\mathrm{Tr}\left[\left(\hat{F}+\frac{2}{h}\hat{\Phi}d\lambda-(\hat{A}+B\lambda)\wedge(\hat{A}+B\lambda)\right)\left(\frac{2}{h}\hat{\Phi}-B\right)\right.\right.\\
&\qquad\qquad\left.\left.+2(\hat{A}+B\lambda)\wedge\hat{D}\left(\frac{\hat{\Phi}}{h}\right)\right]\right)\\
&+\frac{d\tilde{v}}{4}\wedge\mathrm{Tr}\left[\left(\hat{F}+\frac{2}{h}\hat{\Phi}d\lambda-(\hat{A}+B\lambda)\wedge(\hat{A}+B\lambda)\right)\left(\frac{2}{h}\hat{\Phi}-B\right)+2(\hat{A}+B\lambda)\wedge\hat{D}\left(\frac{\hat{\Phi}}{h}\right)\right].
\end{aligned}\tag{3.64}$$

In the last line, $\hat{F}$ can be dropped because $d\tilde{v}\wedge\hat{F}$ is a 4-form annihilated by i_v, and thus has to vanish.[31] For the same reason $d\lambda$ can also be dropped. Continuing this argument, the equality $0 = i_v(-Bd\lambda-\hat{F}) = i_v\hat{D}(-B\lambda-\hat{A}) + i_v((-B\lambda-\hat{A})\wedge(-B\lambda-\hat{A}))$ implies that the last line of (3.64) can be written as

$$\begin{aligned}
&\frac{d\tilde{v}}{4}\wedge\mathrm{Tr}\left[-\hat{D}(\hat{A}+B\lambda)\cdot\left(\frac{2}{h}\hat{\Phi}-B\right)+2(\hat{A}+B\lambda)\wedge\hat{D}\left(\frac{\hat{\Phi}}{h}\right)\right]\\
&\qquad=-d\left(\frac{d\tilde{v}}{4}\wedge\mathrm{Tr}\left[(\hat{A}+B\lambda)\cdot\left(\frac{2}{h}\hat{\Phi}-B\right)\right]\right)
\end{aligned}\tag{3.65}$$

Thus the term in the last line of (3.62) is an integral of a total derivative, and we need to study the potential contributions from regions where some quantities are singular, including the equator where the 't Hooft loop is inserted as well as the north and south poles where $\tilde{v}^\mu\tilde{v}_\mu$ diverges.

Let us denote by $\Sigma(\delta) = \Sigma_{\mathrm{N}}\cup\Sigma_{\mathrm{S}}\cup\Sigma_{\mathrm{eq}}$ the boundary of small neighborhoods that contain the poles and the equator. Specifically, we take a constant $\delta > 0$ and define $\Sigma_{\mathrm{N}} := \{\vartheta = \delta\}$, $\Sigma_{\mathrm{S}} := \{\vartheta = \pi-\delta\}$ and $\Sigma_{\mathrm{eq}} := \{|\vec{x}| = \delta\}$, where $\vec{x}$ denotes the spatial position in the $B_3\times S^1$ coordinate system. Noting that $\hat{A}+B\lambda = A+B\rho$ and that

[30] This expression in fact descends from the Chern-Simons 3-form for a four-dimensional gauge field constructed, via Kronheimer's correspondence, from $\hat{A}$ which can be regarded as a three-dimensional gauge field on $M_3 = S^4/\mathrm{U}(1)$.

[31] Any 4-form α can be expressed as $\alpha = f\,\mathrm{vol}(S^4)$ where f is a function and $\mathrm{vol}(S^4)$ is the volume form on S^4. If i_v annihilates α we have $0 = i_v\alpha = f i_v\mathrm{vol}(S^4)$. Since $i_v\mathrm{vol}(S^4)$ is non-zero, f, and therefore α, have to vanish.

$\hat{\Phi}/h = (\sin^2\vartheta/h)\Phi$, we obtain

$$\begin{aligned}&\int_{S^4}\frac{1}{2h}\left(\left|\left|i_v * \hat{F} - \frac{\Phi}{h}dh\right|\right|^2 + \left|\left|\frac{h}{\sin^2\vartheta}D\left(\frac{\hat{\Phi}}{h}\right)\right|\right|^2\right)\\ &= -\lim_{\delta\to 0}\int_{\Sigma(\delta)}\mathrm{Tr}\left(\left[\left(F + 2\frac{\sin^2\vartheta}{h}\Phi d\rho - (A+B\rho)\wedge(A+B\rho)\right)\left(2\frac{\sin^2\vartheta}{h}\Phi - B\right)\right.\right.\\ &\quad\left.\left. +2(A+B\rho)\wedge D\left(\frac{\sin^2\vartheta}{h}\Phi\right)\right]\wedge\frac{\tilde{v}}{4} + (A+B\rho)\cdot\left(2\frac{\sin^2\vartheta}{h}\Phi - B\right)\wedge\frac{d\tilde{v}}{4}\right).\end{aligned} \tag{3.66}$$

Because the fields must obey the boundary conditions associated to the 't Hooft operator at the equator of S^4, their values on $\Sigma_{\rm eq}$ must approach the background values (3.60), for which the integrand vanishes as $\delta \to 0$. On the hypersurfaces $\Sigma_{\rm N}(\delta)$ and $\Sigma_{\rm S}(\delta)$ where $\vartheta = \delta$ is constant, the integrand[32] vanishes as $\delta \to 0$ for smooth configurations. Hence (3.66) vanishes.

Then the squares in the first line of (3.66) must vanish separately, and we have in particular

$$D\left(\frac{\hat{\Phi}}{h}\right) = 0\,. \tag{3.67}$$

The boundary condition near the operator then requires that $\hat{\Phi} = Bh/2$ up to a gauge transformation, corresponding to the original 't Hooft operator background we started with.

In summary, the most general non-singular solution to the localization equations is the field configuration (3.45).

4 Classical contribution

In this section we calculate the classical contribution to the localization path integral computing the expectation value of a supersymmetric 't Hooft loop in an arbitrary $\mathcal{N} = 2$ gauge theory on S^4. The classical contribution to the path integral is obtained by evaluating the $\mathcal{N} = 2$ gauge theory action on S^4 — including suitable boundary terms — on the saddle point solutions of the localization equations.

Using that the localization equations set to zero the scalar fields in the $\mathcal{N} = 2$ hypermultiplet, the classical contribution to the path integral arises from evaluating the bosonic action of the $\mathcal{N} = 2$ vectormultiplet on S^4 on the Weyl transformed (2.10) saddle point solution (3.45). The relevant part of the $\mathcal{N} = 2$ gauge theory action on S^4 of radius r is given by

$$S_{\mathcal{N}=2} = -\frac{1}{g^2}\int_{S^4}\sqrt{h}\,\mathrm{Tr}\left(\frac{1}{2}F_{\mu\nu}F^{\mu\nu} + D_\mu\Phi_A D^\mu\Phi_A + \frac{R}{6}\Phi_A\Phi_A + K_3^2\right) - \frac{i\theta}{8\pi^2}\int_{S^4}\mathrm{Tr}\,(F\wedge F)\,, \tag{4.1}$$

[32] We can take $\Sigma_{\rm N,S}(\delta)$ to be three-spheres parametrized by ψ and the position on S^2. We note that F, $d\rho$, $D\left(\frac{\sin^2\vartheta}{h}\Phi\right)$, and in particular $A + B\rho$ are regular as differential forms at the north and south poles. Therefore their components not involving the ϑ-direction vanish as $\delta \to 0$. Forms $\tilde{v} = r(d\psi + \omega)$ and $d\tilde{v} = (r/2)\mathrm{vol}(S^2)$ are finite in this limit.

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where we have denoted by $h_{\mu\nu}$ the S^4 metric and $R = 12/r^2$ is the scalar curvature. The classical action (4.1) is invariant under the Weyl transformation (2.10). Therefore, we can calculate the classical contribution to the expectation value of the 't Hooft loop by computing the $\mathcal{N}=2$ gauge theory action on $B_3 \times S^1$ (3.7) evaluated on the background (3.45). The non-topological part of the action is thus[33]

$$-\frac{2\pi \cdot 4\pi}{g^2}\int_0^{2r} dx\, r\left(1-\frac{x^2}{4r^2}\right)x^2\,\mathrm{Tr}\left[\frac{1}{2}F_{ij}F_{ij}+F_{i\hat{4}}F_{i\hat{4}}+D_i\Phi_A D_i\Phi_A+\frac{\Phi_A\Phi_A}{2r^2\left(1-\frac{x^2}{4r^2}\right)}+K_3^2\right], \tag{4.2}$$

while the topological term is[34]

$$\frac{i\theta}{8\pi^2}\cdot 2\pi\cdot 4\pi\int_0^{2r} dx\, r\left(1-\frac{x^2}{4r^2}\right)x^2\epsilon^{ijk}\,\mathrm{Tr}\left[F_{i\hat{4}}F_{jk}\right]. \tag{4.3}$$

Explicit computation using the saddle point configuration (3.45) gives

$$S^{(0)}_{\mathcal{N}=2} = -\frac{8\pi^2}{g^2}r^2\,\mathrm{Tr}\,a^2+\theta\, r\,\mathrm{Tr}(aB)-\mathrm{Tr}\,B^2\left(\frac{4\pi^2}{g^2}+\frac{g^2\theta^2}{16\pi^2}\right)r\int_\delta^{2r}dx\,\frac{1}{x^2}\,. \tag{4.4}$$

The unregulated on-shell action is clearly divergent, as it measures the infinite self-energy of a point-like monopole. This divergence — which is proportional to the length of the curve on which the 't Hooft loop is supported — can be regulated by introducing a cutoff δ in the integration over x, and subtracting terms in the action proportional to $1/\delta$. This subtraction can be implemented by adding to the action (4.1) covariant boundary terms supported on the $x=\delta$ hypersurface Σ_3.

The relevant boundary terms are

$$-\frac{2}{g^2}\int_{\Sigma_3}\mathrm{Tr}\left(\Phi_9\,F\right)\wedge d\tau + i\frac{2}{g^2}\int_{\Sigma_3}\mathrm{Tr}\left(\Phi_0 *_4 F\right)\wedge d\tau\,. \tag{4.5}$$

The first term coincides with the boundary action introduced in [18] to cancel the divergence due to a singular monopole inserted along a loop with an arbitrary shape.[35] The second term in (4.5) is its electric version, and is necessary for non-zero values of the theta angle. Evaluating them on the saddle point solution (3.45) we get

$$-\frac{2}{g^2}\int_{\Sigma_3}\mathrm{Tr}\left(\Phi_9\,F\right)\wedge d\tau = -\frac{2}{g^2}\,\mathrm{Tr}\left(\frac{B}{2\delta}\cdot(-2\pi B)\right)2\pi r = \frac{1}{\delta}\frac{4\pi^2 r}{g^2}\,\mathrm{Tr}\,B^2 \tag{4.6}$$

and

$$\begin{aligned} i\frac{2}{g^2}\int_{\Sigma_3}\mathrm{Tr}\left(\Phi_0 *_4 F\right)\wedge d\tau &= i\frac{2}{g^2}\,\mathrm{Tr}\left(\left(-g^2\theta\frac{B}{16\pi^2}\frac{1}{\delta}+a\right)\cdot\left(ig^2\theta\frac{B}{4\pi}\right)\right)2\pi r = \\ &= \frac{1}{\delta}\frac{g^2\theta^2 r}{16\pi^2}\,\mathrm{Tr}\,B^2-\theta\, r\,\mathrm{Tr}(aB)\,. \end{aligned} \tag{4.7}$$

[33]Where we have used the $\mathrm{SO}(3)\times\mathrm{SO}(2)$ symmetry of the background (3.45) and that the $B_3\times S^1$ metric (3.7) has $R/6 = 1/\left(2r^2\left(1-\frac{x^2}{4r^2}\right)\right)$. x is the radial coordinate in B_3

[34]The volume form is given by $\epsilon^{\hat{4}123}=1$.

[35]Since the Bogomolny equations appear as the Q-variation of the fermion, the Q-exact expression Q(Bogomolny eq., Ψ) contains a positive semi-definite bosonic part $|\text{Bogomolny eq.}|^2$, which can be rewritten as the usual kinetic term plus the boundary term. Thus the boundary term complements the divergent part of the original action to make it locally Q-exact.

The terms proportional to $1/\delta$ in the boundary terms cancel the self-energy divergences in the bulk on-shell action in (4.4). Moreover, the on-shell boundary term (4.7) generates a finite contribution, which precisely cancels the corresponding one appearing in the bulk on-shell action (4.4). Therefore, the leading classical action for the circular 't Hooft loop in the $\mathcal{N}=2$ gauge theory is given by

$$S^{(0)\text{total}}_{\mathcal{N}=2} = -\frac{8\pi^2}{g^2}r^2\,\mathrm{Tr}\,a^2 + \mathrm{Tr}\,B^2\left(\frac{2\pi^2}{g^2}+\frac{g^2\theta^2}{32\pi^2}\right). \tag{4.8}$$

The classical action (4.8) can be split into the sum of two terms, which are the complex conjugate of each other

$$\begin{aligned} S^{(0)\text{total}}_{\mathcal{N}=2} = -\frac{1}{2}r^2\Bigg[&\left(-\frac{8\pi^2}{g^2}+i\theta\right)\mathrm{Tr}\left(ia - ig^2\theta\frac{B}{16\pi^2 r}-\frac{B}{2r}\right)^2 \\ &+\left(-\frac{8\pi^2}{g^2}-i\theta\right)\mathrm{Tr}\left(ia - ig^2\theta\frac{B}{16\pi^2 r}+\frac{B}{2r}\right)^2\Bigg]. \end{aligned} \tag{4.9}$$

This observation leads to an illuminating interpretation. The classical result for the 't Hooft loop path integral on S^4 is captured by the classical contribution to Nekrasov's equivariant instanton and anti-instanton partition functions on $\mathbb{R}^4$ [8] localized at the north and south poles of the S^4 respectively. As we shall see, the classical, one-loop and instanton factors in Nekrasov's equivariant instanton/anti-instanton partition function in $\mathbb{R}^4$ [8] will enter in the computation of the 't Hooft loop on S^4.

We first recall that the classical contribution to the $\mathcal{N}=2$ equivariant instanton partition function in $\mathbb{R}^4$ — or the partition function of the $\mathcal{N}=2$ theory in the Ω-background — is given by [8]

$$Z_{\text{cl}}(\hat{a},q) = \exp\left[\frac{1}{2\varepsilon_1\varepsilon_2}2\pi i\tau\,\mathrm{Tr}\,\hat{a}^2\right]. \tag{4.10}$$

The constant field $\hat{a}\in\mathfrak{t}$ is the equivariant parameter for the action of G-gauge transformations on the moduli space of instantons in $\mathbb{R}^4$, while ε_1 and ε_2 are the equivariant parameters of the $\mathrm{U}(1)_{\varepsilon_1}\times\mathrm{U}(1)_{\varepsilon_2}$ action on $\mathbb{R}^4=\mathbb{C}\oplus\mathbb{C}$

$$\begin{aligned} z_1 &\to e^{i\varepsilon_1}z_1 \\ z_2 &\to e^{i\varepsilon_2}z_2\,. \end{aligned} \tag{4.11}$$

The parameter $q=\exp(2\pi i\tau)$ is the instanton fugacity while $\overline{q}$ is the fugacity for anti-instantons, where τ is the complexified coupling constant of the $\mathcal{N}=2$ gauge theory

$$\tau = \frac{\theta}{2\pi}+\frac{4\pi i}{g^2}\,. \tag{4.12}$$

In section 3.2 we have already mentioned that the supercharge Q with which we localize the 't Hooft loop path integral becomes near the north and south poles of the S^4 the supercharge which localizes the equivariant instanton and anti-instanton partition function in $\mathbb{R}^4$ [8] respectively, with the following equivariant parameters

$$\begin{aligned}\varepsilon_1 = \varepsilon_2 = \varepsilon = \frac{1}{r} \\ \hat{a}(N) = i\Phi_0(N) - \Phi_9(N) \\ \hat{a}(S) = i\Phi_0(S) + \Phi_9(S)\,.\end{aligned} \tag{4.13}$$

Therefore, inspection of the solution of the localization saddle point equations at the north and south poles[36] in (3.45) yields

$$\hat{a}(N) = ia - ig^2\theta\frac{B}{16\pi^2 r} - \frac{B}{2r} \qquad \hat{a}(S) = ia - ig^2\theta\frac{B}{16\pi^2 r} + \frac{B}{2r}\,. \tag{4.14}$$

This implies that the classical equivariant instanton/anti-instanton partition functions arising from the north and south poles are given by

$$Z_{\text{north,cl}} = Z_{\text{cl}}(\hat{a}(N), q) \qquad\qquad Z_{\text{south,cl}} = Z_{\text{cl}}(\hat{a}(S), \overline{q})\,. \tag{4.15}$$

Therefore, the classical expectation value (4.9) for the 't Hooft loop operator with magnetic weight B in any $\mathcal{N} = 2$ gauge theory on S^4 factorizes into a classical contribution associated to the north and south poles respectively

$$\exp\left(-S^{(0)\text{total}}_{\mathcal{N}=2}\right) = Z_{\text{north,cl}} \cdot Z_{\text{south,cl}} = \left| Z_{\text{cl}}\left(ia - ig^2\theta\frac{B}{16\pi^2 r} - \frac{B}{2r}, q\right)\right|^2, \tag{4.16}$$

the south pole contribution being the complex conjugate of the north pole one

$$Z_{\text{south,cl}} = \overline{Z}_{\text{north,cl}}\,. \tag{4.17}$$

The identification of the integrand of the 't Hooft loop path integral with contributions arising from the north and south poles of S^4 will be a recurrent theme in our computation of the 't Hooft loop expectation value. As we shall see, however, an important contribution also arises from the equator of S^4.

5 Instanton contribution

In the previous section we have calculated the classical contribution to the expectation value of a 't Hooft loop with magnetic weight B on S^4 due to the non-singular solutions of the localization equations (besides the obvious singularity created by the insertion of the 't Hooft operator), which are labeled by $a \in \mathfrak{t}$ (3.45). As discussed earlier, however, there exist singular solutions to the localization equations supported at the north and south poles. In this section we determine their contribution to the 't Hooft loop expectation value.

[36] We have evaluated the scalar field Φ_0 and Φ_9 at the north and south poles of S^4. From equation (3.45) we find that the value of the field Φ_9 (Φ_0) at the north and south poles of $B_3 \times S^1$, which are located at $\vec{x} = (0, 0, \pm 2r)$, is $\Phi_9 = \frac{B}{4r}$ ($\Phi_0 = i\frac{a}{2} - ig^2\theta\frac{B}{32\pi^2 r}$). Weyl transforming to S^4 using $\Phi_{S^4} = 2\Phi_{B_3\times S^1}$, we get the formula (4.14).

The localization equations (3.38)-(3.40) at the north and south pole of the S^4 become, respectively, the instanton and anti-instanton equations

$$\text{north}: \quad F^+ = 0 \qquad\qquad \text{south}: \quad F^- = 0\,. \tag{5.1}$$

These equations describe singular field configurations, corresponding to point-like instantons, which are localized at the poles of S^4. The inclusion of these singular field configurations in the localization computation implies that we must enrich the result in section 4 with the contribution of point-like instantons and anti-instantons arising at the north and south pole respectively. We now identify these contributions and include their effect in the computation of the 't Hooft loop path integral.

Nekrasov's equivariant instanton (anti-instanton) partition function in $\mathbb{R}^4$ [8] computes the contribution of instantons (anti-instantons) to the path integral of an $\mathcal{N}=2$ gauge theory in the so-called Ω-background. We denote it by [8]

$$Z_{\text{inst}}(\hat{a}, \tilde{m}_f, \varepsilon_1, \varepsilon_2, q)\,, \tag{5.2}$$

where $(\varepsilon_1, \varepsilon_2, \hat{a}, \tilde{m})$ are the equivariant parameters for the $\mathrm{U}(1)_{\varepsilon_1} \times \mathrm{U}(1)_{\varepsilon_2} \times G \times G_{\mathrm{F}}$ symmetries of the $\mathcal{N}=2$ gauge theory. $\tilde{m}_f$ with $f=1,\ldots,N_{\mathrm{F}}$ denote the equivariant parameters for the flavour symmetry group G_{F} associated to the hypermultiplet and q is the instanton fugacity.

Since the $\mathcal{N}=2$ gauge theory action on S^4 and Q-complex near the poles reduces to those of the $\mathcal{N}=2$ gauge theory in the Ω-background, the contribution of the singular field configurations in our localization computation due to point-like instantons and anti-instantons at the north and south poles respectively, are precisely captured by Nekrasov's instanton and anti-instanton partition function.

As we have already mentioned, the Q-complex of the $\mathcal{N}=2$ theory near the north (south) pole of S^4 reduces to that describing Nekrasov's equivariant instanton (anti-instanton) partition function on $\mathbb{R}^4$ with $\mathrm{U}(1)_{\varepsilon_1} \times \mathrm{U}(1)_{\varepsilon_2}$ equivariant parameters $\varepsilon_1 = \varepsilon_2 = 1/r$. Furthermore, the equivariant parameter $\hat{a} \in \mathfrak{t}$ for the action of the gauge group G on the instanton moduli space is given respectively by equations (4.13), (4.14)

$$\begin{aligned} \hat{a}(N) = i\Phi_0(N) - \Phi_9(N) = ia - ig^2\theta\frac{B}{16\pi^2 r} - \frac{B}{2r} \\ \hat{a}(S) = i\Phi_0(S) + \Phi_9(S) = ia - ig^2\theta\frac{B}{16\pi^2 r} + \frac{B}{2r}\,. \end{aligned} \tag{5.3}$$

Therefore, the contribution to the 't Hooft loop expectation arising from the solutions to the $F^+=0$ equations at the north pole is given by

$$Z_{\text{north,inst}} = Z_{\text{inst}}\left(ia - ig^2\theta\frac{B}{16\pi^2 r} - \frac{B}{2r}, \frac{1}{r} + im_f, \frac{1}{r}, \frac{1}{r}, q\right), \tag{5.4}$$

while that due to the solutions of the $F^-=0$ equations at the south pole is

$$Z_{\text{south,inst}} = Z_{\text{inst}}\left(ia - ig^2\theta\frac{B}{16\pi^2 r} + \frac{B}{2r}, \frac{1}{r} + im_f, \frac{1}{r}, \frac{1}{r}, \overline{q}\right). \tag{5.5}$$

We have used the relation

$$\tilde{m}_f = \frac{\varepsilon_1 + \varepsilon_2}{2} + im_f \qquad f = 1, \ldots, N_{\rm F} \tag{5.6}$$

derived in [19] between the physical mass m_f of a hypermultiplet and the equivariant parameter $\tilde{m}_f$ in Nekrasov's instanton partition function.

Taking into account the following identity obeyed by the instanton partition function [19, 20]

$$Z_{\rm inst}(\hat{a}, \tilde{m}_f, \varepsilon_1, \varepsilon_2, q) = Z_{\rm inst}(-\hat{a}, \varepsilon_1 + \varepsilon_2 - \tilde{m}_f, \varepsilon_1, \varepsilon_2, q)\,, \tag{5.7}$$

we find that the anti-instanton south pole contribution is the complex conjugate of the one in the instanton north pole one

$$Z_{\rm south,inst} = \overline{Z}_{\rm north,inst}\,. \tag{5.8}$$

We can now combine the results of this section with the ones found in the previous one and write down the "classical" contribution to the expectation value of a 't Hooft loop with magnetic weight B. Summing over all saddle points of the localization equations — including both non-singular and singular solutions at the north and south poles — which are labeled by $a \in \mathfrak{t}$, leads to[37]

$$\langle T(B) \rangle \simeq \int da \left| Z_{\rm cl}\left(ia - \frac{B}{2r}, q\right) Z_{\rm inst}\left(ia - \frac{B}{2r}, \frac{1}{r} + im, \frac{1}{r}, \frac{1}{r}, q\right) \right|^2, \tag{5.9}$$

with $Z_{\rm cl}$ and $Z_{\rm inst}$ given in (4.10) and (5.2) respectively.

6 One-loop determinants

The calculation of a path integral using localization enjoys the drastic simplification of reducing the computation to one-loop order with respect to the deformation parameter t, while being exact with respect to the original gauge theory coupling constant. In this section we calculate the relevant determinants required for computing the expectation value of 't Hooft operators on S^4. Computation of the one-loop determinants in the $\mathcal{N} = 2$ gauge fixed action is performed by expanding to quadratic order in all field fluctuations — which include vectomultiplet, hypermultiplet and ghost multiplet fields — the deformation term $\hat{Q} \cdot \hat{V}$ around the saddle point configuration background (3.45). In the gauge fixed theory, the supercharge Q combines with the BRST operator Q_{BRST} as $\hat{Q} = Q + Q_{BRST}$, such that the deformed action $Q \cdot V$ (2.1) together with gauge fixing terms can be written as $\hat{Q} \cdot \hat{V}$, with $\hat{V} = V + V_{\rm ghost}$ [2]. As shown in [2], the saddle points of $\hat{Q} \cdot \hat{V}$ coincide with those of $Q \cdot V$, and we can borrow the saddle point configuration in (3.45) for the calculation of the determinants.

Direct evaluation of the determinants by diagonalization of the quadratic fluctuation operator in the saddle point background is rather complicated. Instead, we calculate the

[37] We have trivially shifted the integration variable $ia \to ia + ig^2\theta \frac{B}{16\pi^2 r}$.

relevant one-loop determinants using an index theorem. More precisely we use the Atiyah-Singer index theorem for transversally elliptic operators [21], which was also used in [2] to compute the partition function of $\mathcal{N}=2$ gauge theories on S^4.

Even though we are considering the physical $\mathcal{N}=2$ gauge theory on S^4 (not a topologically twisted theory), the combined supersymmetry and BRST transformations generated by $\hat{Q}$ can be written in cohomological form [2]. Fields of opposite statistics are paired into doublets under the action of $\hat{Q}$. Schematically, denoting the fields of even and odd statistics with a subindex e and o respectively, we have that

$$\begin{aligned}\hat{Q}\cdot\varphi_{e,o}&=\hat{\varphi}_{o,e}\\ \hat{Q}\cdot\hat{\varphi}_{o,e}&=\mathcal{R}\cdot\varphi_{e,o}\,.\end{aligned}\tag{6.1}$$

Here $\mathcal{R}$ is the generator of the $\mathrm{U}(1)_{J+R}\times G\times G_{\mathrm{F}}$ symmetries discussed in section 3.1, corresponding to the group $\mathrm{U}(1)_{J+R}$ combining the $\mathrm{U}(1)_J$ rotation on S^4 (2.4) with an $\mathrm{SO}(2)_R$ R-symmetry transformation, the G-gauge and the G_{F} flavour symmetries respectively. Therefore, $\hat{Q}$ acts as an equivariant cohomological operator since

$$\hat{Q}^2\cdot\varphi_{e,o}=\mathcal{R}\cdot\varphi_{e,o}\,,\tag{6.2}$$

and $\hat{Q}^2$ is nilpotent on $\mathcal{R}$-invariant field configurations. The invariance of the deformation term $\hat{Q}\cdot\hat{V}$ under the action of $\hat{Q}$ and the pairing of of the fields as in (6.1) leads to cancellations between bosonic and fermionic fluctuations. The remainder of this cancellation is the following ratio of determinants over non-zeromodes [2]

$$\frac{\det_{\mathrm{Coker}D^{\mathrm{vm}}}\mathcal{R}|_o}{\det_{\mathrm{Ker}D^{\mathrm{vm}}}\mathcal{R}|_e}\cdot\frac{\det_{\mathrm{Coker}D^{\mathrm{hm}}}\mathcal{R}|_o}{\det_{\mathrm{Ker}D^{\mathrm{hm}}}\mathcal{R}|_e}\,.\tag{6.3}$$

The differential operators D^{vm} and D^{hm} are obtained from the expansion of the deformation term $\hat{Q}\cdot\hat{V}$ for the vectormultiplet and hypermultiplet fields respectively.

Therefore, the one-loop determinants that appear in the localization computation of the partition function of an $\mathcal{N}=2$ gauge theory on S^4 are given by the product of weights for the group action $\mathcal{R}$ generated by $\hat{Q}^2$ on the vectormultiplet and hypermultiplet fields. Furthermore, the weights appearing in the determinants (6.3) can be determined from the computation of the $\mathcal{R}$-equivariant index

$$\mathrm{ind}\,D=\mathrm{tr}_{\mathrm{Ker}D}e^{\mathcal{R}}-\mathrm{tr}_{\mathrm{Coker}D}e^{\mathcal{R}}\,,\tag{6.4}$$

for $D=D^{\mathrm{vm}}$ and $D=D^{\mathrm{hm}}$. In order to convert the index (Chern character) $\mathrm{ind}\,D$ in (6.4) into a fluctuation determinant (Euler character), we read off the weights $w_\alpha(\varepsilon_1,\varepsilon_2,\hat{a},m_f)$ from the index and combine them to get the determinant according to the rule

$$\sum_j c_j e^{w_j(\varepsilon_1,\varepsilon_2,\hat{a},\hat{m}_f)}\to\prod_j w_j(\varepsilon_1,\varepsilon_2,\hat{a},\hat{m}_f)^{c_j}\,,\tag{6.5}$$

where $(\varepsilon_1,\varepsilon_2,\hat{a},m_f)$ denote the equivariant parameters for $\mathrm{U}(1)_{\varepsilon_1}\times\mathrm{U}(1)_{\varepsilon_2}\times G\times G_{\mathrm{F}}$.[38] The relevant $\mathcal{R}$-equivariant indices can then be calculated from the equivariant Atiyah-Singer index theorem for transversally elliptic operators [21], to which we now turn.

[38] We recall that $\mathrm{U}(1)_{J+R}=(\mathrm{U}(1)_{\varepsilon_1}\times\mathrm{U}(1)_{\varepsilon_2})_{\mathrm{diag}}$.

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The index theorem localizes contributions to the fixed points of the action of $\mathcal{R}$, that is to the north and south poles of S^4. Therefore, the relevant index corresponds to the equivariant index of the vectormultiplet and hypermultiplet complexes of the $\mathcal{N}=2$ theory in the Ω-background, to which the $\mathcal{N}=2$ gauge theory on S^4 reduces at the poles. The presence of a 't Hooft loop, however, introduces a further contribution, arising from the equator, where the operator is supported.

6.1 Review of the Atiyah-Singer equivariant index theory

Consider a pair of vector bundles (E_0, E_1) over a manifold M. Let $V_i = \Gamma(E_i)$ be the space of sections of E_i, $i = 1, 2$.

Let $T = \mathrm{U}(1)^n$ be the maximal torus of a compact Lie group $\mathcal{G}$ acting on M and the bundles E_i, and let $D : V_0 \to V_1$ be an elliptic differential operator commuting with the $\mathcal{G}$-action. In this situation we can define the $\mathcal{G}$-equivariant index of the operator D as a formal character

$$\operatorname{ind} D(t) = \operatorname{tr}_{H^0} t - \operatorname{tr}_{H^1} t \quad t = (t_1, t_2, \dots, t_n) \in T\,, \tag{6.6}$$

where $H^0 = \ker D, H^1 = \operatorname{coker} D$. If D is elliptic and M is compact, H^0 and H^1 are finite dimensional vector spaces.

The index does not depend on small deformations of the operator D and, therefore, is a topological invariant. If the action of $\mathcal{G}$ on M has a discrete set of fixed points, Atiyah and Singer represent the index as a sum over the set of fixed points F

$$\operatorname{ind} D(t) = \sum_{p\in F} \frac{\operatorname{tr}_{E_0(p)} t - \operatorname{tr}_{E_1(p)} t}{\det_{TM_p}(1-t)}\,. \tag{6.7}$$

Each fixed point contribution to the Atiyah-Singer index formula (6.7) is a rational function in t. For an elliptic operator D on a compact manifold M the sum over all of the fixed point contributions to the index is a finite Laurent polynomial in $t = (t_1, \dots, t_n)$, since the spaces H^i are finite dimensional.

The basic example is the equivariant index of the Dolbeault operator $\bar\partial : \Omega^{0,0}(\mathbb{C}) \to \Omega^{0,1}(\mathbb{C})$ from the space of functions to the space of $(0,1)$-forms on the complex plane $M = \mathbb{C}$ under the $T = \mathrm{U}(1)$ action $z \mapsto tz$. Computing the index of $\operatorname{ind}(\bar\partial)(t)$ directly using (6.6) we just need to evaluate the U(1) character on the space of holomorphic functions

$$f(z) = \sum_{k\geq 0} c_k z^k\,, \tag{6.8}$$

since $\operatorname{coker}\bar\partial$ is trivial in $\mathbb{C}$. Under the U(1) action the functions transform as $\tilde f(\tilde z) = f(z)$ for $\tilde z = tz$, that is $\tilde f(z) = f(t^{-1}z)$. Hence $c_k \mapsto \tilde c_k = t^{-k} c_k$. Therefore

$$\operatorname{ind}(\bar\partial)(t) = \sum_{n=0}^{\infty} t^{-n} = \frac{1}{1-t^{-1}}, \tag{6.9}$$

where the last equality should be understood formally since for $|t| = 1$ the series does not actually converge.

On the other hand, we can evaluate $\mathrm{ind}(\overline{\partial})(t)$ using the Atiyah-Singer fixed point theorem (6.7). Since there is a single fixed point at $z = 0$ of the U(1) action, we get[39]

$$\mathrm{ind}(\overline{\partial})(t) = \frac{1-t}{(1-t)(1-t^{-1})} = \frac{1}{1-t^{-1}}\,, \tag{6.10}$$

thus reproducing the previous computation.

The index theory for elliptic operators can be generalized to transversally elliptic operators [21]. Let T be the maximal torus of a Lie group $\mathcal{G}$ that acts on a manifold M. An operator D on M is called transversally elliptic with respect to the $\mathcal{G}$ action on M if it is elliptic in all directions transversal to the $\mathcal{G}$-orbits on M. As in the elliptic case, the index of D possesses the excision property. Therefore the index can be computed as a sum of local contributions, a sum over the fixed points of the $\mathcal{G}$ action. The total index $\mathrm{ind}\,D(t)$ is an infinite formal Laurent series $\sum_n c_n t^n$ with $n \in \mathbb{Z}$, since the cohomology spaces H^i can be infinite dimensional. However, for each c_n, the multiplicity of the representation n in $\oplus(-1)^i H^i$, is finite. Atiyah-Singer theory allows us to find c_n unambiguously since the theory specifies whether each fixed point contribution is to be expanded in powers of t or t^{-1}, after choosing a deformation of the symbol for D.[40]

In the paper [2], the partition function and the Wilson loop expectation value were computed, with the one-loop contributions evaluated using an index theorem. In the set-up of [2] and the current paper, the manifold is $M = S^4$ and the spacetime part of the relevant group $\mathcal{G} = \mathrm{U}(1)_{J+R} \times G \times G_{\mathrm{F}}$ is generated by J (2.4). The differential operators that appear in the quadratic part of $\hat{Q} \cdot \hat{V}$ fail to be elliptic on the equatorial S^3, but they are still transversally elliptic and the generalized index theorem can be applied. In [2] the index is a sum of local contributions from the north and south poles of S^4, which are the fixed points of J.

When we turn on the singular monopole background (3.45), there is an extra complication since some of the fields are singular along the equatorial S^1 where the loop operator is inserted. This gives rise to an extra contribution to the one-loop determinant, associated with the equator of S^4. We believe that the index theorem for transversally elliptic operators can be generalized to the situation where such singular monopoles are present. A similar index theorem was established in [22] using a relation between singular monopoles and U(1)-invariant instantons [14]. Assuming the existence of such an index theorem, we will compute local contributions from the equatorial S^1, for which there is a natural expansion. The specific choice of a deformation of the symbol made in [2] led to the expansion in positive and negative powers of t at the north and south poles, respectively. In the presence of a 't Hooft loop we will apply the same deformation, and therefore obtain the same rules for expansion at the north and south poles.

[39] The fiber $(E_0)_{z=0}$ transforms trivially, the fiber $(E_1)_{z=0}$ transforms as $\tilde{f}_{\tilde{\overline{z}}} = f_{\overline{z}}(d\overline{z}/d\tilde{\overline{z}}) = f_{\overline{z}}\overline{t}^{-1} = f_{\overline{z}}t$ for $|t|^2 = 1$, hence the numerator in the Atiyah-Singer theorem is $(1-t)$. The denominator is $\det_{TM_p}(1-t) = (1-t)(1-t^{-1})$ as $(z, \overline{z}) \mapsto (tz, t^{-1}z)$.

[40] After summing over fixed points, c_n is independent of the choice of deformation.

6.2 North and south pole contributions

We wish to compute the vectormultiplet and hypermultiplet one-loop determinant contributions from the north and south poles, which are the fixed point set of J. The relevant complex for the vectormultiplet calculation is the self-dual complex while for the hypermultiplet it is the Dirac complex. We now consider the associated equivariant indices and one-loop determinants.

Vectormultiplet determinant. As in [2], near the north pole, we consider the complex[41] of vector bundles associated with linearization of the anti-self-dual equation $F^+ = 0$ on $\mathbb{R}^4$

$$D_{\rm SD} : \Omega^0 \xrightarrow{d} \Omega^1 \xrightarrow{d_+} \Omega^{2+} \,, \tag{6.11}$$

where d is the de Rham differential and d_+ is the composition of the de Rham differential and self-dual projection operator. We want to compute the equivariant index of $D_{\rm SD}$ with respect the $T = \mathrm{U}(1)_{\varepsilon_1} \times \mathrm{U}(1)_{\varepsilon_2}$ action which rotates $\mathbb{R}^4 = \mathbb{C} \oplus \mathbb{C}$ as $(z_1, z_2) \mapsto (t_1 z_1, t_2 z_2)$. For the moment we take t_1 and t_2 generic though we will set $t_1 = t_2$ in the end, as $\mathrm{U}(1)_{J+R}$ corresponds to $(\mathrm{U}(1)_{\varepsilon_1} \times \mathrm{U}(1)_{\varepsilon_2})_{\rm diag}$ in the self-dual/anti-self-dual complex at the north/south pole. The Atiyah-Singer formula (6.7) for the complexification of (6.11) gives[42]

$$\begin{aligned} \mathrm{ind}(D_{\rm SD,\mathbb{C}})(t_1,t_2) &= \frac{(t_1 t_2 + t_1^{-1} t_2^{-1} + 2) - (t_1 + t_1^{-1} + t_2 + t_2^{-1})}{(1-t_1)(1-t_1^{-1})(1-t_2)(1-t_2^{-1})} \\ &= \frac{1 + t_1 t_2}{(1-t_1)(1-t_2)} \,. \end{aligned} \tag{6.12}$$

The index for the real complex (6.11) is the half of (6.12).

Unless there is a further input from the transversally elliptic Atiyah-Singer theory, we can expand the function (6.12) in various ways depending on whether we take $|t_i| > 1$ or $|t_i| < 1$. For example, expanding in positive powers of t_1, t_2 we get

$$\frac{1 + t_1 t_2}{(1-t_1)(1-t_2)} = \sum_{n_1, n_2 \geq 0} (1 + t_1 t_2) t_1^{n_1} t_2^{n_2} \,, \tag{6.13}$$

while expanding in negative powers of t_1, t_2 we get

$$\frac{1 + t_1 t_2}{(1-t_1)(1-t_2)} = \sum_{n_1, n_2 \geq 0} (1 + t_1^{-1} t_2^{-1}) t_1^{-n_1} t_2^{-n_2} \,, \tag{6.14}$$

and there are several other available expansions as well.

In order to calculate the one-loop determinant for the $\mathcal{N} = 2$ vectormultiplet, we must consider the self-dual complex (6.11) tensored with the adjoint representation of the gauge

[41] The complex (6.11) can be turned into the two-term complex in (6.6) by "folding" the complex as $D_{\rm SD} : \Omega^1 \xrightarrow{d^* \oplus d_+} \Omega^0 \oplus \Omega^{2+}$, where d^* is the conjugate of d.

[42] The weights of the $\mathrm{U}(1)^2$ action are: $\{(0,0)\}$ for Ω^0, $\{(\pm 1, 0), (0, \pm 1)\}$ for Ω^1, and $\{(0,0),(1,1),(-1,-1)\}$ for Ω^{2+}.

group G, and study the $\mathrm{U}(1)_{\varepsilon_1} \times \mathrm{U}(1)_{\varepsilon_2} \times G \times G_{\mathrm{F}}$-equivariant index for such a complex (see (6.1)). It is given by[43]

$$\operatorname{ind}(D^{\mathrm{vm}}) = \frac{(1+t_1t_2)}{2(1-t_1)(1-t_2)}\chi_{\mathrm{adj}}(g), \qquad g \in G\,, \tag{6.15}$$

where $\chi_{\mathrm{adj}}(g)$ is the character of G in the adjoint representation. More explicitly, let us denote $t_1 = \exp(i\varepsilon_1)$, $t_2 = \exp(i\varepsilon_2)$ and $g = \exp(i\hat{a})$, where $\varepsilon_1, \varepsilon_2$ and $\hat{a}$ are the elements of the Lie algebra of $\mathrm{U}(1)_{\varepsilon_1} \times \mathrm{U}(1)_{\varepsilon_2}$ and of the Cartan subalgebra $\mathfrak{t}$ of G respectively. Denoting by w be the weights of the adjoint representation of G, the index (6.15) can be written as

$$\operatorname{ind}(D^{\mathrm{vm}})(\varepsilon_1, \varepsilon_2, \hat{a}) = \frac{(1+e^{i\varepsilon_1+i\varepsilon_2})}{2(1-e^{i\varepsilon_1})(1-e^{i\varepsilon_2})}\sum_{w\in \mathrm{adj}} e^{iw\cdot\hat{a}}\,. \tag{6.16}$$

As mentioned earlier, the one-loop determinant in the localization computation of the 't Hooft loop path integral can be computed as the product over all the weights of the generator $\mathcal{R}$ of the $\mathrm{U}(1)_{J+R}\times G\times G_{\mathrm{F}}$ action on the space of fields (see (6.1)). Mathematically, the product of weights computes the equivariant Euler class of the normal bundle to the fixed point set. The corresponding index or equivariant Chern character determines the one-loop determinant or equivariant Euler character by taking the weighted product of all weights extracted from the exponents in the Chern character (using (6.5)). Therefore, we will calculate the one-loop determinant of the $\mathcal{N}=2$ vectormultiplet by determining the weights under the action of $\mathrm{U}(1)_{J+R}\times G\times G_{\mathrm{F}}$ from the index (6.16). We remove the terms with $w=0$ because they are independent of $\hat{a}$, so that we are only left with the sum over the roots α of $\mathfrak{g}$.

Let us now consider the north pole contribution to the index and the associated one-loop determinant for the vectormultiplet. As we mentioned earlier, the deformation of the symbol requires that the index in (6.16) be defined by taking the positive expansion for the $\mathrm{U}(1)_{\varepsilon_1} \times \mathrm{U}(1)_{\varepsilon_2}$ weights as in (6.13). This uniquely determines the weights under the action of $\mathrm{U}(1)_{\varepsilon_1} \times \mathrm{U}(1)_{\varepsilon_2} \times G \times G_{\mathrm{F}}$ to be

$$\begin{aligned} n_1\varepsilon_1 + n_2\varepsilon_2 + \alpha\cdot\hat{a} \qquad &\text{for} \qquad n_1, n_2 \geq 0\,,\\ (n_1+1)\varepsilon_1 + (n_2+1)\varepsilon_2 + \alpha\cdot\hat{a} \qquad &\text{for} \qquad n_1, n_2 \geq 0 \end{aligned} \tag{6.17}$$

with multiplicities $1/2$. The one-loop determinant contribution from the north pole of the $\mathcal{N}=2$ vectormultiplet labeled by a root α of the Lie algebra $\mathfrak{g}$ is therefore

$$\prod_{n_1,n_2\geq 0}\left[n_1\varepsilon_1 + n_2\varepsilon_2 + \alpha\cdot\hat{a}\right]^{1/2}\left[(n_1+1)\varepsilon_1 + (n_2+1)\varepsilon_2 + \alpha\cdot\hat{a}\right]^{1/2}. \tag{6.18}$$

In our localization calculation on S^4, we must specialize to the values $\varepsilon_1 = \varepsilon_2 = \varepsilon = 1/r$, which correspond to the $\mathrm{U}(1)_{J+R}$ symmetry. The expression is divergent, and we regularize it by identifying it with the Barnes G-function [23] (see for example section 5.17

[43] Recall that G_{F} acts trivially on vectormultiplet fields.

in [24]). It is an analytic function that has a zero of order n at $x = -n$ for all integers $n > 0$, and can be defined by the infinite product formula

$$G(1+z) = (2\pi)^{z/2} e^{-((1+\gamma)z^2+z)/2} \prod_{n=1}^{\infty} \left(1+\frac{z}{n}\right)^n e^{-z+\frac{z^2}{2n}} . \tag{6.19}$$

Therefore, the corresponding vectormultiplet one-loop determinant is given by[44]

$$Z^{\text{vm}}_{\text{1-loop,pole}}(\hat{a}) = \prod_{\alpha} G^{1/2}\left(\frac{\alpha\cdot\hat{a}}{\varepsilon}\right) G^{1/2}\left(2+\frac{\alpha\cdot\hat{a}}{\varepsilon}\right) . \tag{6.20}$$

At the other fixed point — at the south pole — we need to consider the anti-self-dual complex and an expansion in negative powers of t_1 and t_2. However, the index of the anti-self-dual complex at the south pole coincides with the index of the self-dual complex at the north pole. Relative to the north pole, the difference amounts to the sign change $(\varepsilon_1, \varepsilon_2) \to (-\varepsilon_1, -\varepsilon_2)$, which can be absorbed into the redefinition of roots $\alpha \to -\alpha$, which just exchanges positive and negative roots, yielding once again (6.20).

Therefore, recalling that the equivariant parameters for the G-action at the north and south poles are fixed (4.13), (4.14)

$$\hat{a}(N) = ia - ig^2\theta\frac{B}{16\pi^2 r} - \frac{B}{2r} \qquad \hat{a}(S) = ia - ig^2\theta\frac{B}{16\pi^2 r} + \frac{B}{2r} , \tag{6.21}$$

we obtain that the vectormultiplet one-loop determinant contributions from the north and south poles are

$$Z^{\text{vm}}_{\text{north,1-loop}} = Z^{\text{vm}}_{\text{1-loop,pole}}(\hat{a}(N)) \qquad Z^{\text{vm}}_{\text{south,1-loop}} = Z^{\text{vm}}_{\text{1-loop,pole}}(\hat{a}(S)) , \tag{6.22}$$

with $Z^{\text{vm}}_{\text{1-loop,pole}}(\hat{a})$ given in (6.20). Furthermore, the south pole contribution is the complex conjugate of the north pole

$$Z^{\text{vm}}_{\text{south,1-loop}} = \overline{Z}^{\text{vm}}_{\text{north,1-loop}} , \tag{6.23}$$

precisely the same relation that we found earlier for the classical and instanton contributions.

Let us now compare these results with the computation in [2]. In the absence of a 't Hooft loop we have $\hat{a}(N) = \hat{a}(S) = ia$, and

$$|Z^{\text{vm}}_{\text{1-loop,pole}}(\hat{a})|^2 \tag{6.24}$$

is precisely the one-loop determinant for the vectormultiplet obtained in [2], up to the ghosts-for-ghosts contributions. The ghosts-for-ghosts were introduced to gauge-fix the constant gauge transformations on S^4, and they had the effect of removing the Vandermonde $\prod_{\alpha>0} \alpha\cdot\hat{a}$ from the one-loop factor, while the square of the Vandermonde reappeared as the volume of the adjoint orbit $\{g\hat{a}g^{-1} | g \in G\}$. In the approach of this paper, we do not introduce ghosts-for-ghosts, and the Vandermonde is included in the one-loop factor (6.20).

[44] For asymptotically free gauge theories see discussion after equation (6.32).

Hypermultiplet determinant. The index of the complex for the Dirac operator $D_{\rm Dirac}$ that maps the space of positive-chirality spinors S^+ to the space of negative-chirality spinors S^- in $\mathbb{R}^4$

$$D_{\rm Dirac} : S^+ \to S^- , \tag{6.25}$$

with a suitable inversion of the grading, computes the contribution of a hypermultiplet to the one-loop determinant [2]. By applying the fixed-point formula (6.7) to the Dirac complex, we obtain[45]

$$\begin{aligned} \mathrm{ind}\, D_{\rm Dirac} &= \frac{t_1^{1/2}t_2^{1/2} + t_1^{-1/2}t_2^{-1/2} - (t_1^{1/2}t_2^{-1/2} + t_1^{-1/2}t_2^{1/2})}{(1-t_1)(1-t_1^{-1})(1-t_2)(1-t_2^{-1})} \\ &= \frac{t_1^{1/2}t_2^{1/2}}{(1-t_1)(1-t_2)} . \end{aligned} \tag{6.26}$$

The kinetic operator for a hypermultiplet in the adjoint representation of the gauge group and the one-loop factor were analyzed in [2] in detail. The corresponding index is given by tensoring the Dirac bundle with the adjoint bundle. We also need to remember that the $G_{\rm F} = \mathrm{SU}(2)$ flavour symmetry associated to an adjoint hypermultiplet acts on the bundle. Therefore the $\mathrm{U}(1)_{\varepsilon_1} \times \mathrm{U}(1)_{\varepsilon_2} \times G \times G_{\rm F}$ equivariant index for this complex, taking into account the inversion of the grading, is given by

$$\mathrm{ind}\, D^{\rm hm}_{\rm adj}(\varepsilon_1, \varepsilon_2, \hat{a}, \hat{m}) = -\frac{e^{\frac{1}{2}(i\varepsilon_1 + i\varepsilon_2)}}{(1-e^{i\varepsilon_1})(1-e^{i\varepsilon_2})} \frac{e^{i\hat{m}} + e^{-i\hat{m}}}{2} \sum_{w \in {\rm adj}} e^{iw\cdot\hat{a}} . \tag{6.27}$$

We recall that the equivariant parameter $\hat{m} = im$ for the SU(2) flavour symmetry, which takes values in the SU(2) Cartan subalgebra, is interpreted as the mass m of the adjoint hypermultiplet.

Given the formula for the equivariant index for the hypermultiplet in the adjoint representation, group theory completely determines the corresponding index for an arbitrary representation R of the gauge group. To explain this claim, let us recall that the precise flavour symmetry depends on the type of matter representation, and that in general we need to consider half-hypermultiplets although in the end half-hypermultiplets pair up into full hypermultiplets. For a complex irreducible representation R, half-hypermultiplets always appear as copies of conjugate pairs $N_{\rm F} \cdot (R \oplus \overline{R})$, and the flavour symmetry is $\mathrm{U}(N_{\rm F})$. Half-hypermultiplets in a real irreducible representation R can only arise in an even number $2N_{\rm F}$, in which case the flavour symmetry is enhanced to $\mathrm{Sp}(2N_{\rm F})$.[46] If the irreducible representation R is pseudo-real, classically an arbitrary number n of half-hypermultiplets

[45]The index can also be obtained by noting that the Dirac complex in $\mathbb{C}^2 = \mathbb{R}^4$ is related to the Dolbeault operator $\overline{\partial} : \Omega^{0,0} \to \Omega^{0,1} \to \Omega^{0,2}$. The bundle S^+ is given by $\Omega^{0,0} \oplus \Omega^{0,2}$ twisted by $K^{1/2}$ while S^- is given by $\Omega^{0,1}$ twisted by $K^{1/2}$, where K is the canonical bundle. We want to compute the equivariant index of $D_{\rm Dirac}$ with respect the $T = \mathrm{U}(1)_{\varepsilon_1} \times \mathrm{U}(1)_{\varepsilon_2}$ action $(z_1, z_2) \mapsto (t_1 z_1, t_2 z_2)$. Hence up to the twist by $K^{1/2}$, which contributes a factor of $(t_1 t_2)^{1/2}$ to the index, the Dirac complex (6.25) is isomorphic to standard Dolbeault complex in $\mathbb{C}^2$. The relative factor $t_1^{-1/2} t_2^{-1/2}$ between (6.26) and the Dolbeault index $t_1 t_2 / [(1-t_1)(1-t_2)]$ accounts for the twist by $K^{1/2}$.

[46] In our convention $\mathrm{Sp}(2N)$ has rank N. Also $\mathrm{Sp}(2) = \mathrm{SU}(2)$.

can appear with $\mathrm{SO}(n)$ as the flavour symmetry group, but for odd n an anomaly renders the theory inconsistent [25]. Thus $n = 2N_\mathrm{F}$ has to be even and the flavour symmetry is enhanced to $\mathrm{SO}(2N_\mathrm{F})$. In every case, the flavour symmetry group acts in the defining representation and there are N_F mass parameters $\hat{m}_f = im_f$ with $f = 1, \dots N_\mathrm{F}$ parametrizing the Cartan subalgebra of G_F. As shown in appendix E, the following expression for the index holds for a hypermultiplet in an arbitrary representation R of G:

$$\mathrm{ind}\, D_R^{\mathrm{hm}}(\varepsilon_1, \varepsilon_2, \hat{a}, \hat{m}_f) = -\frac{e^{\frac{1}{2}(i\varepsilon_1 + i\varepsilon_2)}}{2(1-e^{i\varepsilon_1})(1-e^{i\varepsilon_2})} \sum_{f=1}^{N_\mathrm{F}} \sum_{w\in R} \left(e^{iw\cdot\hat{a} - i\hat{m}_f} + e^{-iw\cdot\hat{a}+i\hat{m}_f} \right) . \quad (6.28)$$

At the north and south poles, we expand the index (6.28) in positive and negative powers of (t_1, t_2) respectively, from which we read the weights of the the $\mathrm{U}(1)_{\varepsilon_1} \times \mathrm{U}(1)_{\varepsilon_2} \times G \times G_\mathrm{F}$ action. Both expansions give rise to identical one-loop determinants, given in terms of the weights by (6.5).

The relevant hypermultiplet one-loop determinant of the theory on S^4 is obtained by setting $\varepsilon_1 = \varepsilon_2 = \varepsilon = 1/r$, the G-equivariant parameters at the north and south poles to (6.21) and $\hat{m}_f = im_f$, where m_f with $f = 1, \dots, N_\mathrm{F}$ are the masses of the N_F hypermultiplets. Therefore the one-loop determinants of N_F massive hypermultiplets in a representation R of G arising from the north and south poles are given by

$$Z^{\mathrm{hm}}_{\text{north,1-loop}} = Z^{\mathrm{hm}}_{\text{1-loop,pole}}(\hat{a}(N), im_f) \qquad\qquad Z^{\mathrm{hm}}_{\text{south,1-loop}} = Z^{\mathrm{hm}}_{\text{1-loop,pole}}(\hat{a}(S), im_f) . \quad (6.29)$$

with [44]

$$Z^{\mathrm{hm}}_{\text{1-loop,pole}}(\hat{a}, \hat{m}_f) = \prod_{f=1}^{N_\mathrm{F}} \prod_{w\in R} G^{-1/2}\left(1 + \frac{w\cdot\hat{a}}{\varepsilon} - \frac{\hat{m}_f}{\varepsilon}\right) G^{-1/2}\left(1 - \frac{w\cdot\hat{a}}{\varepsilon} + \frac{\hat{m}_f}{\varepsilon}\right) , \quad (6.30)$$

where w are the weights of the representation R. We note that for an arbitrary representation R, the hypermultiplet one-loop determinant at the south pole is the complex conjugate of the determinant at the north pole[47]

$$Z^{\mathrm{hm}}_{\text{south,1-loop}} = \overline{Z}^{\mathrm{hm}}_{\text{north,1-loop}} . \quad (6.31)$$

We can now start gathering the results obtained until now. Combining the vectormultiplet and hypermultiplet determinants given in (6.20) and (6.30), we conclude that the pole contribution to the one-loop determinant for an arbitrary $\mathcal{N} = 2$ Lagrangian theory in S^4 in the presence of a 't Hooft operator can be written in terms of

$$Z_{\text{1-loop,pole}}(\hat{a}, \hat{m}_f) = \frac{\prod_\alpha \left[G\left(r\alpha\cdot\hat{a}\right) G\left(2 + r\alpha\cdot\hat{a}\right)\right]^{1/2}}{\prod_{f=1}^{N_\mathrm{F}} \prod_{w\in R} \left[G\left(1 + rw\cdot\hat{a} - r\hat{m}_f\right) G\left(1 - rw\cdot\hat{a} + r\hat{m}_f\right)\right]^{1/2}} , \quad (6.32)$$

where we recall that $\varepsilon = 1/r$. Formula (6.32) holds for an arbitrary $\mathcal{N} = 2$ gauge theory admitting a Lagrangian description, and can be explicitly calculated given the choice of gauge group G and of a representation R of G under which the hypermultiplet transforms.

[47] The expression $|Z^{\mathrm{hm}}_{\text{north,1-loop}}|^2$ reproduces the one-loop determinant obtained in [2] when there is no 't Hooft loop, corresponding to $\hat{a}(N) = \hat{a}(S) = ia, \hat{m}_f = im_f$.

For asymptotically free gauge theories, the localization calculation is most accurately performed by embedding such a theory into one that is ultraviolet finite, which then flows to the asymptotically free theory upon taking the mass parameters of the finite theory to be very large. As a prototype of this construction, $\mathcal{N}=2$ pure super Yang-Mills with arbitrary gauge group G can be regulated by embedding it in the $\mathcal{N}=2^*$ theory, consisting of a vectormultiplet and massive hypermultiplet in the adjoint representation of G, by then taking the mass of the hypermultiplet to be very large. This construction exists for an arbitrary asymptotically free four dimensional $\mathcal{N}=2$ gauge theory. Given an asymptotically free $\mathcal{N}=2$ gauge theory, the end result of this procedure in the localization computation is that the one-loop determinants are given by (6.32) for the field content of the asymptotically free theory, together with the replacement of the bare coupling constant τ of the theory with the familiar one-loop corrected running coupling constant τ_{ren}.

The complete one-loop determinants in our localization computation arising at the north and south poles are thus given by

$$Z_{\text{north,1-loop}} = Z_{\text{1-loop,pole}}(\hat{a}(N), im_f) \qquad Z_{\text{south,1-loop}} = Z_{\text{1-loop,pole}}(\hat{a}(S), im_f)\,, \tag{6.33}$$

with $\hat{a}(N)$ and $\hat{a}(S)$ in (6.21). Combining the one-loop result with the classical and instanton contributions computed in the previous two sections, we have that the expectation value of a 't Hooft loop labeled by a coweight B in an $\mathcal{N}=2$ gauge theory with gauge group G and N_{F} massive hypermultiplets in a representation R of G is given by[48]

$$\begin{aligned}&\langle T(B)\rangle \simeq \\ &\quad\int da \left| Z_{\text{cl}}\left(ia-\frac{B}{2r},q\right) Z_{\text{1-loop,pole}}\left(ia-\frac{B}{2r},im_f\right) Z_{\text{inst}}\left(ia-\frac{B}{2r},\frac{1}{r}+im_f,\frac{1}{r},\frac{1}{r},q\right)\right|^2\end{aligned} \tag{6.34}$$

with Z_{cl}, $Z_{\text{1-loop,pole}}$ and Z_{inst} given in (4.10), (6.32) and (5.2) respectively.

Therefore, the path integral completely factorizes into north and south pole contributions as

$$\langle T(B)\rangle \simeq \int da\, Z_{\text{north}}\cdot Z_{\text{south}} = \int da\, \left|Z_{\text{north}}\right|^2\,, \tag{6.35}$$

with

$$\begin{aligned} Z_{\text{north}} &= Z_{\text{cl}}\left(\hat{a}(N),q\right) Z_{\text{1-loop,pole}}(\hat{a}(N),im_f) Z_{\text{inst}}\left(\hat{a}(N),\frac{1}{r}+im_f,\frac{1}{r},\frac{1}{r},q\right)\\ Z_{\text{south}} &= Z_{\text{cl}}\left(\hat{a}(S),\overline{q}\right) Z_{\text{1-loop,pole}}(\hat{a}(S),im_f) Z_{\text{inst}}\left(\hat{a}(S),\frac{1}{r}+im_f,\frac{1}{r},\frac{1}{r},\overline{q}\right)\,,\end{aligned} \tag{6.36}$$

which furthermore are complex conjugate to each other

$$Z_{\text{south}} = \overline{Z}_{\text{north}}\,. \tag{6.37}$$

When the gauge theory is asymptotically free, we must replace the bare instanton fugacity q by the renormalized one q_{ren} in Z_{north} and Z_{south}. The $\simeq$ symbol is used in (6.34) and (6.35)

[48] After trivially shifting the integration variable $ia \to ia + ig^2\theta\frac{B}{16\pi^2 r}$.

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since in the presence of a 't Hooft loop operator, an extra contribution supported on the loop must be included, to which we now turn.[49]

6.3 Equator contribution

In the absence of a 't Hooft loop, the index is a sum of local contributions from the north and south poles [2], which are the fixed points of J. In defining the 't Hooft loop path integral in gauge theory, we must impose boundary conditions along the loop compatible with the field configuration of a singular monopole. In this subsection we calculate the contributions to the vectormultiplet and hypermultiplet indices as well as one-loop determinants from the equatorial S^1 where the 't Hooft loop is located, which are functions of the weights for the group action $\mathrm{U}(1)_{J+R} \times G \times G_{\mathrm{F}}$ generated by Q^2 (also by $\hat{Q}^2$).

Let us recall from section 3.1 that the isometry generator J in Q^2 acts on $B_3 \times S^1$ as a spatial rotation along the x_3-axis as well as a shift in the periodic coordinate τ. The conformal killing spinor ϵ_Q in (3.14) with which we localize the 't Hooft loop path integral can be written as

$$\epsilon_Q = e^{-\tau \frac{\Gamma^{56}+\Gamma^{78}}{4}} \left(1 - i\frac{x^i}{2r}\tilde{\Gamma}_i \Gamma^{120}\right) \hat{\varepsilon}_s \,. \tag{6.38}$$

Note that ϵ_Q changes its sign when going around the S^1, under $\tau \to \tau + 2\pi$. Therefore while all the bosons are periodic, all the fermions in the vielbein basis are antiperiodic. In particular, within each supermultiplet bosons and fermions obey different boundary conditions around S^1.

Recall that Q^2 also includes the $\mathrm{U}(1)_R$ transformation (see (3.20)), which is generated by $J_{56} + J_{78}$. When we apply the index theorem it is convenient to redefine fields of the theory and ϵ_Q using $\mathrm{U}(1)_R$ as[50]

$$\begin{aligned}
\epsilon_Q &\to e^{\tau \frac{\Gamma^{56}+\Gamma^{78}}{4}} \epsilon_Q \,, \\
A_M &\to \left(e^{\tau \frac{J_{56}+J_{78}}{2}}\right)_{MN} A_N \,, \\
\Psi &\to e^{\tau \frac{\Gamma^{56}+\Gamma^{78}}{4}} \Psi \,, \\
\chi &\to \chi \,,
\end{aligned} \tag{6.39}$$

where we have normalized the ten-dimensional Lorentz generators in the vector representation as $(J_{MN})_{PQ} = \delta_{MP}\delta_{NQ} - \delta_{MQ}\delta_{NP}$ and used that $\mathrm{U}(1)_R$ is generated by $\Gamma^{56}+\Gamma^{78}$ when acting on spinors. After the field redefinition, the whole vectormultiplet is periodic, and all fields in the hypermultiplet are antiperiodic.[51] This redefinition makes the spinor ϵ_Q

[49] In section 7 we will include yet another contribution due to monopole screening, which is non-perturbative in nature.

[50] Here we are using ten dimensional notation for the bosonic fields of the $\mathcal{N}=2$ theory, so that $A_M = \{A_\mu, q, \tilde{q}, \Phi_9, \Phi_0\}$ with $M = 1, \ldots, 9, 0$.

[51] The R-symmetry group $\mathrm{U}(1)_R$ acts non-trivially on the fermions in the vectormultiplet and on the scalars in the hypermultiplet.

independent of τ. The shift in τ now induces an R-symmetry transformation[52] in addition to the S^1 part of isometry J.

Vectormultiplet determinant. As we saw in section 3.2, the localization equations near the location of the 't Hooft operator — which wraps the S^1 at the origin in B_3 — are approximately the Bogomolny equations

$$*_3 F = D\Phi \tag{6.40}$$

in $B_3 \times S^1$. The differential operator that appears in the kinetic term for the vector-multiplet is obtained by linearizing the Bogomolny equations. Linearization of the gauge transformation and the Bogomolny equations is described by the complex[53]

$$D_{\text{Bogo}} : \Omega^0 \to \Omega^1 \oplus \Omega^0 \to \Omega^2 \tag{6.41}$$

in $\mathbb{R}^3 - \{0\}$. In appendix F, we explain Kronheimer's observation that the Bogomolny equations in $\mathbb{R}^3$ with a monopole singularity at the origin — where the 't Hooft operator resides — is equivalent to the anti-self-duality equations for gauge fields in $\mathbb{R}^4$ invariant under the action of a spacetime symmetry group $\mathrm{U}(1)_K$. Using Kronheimer's correspondence, we can obtain this complex by projecting the self-dual complex (6.11) to the $\mathrm{U}(1)_K$-invariant sections. We can compute the index of the complex (6.41) by averaging the index of the self-dual complex over the $\mathrm{U}(1)_K$ action, picking up the contributions only from the $\mathrm{U}(1)_K$ invariant sections.[54]

In equation (6.9), the index for the Dolbeault operator $\overline{\partial}$ on $\mathbb{C}$ was obtained as the $\mathrm{U}(1)$ character on the space of holomorphic functions. In this toy example the index is an infinite power series corresponding to infinitely many monomials. The same logic can be used to derive the index (6.12) for the complex (6.11) through an expansion in a basis of local sections. Among such sections, those which are invariant under $\mathrm{U}(1)_K$ correspond to the ordinary spherical harmonics for the bundles in three dimensions. We can keep track of the original expansion by introducing an infinitesimal positive parameter $\delta > 0$:

$$\mathrm{ind}_\delta(D_{\mathrm{SD}})(t_1, t_2) = \frac{(1 + t_1^{-1} t_2^{-1})(1 - t_1)(1 - t_2)}{2(1 - e^{-\delta} t_1)(1 - e^{-\delta} t_1^{-1})(1 - e^{-\delta} t_2)(1 - e^{-\delta} t_2^{-1})}\,. \tag{6.42}$$

We now parametrize the $\mathrm{U}(1) \times \mathrm{U}(1)$ weights as

$$t_1 = e^{-i\nu + i\frac{1}{2}\varepsilon}\,, \qquad t_2 = e^{i\nu + i\frac{1}{2}\varepsilon}\,, \tag{6.43}$$

where ν is the parameter for the group $\mathrm{U}(1)_K$ used in Kronheimer's construction: $(\mathbb{C}^2 - \{0\})/\mathrm{U}(1)_K \simeq \mathbb{R}^3 - \{0\}$. The parameter ε is the angle for a rotation along the x_3-axis

[52] To be precise, the R-symmetry accounted for by a shift in τ is half the full amount. The rest combines, as in topological twist, with the spatial part of J so that gauginos transform as 0- and 1-forms, and the hypermultiplet scalars as spinors under $\mathrm{U}(1)_{J+R}$.

[53] This complex can also be turned into a two-term complex as in (6.6) by folding the complex.

[54] A similar computation was done in [22], where more than one singular monopole was considered on a compact manifold. While our integrand to be averaged is a rational function with poles on the integration contour, the integrand in [22] was a polynomial due to cancellations among singular monopoles.

in $\mathbb{R}^3$, and the factors of $1/2$ ensure that for $\varepsilon = 2\pi$ this rotation acts as -1 on $\mathbb{C}^2$ even though it acts as $+1$ on $\mathbb{R}^3$.

In order to describe the singular monopole background due to the 't Hooft operator, we also need to twist by the adjoint gauge bundle on which the gauge group G and $\mathrm{U}(1)_K$ act as $e^{\hat{a}+B\nu}$, with B being the magnetic weight labeling the operator. The four-dimensional sections invariant under $\mathrm{U}(1)_K$ can be identified with the monopole harmonics [26] of the corresponding bundles over $\mathbb{R}^3 - \{0\}$. The index for the self-dual complex twisted by the gauge bundle is given by

$$\begin{aligned}\mathrm{ind}_\delta(D_{\mathrm{SD}})(\nu,\varepsilon,\hat{a}) &= \frac{(1+e^{-i\varepsilon})(1-e^{-i\nu+i\varepsilon/2})(1-e^{i\nu+i\varepsilon/2})}{2(1-e^{-\delta}e^{i\nu-i\varepsilon/2})(1-e^{-\delta}e^{-i\nu+i\varepsilon/2})(1-e^{-\delta}e^{-i\nu-i\varepsilon/2})(1-e^{-\delta}e^{i\nu+i\varepsilon/2})} \\ &\quad\times \sum_{w\in\mathrm{adj}} e^{iw\cdot\hat{a}+iw\cdot B\nu}. \end{aligned} \tag{6.44}$$

By averaging over $\mathrm{U}(1)_K$, we get the desired index for the complex (6.41)

$$\begin{aligned}\mathrm{ind}(D_{\mathrm{Bogo}}) &= \lim_{\delta\to 0}\int_0^{2\pi}\frac{d\nu}{2\pi}\mathrm{ind}_\delta(D_{\mathrm{SD}})(\nu,\varepsilon,\hat{a}) \\ &= \lim_{\delta\to 0}\oint_{|z|=1}\frac{dz}{2\pi i}\frac{(1+e^{-i\varepsilon})(z-e^{i\varepsilon/2})(1-e^{i\varepsilon/2}z)}{2(1-e^{-\delta}e^{-i\varepsilon/2}z)(z-e^{-\delta}e^{i\varepsilon/2})(z-e^{-\delta}e^{-i\varepsilon/2})(1-e^{-\delta}e^{i\varepsilon/2}z)} \\ &\quad\times\sum_{w\in\mathrm{adj}} e^{-iw\cdot\hat{a}}z^{-w\cdot B}, \end{aligned} \tag{6.45}$$

where we have renamed w as $w \to -w$. We can evaluate the integral by summing over residues for the poles inside the unit circle. For $w\cdot B > 0$ a pole at $z = 0$ contributes

$$\begin{aligned}&\sum_{w\cdot B>0} e^{-iw\cdot\hat{a}}\frac{(1+e^{-i\varepsilon})}{2(w\cdot B-1)!}\left(\frac{\partial}{\partial z}\right)^{w\cdot B-1}\Bigg|_{z=0}\frac{1}{(1-e^{-i\varepsilon/2}z)(z-e^{-i\varepsilon/2})} \\ &= -\frac{e^{i\varepsilon/2}+e^{-i\varepsilon/2}}{2}\sum_{w\cdot B>0} e^{-iw\cdot\hat{a}}\left(e^{i\frac{w\cdot B-1}{2}\varepsilon}+e^{i\frac{w\cdot B-3}{2}\varepsilon}+\ldots+e^{-i\frac{w\cdot B-1}{2}\varepsilon}\right) \\ &= -\frac{e^{i\varepsilon/2}+e^{-i\varepsilon/2}}{2}\sum_{w\cdot B>0} e^{-iw\cdot\hat{a}}\frac{e^{i(w\cdot B)\varepsilon/2}-e^{-i(w\cdot B)\varepsilon/2}}{e^{i\varepsilon/2}-e^{-i\varepsilon/2}}. \end{aligned} \tag{6.46}$$

In addition there are always two poles at $z = e^{-\delta}e^{i\varepsilon/2}, e^{-\delta}e^{-i\varepsilon/2}$. In the limit $\delta \to 0$, the contribution of the pole at $z = e^{-\delta}e^{i\varepsilon/2}$ is given by

$$\begin{aligned}&\frac{(1+e^{-i\varepsilon})(e^{-\delta}e^{i\varepsilon/2}-e^{i\varepsilon/2})(1-e^{i\varepsilon}e^{-\delta})}{2(1-e^{-2\delta})(e^{-\delta}e^{i\varepsilon/2}-e^{-\delta}e^{-i\varepsilon/2})(1-e^{-2\delta}e^{i\varepsilon})}\sum_{w\in\mathrm{adj}} e^{-iw\cdot\hat{a}}e^{-w\cdot B(-\delta+i\varepsilon/2)} \\ &\to -\frac{1}{4}\frac{e^{i\varepsilon/2}+e^{-i\varepsilon/2}}{e^{i\varepsilon/2}-e^{-i\varepsilon/2}}\sum_{w\in\mathrm{adj}} e^{-iw\cdot\hat{a}}e^{-iw\cdot B\varepsilon/2}, \end{aligned} \tag{6.47}$$

while the pole at $z = e^{-\delta}e^{-i\varepsilon/2}$ contributes

$$
\begin{aligned}
&\frac{(1+e^{-i\varepsilon})(e^{-\delta}e^{-i\varepsilon/2}-e^{i\varepsilon/2})(1-e^{-\delta})}{2(1-e^{-2\delta}e^{-i\varepsilon})(e^{-\delta}e^{-i\varepsilon/2}-e^{-\delta}e^{i\varepsilon/2})(1-e^{-2\delta})}\sum_{w\in\mathrm{adj}} e^{-iw\cdot\hat{a}}e^{-w\cdot B(-\delta-i\varepsilon/2)} \\
&\to \frac{1}{4}\frac{e^{i\varepsilon/2}+e^{-i\varepsilon/2}}{e^{i\varepsilon/2}-e^{-i\varepsilon/2}}\sum_{w\in\mathrm{adj}} e^{-iw\cdot\hat{a}}e^{i(w\cdot B)\varepsilon/2}\,.
\end{aligned}\tag{6.48}
$$

Combining the residues we get

$$
\begin{aligned}
\mathrm{ind}(D_{\mathrm{Bogo}}) &= -\frac{e^{i\varepsilon/2}+e^{-i\varepsilon/2}}{2}\sum_{w\cdot B>0} e^{-iw\cdot\hat{a}}\frac{e^{i(w\cdot B)\varepsilon/2}-e^{-i(w\cdot B)\varepsilon/2}}{e^{i\varepsilon/2}-e^{-i\varepsilon/2}} \\
&\qquad +\frac{1}{4}(e^{i\varepsilon/2}+e^{-i\varepsilon/2})\sum_{w\cdot B\neq 0} e^{-iw\cdot\hat{a}}\frac{e^{i(w\cdot B)\varepsilon/2}-e^{-i(w\cdot B)\varepsilon/2}}{e^{i\varepsilon/2}-e^{-i\varepsilon/2}} \\
&= -\frac{1}{4}(e^{i\varepsilon/2}+e^{-i\varepsilon/2})\sum_{\alpha>0}(e^{i\alpha\cdot\hat{a}}+e^{-i\alpha\cdot\hat{a}})\frac{e^{i(\alpha\cdot B)\varepsilon/2}-e^{-i(\alpha\cdot B)\varepsilon/2}}{e^{i\varepsilon/2}-e^{-i\varepsilon/2}}\,.
\end{aligned}\tag{6.49}
$$

In the last line we replaced the sum over the adjoint weights satisfying $w\cdot B>0$ by the sum over positive roots $\alpha>0$. This is possible because by taking B to be in the Weyl chamber all such w's are positive roots.

For the vectormultiplet one-loop determinant computation, we also need to tensor with the space of periodic functions on S^1. Thus we need to compute $\sum_{n\in\mathbb{Z}} e^{in\varepsilon}\mathrm{ind}(D_{\mathrm{Bogo}})$. A simplification arises because the parameter n is summed over, and can be shifted by an integer freely. Finally, the equatorial index for the vectormultiplet is

$$
\begin{aligned}
\mathrm{ind}(D^{\mathrm{vm}}_{\mathrm{eq}})(\varepsilon,\hat{a}) &= \sum_{n\in\mathbb{Z}} e^{in\varepsilon}\,\mathrm{ind}(D_{\mathrm{Bogo}}) \\
&= -\sum_{n\in\mathbb{Z}} e^{in\varepsilon}\frac{e^{i\varepsilon/2}+e^{-i\varepsilon/2}}{4}\sum_{\alpha>0}(e^{i\alpha\cdot\hat{a}}+e^{-i\alpha\cdot\hat{a}})\left(e^{i\frac{\alpha\cdot B-1}{2}\varepsilon}+e^{i\frac{\alpha\cdot B-3}{2}\varepsilon}+\ldots+e^{-i\frac{\alpha\cdot B-1}{2}\varepsilon}\right) \\
&= -\sum_{\alpha>0}(\alpha\cdot B)\frac{e^{i\alpha\cdot\hat{a}}+e^{-i\alpha\cdot\hat{a}}}{2}\times\sum_{n\in\mathbb{Z}}\begin{cases} e^{in\varepsilon} & \text{if } \alpha\cdot B \text{ is even,} \\ e^{i(n+1/2)\varepsilon} & \text{if } \alpha\cdot B \text{ is odd.}\end{cases}
\end{aligned}\tag{6.50}
$$

Note that we can write the last sum as $\sum_{n\in\mathbb{Z}} e^{i(n+\alpha\cdot B/2)\varepsilon}$ in both cases.[55] Applying the rule (6.5) to the index (6.50), we obtain the one-loop determinant[56]

$$
\begin{aligned}
Z^{\mathrm{vm}}_{\text{1-loop,eq}}(\hat{a},B) &= \prod_{n\in\mathbb{Z}}\prod_{\alpha>0}\left(n\varepsilon+\frac{\alpha\cdot B}{2}\varepsilon+\alpha\cdot\hat{a}\right)^{-\alpha\cdot B/2}\left(n\varepsilon+\frac{\alpha\cdot B}{2}\varepsilon-\alpha\cdot\hat{a}\right)^{-\alpha\cdot B/2} \\
&= \prod_{\alpha>0}\left[\sin\left(\pi\alpha\cdot\left(\frac{\hat{a}}{\varepsilon}+\frac{B}{2}\right)\right)\right]^{-\alpha\cdot B}\,.
\end{aligned}\tag{6.51}
$$

[55] Physically, half-odd integer coefficients appear in the exponential for odd $\alpha\cdot B$ because the relation between the angular momentum and statistics is reversed when the monopole charge is odd [26].

[56] We regulate the product by identifying it with the product representation of the sine function.

The equivariant parameter $\hat{a}$ to be used for the gauge group action at the equator is the gauge parameter $\Lambda(E)$ given in (3.26). To evaluate it explicitly, note that the field strength given in (3.9) leads to the gauge potential

$$A_\tau = ig^2\theta \frac{B}{16\pi^2}\frac{r}{|\vec{x}|}\left(1+\frac{|x|^2}{4r^2}\right)+C \tag{6.52}$$

where C is a constant. As one sees from (3.5) the S^1 parametrized by τ shrinks at $|\vec{x}|=2r$. Therefore the component A_τ has to vanish at $|\vec{x}|=2r$, and this fixes the value of C to $-ig^2\theta\frac{B}{16\pi^2}$. Then

$$\hat{a}(E) = \lim_{|\vec{x}|\to 0}\left(v^0\Phi_0+v^4A_\tau\right) = ia - ig^2\theta\frac{B}{16\pi^2 r}\,, \tag{6.53}$$

where we used the values of v^0 and v^4 given in (3.27). The singular terms in Φ_0 and A_τ canceled out to leave a finite quantity. Setting $\varepsilon = 1/r$, the equatorial one-loop determinant for the vectormultiplet is given by

$$Z^{\text{vm}}_{\text{equator}} = Z^{\text{vm}}_{\text{1-loop,eq}}\left(ia - ig^2\theta\frac{B}{16\pi^2 r}, B\right)\,. \tag{6.54}$$

Hypermultiplet determinant. We deal with the hypermultiplet in a similar way. The relevant differential operator is the Dirac operator plus a coupling to the Higgs field Φ_9. In Kronheimer's correspondence, this lifts simply to the Dirac operator on $\mathbb{C}^2$ given in (6.26). We regularize the index (6.26) by specifying the expansion in a local basis as

$$\text{ind}_\delta(D_{\text{Dirac}})(t_1,t_2) = \frac{t_1^{-1/2}t_2^{-1/2}(1-t_1)(1-t_2)}{(1-e^{-\delta}t_1^{-1})(1-e^{-\delta}t_1)(1-e^{-\delta}t_2^{-1})(1-e^{-\delta}t_2)}\,. \tag{6.55}$$

We can twist the Dirac complex by a vector bundle whose sections transform in representation R of the gauge group. Including the action of the gauge and flavour groups $G\times G_{\text{F}}$ as in (6.28), and then averaging over $\text{U}(1)_K$, we obtain

$$\begin{aligned}\text{ind}(D_{\text{DH}}) &= \lim_{\delta\to 0}\int_0^{2\pi}\frac{d\nu}{2\pi}\text{ind}_\delta(D_{\text{Dirac}})(t_1,t_2,\hat{a},\hat{m}_f)\\ &= -\frac{1}{4}\sum_{f=1}^{N_{\text{F}}}\sum_{w\in R, w\cdot B>0}\left(e^{iw\cdot\hat{a}-i\hat{m}_f}+e^{-iw\cdot\hat{a}+i\hat{m}_f}\right)\frac{e^{i(w\cdot B)\varepsilon/2}-e^{-i(w\cdot B)\varepsilon/2}}{e^{i\varepsilon/2}-e^{-i\varepsilon/2}}\\ &\quad+\frac{1}{4}\sum_{f=1}^{N_{\text{F}}}\sum_{w\in R, w\cdot B<0}\left(e^{iw\cdot\hat{a}-i\hat{m}_f}+e^{-iw\cdot\hat{a}+i\hat{m}_f}\right)\frac{e^{i(w\cdot B)\varepsilon/2}-e^{-i(w\cdot B)\varepsilon/2}}{e^{i\varepsilon/2}-e^{-i\varepsilon/2}}\,.\end{aligned} \tag{6.56}$$

We noted above that the hypermultiplet fields are antiperiodic in τ. Thus we must tensor with the space of anti-periodic functions on S^1, and change the sign for the index because we shift the degrees for physical fields in the complex (as we did already for the hypermultiplet contribution at the poles). The equatorial index for the hypermultiplet is thus

$$\begin{aligned}\operatorname{ind}(D^{\text{hm}}_{R,\text{eq}})(\varepsilon,\hat a,\hat m_f) &= -\sum_{n\in\mathbb{Z}} e^{i(n+1/2)\varepsilon}\operatorname{ind}(D_{\text{DH}})\\ &= \frac14\sum_{f=1}^{N_{\text{F}}}\sum_{w\in R}|w\cdot B|(e^{iw\cdot\hat a-i\hat m_f}+e^{-iw\cdot\hat a+i\hat m_f})\times\sum_{n\in\mathbb{Z}}\begin{cases}e^{in\varepsilon} & \text{if } w\cdot B \text{ is even,}\\ e^{i(n+1/2)\varepsilon} & \text{if } w\cdot B \text{ is odd.}\end{cases}\end{aligned}\tag{6.57}$$

Therefore, the one-loop determinant contribution from the equator of N_{F} hypermultiplets in a representation R of the gauge group is

$$Z^{\text{hm}}_{\text{equator}} = Z^{\text{hm}}_{\text{1-loop},R,\text{eq}}(ia, im_f, B)\,,\tag{6.58}$$

where

$$Z^{\text{hm}}_{\text{1-loop},R,\text{eq}}(\hat a,\hat m_f,B) = \prod_{f=1}^{N_{\text{F}}}\prod_{w\in R}\left[\sin\left(\pi w\cdot\left(\frac{\hat a}{\varepsilon}+\frac{B}{2}\right)-\pi\frac{\hat m_f}{\varepsilon}\right)\right]^{|w\cdot B|/2}.\tag{6.59}$$

Combining the vectormultiplet (6.51) and hypermultiplet (6.59) determinants, the complete equator contribution is given by

$$Z^{\text{1-loop}}_{\text{equator}} = Z_{\text{1-loop,eq}}\left(ia - ig^2\theta\frac{B}{16\pi^2 r}, im_f, B\right)\tag{6.60}$$

with[57]

$$Z_{\text{1-loop,eq}}(\hat a,\hat m_f,B) = \frac{\prod_{f=1}^{N_{\text{F}}}\prod_{w\in R}\left[\sin\left(\pi w\cdot\left(\frac{\hat a}{\varepsilon}+\frac{B}{2}\right)-\pi\frac{\hat m_f}{\varepsilon}\right)\right]^{|w\cdot B|/2}}{\prod_{\alpha>0}\left[\sin\left(\pi\alpha\cdot\left(\frac{\hat a}{\varepsilon}+\frac{B}{2}\right)\right)\right]^{|\alpha\cdot B|}}.\tag{6.61}$$

We are now in the position of writing the exact expectation value of a 't Hooft loop in an $\mathcal{N}=2$ gauge theory on S^4 with magnetic weight B. Multiplying the contributions associated to the poles and the equator, we have that[58]

$$\langle T(B)\rangle = \int da\, Z_{\text{north}}\cdot Z_{\text{south}}\cdot Z^{\text{1-loop}}_{\text{equator}} = \int da\,|Z_{\text{north}}|^2\cdot Z^{\text{1-loop}}_{\text{equator}}\tag{6.62}$$

where

$$\begin{aligned}Z_{\text{north}} &= Z_{\text{cl}}\left(a(N),q\right)Z_{\text{1-loop,pole}}(a(N),im_f)Z_{\text{inst}}\left(a(N),\frac1r+im_f,\frac1r,\frac1r,q\right)\\ Z_{\text{south}} &= Z_{\text{cl}}\left(a(S),\overline{q}\right)Z_{\text{1-loop,pole}}(a(S),im_f)Z_{\text{inst}}\left(a(S),\frac1r+im_f,\frac1r,\frac1r,\overline{q}\right)\\ Z^{\text{1-loop}}_{\text{equator}} &= Z_{\text{1-loop,eq}}(a(E),im_f,B)\,,\end{aligned}\tag{6.63}$$

with $Z_{\text{cl}}, Z_{\text{1-loop,pole}}, Z_{\text{inst}}$ and $Z_{\text{1-loop,eq}}$ given in (4.10), (6.32), (5.2) and (6.61).

In section 7 we will identify further non-perturbative corrections to this result arising due to monopole screening.

[57] Up to a phase, this expression is valid even if B is not in the Weyl chamber.

[58] Shitfing variables $ia\to ia+ig^2\theta\frac{B}{16\pi^2 r}$.

6.4 Examples

The formulae we have found for the one-loop determinants in the localization computation is valid for an arbitrary $\mathcal{N}=2$ gauge theory on S^4 admitting a Lagrangian description. Combining the contributions from the north pole, south pole and equator we get for a 't Hooft operator of magnetic weight B

$$Z_{\text{1-loop,pole}}\left(ia-\frac{B}{2},im\right)Z_{\text{1-loop,pole}}\left(ia+\frac{B}{2},im\right)Z_{\text{1-loop,eq}}(ia,im)\,. \tag{6.64}$$

The choice of gauge group G and representation R characterizing the $\mathcal{N}=2$ theory is encoded in the one-loop determinant formulae (6.32) and (6.61) in the choice of the root system $\{\alpha\}$, which characterizes the gauge group, and of the weights $\{w\}$ of R. Here we write explicitly these formulae for two simple $\mathcal{N}=2$ gauge theories with $G=\mathrm{SU}(N)$: $\mathcal{N}=2^*$ and $\mathcal{N}=2$ conformal SQCD. We also consider $\mathcal{N}=4$ super Yang-Mills, which is a special case of $\mathcal{N}=2^*$. From now on we set $\varepsilon=r=1$.

The $\mathcal{N}=2^*$ SU(N) theory. For this theory the hypermultiplet is in the adjoint representation and has mass m. We parametrize

$$a=i\,\mathrm{diag}(a_1,\dots,a_N)\,, \tag{6.65}$$

with $\sum_i a_i=0$. The magnetic weight B of an arbitrary 't Hooft loop is

$$B=i\,\mathrm{diag}(n_1,\dots,n_N)-i1_{N\times N}\frac{1}{N}\sum_i n_i \qquad n_i\in\mathbb{Z}\,. \tag{6.66}$$

Therefore, the pole one-loop contribution (6.32) is given by

$$Z_{\text{1-loop,pole}}(\hat{a},\hat{m})=\left(\prod_{i\neq j}\frac{G(\hat{a}_i-\hat{a}_j)G(2+\hat{a}_i-\hat{a}_j)}{G(1+\hat{a}_i-\hat{a}_j-\hat{m})G(1+\hat{a}_i-\hat{a}_j+\hat{m})}\right)^{1/2}. \tag{6.67}$$

Up to a phase, we have for the equator one-loop contribution (6.61)

$$Z_{\text{1-loop,eq}}(ia,im,B)$$
$$=\left(\prod_{i<j}\frac{\sinh\left[\pi(a_i-a_j)-\pi m-\pi i\frac{n_i-n_j}{2}\right]\sinh\left[\pi(a_i-a_j)+\pi m-\pi i\frac{n_i-n_j}{2}\right]}{\sinh^2\left[\pi(a_i-a_j)-\pi i\frac{n_i-n_j}{2}\right]}\right)^{\frac{|n_i-n_j|}{2}} \tag{6.68}$$

If we further restrict to the special case of $G=\mathrm{SU}(2)$, so that

$$a=i\,\mathrm{diag}(a,-a) \qquad B=i\,\mathrm{diag}(p/2,-p/2)\,,\quad \alpha=i\,\mathrm{diag}(1,-1)\,, \tag{6.69}$$

we have that $\alpha\cdot B\equiv-\mathrm{Tr}(\alpha B)=p$. Here the new a is a real number, and p is a non-negative integer (it is twice the usual SU(2) spin). The pole contribution (6.61) is thus

$$Z_{\text{1-loop,pole}}(\hat{a},\hat{m})=\left(\frac{G(2\hat{a})G(2+2\hat{a})G(-2\hat{a})G(2-2\hat{a})}{G(1+2\hat{a}+\hat{m})G(1+2\hat{a}-\hat{m})G(1-2\hat{a}+\hat{m})G(1-2\hat{a}-\hat{m})}\right)^{1/2}, \tag{6.70}$$

while the equator contribution (6.61) is

$$Z_{\text{1-loop,eq}}(ia, im, p) = \begin{cases} \dfrac{\sinh^{p/2}[\pi(2a+m)]\sinh^{p/2}[\pi(2a-m)]}{\sinh^p(2\pi a)} & \text{for } p \text{ even,} \\ \dfrac{\cosh^{p/2}[\pi(2a+m)]\cosh^{p/2}[\pi(2a-m)]}{\cosh^p(2\pi a)} & \text{for } p \text{ odd.} \end{cases} \tag{6.71}$$

The $\mathcal{N} = 4$ SU(N) theory. We note that for the $\mathcal{N} = 4$ super Yang-Mills theory, obtained by setting $m = 0$ in the $\mathcal{N} = 2^*$ expressions, the equatorial one-loop contribution (6.61) becomes trivial for arbitrary gauge group G. Furthermore, in $\mathcal{N} = 4$ super Yang-Mills, the one-loop pole contribution (6.32) reduces to the Vandermonde determinant corresponding to the gauge group G

$$\prod_{\alpha>0} \alpha\cdot\hat{a}\,. \tag{6.72}$$

For $\mathcal{N} = 4$ super Yang-Mills, the one-loop factors trivialize. This result was already demonstrated in the perturbative computation of the 't Hooft loop path integral in [9] (see also [18, 27]).

Conformal SQCD. This theory has gauge group SU(N) and $N_\text{F} = 2N$ massive hypermultiplets in the fundamental representation of SU(N) with masses m_f with $f = 1,\ldots,2N$. We are interested in the 't Hooft loop specified by the magnetic weight

$$B = i\,\text{diag}(n_1,\ldots,n_N) \qquad \sum_i n_i = 0\,. \tag{6.73}$$

Dirac quantization requires that $n_i \in \mathbb{Z}$. The one-loop pole contribution (6.61) is given by

$$Z_{\text{1-loop,pole}}(\hat{a},\hat{m}_f) = \left(\frac{\prod_{i\neq j} G(\hat{a}_i-\hat{a}_j)G(2+\hat{a}_i-\hat{a}_j)}{\prod_{f=1}^{N_\text{F}}\prod_{i=1}^N G(1+\hat{a}_i-\hat{m}_f)G(1-\hat{a}_i+\hat{m}_f)}\right)^{1/2}. \tag{6.74}$$

Up to a phase, the equatorial one-loop contribution (6.61) is given by

$$Z_{\text{1-loop,eq}}(ia, im_f, B) = \frac{\prod_{f=1}^{2N}\prod_{j=1}^N\left(\sinh\left[\pi a_j - \pi m_f - \pi i\frac{n_j}{2}\right]\right)^{|n_j|/2}}{\prod_{i<j}\left(\sinh\left[\pi(a_i-a_j) - \pi i\frac{n_i-n_j}{2}\right]\right)^{|n_i-n_j|}}\,. \tag{6.75}$$

As in (6.71), each sinh becomes cosh when n_j in the numerator or $n_i - n_j$ in the denominator is odd.

Specializing further to $G = \text{SU}(2)$, we have $N_\text{F} = 4$ fundamental hypermultiplets. With the same parametrization as in the $\mathcal{N} = 2^*$ case, $p = 2n$ needs to be even for Dirac quantization. Up to a phase, the one-loop factor (6.32) is

$$\begin{aligned} &Z_{\text{1-loop,pole}}(\hat{a},\hat{m}_f) \\ &\quad= \left(\frac{G(2\hat{a})G(2+2\hat{a})G(-2\hat{a})G(2-2\hat{a})}{\prod_{f=1}^4 G(1+\hat{a}-\hat{m}_f)G(1-\hat{a}-\hat{m}_f)G(1-\hat{a}+\hat{m}_f)G(1+\hat{a}+\hat{m}_f)}\right)^{1/2} \end{aligned} \tag{6.76}$$

for the north and south poles, and

$$Z_{\text{1-loop,eq}}(ia, im_f, 2n) = \begin{cases} \dfrac{\prod_{f=1}^{4} \sinh^{n/2}[\pi(a+m_f)] \sinh^{n/2}[\pi(a-m_f)]}{\sinh^{2n}(2\pi a)} & \text{for } n \text{ even}, \\ \dfrac{\prod_{f=1}^{4} \cosh^{n/2}[\pi(a+m_f)] \cosh^{n/2}[\pi(a-m_f)]}{\sinh^{2n}(2\pi a)} & \text{for } n \text{ odd} \end{cases} \tag{6.77}$$

for the equator.

We note that for real values of a_i and m_f, one encounters no branch point upon integrating over a_i in (6.62). When the exponent of a sinh is a half-odd integer, the sinh actually becomes a cosh and has no zero.

7 Non-perturbative effects of monopole screening

7.1 Physical picture of monopole screening

In the absence of a 't Hooft loop, Q-invariance requires the curvature F to vanish everywhere on S^4, except at the north and south poles.[59] If we allowed only smooth configurations, we would conclude that only trivial gauge field configurations contribute. As shown in [2], however, localization permits instanton corrections at the north and south poles, which are precisely captured by the Nekrasov partition function.

One can argue in two steps that such corrections are necessary [2]. First, Q-invariance requires that the field strength F vanish only away from the north and south poles. If singular configurations arise as a limit of smooth configurations, there can be contributions to the path integral localized at the poles. Second, the localization Lagrangian $Q \cdot V$ in the neighborhood of the poles is approximately that of the twisted $\mathcal{N}=2$ Lagrangian in the Ω-background in $\mathbb{R}^4$ with the specific values of the equivariant parameters $\varepsilon_1 = \varepsilon_2 = 1/r$. The approximation becomes exact at the poles. Building on the earlier work [28–30], Nekrasov showed that the path integral of such a theory computes the equivariant integral of certain differential forms defined on the instanton moduli space [8]. The integral can be computed by a localization formula as a sum over fixed points. The fixed points in the moduli space of instantons indeed correspond to gauge field configurations that are non-trivial only at the origin of $\mathbb{R}^4$. We studied these instanton corrections in the presence of a 't Hooft loop in section 5 and found that the instanton contributions are given by the Nekrasov partition function at the north and south poles with its arguments shifted due to the insertion of the 't Hooft operator at the equator.

In this section we study another type of non-perturbative corrections due to the screening of magnetic charge associated to a 't Hooft operator. We begin by explaining how such corrections arise in our localization framework.

Monopole screening. As we showed in section 3.3 the only possible field configurations that can contribute to the path integral are those of the form (3.9) in the bulk of S^4. They are only allowed to deviate from (3.9) in an infinitesimal neighborhood of either the poles

[59] This is a special case of the vanishing theorem in section 3.3.

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or the equator. The deviations at the poles were considered in [2] and have been reviewed in section 5; they are the small instanton solutions of the anti-self-dual/self-dual equations that approximate the Q-invariance equations near the poles. In the neighborhood of the loop, we saw in section 3 that the Q-invariance equations are approximately the Bogomolny equations. Therefore we should study the monopole moduli space and look for the analog of small instanton field configurations.

The monopole moduli space $\mathcal{M}_{\text{mono}}$ relevant for us is the space of solutions on $\mathbb{R}^3$, up to gauge transformations, to the Bogomolny equations with a prescribed singularity at the origin corresponding to the insertion of a 't Hooft operator. Since we are only interested in the behavior at the origin, the boundary condition at the infinity of $\mathbb{R}^3$ is irrelevant. It is simplest to consider the solutions that have a vanishing Higgs expectation value at infinity. The vanishing Higgs vev will allow us to use the ADHM construction of instantons to describe the monopole moduli space in section 7.2.[60] We will describe the moduli space explicitly in the case $G = \mathrm{SU}(2)$. For the moment we proceed assuming that G is a general Lie group.

The magnetic charge of the singular monopole configuration created by the 't Hooft operator is specified by a coweight $w \equiv B$. Generally, smooth monopoles that surround the singular monopole screen its magnetic charge so that the asymptotic behavior of the fields at infinity is that of the background configuration (3.9) with the coweight w replaced by a smaller coweight v. This is because the magnetic charge of a smooth monopole is labeled by a coroot of G, and can screen the charge of the singular monopole by that amount. The coweight v is such that its corresponding weight appears in the irreducible ${}^{L}G$-representation specified by the highest weight corresponding to w. In the terminology of [31], such v is said to be associated to w. One can show that the magnetic charge v seen at infinity must have a smaller norm than w by applying a method similar to the one we used to prove completeness of solutions in section 3.3.[61]

Denoting by $\mathcal{M}(w;v)$ the moduli space of solutions whose asymptotic magnetic charge is given by v, we have that the relevant moduli space to consider is

$$\mathcal{M}_{\text{mono}}(w) = \bigcup_{v} \mathcal{M}(w;v)\,, \tag{7.1}$$

where the union is over coweights v such that (if we identify coweights with weights using a metric) v is a weight that appears in the highest weight representation specified by w.

The spaces $\mathcal{M}_{\text{mono}}(w)$ and $\mathcal{M}(w;v)$ are in general singular. To understand the nature of the singularities in these spaces, it is useful to recall the situation with instantons. The Uhlenbeck compactified instanton moduli space $\mathcal{M}_{\text{inst}}$ [32] is singular due to instantons of zero size. For $G = \mathrm{U}(N)$ the moduli space $\mathcal{M}_{\text{inst}}$ can be conveniently resolved by turning on a Fayet-Illiopolous parameter for the real ADHM equation. The resolved space $\overline{\mathcal{M}}_{\text{inst}}$ is known to be isomorphic to the moduli space of non-commutative instantons [33], or the Gieseker resolution [34] in terms of torsion free sheaves.

[60] When the gauge group G is a classical group the moduli space can be constructed using the ADHM construction. In this paper we focus on the case where G is $\mathrm{U}(N)$ or $\mathrm{SU}(N)$.

[61] The difference $w\cdot w - v\cdot v$ can be expressed in terms of the integral of the instanton density, upon lifting the field configuration to instantons in $\mathbb{C}^2$ using Kronheimer's correspondence explained in appendix F.

As explained in [31], a natural resolution $\overline{\mathcal{M}}_{\text{mono}}$ of the moduli space of monopoles with a monopole singularity labeled by w at the origin involves all the coweights w' associated to the coweight w. The coweights w' represent the magnetic charge at the origin reduced by the smooth monopoles that are attracted to the singular monopole there. This effect was called *monopole bubbling* in [31]. This means that a natural resolution $\overline{\mathcal{M}}(w; v)$ of a component in (7.1) contains smaller moduli spaces in its boundary,

$$\partial \overline{\mathcal{M}}(w; v) \supset \bigcup_{w'} \overline{\mathcal{M}}(w'; v) \,, \tag{7.2}$$

with w' being the coweights such that w' is associated to w while v is associated to w'. In the case $G = \mathrm{U}(N)$, one can see this structure explicitly using the ADHM construction. Thus we have the resolution of the whole moduli space

$$\overline{\mathcal{M}}_{\text{mono}}(w) = \bigcup_{v} \overline{\mathcal{M}}(w; v) \,, \tag{7.3}$$

where the union is over the coweights v associated to w.

We only need to study the neighborhood of the monopole bubbling locus, where all the smooth monopoles are almost on top of the singular monopole, because only these would be the approximate solutions to our genuine Q-invariance equation. For example, for gauge group SU(2) and for a 't Hooft operator with $w = (1, -1)$ (spin 1) and $v = (0, 0)$ (spin 0) the bubbling locus is the zero-section $\mathbb{P}^1$ in the resolved A_1 space $\overline{\mathcal{M}}(w; v) = T^*\mathbb{P}^1$ (see section 7.3 for details). Because Q-invariance implies in particular the invariance under the $\mathrm{U}(1)_{J+R}$ generated by Q^2 we are only interested in the $\mathrm{U}(1)_{J+R}$ fixed points. Such fixed points are necessarily in the bubbling locus because when lifted by one dimension so that monopoles become instantons, the fixed points of the $\mathrm{U}(1)_{J+R} \times \mathrm{U}(1)_K$ action sit in the small-instanton locus; see section 7.2. Thus these fixed points represent all the subleading saddle points of the original gauge theory path integral. Upon evaluating the path integral, we need to *sum over the fixed points*.

At each fixed point of $\overline{\mathcal{M}}(w; v)$, we need to compute the fluctuation determinants. The common factor that appears for fixed magnetic weight v was computed in section 6.3, where it was called $Z_{\text{1-loop,eq}}(ia, im_f, v)$. Let us denote by $Z_{\text{mono}}(ia, im_f; w; v)$ the sum of contributions from fluctuations at the fixed points in a single component $\overline{\mathcal{M}}(w; v)$ divided by $Z_{\text{1-loop,eq}}(ia, im_f, v)$. The function $Z_{\text{mono}}(ia, im_f; w; v)$ is the monopole analog of the Nekrasov instanton partition function, whose computation is reviewed in appendix G from a related point of view.

Therefore, given the decomposition of the moduli space in (7.3), the expectation value of the 't Hooft loop operator with magnetic charge $B = w$ takes the form

$$\begin{aligned} \langle T(w) \rangle = \int da \sum_{v} Z_{\text{mono}}(ia, im_f; w, v) Z_{\text{1-loop,eq}}(ia, im_f, v) \\ \times \left| Z_{\text{cl}}\left(ia - \frac{v}{2r}, q\right) Z_{\text{1-loop,pole}}\left(ia - \frac{v}{2r}, im_f\right) Z_{\text{inst}}\left(ia - \frac{v}{2r}, \frac{1}{r} + im_f, \frac{1}{r}, \frac{1}{r}, q\right) \right|^2 . \end{aligned} \tag{7.4}$$

Except $Z_{\text{mono}}(ia, im_f; w; v)$, all the expressions in the integrand were already calculated in the previous sections. Our remaining task is to compute this factor.

7.2 ADHM construction of the monopole moduli space

To perform explicit calculations we need an efficient way to describe the monopole moduli space $\overline{\mathcal{M}}_{\text{mono}}$. The connection between monopoles and instantons [14] reviewed in appendix F, combined with the ADHM construction of instantons [35], provides a useful method to manipulate the monopole moduli space.

Let us briefly review the ADHM construction of instantons in $\mathbb{C}^2$. For simplicity we will take the gauge group G to be $\mathrm{U}(N)$. The basic data in the construction are encoded in the complex

$$0 \to \mathcal{H} \overset{\alpha(z)}{\to} \mathcal{H} \otimes U \oplus E_\infty \overset{\beta(z)}{\to} \mathcal{H} \otimes \wedge^2 U \to 0\,, \tag{7.5}$$

where $\mathcal{H} \simeq \mathbb{C}^k, U \simeq \mathbb{C}^2, E_\infty \simeq \mathbb{C}^N$. On U the $\mathrm{U}(1)_K$ acts as $(z_1, z_2) \mapsto (e^{-i\nu} z_1, e^{i\nu} z_2)$. The z-dependent maps $\alpha(z)$ and $\beta(z)$ are given by

$$\alpha(z) = \begin{pmatrix} z_2 - B_2 \\ -z_1 + B_1 \\ -J \end{pmatrix}\,, \qquad \beta(z) = (z_1 - B_1, z_2 - B_2, -I)\,, \tag{7.6}$$

and their cohomology $E_z = \operatorname{Ker} \beta(z)/\operatorname{Im} \alpha(z)$ is identified with the fiber of the gauge bundle (in the fundamental representation). We are particularly interested in $E_{z=0}$ since it encodes the singularity of the 't Hooft loop. The $\mathrm{U}(1)_K$ action on a vector space V is specified by the character $\chi(V)$, which is a Laurent polynomial of $e^{i\nu} \in \mathrm{U}(1)_K$. The 't Hooft loop with charge $w = i\text{diag}(p_1, \ldots, p_N)$ corresponds to the case

$$\chi(E_0) = \sum_{i=1}^{N} e^{ip_i\nu}. \tag{7.7}$$

The $\mathrm{U}(1)_K$ action on (z_1, z_2) implies that

$$\chi(U) = e^{i\nu} + e^{-i\nu}\,. \tag{7.8}$$

The characters of $\mathcal{H}$ and E_∞ take the form

$$\chi(\mathcal{H}) = \operatorname{Tr} e^{iK\nu}\,, \qquad\qquad \chi(E_\infty) = \operatorname{Tr} e^{iM\nu}\,, \tag{7.9}$$

where K is a diagonal $k \times k$ matrix and $M = \text{diag}(q_1, \ldots, q_N)$ is a diagonal $N \times N$ matrix related to the coweight $v = i(q_1, \ldots, q_N)$ corresponding to the magnetic charge at infinity. Both K and M have integer entries that we choose to be in the descending order. The characters of various spaces are related as

$$\chi(E_0) = \chi(E_\infty) + (\chi(U) - 2)\chi(\mathcal{H})\,, \tag{7.10}$$

For given $w = i(p_1, \ldots, p_N)$, the choice of K and M is not necessarily unique, but we have the non-trivial condition that the whole right hand side of (7.10) has only positive coefficients.

The moduli space $\mathcal{M}(w;v)$ is given as a hyperKähler quotient of the space of $\mathrm{U}(1)_K$-invariant ADHM data (B_1, B_2, I, J). The action of $\mathrm{U}(1)_K$ on (B_1, B_2, I, J) can be read off from the complex (7.5) and the action on (z_1, z_2). In the usual ADHM construction of the instanton moduli space, we take a quotient by a certain action of the $\mathrm{U}(k)$ group. This action of $\mathrm{U}(k)$ on the ADHM data is induced from its natural action on $\mathcal{H} \simeq \mathbb{C}^k$. The choice of K breaks the $\mathrm{U}(k)$ symmetry into the commutant subgroup $\prod_r \mathrm{U}(k_r)$, where $k = \sum_r k_r$ and k_r is the number of entries of the r-th largest integer in the diagonal of K. Thus the moduli space is given as the hyperKähler quotient

$$\mathcal{M}(w,v) = \left\{ (B_1, B_2, I, J) \,\middle|\, \begin{array}{rl} -B_1 + [K, B_1] &= 0 \\ B_2 + [K, B_2] &= 0 \\ KI - IM &= 0 \\ MJ - JK &= 0 \end{array} \right\} /\!/\!/ \prod_r \mathrm{U}(k_r)\,. \tag{7.11}$$

The hyperKähler quotient denoted by "///" can be implemented by imposing the ADHM equations

$$\mu_{\mathbb{C}} \equiv [B_1, B_2] + IJ = 0\,, \tag{7.12}$$

$$\mu_{\mathbb{R}} \equiv [B_1^\dagger, B_1] + [B_2^\dagger, B_2] + II^\dagger - J^\dagger J = 0 \tag{7.13}$$

and then considering the solutions up to the action of $\prod_r \mathrm{U}(k_r)$. Or alternatively, if we are only interested in the complex structure, we can drop the real equation $\mu_{\mathbb{R}} = 0$ and divide by the complexified group $\prod_r \mathrm{U}(k_r)_{\mathbb{C}}$. A resolution $\overline{\mathcal{M}}(w,v)$ of the moduli space can be achieved by setting $\mu_{\mathbb{R}}$ to a non-zero constant matrix instead of requiring it to vanish.

The $\mathrm{U}(1)_{J+R}$-fixed points can be found by demanding that for any $e^{i\varepsilon} \in \mathrm{U}(1)_{J+R}$ there exists $e^{\phi} \in \prod_r \mathrm{U}(k_r)$ such that[62]

$$\begin{aligned} e^{i\varepsilon/2} e^{\phi} B_s e^{-\phi} &= B_s\,, \qquad\qquad s = 1, 2\,, \\ e^{i\varepsilon/2} e^{\phi} I &= I\,, \\ e^{i\varepsilon/2} J e^{-\phi} &= J\,. \end{aligned} \tag{7.14}$$

By construction, the fixed points of $\mathrm{U}(1)_{J+R}$ in $\overline{\mathcal{M}}(w;v)$ automatically correspond to the fixed points of $\mathrm{U}(1)_K \times \mathrm{U}(1)_{J+R}$ in the instanton moduli space. The fixed points in the instanton moduli space were classified in [36], and they were found to sit on the boundary components of the moduli space corresponding to small instantons. This in turn implies that the $\mathrm{U}(1)_{J+R}$-fixed points on the monopole moduli space sit on the bubbling locus. We also know from the experience with instantons that the fixed points of $\mathrm{U}(1)_K \times \mathrm{U}(1)_{J+R} \times G \times G_{\mathrm{F}}$ coincide with the fixed points of $\mathrm{U}(1)_K \times \mathrm{U}(1)_{J+R}$.

At each fixed point, the ratio $Z_{\text{1-loop}}(w;v)/Z_{\text{1-loop}}(v;v)$ can be calculated from the weights of the equivariant group action on the tangent space and the Dirac zeromodes. The ADHM construction provides a concrete procedure to derive such weights.

[62] In our convention the $\mathrm{U}(1)_{J+R}$ acts both on I and J as $e^{i\varepsilon/2}$, implying that it also acts on E_∞ as $e^{i\varepsilon/2}$.

The tangent space can be described by considering the linearization of the ADHM system. Namely, let us consider the complex

$$0 \to \left\{ \delta\phi \in \mathrm{Lie}\left(\prod_r \mathrm{U}(k_r)_{\mathbb{C}}\right)\right\} \xrightarrow{h_1} \left\{ (\delta B_1, \delta B_2, \delta I, \delta J) \left| \begin{array}{rl} -\delta B_1 + K\delta B_1 - (\delta B_1)K &= 0 \\ \delta B_2 + K\delta B_2 - (\delta B_2)K &= 0 \\ K\delta I - (\delta I)M &= 0 \\ M\delta J - (\delta J)K &= 0 \end{array} \right. \right\}$$
$$\xrightarrow{h_2} \{\delta\mu_{\mathbb{C}} \in \mathrm{End}\,\mathcal{H} \otimes \wedge^2 U \,|\, [K, X] = 0\} \to 0\,, \tag{7.15}$$

where the two maps h_1 and h_2 are the linearizations of the $\prod_r \mathrm{U}(k_r)_{\mathbb{C}}$ transformation and the complex ADHM equation $\mu_{\mathbb{C}} = 0$:

$$\begin{aligned} h_1(\delta\phi) &= ((\delta\phi)B_1 - B_1\delta\phi, (\delta\phi)B_2 - B_2\delta\phi, (\delta\phi)I, -J\delta\phi)\,, \\ h_2(\delta B_1, \delta B_2, \delta I, \delta J) &= [\delta B_1, B_2] + [B_1, \delta B_2] + (\delta I)J + I\delta J\,. \end{aligned} \tag{7.16}$$

The tangent space of the moduli space at the point (B_1, B_2, I, J) is given by the cohomology $\ker h_2/\mathrm{im}\, h_1$. The fixed-point equations for the action of $\mathrm{U}(1)_{J+R} \times G \times \prod_r \mathrm{U}(k_r)$

$$\begin{aligned} i\frac{\varepsilon}{2}B_s + [\phi, B_s] &= 0\,, \qquad\qquad s = 1, 2\,, \\ i\frac{\varepsilon}{2}I + \phi I - I\hat{a} &= 0\,, \\ i\frac{\varepsilon}{2}J - J\phi + \hat{a}J &= 0\,, \end{aligned} \tag{7.17}$$

determine ϕ as a function of ε and $\hat{a}$, i.e., they define a homomorphism $\mathrm{U}(1)_{J+R} \times G \to \prod_r \mathrm{U}(k_r)$ at each fixed point. Thus we have an action of $\mathrm{U}(1)_{J+R} \times G$ on the complex (7.15), and the character on the tangent space is given as

$$-\mathrm{Tr}_{V_1}(g) + \mathrm{Tr}_{V_2}(g) - \mathrm{Tr}_{V_3}(g)\,, \tag{7.18}$$

where V_1, V_2, V_3 are the three vector spaces that appear in (7.15) and $g = (e^{i\varepsilon}, e^{\hat{a}}) \in \mathrm{U}(1)_{J+R} \times G$.

There is another method, heuristic but efficient, which can be used to compute the weights on the tangent space based on the character on the space of holomorphic functions. It is best explained in the example we consider next.

7.3 Example: SU(2) $\mathcal{N} = 2^*$ theory

For $G = \mathrm{SU}(2)$, we can label the coweights with integers (corresponding to twice the spin). Also we slightly modify the ADHM construction above and allow w, v and K to have half odd integers. We define the integers $p \geq 0$ and q by[63]

$$w = i(p/2, -p/2)\,, \qquad v = i(q/2, -q/2)\,. \tag{7.19}$$

[63]Denoting reduced magnetic charge by q should not cause confusion with the instanton parameter $e^{2\pi i\tau}$ as the latter does not appear in this subsection.

Since v is associated to w, $p-q$ is non-negative and even. The constraint (7.10) then implies that $k=p-1$ and also that

$$\chi(\mathcal{H}) = \frac{e^{i\frac{p}{2}\nu}+e^{-i\frac{p}{2}\nu}-e^{i\frac{q}{2}\nu}-e^{-i\frac{q}{2}\nu}}{(e^{i\frac{1}{2}\nu}-e^{-i\frac{1}{2}\nu})^2}\,. \tag{7.20}$$

As a character, $\chi(\mathcal{H})$ is a polynomial with positive coefficients for $-p\le q\le p$. For ease of writing we will assume $q\ge 0$, and sum over $q<0$ in the end remembering that the Weyl group acts as $q\to -q, \hat{a}\to -\hat{a}$. We can then write

$$e^{iK\nu} = \chi(\mathcal{H}) = e^{i(\frac{p}{2}-1)\nu}+\ldots+\frac{p-q}{2}e^{i\frac{q}{2}\nu}+\ldots+\frac{p-q}{2}e^{-i\frac{q}{2}\nu}+\ldots+e^{-i(\frac{p}{2}-1)\nu}\,, \tag{7.21}$$

where in the last expression the coefficient of the exponential increases from 1 to $\frac{p-q}{2}$ monotonically, stays constant, and then decreases monotonically to 1.

In order to illustrate the analysis, we start with the simplest non-trivial case that involves monopole screening, namely $w=i(1,-1), v=(0,0)$ corresponding to $p=2, q=0$. We now explicitly work out the details of calculations involving $\overline{\mathcal{M}}(2;0)$. In this case the constraint (7.10) is solved by

$$\chi(E_0) = e^{i\nu}+e^{-i\nu}\,, \qquad \chi(E_\infty)=2\,, \qquad \chi(\mathcal{H})=1\,. \tag{7.22}$$

Let us write $B_1=(b_1), B_2=(b_2), I=(i_1,i_2), J=(j_1,j_2)^T$. The non-trivial $\mathrm{U}(1)_K$ action is given by $b_1\to e^{-i\nu}b_1, b_2\to e^{i\nu}b_2$. Thus a $\mathrm{U}(1)_K$ invariant instanton has to be centered at the origin, i.e., $b_1=b_2=0$. The remaining variables satisfy the ADHM equations

$$i_1j_1+i_2j_2 = \xi_{\mathbb{C}}\,, \tag{7.23}$$

$$|i_1|^2+|i_2|^2-|j_1|^2-|j_2|^2 = \xi_{\mathbb{R}}\,, \tag{7.24}$$

and are subject to the $\mathrm{U}(k)=\mathrm{U}(1)$ equivalence relation

$$(i_1,i_2,j_1,j_2)\sim(e^{i\phi}i_1,e^{i\phi}i_2,e^{-i\phi}j_1,e^{-i\phi}j_2)\,. \tag{7.25}$$

We have introduced the deformation parameters $\xi=(\xi_{\mathbb{C}},\xi_{\mathbb{R}})$.

The moduli space $\mathcal{M}(2;0)$ can be smoothed by turning on ξ. Using a hyperKähler rotation we can set $\xi_{\mathbb{C}}=0$ and $\xi_{\mathbb{R}}>0$. Then (i_1,i_2) cannot vanish. The equation (7.23) can be solved by introducing a charge-(-2) variable μ via $(j_1,j_2)=\mu(i_2,-i_1)/\sqrt{|i_1|^2+|i_2|^2}$. We see that (i_1,i_2,μ) are essentially the variables for $T^*\mathbb{P}^1$ that appear in the gauged linear sigma model description [37].

The fixed points of the $\mathrm{U}(1)_{J+R}\times G$ action are found by demanding that the ADHM data are invariant up to (7.25):

$$(i_1,i_2,j_1,j_2) = (e^{i(\phi+\frac{\varepsilon}{2}-\hat{a})}i_1, e^{i(\phi+\frac{\varepsilon}{2}+\hat{a})}i_2, e^{i(-\phi+\frac{\varepsilon}{2}+\hat{a})}j_1, e^{i(-\phi+\frac{\varepsilon}{2}-\hat{a})}j_2)\,, \tag{7.26}$$

where $e^{i\varepsilon}\in\mathrm{U}(1)_{J+R}$ and $\mathrm{diag}(e^{i\hat{a}},e^{-i\hat{a}})\in G=\mathrm{SU}(2)$. We find two fixed points P_1 and P_2:

$$\begin{aligned} P_1: & i_2\neq 0\,, \qquad i_1=j_1=j_2=0\,, \qquad \phi=-\hat{a}-\frac{\varepsilon}{2}\,,\\ P_2: & i_1\neq 0\,, \qquad i_2=j_1=j_2=0\,, \qquad \phi=\hat{a}-\frac{\varepsilon}{2}\,. \end{aligned} \tag{7.27}$$

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At each fixed point, we have the complex (7.15) with the vector spaces $V_1 = \{\delta\phi\} \simeq \mathbb{C}, V_2 = \{(\delta i_1, \delta i_2, \delta j_1, \delta j_2)\} \simeq \mathbb{C}^4, V_3 \simeq \mathbb{C}$ representing the tangent space. The weights of $\mathrm{U}(1)_{J+R} \times G$ are given by

$$\mathrm{Tr}_{V_1}(g) = 1\,, \tag{7.28}$$

$$\mathrm{Tr}_{V_2}(g) = \begin{cases} e^{-2i\hat{a}} + 1 + e^{2i\hat{a}+i\varepsilon} + e^{i\varepsilon} & \text{at } P_1\,, \\ 1 + e^{2i\hat{a}} + e^{i\varepsilon} + e^{i\varepsilon-2i\hat{a}} & \text{at } P_2\,, \end{cases} \tag{7.29}$$

$$\mathrm{Tr}_{V_3}(g) = e^{i\varepsilon}\,. \tag{7.30}$$

Thus at P_1 the character on the tangent space is $e^{-2i\hat{a}} + e^{2i\hat{a}+i\varepsilon}$, corresponding to the weights $(e^{-2i\hat{a}}, e^{2i\hat{a}+i\varepsilon})$. At P_2 the weights are $(e^{2i\hat{a}}, e^{i\varepsilon-2i\hat{a}})$.

At P_1, we get an extra contribution to the index $\mathrm{ind}(D_{\mathrm{Bogo}})$ in (6.49):

$$\mathrm{ind}(D_{\mathrm{Bogo}}) \to \mathrm{ind}(D_{\mathrm{Bogo}}) + \mathrm{ind}(D_{\mathrm{Bogo}})_{\mathrm{mono}}\,, \tag{7.31}$$

$$\text{where} \qquad \mathrm{ind}(D_{\mathrm{Bogo}})_{\mathrm{mono}} = -\frac{1+e^{-i\varepsilon}}{2}(e^{-2i\hat{a}} + e^{2i\hat{a}+i\varepsilon})\,. \tag{7.32}$$

Here the factor $(1+e^{-i\varepsilon})/2$ has the same origin as in (6.44).

We also get the extra contribution for the adjoint hypermultiplet. To understand this, note the relations among the indices of the Dirac, self-dual, and Dolbeault complexes in four dimensions

$$\mathrm{ind}(D_{\mathrm{SD}}) = \frac{1+e^{-i\varepsilon_1-i\varepsilon_2}}{2}\mathrm{ind}(\overline{D})\,, \tag{7.33}$$

$$\mathrm{ind}(D_{\mathrm{Dirac}}) = e^{-\frac{i}{2}(\varepsilon_1+\varepsilon_2)}\frac{e^{i\hat{m}}+e^{-i\hat{m}}}{2}\mathrm{ind}(\overline{D})\,. \tag{7.34}$$

Since the indices for the Bogomolny and Dirac-Higgs complexes are obtained from $\mathrm{ind}(D_{\mathrm{SD}})$ and $\mathrm{ind}(D_{\mathrm{Dirac}})$ by averaging over $\mathrm{U}(1)_K$ respectively, they are related as

$$\left(e^{\frac{i}{2}\varepsilon} + e^{-\frac{i}{2}\varepsilon}\right)\mathrm{ind}(D_{\mathrm{DH}}) = \left(e^{i\hat{m}} + e^{-i\hat{m}}\right)\mathrm{ind}(D_{\mathrm{Bogo}})\,. \tag{7.35}$$

The index that leads to the fluctuation determinants is

$$\begin{aligned}\sum_{n\in\mathbb{Z}} e^{in\varepsilon}\mathrm{ind}(D_{\mathrm{Bogo}}) &- \sum_{n\in\mathbb{Z}} e^{i(n+1/2)\varepsilon}\mathrm{ind}(D_{\mathrm{DH}}) \\ &= \sum_{n\in\mathbb{Z}} e^{in\varepsilon}\left(1 - \frac{e^{i\hat{m}}+e^{-i\hat{m}}}{2}\right)\mathrm{ind}(D_{\mathrm{Bogo}})\end{aligned} \tag{7.36}$$

According to the rule $\sum c_j e^{w_j(\hat{a},\hat{m},\varepsilon)} \to \prod w_j(\hat{a},\hat{m},\varepsilon)^{c_j}$, this leads to the one-loop determinant

$$\begin{aligned}\prod_{n\in\mathbb{Z}} &\frac{[(n\varepsilon+\hat{m}+2\hat{a})(n\varepsilon+\hat{m}-2\hat{a})(n\varepsilon-\hat{m}+2\hat{a})(n\varepsilon-\hat{m}-2\hat{a})]^{1/2}}{(n\varepsilon+2\hat{a})(n\varepsilon-2\hat{a})} \\ &= \frac{\sin(2\pi r\hat{a}+\pi r\hat{m})\sin(2\pi r\hat{a}-\pi r\hat{m})}{\sin^2(2\pi r\hat{a})}\end{aligned} \tag{7.37}$$

where we have used that $\varepsilon = 1/r$. The second fixed point P_2 contributes the same amount. Thus

$$Z_{\text{mono}}(\hat{a}, \hat{m}; 2, 0) = 2\frac{\sin[\pi r(2\hat{a}+\hat{m})]\sin[\pi r(2\hat{a}-\hat{m})]}{\sin^2[2\pi r\hat{a}]} \,. \tag{7.38}$$

There is another method based on contour integrals as applied in [38, 39] to instantons. Let us temporarily ignore the matter contribution. In this approach,[64] we compute the character of the space holomorphic functions on the moduli space $\mathcal{M} = \mathcal{M}(p;q)$, identify it with the index of the Dolbeault operator on the resolved moduli space and read off the weights. The holomorphic functions depend on B_1, B_2, I, J, and we need to take into account the complex ADHM equation (7.12) and the quotient by the group $\prod_r \mathrm{U}(k_r)$. Schematically, the character is computed by averaging over $h \in \prod_r \mathrm{U}(k_r)$,

$$\mathrm{ch}(g) = \frac{1}{\mathrm{Vol}}\int dh \frac{\det_{\text{equations}}(1-e^{\text{weight}}h)}{\det_{\text{variables}}(1-e^{\text{weight}}h)} \,, \tag{7.39}$$

where the determinants are taken in the spaces of equations and variables and Vol is the volume of $\prod_r \mathrm{U}(k_r)$. For $\mathcal{M}(2;0)$,

$$\mathrm{ch}(g) = \int_0^{2\pi}\frac{d\phi}{2\pi}\frac{1-e^{i\varepsilon}}{(1-e^{i\frac{1}{2}\varepsilon-i\hat{a}+i\phi})(1-e^{i\frac{1}{2}\varepsilon+i\hat{a}+i\phi})(1-e^{i\frac{1}{2}\varepsilon+i\hat{a}-i\phi})(1-e^{i\frac{1}{2}\varepsilon-i\hat{a}-i\phi})} \,. \tag{7.40}$$

To evaluate the integral by residues we need to specify the precise contour. Following [38] we assume that $\mathrm{Im}\,\varepsilon > 0$ and treat ϕ and $\hat{a}$ as real variables. we find two poles in $z = e^{i\phi}$, and the character is given as

$$\mathrm{ch}(g) = \frac{1}{(1-e^{-2i\hat{a}})(1-e^{i\varepsilon+2i\hat{a}})} + \frac{1}{(1-e^{2i\hat{a}})(1-e^{i\varepsilon-2i\hat{a}})} \,. \tag{7.41}$$

Given the weights we found above, (7.41) is consistent with the identification of the character with the index

$$\begin{aligned} \mathrm{ind}(\overline{\partial}) &\equiv \sum_{k=0}^{\dim\mathcal{M}} (-1)^k \,\mathrm{Tr}_{H^{0,k}_{\overline{\partial}}(\mathcal{M})}(g) \\ &= \sum_{\substack{P:\text{ fixed}\\ \text{points}}} \frac{1}{\prod_j (1-e^{w_j(P)})} \,, \end{aligned} \tag{7.42}$$

where j runs over the holomorphic tangent directions.

After this practice, let us now include the matter contribution. It is convenient to consider the so-called χ_y-genus:[65]

$$\begin{aligned} \chi_y(\mathcal{M}) &= \sum_{k,l\geq 0} y^k(-1)^l \,\mathrm{Tr}_{H^{k,l}_{\overline{\partial}}(\mathcal{M})}(g) \\ &= \sum_{\substack{P:\text{ fixed}\\ \text{points}}} \prod_j \frac{1-ye^{w_j(P)}}{1-e^{w_j(P)}} \,, \end{aligned} \tag{7.43}$$

[64] It requires no explicit resolution of singularities, and therefore can be applied to any group that admits an ADHM construction.

[65] The χ_y-genus also appeared in the instanton calculus for $\mathcal{N} = 2^*$ theory [40].

with $y = e^{i\hat{m}}$. Each weight $w_j(P)$ will be of the form

$$w_j = in_j\hat{a} + \frac{i}{2}l_j\varepsilon\,, \tag{7.44}$$

where n_j and l_j are integers. (7.33) implies that the contribution to $\mathrm{ind}(D_{\mathrm{Bogo}})$ at the fixed point P is given by $-\frac{1+e^{-i\varepsilon}}{2}\sum_j e^{w_j}$. Then the contribution to (7.36) is given by

$$\sum_{n\in\mathbb{Z}} e^{in\varepsilon}\left(\frac{e^{i\hat{m}}+e^{-i\hat{m}}}{2}-1\right)\sum_j e^{in_j\hat{a}+\frac{i}{2}l_j\varepsilon}\,. \tag{7.45}$$

Summing over the fixed points P, the contribution to the path integral is

$$\begin{aligned} Z_{\mathrm{mono}}(\hat{a},\hat{m};w,v) \;&=\; \sum_{\substack{P:\text{ fixed}\\ \text{points}}}\prod_j\prod_{n\in\mathbb{Z}}\frac{(n\varepsilon+\hat{m}+n_j\hat{a}+l_j\varepsilon/2)^{1/2}(n\varepsilon-\hat{m}+n_j\hat{a}+l_j\varepsilon/2)^{1/2}}{(n\varepsilon+n_j\hat{a}+l_j\varepsilon/2)} \\ &=\; \sum_{\substack{P:\text{ fixed}\\ \text{points}}}\prod_j\frac{\sin^{1/2}[\pi(n_jr\hat{a}+r\hat{m}+l_j/2)]\sin^{1/2}[\pi(n_jr\hat{a}-r\hat{m}+l_j/2)]}{\sin[\pi(n_jr\hat{a}+l_j/2)]}\,, \end{aligned} \tag{7.46}$$

where we recall that $\varepsilon = 1/r$. On the other hand, the χ_y genus in (7.43) can be written as

$$\chi_y(\mathcal{M}) = e^{\frac{i}{2}(\dim_{\mathbb{C}}\mathcal{M})\hat{m}}\sum_{\substack{P:\text{ fixed}\\ \text{points}}}\prod_j\frac{\sin^{1/2}[\frac{1}{2}(n_j\hat{a}+\hat{m}+l_j\varepsilon/2)]\sin^{1/2}[\frac{1}{2}(n_j\hat{a}-\hat{m}+l_j\varepsilon/2)]}{\sin[\frac{1}{2}(n_j\hat{a}+l_j\varepsilon/2)]}\,. \tag{7.47}$$

Thus we find that

$$Z_{\mathrm{mono}}(\hat{a};w,v) \;=\; e^{-\frac{i}{2}(\dim_{\mathbb{C}}\mathcal{M})\hat{m}}\chi_y(\mathcal{M}(w;v))\Big|_{(\varepsilon,\hat{m},\hat{a})\to(2\pi,2\pi r\hat{m},2\pi r\hat{a})}\,. \tag{7.48}$$

This is why the χ_y-genus is useful for us.

We now calculate the χ_y genus using the ADHM construction of the monopole moduli space. Locally at the origin of the space of ADHM data, the space of holomorphic sections is the tensor product of the space of holomorphic functions and the space of Dirac zeromodes. (7.43) corresponds to $\mathrm{Tr}[\det(1-yg)]$, where the trace is over the holomorphic functions and the determinant is over the zeromodes. Since the space of zeromodes is given by the cohomology of the complex (7.15), the determinant over the zeromodes is given by $\det_{V_2}(\cdot)/\det_{V_1}(\cdot)\det_{V_3}(\cdot)$. Thus

$$\begin{aligned} &\chi_y(\mathcal{M}) \\ &\quad= \frac{1}{\mathrm{Vol}}\int dh\frac{\det_{\mathrm{equations}}(1-e^{\mathrm{weight}}h)}{\det_{\mathrm{variables}}(1-e^{\mathrm{weight}}h)}\frac{\det_{V_2}(1-ye^{\mathrm{weight}}h)}{\det_{V_1}(1-ye^{\mathrm{weight}}h)\det_{V_3}(1-ye^{\mathrm{weight}}h)}\,. \end{aligned} \tag{7.49}$$

In the case of $\mathcal{M}(2;0)$,

$$
\begin{aligned}
&\chi_y(\mathcal{M}(2;0)) \\
&\quad = \int_0^{2\pi} \frac{d\phi}{2\pi} \frac{(1-e^{i\varepsilon})(1-e^{i\hat{m}+i\frac{1}{2}\varepsilon-i\hat{a}+i\phi})(1-e^{i\hat{m}+i\frac{1}{2}\varepsilon+i\hat{a}+i\phi})}{(1-e^{i\frac{1}{2}\varepsilon-i\hat{a}+i\phi})(1-e^{i\frac{1}{2}\varepsilon+i\hat{a}+i\phi})(1-e^{i\frac{1}{2}\varepsilon+i\hat{a}-i\phi})(1-e^{i\frac{1}{2}\varepsilon-i\hat{a}-i\phi})} \\
&\qquad \times \frac{(1-e^{i\hat{m}+i\frac{1}{2}\varepsilon+i\hat{a}-i\phi})(1-e^{i\hat{m}+i\frac{1}{2}\varepsilon-i\hat{a}-i\phi})}{(1-e^{i\hat{m}})(1-e^{i\hat{m}+i\varepsilon})} \\
&\quad = \frac{(1-e^{i\hat{m}-2i\hat{a}})(1-e^{i\hat{m}+i\varepsilon+2i\hat{a}})}{(1-e^{-2i\hat{a}})(1-e^{i\varepsilon+2i\hat{a}})} + \frac{(1-e^{i\hat{m}+2i\hat{a}})(1-e^{i\hat{m}+i\varepsilon-2i\hat{a}})}{(1-e^{2i\hat{a}})(1-e^{i\varepsilon-2i\hat{a}})} \,.
\end{aligned} \tag{7.50}
$$

We note that (7.38) is indeed obtained from (7.50) using the relation (7.48).

The magnetic charge p can be screened by monopoles and get reduced to q, also an even integer. We set $l := p - q$. The moduli space $\mathcal{M}(p;q)$ can be described using the ADHM construction as follows. The action of $\mathrm{U}(1)_K$ is specified by the matrix

$$
K = \begin{pmatrix}
\frac{p-2}{2}\mathbb{I}_{1\times 1} & & & & & & & & \\
& \frac{p-4}{2}\mathbb{I}_{2\times 2} & & & & & & & \\
& & \ddots & & & & & & \\
& & & \frac{p-l}{2}\mathbb{I}_{\frac{l}{2}\times\frac{l}{2}} & & & & & \\
& & & & \ddots & & & & \\
& & & & & -\frac{p-l}{2}\mathbb{I}_{\frac{l}{2}\times\frac{l}{2}} & & & \\
& & & & & & \ddots & & \\
& & & & & & & -\frac{p-4}{2}\mathbb{I}_{2\times 2} & \\
& & & & & & & & -\frac{p-2}{2}\mathbb{I}_{1\times 1}
\end{pmatrix}, \tag{7.51}
$$

and the action of G by

$$
M = \begin{pmatrix} \frac{q}{2} & 0 \\ 0 & -\frac{q}{2} \end{pmatrix} . \tag{7.52}
$$

The conditions of $\mathrm{U}(1)_K$-invariance

$$
\begin{aligned}
B_1 + [K, B_1] = 0\,, &\qquad -B_2 + [K, B_2] = 0\,, \\
KI - IM = 0\,, &\qquad MJ - JK = 0
\end{aligned} \tag{7.53}
$$

require that the ADHM matrices take the following form:

$$B_1 = \begin{pmatrix} 0 & & & & \\ B_{21} & 0 & & & \\ 0 & B_{32} & 0 & & \\ & \ddots & \ddots & \ddots & \\ & & 0 & B_{p-1,p-2} & 0 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 0 & \tilde{B}_{12} & 0 & & \\ & 0 & \tilde{B}_{23} & \ddots & \\ & & 0 & \ddots & 0 \\ & & & \ddots & \tilde{B}_{p-2,p-1} \\ & & & & 0 \end{pmatrix}, \tag{7.54}$$

$$I = \begin{pmatrix} 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \\ I_{l/2,1} & 0 \\ \vdots & \vdots \\ 0 & I_{p-l/2,2} \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{pmatrix}, \qquad J = \begin{pmatrix} 0 & \cdots & J_{1,l/2} & \cdots & 0 & \cdots & 0 \\ 0 & \cdots & 0 & \cdots & J_{2,p-l/2} & \cdots & 0 \end{pmatrix}. \tag{7.55}$$

We consider the space of solutions to the complex ADHM equation

$$[B_1, B_2] + IJ = 0 \tag{7.56}$$

and then take the quotient by the complexification of the group $\prod_{r=1}^{p-1} \mathrm{U}(k_r)$ with

$$(k_1, k_2, \ldots, k_{p-1}) := (1, 2, \ldots, l/2, \ldots, l/2, \ldots, 1)\,. \tag{7.57}$$

Counting shows that the resulting space $\mathcal{M}(p;q)$ has complex dimension l. This is the singular moduli space of monopoles on $\mathbb{R}^3$ in the presence of a singular monopole of charge p at the origin with the boundary condition that the fields look like the charge $p-l$ monopole at infinity. The group $\mathrm{U}(1)_{J+R}$ generated by Q^2 and the maximal torus $\mathrm{U}(1)_\infty \subset \mathrm{SU}(2)$ of the group of global gauge transformations act on $\mathcal{M}(p;q)$ according to

$$B_s \to e^{i\frac{1}{2}\varepsilon} B_s\,, \qquad s = 1, 2\,, \tag{7.58}$$

$$I \to e^{i\frac{1}{2}\varepsilon} I \begin{pmatrix} e^{-i\hat{a}} & \\ & e^{i\hat{a}} \end{pmatrix}, \tag{7.59}$$

$$J \to e^{i\frac{1}{2}\varepsilon} \begin{pmatrix} e^{i\hat{a}} & \\ & e^{-i\hat{a}} \end{pmatrix} J\,. \tag{7.60}$$

We thus obtain a contour integral expression for the χ_y-genus of $\mathcal{M}(p;q)$:

$$\begin{aligned}\chi_y(\mathcal{M}) &= \frac{1}{\prod_{r=1}^{p-1} k_r!} \oint \prod_{r=1}^{p-1} \prod_{i=1}^{k_r} \frac{dz_{r,i}}{2\pi i z_{r,i}} \prod_{i<j} \frac{-z_{r,i}}{z_{r,j}} (1 - z_{r,j}/z_{r,i})^2 \\ &\times \prod_{r=1}^{p-1} \prod_{i,j} \frac{(1 - e^{i\varepsilon} z_{r,i}/z_{r,j})}{(1 - e^{i\hat{m}} z_{r,i}/z_{r,j})(1 - e^{i\hat{m}} e^{i\varepsilon} z_{r,i}/z_{r,j})} \\ &\times \frac{\prod_{i=1}^{k_{\frac{l-2}{2}}} (1 - e^{i\hat{m}} e^{i\hat{a}+i\frac{1}{2}\varepsilon}/z_{\frac{l-2}{2},i}) \left(1 - e^{im} e^{i\frac{1}{2}\varepsilon - i\hat{a}} z_{\frac{l-2}{2},i}\right)}{\prod_{i=1}^{k_{\frac{l-2}{2}}} (1 - e^{i\hat{a}+i\frac{1}{2}\varepsilon}/z_{\frac{l-2}{2},i}) \left(1 - e^{i\frac{1}{2}\varepsilon - i\hat{a}} z_{\frac{l-2}{2},i}\right)} \\ &\times \frac{\prod_{i=1}^{k_{p-\frac{l}{2}-1}} \left(1 - e^{im} e^{i\hat{a}+i\frac{1}{2}\varepsilon} z_{p-1-l/2,i}\right)\left(1 - e^{i\hat{m}} e^{i\frac{1}{2}\varepsilon - i\hat{a}}/z_{p-1-l/2,i}\right)}{\prod_{i=1}^{k_{p-\frac{l}{2}-1}} \left(1 - e^{i\hat{a}+i\frac{1}{2}\varepsilon} z_{p-1-l/2,i}\right)\left(1 - e^{i\frac{1}{2}\varepsilon - i\hat{a}}/z_{p-1-l/2,i}\right)} \\ &\times \frac{\prod_{r=1}^{p-2}\prod_{i=1}^{k_r}\prod_{j=1}^{k_{r-1}} \left(1 - e^{i\hat{m}} e^{i\frac{1}{2}\varepsilon} z_{r,i}/z_{r-1,j}\right) \prod_{r=1}^{p-2}\prod_{i=1}^{k_r}\prod_{j=1}^{k_{r+1}} \left(1 - e^{i\hat{m}} e^{i\frac{1}{2}\varepsilon} z_{r,i}/z_{r+1,j}\right)}{\prod_{r=1}^{p-2}\prod_{i=1}^{k_r}\prod_{j=1}^{k_{r-1}} \left(1 - e^{i\frac{1}{2}\varepsilon} z_{r,i}/z_{r-1,j}\right) \prod_{r=1}^{p-2}\prod_{i=1}^{k_r}\prod_{j=1}^{k_{r+1}} \left(1 - e^{i\frac{1}{2}\varepsilon} z_{r,i}/z_{r+1,j}\right)} .\end{aligned} \tag{7.61}$$

The first line on the right hand side represents the Haar measure on $\prod_r \mathrm{U}(k_r)$, which would be clearer if the integral is written in terms of $\phi_{r,i}$ such that $z_{r,i} = e^{i\phi_{r,i}}$. We again choose to use the prescription where we integrate over each $z_{r,i}$ along the unit circle $|z_{r,i}| = 1$, assuming that $\hat{a} \in \mathbb{R}$ and $\operatorname{Im} \varepsilon > 0$. The integral can be evaluated by residues, and the computation can be automated as a Mathematica code. Applying the rule (7.48), we find experimentally[66] that

$$Z_{\text{mono}}(\hat{a}, \hat{m}; p, q) = \frac{p!}{(\frac{p-q}{2})!(\frac{p+q}{2})!} \times \begin{cases} \dfrac{\cos^{\frac{p-q}{2}}[\pi r(2\hat{a}+\hat{m})] \cos^{\frac{p-q}{2}}[\pi r(2\hat{a}-\hat{m})]}{\cos^{p-q}[2\pi r\hat{a}]} & \text{for } p \text{ odd,} \\ \dfrac{\sin^{\frac{p-q}{2}}[\pi r(2\hat{a}+\hat{m})] \sin^{\frac{p-q}{2}}[\pi r(2\hat{a}-\hat{m})]}{\sin^{p-q}[2\pi r\hat{a}]} & \text{for } p \text{ even.} \end{cases} \tag{7.62}$$

Combined with (6.71), the dependence of $Z_{\text{mono}}(ia, im; p, q) Z_{\text{1-loop,eq}}(ia, im; q)$ on q is in fact only in the binomial coefficient.

[66] We have checked this for $(p, q) = (2, 0), (3, 1), (4, 2), (5, 3), (4, 0), (6, 2)$, and $(6, 0)$.

We now put everything together. Including the terms with $q \le 0$, we get

$$\langle T_p \rangle = \sum_{q=p,p-2,\dots,-p} \frac{p!}{(\frac{p-q}{2})!(\frac{p+q}{2})!} \int da \left| Z_{\text{1-loop,pole}} \left(ia - \frac{q}{4r} \right) Z_{\text{cl}} \left(ia - \frac{q}{4r} \right) \right.$$
$$\left. \times Z_{\text{inst}} \left(ia - \frac{q}{4r} \right) \right|^2 \times \begin{cases} \dfrac{\cosh^{\frac{p}{2}}[\pi r(2a+m)] \cosh^{\frac{p}{2}}[\pi r(2a-m)]}{\cosh^p[2\pi r a]} & \text{for } p \text{ odd,} \\ \dfrac{\sinh^{\frac{p}{2}}[\pi r(2a+m)] \sinh^{\frac{p}{2}}[\pi r(2a-m)]}{\sinh^p[2\pi r a]} & \text{for } p \text{ even.} \end{cases} \tag{7.63}$$

This is the complete gauge theory result for 't Hooft loops in SU(2) $\mathcal{N} = 2^*$ theory. We checked numerically for low values of p and generic values of τ and m that (7.63) agrees with the expectation value of the S-dual Wilson loops.

This analysis, with the philosophy described, can be extended to other gauge theories.

8 Gauge theory computation vs Toda CFT

In this section we compare the results of our gauge theory analysis for the expectation value of 't Hooft loop operators in $\mathcal{N} = 2$ gauge theories on S^4 with formulae in [10–12], which were obtained from computations in two dimensional Liouville/Toda CFT. As we shall see, for the theories for which we explicitly carry out the comparison, we find beautiful agreement.

In [10, 11] a dictionary was put forward relating the exact expectation value of gauge theory loop operators in $\mathcal{N} = 2$ gauge theories on S^4 and Liouville/Toda correlation functions in the presence of Liouville/Toda loop operators (topological defects). This enriches the AGT correspondence [20], which identifies the gauge theory partition function with a correlation function in Liouville/Toda (see also [41]), to encompass more general observables. The identification in [10, 11] has yielded explicit predictions for the exact expectation value of 't Hooft loop operators in $\mathcal{N} = 2$ gauge theories on S^4.

We compare the Liouville/Toda results for 't Hooft operators in $\mathcal{N} = 2^*$ with the corresponding gauge theory computations for both the one-loop determinants as well as for the non-perturbative contributions due to monopole screening.

8.1 't Hooft loop determinants from Toda CFT

We now explicitly compare the results obtained for 't Hooft operators in the $N = 2^*$ theory — corresponding to an $\mathcal{N} = 2$ SU(N) vectormultiplet with a massive hypermultiplet in the adjoint representation — with loop operator computations in Toda CFT on the once-punctured torus. For a 't Hooft loop labeled by a magnetic weight $B = h_1$ — corresponding to the fundamental representation of SU(N) — the Toda CFT calculation yields [12]

$$\int da\, C(ia, im) \overline{Z_{\text{cl}}(ia, q) Z_{\text{inst}}(ia, 1{+}im, q)} \sum_{k=1}^{N} T_k(ia, im) Z_{\text{cl}}(ia{-}h_k, q) Z_{\text{inst}}(ia{-}h_k, 1{+}im, q)\,, \tag{8.1}$$

where

$$T_k(ia, im) = \frac{1}{N} \prod_{1 \le j \le N}^{j \ne k} \frac{\Gamma(i(a_j - a_k))\Gamma(2 + i(a_j - a_k))}{\Gamma(1 + i(a_j - a_k) - im)\Gamma(1 + i(a_j - a_k) + im)}\,,$$

m is the mass of the adjoint hypermultiplet and h_i are the N weights of the fundamental representation of $\mathrm{SU}(N)$.[67] The result in (8.1) is expressed as much as possible in terms of gauge theory quantities introduced in previous sections. The factor $Z_{\rm cl}(ia,q)$ is the classical contribution to Nekrasov's equivariant instanton partition function (4.10)

$$Z_{\rm cl}(ia,q) = \exp\left[\pi i\tau a\cdot a\right] = \exp\left[\pi i\tau\sum_{l=1}^N a_l^2\right]\,, \tag{8.2}$$

while $Z_{\rm inst}(ia,1+im,q)$ is the instanton contribution (5.2).[68] Finally $C(ia,im)$ is the Toda CFT three-point function[69] relevant for the once-punctured torus description of $\mathcal{N}=2^*$ (à la [42])

$$C(ia,im) = \frac{\prod_{\alpha>0}\Upsilon_{b=1}(-i\alpha\cdot a)\Upsilon_{b=1}(i\alpha\cdot a)}{\prod_{i,j=1}^N\Upsilon_{b=1}(1+i(h_i-h_j)\cdot a+im)}\,, \tag{8.3}$$

with α the roots of the $\mathrm{SU}(N)$ Lie algebra. Since $\Upsilon_{b=1}(x)=G(x)G(2-x)/2\pi$, with $G(x)$ being the Barnes G-function (6.19) and because $\alpha=h_i-h_j$ for $j>i$ if $\alpha>0$ we obtain[70]

$$C(ia,im) = \frac{\prod_\alpha G(i\alpha\cdot a)G(2+i\alpha\cdot a)}{\prod_{\alpha>0}\prod_{\pm,\pm}G(1\pm i\alpha\cdot a\pm im)}\,. \tag{8.4}$$

We note that $C(ia,im)$ is precisely given by the square of the one-loop factor in Nekrasov's partition function of $\mathcal{N}=2^*$ in $\mathbb{R}^4$ (see (6.32) and (6.67))

$$C(ia,im) = |Z_{\text{1-loop,pole}}(ia,im)|^2\,, \tag{8.5}$$

with

$$\begin{aligned} Z_{\text{1-loop,pole}}(ia,im) &= Z_{\text{1-loop,pole}}(-ia,-im) = \overline{Z_{\text{1-loop,pole}}(ia,im)} \\ &= \left[\frac{\prod_\alpha G(i\alpha\cdot a)G(2+i\alpha\cdot a)}{\prod_{\alpha>0}\prod_{\pm,\pm}G(1\pm i\alpha\cdot a\pm im)}\right]^{1/2}\,. \end{aligned} \tag{8.6}$$

Thus we can write the Toda loop correlator as

$$\begin{aligned} &\int da\,|Z_{\text{1-loop,pole}}(ia,im)|^2\,\overline{Z_{\rm cl}(ia,q)Z_{\rm inst}(ia,1+im,q)} \\ &\qquad\times\sum_{k=1}^N T_k(ia,m)Z_{\rm cl}(ia-h_k,q)Z_{\rm inst}(ia-h_k,1+im,q)\,. \end{aligned} \tag{8.7}$$

We note that the result is given by the sum of N terms, associated to the N weights of the fundamental representation of $\mathrm{SU}(N)$. Each of the N weights yields an identical contribution, and therefore we can focus on the contribution of the highest weight term, labeled by

[67] Explicitly, $h_i=(\delta_{ij}-1/N)_{j=1}^N$.

[68] To lighten notation we have set $r=1$, have omitted the $\varepsilon_1,\varepsilon_2$ dependence of the instanton partition function $Z_{\rm inst}(\hat a,\hat m,\varepsilon_1,\varepsilon_2,q)\to Z_{\rm inst}(\hat a,\hat m,q)$ and also used that $\hat m=1+im$ (5.6).

[69] This is the three-point function of two non-degenerate and one semi-degenerate primary operators in Toda CFT when the background charge $b=1$.

[70] In this section, in order to avoid cluttering formulas, we drop inessential overall numerical factors.

h_1. It is important to remark at this point that genuine new contributions appear for loop operators labeled by a representation with highest weight B for which not all weights are in the Weyl orbit of B (non-minuscule representations). These contributions correspond precisely to the non-perturbative contributions due to monopole screening encountered in our gauge theory analysis! Section 8.2 demonstrates for 't Hooft loops with higher magnetic weight B that Liouville theory precisely reproduces the non-perturbative screening contributions discussed in section 7.2.

Focusing on the highest weight vector contribution, we trivially rewrite the answer as

$$\begin{aligned}\int da\, & Z_{\text{1-loop,pole}}(-ia,-im)Z_{\text{cl}}(-ia,\bar{q})Z_{\text{inst}}(-ia,1-im,\bar{q}) \times T_1(ia,im) \\ & \times Z_{\text{1-loop,pole}}(ia,im)Z_{\text{cl}}(ia-h_1,q)Z_{\text{inst}}(ia-h_1,1+im,q)\,.\end{aligned} \tag{8.8}$$

Without encountering any residues, we now shift the contour of integration $ia \to ia + h_1/2$ to express the answer in a more symmetric form

$$\begin{aligned}\int da\, & |Z_{\text{cl}}(ia-h_1/2,q)Z_{\text{inst}}(ia-h_1/2,1+im,q)|^2 \\ & Z_{\text{1-loop,pole}}(-ia-h_1/2,-im)Z_{\text{1-loop,pole}}(ia+h_1/2,im)\; T_1(ia+h_1/2,im)\,.\end{aligned} \tag{8.9}$$

Our next goal is to rewrite the second line in (8.9) as a complete square of a function with the same shifted argument $ia - h_1/2$ as in the first line times a remainder, which we denote by $E(ia,im)$

$$\int da\, |Z_{\text{cl}}(ia-h_1/2,q)Z_{\text{1-loop,pole}}(ia-h_1/2,im)Z_{\text{inst}}(ia-h_1/2,1+im,q)|^2 \times E(ia,im)\,. \tag{8.10}$$

To anticipate where this path will leads us when comparing with our gauge theory analysis, the complete square contributions reproduce the classical, one-loop and instanton contributions that arise from the north and south poles of S^4, while the remainder captures the contribution from the equator!

In order to determine $E(ia,im)$ in (8.10) we need to calculate

$$E(ia,im) = \frac{Z_{\text{1-loop,pole}}(ia+h_1/2,im)}{Z_{\text{1-loop,pole}}(ia-h_1/2,im)}\, T_1(ia+h_1/2,im)\,. \tag{8.11}$$

The ratio of one-loop factors can be determined by recalling that $a_j = a \cdot h_j$, so that the shifts $ia \pm h_1/2$ in the arguments in (8.11) are given by (since $h_i \cdot h_j = \delta_{ij} - 1/N$)

$$\begin{aligned} & ia_j \to ia_j \mp 1/N \qquad j \neq 1 \\ & ia_1 \to ia_1 \pm 1/2 \mp 1/N\,. \end{aligned} \tag{8.12}$$

Therefore, only $a_{1j} \equiv a_1 - a_j$ shifts, by $ia_{1j} \to ia_{1j} \pm 1/2$. Since $\alpha = h_i - h_j$ for $j > i$ if $\alpha > 0$, we decompose the product over positive roots appearing in (8.6)

$$\prod_{\alpha>0} \cdot = \prod_{j=2}^{N} \cdot \prod_{2\leq i<j\leq N}\,, \tag{8.13}$$

comprising the splitting of positive roots into $h_1 - h_j$ and the rest. Therefore only the factors $\prod_{j=2}^N \cdot$ shift, the rest cancel between the numerator and denominator in (8.11). We find

$$\frac{Z_{\text{1-loop,pole}}(ia+h_1/2,im)}{Z_{\text{1-loop,pole}}(ia-h_1/2,im)} = \prod_{j=2}^{N}\left[\prod_{\pm}\frac{G(\frac{1}{2}+ia_{1j}\pm im)G(\frac{3}{2}-ia_{1j}\pm im)}{G(\frac{3}{2}+ia_{1j}\pm im)G(\frac{1}{2}-ia_{1j}\pm im)}\right]^{1/2}$$
$$\left[\frac{G(\frac{1}{2}+ia_{1j})G(\frac{5}{2}+ia_{1j})G(-\frac{1}{2}-ia_{1j})G(\frac{3}{2}-ia_{1j})}{G(-\frac{1}{2}+ia_{1j})G(\frac{3}{2}+ia_{1j})G(\frac{1}{2}-ia_{1j})G(\frac{5}{2}-ia_{1j})}\right]^{1/2}, \tag{8.14}$$

which since $G(z+1)/G(z) = \Gamma(z)$ equals

$$\prod_{j=2}^{N}\left[\prod_{\pm}\frac{\Gamma(\frac{1}{2}-ia_{1j}\pm im)}{\Gamma(\frac{1}{2}+ia_{1j}\pm im)}\right]^{1/2}\left[\frac{\Gamma(-\frac{1}{2}+ia_{1j})\Gamma(\frac{3}{2}+ia_{1j})}{\Gamma(-\frac{1}{2}-ia_{1j})\Gamma(\frac{3}{2}-ia_{1j})}\right]^{1/2}. \tag{8.15}$$

Using the explicit form of the monodromy operators $T_k(ia,m)$ in (8.1) we arrive at

$$E(ia,im) = \prod_{j=2}^{N}\left[\frac{\Gamma(-\frac{1}{2}+ia_{1j})\Gamma(\frac{3}{2}+ia_{1j})\Gamma(-\frac{1}{2}-ia_{1j})\Gamma(\frac{3}{2}-ia_{1j})}{\prod_{\pm}\Gamma(\frac{1}{2}+ia_{1j}\pm im)\Gamma(\frac{1}{2}-ia_{1j}\pm im)}\right]^{1/2}, \tag{8.16}$$

which by Euler's reflection formula $\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}$ yields

$$E(ia,im) = \prod_{j=2}^{N}\left[\frac{\sin(\pi(\frac{1}{2}+ia_{1j}-im))\sin(\pi(\frac{1}{2}-ia_{1j}-im))}{\sin(\pi(\frac{1}{2}+ia_{1j}))\sin(\pi(\frac{1}{2}+ia_{1j}))}\right]^{1/2}. \tag{8.17}$$

The result can be written in a more covariant form to arrive at the final answer

$$\begin{aligned} E(ia,im) &= \prod_{\alpha>0}\frac{\sin^{\frac{|\alpha\cdot B|}{2}}\left(\pi\left[\frac{\alpha\cdot B}{2}+i\alpha\cdot a-im\right]\right)\sin^{\frac{|\alpha\cdot B|}{2}}\left(\pi\left[\frac{\alpha\cdot B}{2}-i\alpha\cdot a-im\right]\right)}{\sin^{|\alpha\cdot B|}\left(\pi\left[\frac{\alpha\cdot B}{2}+i\alpha\cdot a\right]\right)} \\ &= \frac{\prod_{w\in\text{adj}}\sin^{\frac{|w\cdot B|}{2}}\left(\pi\left[\frac{w\cdot B}{2}+iw\cdot a-im\right)\right]}{\prod_{\alpha>0}\sin^{|\alpha\cdot B|}\left(\pi\left[\frac{\alpha\cdot B}{2}+i\alpha\cdot a\right]\right)}. \end{aligned} \tag{8.18}$$

This is precisely the gauge theory formula for the equatorial one-loop determinant (6.61).

This shows that the Toda prediction for the expectation of the 't Hooft loop operator labeled by the fundamental representation in the $\mathcal{N}=2^*$ theory with SU(N) gauge group precisely agrees with our gauge theory computation. We identify in the Toda correlator the factor $|Z_{\text{cl}}(ia-h_1/2,q)Z_{\text{1-loop,pole}}(ia-h_1/2,im)Z_{\text{inst}}(ia-h_1/2,1+im,q)|^2$ in (8.10) with the gauge theory contributions arising from the north and south poles of S^4 (see (6.34)), while comparison of (8.18) with (6.61) demonstrates that indeed $E(ia,m)$ precisely captures the gauge theory contribution from the equator, so that

$$E(ia,im) = Z_{\text{1-loop,eq}}(ia,im,h_1)\,. \tag{8.19}$$

The Toda calculation of [12] can be extended to describe 't Hooft operators with higher magnetic weight in $\mathcal{N}=2^*$. We have checked that the Toda calculation for $B=2h_1$ also exactly reproduces the gauge theory prediction.

8.2 Monopole screening from Liouville theory

We now specialize to the A_1 Toda theory, i.e., Liouville theory. We compare the results in [10, 11], which are the special case of the general Toda calculations above, with the non-perturbative contributions from monopole screening in SU(2) $\mathcal{N} = 2^*$ theory computed in section 7.3

In Liouville theory, the 't Hooft loop expectation value is given in terms of shifted conformal blocks. To simplify formulas we set $r = 1$ without loss of generality, also set b to 1 and adapt the normalization of [10]

$$Z_{\mathrm{L}}(\hat{a}, im, \tau) \equiv e^{-2\pi i\tau\,\hat{a}^2}\mathcal{F}\left(1+\hat{a}, 1+im, \tau\right) , \tag{8.20}$$

where $\mathcal{F}(\alpha, \alpha_e, \tau)$ is the conformal block of the 1-punctured torus with modulus τ in the standard normalization [43], with internal and external Liouville momenta α and α_e.[71] Up to a normalization constant, it was shown in [10, 11] that the loop operator expectation value is given by[72]

$$\begin{aligned}
&\langle(\mathcal{L}_{1,0})^p\rangle \\
&\quad = \int_{\hat{a}\in i\mathbb{R}} d\hat{a}\, C(1+\hat{a}, 1-\hat{a}, 1+im)\overline{Z_{\mathrm{L}}(\hat{a}, im, \tau)}[(\mathcal{L}_{1,0})^p\cdot Z_{\mathrm{L}}](\hat{a}, im, \tau) \\
&\quad = \int_{\hat{a}\in i\mathbb{R}} d\hat{a}\, C(1+\hat{a}, 1-\hat{a}, 1+im) Z_{\mathrm{L}}(-\hat{a}, -im, -\overline{\tau})[(\mathcal{L}_{1,0})^p\cdot Z_{\mathrm{L}}](\hat{a}, im, \tau)
\end{aligned} \tag{8.21}$$

The Liouville loop operator $\mathcal{L}_{1,0}$ acts as a difference operator. For any meromorphic function $f(\hat{a})$, let us define the operators $\hat{h}_\pm$ as multiplication by the functions $h_\pm(\hat{a})$:

$$\begin{aligned}
(\hat{h}_+\cdot f)(\hat{a}) &\equiv \frac{\Gamma(-2\hat{a})\Gamma(2-2\hat{a})}{\Gamma(-2\hat{a}+1+im)\Gamma(-2\hat{a}+1-im)} f(\hat{a}) \equiv h_+(\hat{a})f(\hat{a})\,, \\
(\hat{h}_-\cdot f)(\hat{a}) &\equiv \frac{\Gamma(2\hat{a})\Gamma(2+2\hat{a})}{\Gamma(2\hat{a}+1+im)\Gamma(2\hat{a}+1-im)} f(\hat{a}) \equiv h_-(\hat{a})f(\hat{a})\,.
\end{aligned} \tag{8.22}$$

We also define the shift operator Δ:

$$(\Delta\cdot f)(\hat{a}) := f(\hat{a}-1/4)\,. \tag{8.23}$$

With these definitions the Liouville loop operator is defined by

$$\mathcal{L}_{1,0} = \hat{h}_+\Delta^2 + \hat{h}_-\Delta^{-2}\,, \tag{8.24}$$

and the higher powers of $\mathcal{L}_{1,0}$ take the form

$$(\mathcal{L}_{1,0})^p = \sum_{q=p,p-2,\ldots,-p} \hat{h}_{p,q}\,\Delta^{2q}\,, \tag{8.25}$$

[71] The shift by 1 in $\alpha_e = 1 + im$ was clarified in [19].

[72] The complex conjugate of $\overline{\tau}$ appears with a minus sign because $\overline{\tau}$ enters into $\overline{Z_{\mathrm{L}}}$ through $\overline{e^{2\pi i\tau}} = e^{-2\pi i\overline{\tau}}$. In this subsection we avoid using the symbol q to denote $e^{2\pi i\tau}$, in order to avoid confusion with screened magnetic charge q.

where $\hat{h}_{p,q}$ is multiplication by a function $h_{p,q}(\hat{a})$. This function can be determined by the recursion relation

$$h_{p+1,q}(\hat{a}) = h_+(\hat{a})[\Delta^2 \cdot h_{p,q-1}](\hat{a}) + h_-(\hat{a})[\Delta^{-2} \cdot h_{p,q+1}](\hat{a})\,. \tag{8.26}$$

The solution is given by

$$\begin{aligned} h_{p,q}(\hat{a}) = &\left(\prod_{r=1}^{|q|} \frac{\Gamma(-2\mathrm{sgn}(q)\hat{a}+r-1)\Gamma(-2\mathrm{sgn}(q)\hat{a}+r+1)}{\Gamma(-2\mathrm{sgn}(q)\hat{a}+r+im)\Gamma(-2\mathrm{sgn}(q)\hat{a}+r-im)}\right) \\ &\times \frac{p!}{\left(\frac{p+q}{2}\right)!\left(\frac{p-q}{2}\right)!} \frac{\sin^{\frac{p-|q|}{2}}(2\pi\hat{a}+\pi im)\sin^{\frac{p-|q|}{2}}(2\pi\hat{a}-\pi im)}{\sin^{p-|q|}(2\pi\hat{a})}\,. \end{aligned} \tag{8.27}$$

Up to $\hat{a}$-independent factors, the Liouville three-point function is related to the gauge theory one-loop determinant as

$$\begin{aligned} C(1+\hat{a},1-\hat{a},1+im) &= |Z_{\text{1-loop,pole}}(\hat{a},im)|^2 \\ &= Z_{\text{1-loop,pole}}(-\hat{a},-im)Z_{\text{1-loop,pole}}(\hat{a},im) \end{aligned} \tag{8.28}$$

for $\hat{a} \in i\mathbb{R}$, where $Z_{\text{1-loop,pole}}(\hat{a},\hat{m})$ is given in (6.70). Thus the Liouville correlator (8.21) becomes

$$\begin{aligned} \langle(\mathcal{L}_{1,0})^p\rangle = &\int_{\hat{a}\in i\mathbb{R}} d\hat{a}\, Z_{\text{1-loop,pole}}(-\hat{a},-im)Z_{\mathrm{L}}(-\hat{a},-im,-\overline{\tau}) \\ &\times Z_{\text{1-loop,pole}}(\hat{a},im)\sum_q \left[\hat{h}_{p,q}\Delta^{2q}\cdot Z_{\mathrm{L}}\right](\hat{a},im,\tau)\,. \end{aligned} \tag{8.29}$$

Assuming that we can shift the contour without picking up residues,[73] we can write this as

$$\begin{aligned} \langle(\mathcal{L}_{1,0})^p\rangle = &\sum_q \int_{\hat{a}\in i\mathbb{R}} d\hat{a}\, [\Delta^{-q}\cdot h_{p,q}](\hat{a}) \frac{[\Delta^{-q}\cdot Z_{\text{1-loop,pole}}](\hat{a},im)}{[\Delta^{q}\cdot Z_{\text{1-loop,pole}}](\hat{a},im)} \\ &\times [\Delta^q\cdot Z_{\text{1-loop,pole}}](-\hat{a},-im)[\Delta^q\cdot Z_{\mathrm{L}}](-\hat{a},-im,-\overline{\tau}) \\ &\times [\Delta^q\cdot Z_{\text{1-loop,pole}}](\hat{a},im)[\Delta^q\cdot Z_{\mathrm{L}}](\hat{a},im,\tau)\,. \end{aligned} \tag{8.30}$$

Using (6.70) and (8.27), we can calculate the combination in the first line and obtain a simple result:

$$\begin{aligned} &[\Delta^{-q}\cdot h_{p,q}](\hat{a})\frac{[\Delta^{-q}\cdot Z_{\text{1-loop,pole}}](\hat{a})}{[\Delta^{q}\cdot Z_{\text{1-loop,pole}}](\hat{a})} \\ &= \frac{p!}{\left(\frac{p+q}{2}\right)!\left(\frac{p-q}{2}\right)!}\times\begin{cases} \dfrac{\sin^{\frac{p}{2}}(2\pi\hat{a}+\pi im)\sin^{\frac{p}{2}}(2\pi\hat{a}-\pi im)}{\sin^p(2\pi\hat{a})} & \text{for } p \text{ odd}, \\ \dfrac{\cos^{\frac{p}{2}}(2\pi\hat{a}+\pi im)\cos^{\frac{p}{2}}(2\pi\hat{a}-\pi im)}{\cos^p(2\pi\hat{a})} & \text{for } p \text{ even}, \end{cases} \end{aligned} \tag{8.31}$$

Comparing this with (6.71) and (7.62), this is precisely $Z_{\text{mono}}(ia,im;p,q)Z_{\text{1-loop,eq}}(ia,im,q)$.

[73] We have checked the validity of the contour deformation numerically by comparing with the S-dual Wilson loop expectation values.

Using the relation [20]

$$Z_{\rm L}(\hat{a}, im, \tau) = Z_{\rm cl}(\hat{a}, e^{2\pi i\tau}) Z_{\rm inst}(\hat{a}, 1+im, e^{2\pi i\tau})\,, \tag{8.32}$$

we thus obtain

$$\begin{aligned}\langle(\mathcal{L}_{1,0})^p\rangle = &\sum_{q=p,p-2,\ldots,-p} \frac{p!}{\left(\frac{p+q}{2}\right)!\left(\frac{p-q}{2}\right)!} \int da\, Z_{\text{1-loop,eq}}(ia, im, p) \\ &\times \left| Z_{\rm cl}\left(ia-\frac{q}{4}, e^{2\pi i\tau}\right) Z_{\text{1-loop,pole}}\left(ia-\frac{q}{4}, im\right) Z_{\rm inst}\left(ia-\frac{q}{4}, im, e^{2\pi i\tau}\right)\right|^2 .\end{aligned} \tag{8.33}$$

After reintroducing the dimensionful parameter r, the gauge theory result (7.63) for $\langle T_p\rangle$ and the Liouville theory expression (8.33) for $\langle(\mathcal{L}_{1,0})^p\rangle$ precisely agree, including the monopole screening contributions!

We note that the charge p 't Hooft loop T_p corresponds to the Liouville operator $(\mathcal{L}_{1,0})^p$. Thus our charge p 't Hooft loop T_p equals the power $(T_1)^p$ of the 't Hooft loop that is S-dual to the spin 1/2 Wilson loop, and differs from the S-dual of the spin $p/2$ Wilson loop. The origin of the power is in the natural resolution of the Bogomolny moduli space. As explained in [31], the moduli space of solutions describing an array of p minimal 't Hooft loops $T_{p=1}$ develops a singularity when two of the loop operators collide. In the limit that all of them are on top of each other, the magnetic charge of the 't Hooft loop is p. Said another way, the singularity of the moduli space can be resolved by replacing the charge p 't Hooft loop with a collection of slightly displaced minimal 't Hooft loops.[74]

9 Conclusions

We performed an exact localization calculation for the expectation value of supersymmetric 't Hooft loop opertors in $\mathcal{N}=2$ supersymmetric gauge theories on S^4. These results combined with the exact computation of Wilson loop expectation values [2] constitute a suite of exact calculations for the simplest loop operators in these gauge theories and allow for a quantitative study of S-duality for this rich class of gauge theory observables.

A 't Hooft loop was defined by specifying a boundary condition of the fields in the path integral. We integrated over the non-singular and singular solutions to the saddle point equations in the localization computation.

In the leading classical approximation the expectation value was obtained by evaluating the on-shell action in the non-singular background (3.9), and the only perturbative quantum corrections[75] in the localization path integral are the one-loop determinants computed using the Atiyah-Singer index theorem, arising from the north pole, south pole and equator. The 't Hooft loop expectation value receives two types of non-perturbative corrections. The first is from instantons and anti-instantons localized at the north and south poles as in [2],

[74] As noted in [2], the localization supercharge Q is indeed compatible with parallel loop operators each located at a fixed latitude.

[75] The one-loop determinants are the unique perturbative corrections with respect to the localization action $Q\cdot V$. All the perturbative corrections with respect to the physical action [9] are reproduced by integrating over the zero-mode a.

arising because our localization saddle point equations become $F^+ = 0$ and $F^- = 0$ there. One new feature in our calculation is that the Nekrasov instanton partition functions at the poles have their argument shifted due to the 't Hooft loop background. The second type of non-perturbative correction occurs as new saddle point field configurations, where smooth monopoles in the bulk of S^4 screen the charge of the singular monopole inserted along the loop. These arise from non-abelian solutions to the Bogomolny equations $D\Phi = *F$, which describe the saddle point equations in the equator. The field configurations were identified as the fixed points of an equivariant group action on the moduli space of solutions of the Bogomolny equations.

In this paper we have focused on the computation of 't Hooft operators for which the magnetic charge and the electric charge vectors are parallel, where the electric charge is acquired by the Witten effect, due to the non-vanishing topological angle θ. The techniques introduced here, however, can be used to compute general dyonic Wilson-'t Hooft operators. The new ingredient for a dyonic operator is the insertion of a Wilson loop for the unbroken gauge group preserved by the singular monopole background.

We compared our gauge theory calculations with some of the predictions in [10–12] obtained from computations with topological defects in Liouville and Toda field theories, and found a perfect match for all comparisons we have performed. The physical observables in Liouville/Toda theory are known to be invariant under the modular transformations (or more generally under the Moore-Seiberg groupoid) that are identified with the S-duality transformations in gauge theory. Thus our results prove S-duality invariance of the $\mathcal{N} = 2$ gauge theories, in the sector of physical observables involving Wilson and 't Hooft loop operators. In turn, the progress we made on gauge theory loop operators provides motivation to study in more depth the two-dimensional observables. In particular the computational techniques for *topological webs*, — the defects involving trivalent vertices — are to be developed in order to make a useful comparison with more complicated loop operators in higher-rank gauge theories.

In our study an important role was played by the equivariant index for the moduli space of solutions of the Bogomolny equations in the presence of a singular monopole background, created by the 't Hooft operator. The analysis was similar to that of the instanton moduli space that led to the Nekrasov partition function, and we defined the quantity Z_{mono} which is an analogous physical quantity in the monopole case. It is possible to generalize and formalize the definition of Z_{mono} by setting up a localization scheme on $S^1 \times \mathbb{R}^3$ [44].

The localization techniques we developed for 't Hooft operators should also admit generalizations to other supersymmetric disorder operators, such as monopole and vortex loop operators in three dimensions and surface operators in four dimensions. For example, it would be interesting to formulate a path integral framework that realizes the mathematical calculations [45–48] for instantons in the presence of singularities representing surface operators. Also, the localization framework for $\mathcal{N} = 2$ theories on S^4 should apply to surface operators preserving two-dimensional $\mathcal{N} = (2, 2)$ supersymmetry. Localization calculations for such observables should help understand disorder operators in the broad duality web involving quantum field theories in diverse dimensions.

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A Supersymmetry and killing spinors

The spinors in this paper transform in a representation of Spin(10), whose generators are constructed from the Clifford algebra $Cl(10)$

$$\{\gamma^M, \gamma^N\} = 2\eta^{MN} \qquad \text{where} \quad M = 1, \ldots, 9, 0\,. \tag{A.1}$$

We take the Euclidean metric $\eta^{MN} = \delta^{MN}$. In the chiral representation

$$\gamma^M = \begin{pmatrix} 0 & \tilde{\Gamma}^M \\ \Gamma^M & 0 \end{pmatrix}, \tag{A.2}$$

where $\tilde{\Gamma}^M \equiv (\Gamma^1, \ldots, \Gamma^9, -\Gamma^0)$, and $\Gamma^M, \tilde{\Gamma}^M$ are 16×16 matrices which satisfy

$$\tilde{\Gamma}^M \Gamma^N + \tilde{\Gamma}^N \Gamma^M = 2\delta^{MN}, \qquad \Gamma^M \tilde{\Gamma}^N + \Gamma^N \tilde{\Gamma}^M = 2\delta^{MN}\,. \tag{A.3}$$

The matrices $\tilde{\Gamma}^M$ and Γ^M act respectively on the negative and positive chirality spinors of Spin(10) since

$$\gamma^{(10)} \equiv -i\gamma^1 \ldots \gamma^9 \gamma^0 = \begin{pmatrix} -i\tilde{\Gamma}^1\Gamma^2 \ldots \tilde{\Gamma}^9\Gamma^0 & 0 \\ 0 & -i\Gamma^1\tilde{\Gamma}^2 \ldots \Gamma^9\tilde{\Gamma}^0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{A.4}$$

In Euclidean signature, which we use in this paper, ten dimensional spinors are complex. We choose a basis in which $\Gamma^1, \ldots, \Gamma^9$ are real and Γ^0 imaginary. To describe Γ^M explicitly it is convenient to break SO(10) to SO(8) $\times$ SO(2) and use the octonionic construction of the Clifford algebra $Cl(10)$. For the explicit expressions which are needed for explicit construction of the supersymmetry equations in components we use matrices as defined in appendix A of [2] with a certain permutation of spacetime indices. If $\underline{\Gamma}^M$ are the matrices in [2], then the present Γ^M are given by

$$\begin{aligned}\Gamma^M &= \underline{\Gamma}^{M+1} \quad \text{for} \quad M = 1, 2, 3, 5, 6, 7 \\ \Gamma^4 = \underline{\Gamma}^1 \qquad \Gamma^8 &= \underline{\Gamma}^5 \qquad \Gamma^9 = \underline{\Gamma}^9 \qquad \Gamma^0 = i\underline{\Gamma}^0 .\end{aligned} \tag{A.5}$$

The factor of i appears in the relation to Γ^0 because our present conventions use the Euclidean metric $\eta^{MN} = \delta^{MN}$, while [2] used the Lorentz metric with $\eta^{00} = -1$.

The supersymmetry parameter ϵ and gaugino Ψ in the $\mathcal{N} = 2$ vectormultiplet are positive chirality spinors of Spin(10), while hyperino χ in the $\mathcal{N} = 2$ hypermultiplet is a negative chirality spinor; they are subject to the projections

$$\Gamma^{5678}\epsilon = -\epsilon\,, \qquad \Gamma^{5678}\Psi = -\Psi\,, \qquad \Gamma^{5678}\chi = \chi\,, \tag{A.6}$$

where $\Gamma^{5678}\epsilon = \tilde{\Gamma}^5\Gamma^6\tilde{\Gamma}^7\Gamma^8\epsilon$.

The conformal Killing spinor equation (3.10) in the $B_3 \times S^1$ metric (3.7) is

$$\nabla_\mu \epsilon = \tilde{\Gamma}_\mu \tilde{\epsilon} \tag{A.7}$$

$$\tilde{\Gamma}^\mu \nabla_\mu \tilde{\epsilon} = -\frac{1}{4r^2}\frac{1}{\left(1 - \frac{|\vec{x}|^2}{4r^2}\right)}\epsilon\,. \tag{A.8}$$

In the vielbein basis $e^{\hat{i}} = e^i = dx^i$ and $e^{\hat{4}} = r\left(1 - \frac{|\vec{x}|^2}{4r^2}\right)d\tau$, the non-zero components of the spin connection are

$$w^{\hat{4}i} = -w^{i\hat{4}} = -\frac{x^i}{2r}d\tau \qquad i = 1, \ldots, 3\,. \tag{A.9}$$

Equation (A.8) implies that $\tilde{\epsilon} = \epsilon_c(\tau)$ while the first three equations in (A.7) imply that $\epsilon = \epsilon_s(\tau) + x^i\tilde{\Gamma}_i\epsilon_c(\tau)$. The solution to the equation

$$\nabla_\tau \epsilon = \tilde{\Gamma}_\tau \tilde{\epsilon} \tag{A.10}$$

is (3.14)

$$\epsilon = \cos(\tau/2)\left(\hat{\varepsilon}_s + x^i\tilde{\Gamma}_i\,\hat{\varepsilon}_c\right) + \sin(\tau/2)\,\tilde{\Gamma}^4\left(2r\,\hat{\varepsilon}_c + \frac{x^i}{2r}\Gamma_i\,\hat{\varepsilon}_s\right)\,, \tag{A.11}$$

with $\hat{\varepsilon}_s$ and $\hat{\varepsilon}_c$ two constant ten dimensional Weyl spinors of opposite chirality.

B Lie algebra conventions

Let G be a compact Lie group and $\mathfrak{g}$ the Lie algebra of G. As a vector space $\mathfrak{g}$ is isomorphic to $\mathbb{R}^{\dim G}$. In our conventions, for a gauge theory with gauge group G, the fields $A_\mu, F_{\mu\nu}$ and Φ_A $(A = 0, 9)$ of the vectormultiplet take values in $\mathfrak{g}$. In particular, we write the covariant derivative as $D \equiv D_A = d + A$ and the curvature as $F_{\mu\nu} = [D_\mu, D_\nu]$. If G is U(N) or SU(N), the basis $\{T_\alpha\}$ of the Lie algebra $\mathfrak{g}$ can be represented by $N \times N$ antihermitian matrices. Given the basis, the real coordinates a^α of an element $a \in \mathfrak{g}$ are defined by the expansion $a = a^\alpha T_\alpha$. Let $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \otimes \mathbb{C}$ be the complexification of $\mathfrak{g}$. An element $a = x + iy$

of $\mathfrak{g}_\mathbb{C}$, where $x, y \in \mathfrak{g}$, can be written as $a = a^\alpha T_\alpha$ with a^α being complex numbers. We say that an element a of $\mathfrak{g}_\mathbb{C}$ is real if the coordinates a^α are real. Complex conjugation acts by conjugating the coefficients: $a = a^\alpha T_\alpha \to \overline{a^\alpha} T_\alpha$.

If G is a compact Lie group, then the Lie algebra $\mathfrak{g}$ of G can be equipped with a positive definite bilinear form $(\bullet, \bullet) : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ invariant under the adjoint action of G. Such a bilinear form $\mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ is defined uniquely up to a scaling, and extends holomorphically to $\mathfrak{g}_\mathbb{C} \times \mathfrak{g}_\mathbb{C} \to \mathbb{C}$. For $\mathfrak{g} = \mathfrak{u}(N)$ and $\mathfrak{g} = \mathfrak{su}(N)$, we choose $(\bullet, \bullet)$, also donoted by $\bullet \cdot \bullet$, to be given by minus the trace in the fundamental representation: $(a, b) = -\operatorname{Tr} ab$.

The basis elements T^α in the Cartan algebra $\mathfrak{t}$ of $\mathfrak{u}(N)$ can be represented by the diagonal $N \times N$ matrices

$$T_\alpha = i \operatorname{diag}(0, \ldots, 0, 1, 0, \ldots, 0)\,, \tag{B.1}$$

where 1 is at the position α. Since in this basis the bilinear form is the identity matrix, $-\operatorname{Tr} T_\alpha T_\beta = \delta_{\alpha\beta}$, we do not distinguish between contravariant and covariant Lie algebra indices. For an element $a = a^\alpha T_\alpha$ of $\mathfrak{t}$ we refer to a using the following equivalent notations

$$(a_1, \ldots, a_N) \leftrightarrow a \leftrightarrow a_\alpha T_\alpha = \begin{pmatrix} ia_1 & 0 & \ldots \\ 0 & ia_2 & \ldots \\ \vdots & \vdots & \ddots \end{pmatrix}, \tag{B.2}$$

where a_α are real. When dealing with complexification $\mathfrak{t}_\mathbb{C}$ we allow a_α to be complex. The notation (B.2) is also used for $\mathfrak{g} = \mathfrak{su}(N)$. For example, the Nekrasov instanton partition function Z_inst takes a complex element $\hat{a}$ of $\mathfrak{t}_\mathbb{C}$, i.e. the Coulomb parameter, as one of its arguments. We use equivalently the following forms referring to Z_inst evaluated at $\hat{a}$

$$Z_\text{inst}(\hat{a}; \varepsilon_1, \varepsilon_2) = Z_\text{inst}((\hat{a}_1, \ldots, \hat{a}_N), \varepsilon_1, \varepsilon_2) = Z_\text{inst}(\hat{a}_1, \ldots, \hat{a}_N; \varepsilon_1, \varepsilon_2)\,. \tag{B.3}$$

It should be clear from the context what $\hat{a}$ refers to in the main text.

C Coordinates and Weyl transformations on S^4

The SO(5) isometry of the round metric on S^4 is made manifest by the induced metric on the following hypersurface in $\mathbb{R}^5$

$$X_1^2 + \ldots + X_5^2 = r^2\,. \tag{C.1}$$

In this paper, a certain $\mathrm{U}(1)_J \subset \mathrm{SO}(5)$ isometry of S^4 generated by the generator J plays a key role. It acts on the embedding coordinates as

$$\begin{aligned} X_1 + iX_2 &\to e^{i\varepsilon}(X_1 + iX_2) \\ X_3 + iX_4 &\to e^{i\varepsilon}(X_3 + iX_4)\,, \end{aligned} \tag{C.2}$$

and its fixed points $X_5 = \pm r$ define the north and south pole of S^4. The following coordinates are of use in the paper:

Latitude coordinates: the metric is given by

$$ds^2 = r^2(d\vartheta^2 + \sin^2\vartheta d\Omega_3) \tag{C.3}$$

where $d\Omega_n$ is the metric on the unit S^n and ϑ is the latitude angle on S^4, with $\vartheta = 0, \pi/2$ and π corresponding to the north pole, equator and south pole respectively. The embedding coordinates are

$$\begin{aligned} X_a &= r\sin\vartheta\, n_a \qquad a = 1,2,3,4 \\ X_5 &= r\cos\vartheta\,, \end{aligned} \tag{C.4}$$

where n_a is a unit vector in $\mathbb{R}^4$ parametrizing S^3.

The $\mathrm{U}(1)_J$ action induced by J is realized by the Hopf fibration. Consider $S^3 : |w_1|^2 + |w_2|^2 = 1$, $(w_1, w_2) \in \mathbb{C}^2$ and the U(1) action $(w_1, w_2) \mapsto (e^{i\varepsilon}w_1, e^{i\varepsilon}w_2)$. Introduce angular coordinates on $\mathbb{C}^2$: $w_1 = \rho\cos\frac{\eta}{2}e^{i\psi}$ and $w_2 = \rho\sin\frac{\eta}{2}e^{i\psi+i\varphi}$ so that the $\mathrm{U}(1)_J$ acts by shifts $\psi \to \psi + \varepsilon$, and consider the map $\overline{\mathbb{C}}^2 \otimes \mathbb{C}^2 \to \mathbb{R}^3$

$$\vec{x} = \overline{w}\vec{\sigma}w = \rho^2(\sin\eta\cos\varphi, \sin\eta\sin\varphi, \cos\eta), \tag{C.5}$$

so that (ρ^2, η, φ) are the spherical coordinates on $\mathbb{R}^3$. Rewriting the flat metric on $\mathbb{C}^2$ in the $(\rho, \eta, \varphi, \psi)$ coordinates we get

$$ds^2 = d\rho^2 + \rho^2\left(\frac{1}{4}d\eta^2 + \frac{1}{2}(1-\cos\eta)d\varphi^2 + (1-\cos\eta)d\varphi d\psi + d\psi^2\right). \tag{C.6}$$

The unit S^3 is at $\rho = 1$ with metric

$$d\Omega_3 = \frac{1}{4}d\eta^2 + \frac{1}{2}(1-\cos\eta)d\varphi^2 + (1-\cos\eta)d\varphi d\psi + d\psi^2 = \frac{1}{4}d\Omega_2 + (d\psi+\omega)^2\,, \tag{C.7}$$

where

$$d\Omega_2 = d\eta^2 + \sin^2\eta\, d\varphi^2 \tag{C.8}$$

and

$$\omega = \frac{1}{2}(1-\cos\eta)d\varphi. \tag{C.9}$$

In these coordinates, the $\mathrm{U}(1)_J$ vector field is $v = \frac{1}{r}\frac{\partial}{\partial\psi}$, and the dual 1-form used in section 3.3 is

$$\tilde{v} = \frac{dx^\mu h_{\mu\nu}v^\nu}{v^\mu v_\mu} = r(d\psi + \omega)\,. \tag{C.10}$$

The 1-form ω satisfies $d\omega = \frac{1}{2}\mathrm{vol}(S^2)$.

$S^2 \times S^1$ foliation coordinates: the metric is given by

$$ds^2 = r^2(d\xi^2 + \sin^2\xi d\Omega_2 + \cos^2\xi d\tau^2) \tag{C.11}$$

where τ is the coordinate on S^1 and $0 \le \xi \le \pi/2$. The embedding coordinates are given by

$$\begin{aligned} X_1 + iX_2 &= r\cos\xi e^{i\tau} \\ X_3 + iX_4 &= r\sin\xi\sin\alpha e^{i\phi} \\ X_5 &= r\sin\xi\cos\alpha\,, \end{aligned} \tag{C.12}$$

where $d\Omega_2 = d\alpha^2 + \sin^2\alpha d\phi^2$. The $\mathrm{U}(1)_J$ symmetry generator J acts by shifts

$$\begin{aligned} \tau &\to \tau + \varepsilon \\ \phi &\to \phi + \varepsilon \,. \end{aligned} \tag{C.13}$$

In these coordinates the north and south pole are at $(\xi = \pi/2, \alpha = 0)$ and $(\xi = \pi/2, \alpha = \pi)$ respectively.

$B_3 \times S^1$ foliation coordinates: the metric is given by

$$ds^2 = \frac{\sum_{i=1}^3 dx_i^2}{\left(1 + \frac{|\vec{x}|^2}{4r^2}\right)^2} + r^2 \frac{\left(1 - \frac{|\vec{x}|^2}{4r^2}\right)^2}{\left(1 + \frac{|\vec{x}|^2}{4r^2}\right)^2} d\tau^2 \,, \tag{C.14}$$

and $|\vec{x}|^2 \leq 4r^2$ defines the three-ball B_3. The embedding coordinates are given by

$$\begin{aligned} X_1 + iX_2 &= r\frac{\left(1 - \frac{|\vec{x}|^2}{4r^2}\right)}{\left(1 + \frac{|\vec{x}|^2}{4r^2}\right)} e^{i\tau} \\ X_I &= \frac{x_{I-2}}{\left(1 + \frac{|\vec{x}|^2}{4r^2}\right)} \qquad I = 3,4,5 \,. \end{aligned} \tag{C.15}$$

The $\mathrm{U}(1)_J$ symmetry generator J acts by

$$\begin{aligned} x_1 + ix_2 &\to e^{i\varepsilon}(x_1 + ix_2) \\ \tau &\to \tau + \varepsilon \,. \end{aligned} \tag{C.16}$$

In these coordinates the north and south pole are at $\vec{x} = (0,0,2r)$ and $\vec{x} = (0,0,-2r)$ respectively.

D Q-invariance of the 't Hooft loop background

The background created by a circular 't Hooft loop with magnetic weight B located at $\vec{x} = 0$ in the $B_3 \times S^1$ metric (3.7) takes the same form as that of a static 't Hooft line in flat spacetime (3.8) (for $\theta = 0$)

$$\begin{aligned} F_{jk} &= -\frac{B}{2}\epsilon_{ijk}\frac{x_i}{|\vec{x}|^3} \\ \Phi_9 &= \frac{B}{2|\vec{x}|} \,. \end{aligned} \tag{D.1}$$

Since $B \in \mathfrak{t}$ takes values in the Cartan subalgebra of the gauge group G, the singularity is abelian in nature.

We can verify that the the deformed monopole equations (3.38), (3.39), (3.40) are solved by the 't Hooft loop background (D.1). For example, let's consider the first spatial equation (3.38). In the background (D.1) $F_{1\hat{4}}, F_{3\hat{4}}, K_1, D_{\hat{4}}\Phi_9$ vanish. We group the remaining terms to make the structure of cancellation obvious using that (D.1) satisfies

$$D_i\Phi_9 = \partial_i\Phi_9 = -\frac{Bx_i}{2|\vec{x}|^3} = \frac{1}{2}\epsilon_{ijk}F_{jk} \qquad i,j,k = 1,2,3 \,, \tag{D.2}$$

where the first equality is due to the abelian nature of the background (D.1). Evaluation yields for (3.38)

$$
\begin{aligned}
&4r^2(-D_1\Phi_9+F_{23})+(-x_1^2+x_2^2+x_3^2)(D_1\Phi_9+F_{23})+(-2x_1x_2)(D_2\Phi_9-F_{13})\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad +(-2x_1x_3)(D_3\Phi_9+F_{12})+-2x_1\Phi_9\\
&=0+(-x_1^2+x_2^2+x_3^2)\frac{-Bx_1}{|\vec{x}|^3}+(-2x_1x_2)\frac{-Bx_2}{|\vec{x}|^3}+(-2x_1x_3)\frac{-Bx_3}{|\vec{x}|^3}-\frac{Bx_1}{|\vec{x}|}\\
&=\frac{Bx_1x^2}{|\vec{x}|^3}-\frac{Bx_1}{|\vec{x}|}=0\,.
\end{aligned}
\tag{D.3}
$$

The cancellation in the second deformed monopole equation (3.39) is exactly the same with replacement of indices $1\to 2$. In the third equation (3.40), the relative signs are different, but again all terms cancel similarly. In analyzing the last equation (3.40) we also find that Φ_0 can be turned on as long as

$$
K_3=-\frac{\Phi_0/r}{1+\frac{|\vec{x}|^2}{4r^2}}\,. \tag{D.4}
$$

This observation plays an important role in finding the most general solution to the saddle point equations, as discussed in section 3.3.

Similarly, it is very easy to show that the invariance equations (3.37) are satisfied by the background (D.1). For these only $D_i\Phi_9$ and F_{jk} contribute and cancel elementarily due to formula (D.2). These equations also exhibit that Φ_0 has a zeromode, given by

$$
\Phi_0=\frac{a}{1+\frac{|\vec{x}|^2}{4r^2}}\,, \tag{D.5}
$$

and therefore, due to (D.4)

$$
K_3=-\frac{a/r}{\left(1+\frac{|\vec{x}|^2}{4r^2}\right)^2}\,, \tag{D.6}
$$

where $a\in\mathfrak{t}$ is constant. In comparison with [2] the profile of Φ_0 is not constant in $B_3\times S^1$. However, since the metric on $B_3\times S^1$ and S^4 are related by a Weyl transformation with $\Omega=\left(1+\frac{|\vec{x}|^2}{4r^2}\right)$, it follows from (2.10) that the Weyl transformation makes Φ_0 constant in S^4, as found in [2] .

It is straightforward to show that the background created by the 't Hooft loop when $\theta\neq 0$ (3.9)

$$
\begin{aligned}
F_{jk}&=-\frac{B}{2}\epsilon_{ijk}\frac{x_i}{|\vec{x}|^3}\,, &\qquad F_{i\hat{4}}&=-ig^2\theta\frac{B}{16\pi^2}\frac{x_i}{|\vec{x}|^3}\,,\\
\Phi_9&=\frac{B}{2|\vec{x}|}\,, &\qquad \Phi_0&=-g^2\theta\frac{B}{16\pi^2}\frac{1}{|\vec{x}|}\,,
\end{aligned}
\tag{D.7}
$$

solves the localization equations $Q\cdot\Psi=0$ As we have already demonstrated that the terms involving Φ_0 and F_{jk} cancel in the invariance equations (3.37) and deformed monopole equations (3.38)–(3.40), we just have to exhibit cancellation of the terms involving Φ_0 and F_{i4}. Since the 't Hooft loop background is τ independent and abelian (i.e. $[\Phi_0,\Phi_9]=0$), we are just left to verify from the invariance equations (3.37) that

$$\begin{aligned} \frac{1}{2r}F_{i4} + \left[D_i, \frac{i}{2}\left(1+\frac{|\vec{x}|^2}{4r^2}\right)\Phi_0\right] &= 0 \qquad i=1,2,3 \\ x_1 F_{42} - x_2 F_{41} &= 0\,. \end{aligned} \tag{D.8}$$

Using that $F_{i4} = r\left(1-\frac{|\vec{x}|^2}{4r^2}\right)F_{i\hat{4}}$ and $D_i\left(\left(1+\frac{|\vec{x}|^2}{4r^2}\right)\Phi_0\right) = -\frac{x_i}{|\vec{x}|^2}\left(1-\frac{|\vec{x}|^2}{4r^2}\right)\Phi_0 = \frac{i}{r}F_{i4}$, we conclude that (D.7) solves the equations (3.37).

We now verify that the deformed monopole equations (3.38)–(3.39), 3.40) are solved by Φ_0 and $F_{i\hat{4}}$ of the 't Hooft loop background (D.7). From (3.38) and (3.39) we get

$$-x_3 F_{i\hat{4}} + x_i F_{3\hat{4}} = 0 \qquad i=1,2\,, \tag{D.9}$$

which is trivially satisfied by (D.7). From (3.40) the relevant equation is

$$i\Phi_0 - x_1 F_{1\hat{4}} - x_2 F_{2\hat{4}} - x_3 F_{3\hat{4}} = 0 \tag{D.10}$$

which is indeed solved by the 't Hooft loop background (D.7).

This concludes the explicit check that the direct sum of the monopole background configuration (D.7) and the Φ_0 zeromode profile (D.5) with the associated auxiliary field K_3 (D.6) solve the localization equations $Q\cdot\Psi = 0$.

E Hypermultiplets in general representations

In this appendix we will derive the formula (6.28) of the one-loop index for hypermultiplets in an arbitrary representation.

We will do this by generalizing, and also applying in a suitable way, the formula (6.27) that is valid for the adjoint representation. Let us begin with $\mathcal{N}=2^*$ theory in flat space which we regard as a dimensional reduction of the super Yang-Mills in ten dimensions. The group SO(4) that rotates the 5678 directions factorizes into the product of the R-symmetry group $\mathrm{SU}(2)_R$ and the flavour symmetry group $\mathrm{SU}(2)_{\mathrm{F}}$.

In order to derive (6.28) for complex and real representations, let us take the gauge group to be U(2). Applying the adjoint formula (6.27) to this case, we find the index[76]

$$-\frac{e^{\frac{1}{2}(i\varepsilon_1+i\varepsilon_2)}}{(1-e^{i\varepsilon_1})(1-e^{i\varepsilon_2})}\frac{e^{i\hat{m}}+e^{-i\hat{m}}}{2}\left(e^{i(\hat{a}_1-\hat{a}_2)}+e^{-i(\hat{a}_1-\hat{a}_2)}\right), \tag{E.1}$$

where $\mathrm{diag}(e^{i\hat{a}_1}, e^{i\hat{a}_2})$ and $\mathrm{diag}(e^{i\hat{m}}, e^{-i\hat{m}})$ parametrize the maximal tori of U(2) and $\mathrm{SU}(2)_{\mathrm{F}}$ that we denote by $\mathrm{U}(1)_1\times\mathrm{U}(1)_2$ and $\mathrm{U}(1)_{\mathrm{F}}$ respectively. Under $\mathrm{U}(1)_1\times\mathrm{U}(1)_2\times\mathrm{U}(1)_{\mathrm{F}}$, off-diagonal fields in the hypermultiplet transform in representations with charges $(+1,-1,\pm1)$ and their complex conjugate. The trick is to consider a new $\mathcal{N}=2$ theory that is obtained by setting all the off-diagonal components of the U(2) adjoint fields in the vector multiplet to zero, regarding $G' = \mathrm{U}(1)_1$ as a new gauge group. We also project the hypermultiplet fields onto those with charges $(+1,-1,+1)$ and their conjugate, and regard

[76] We neglect the terms with zero weights.

$\mathrm{U}(1)'_\mathrm{F} \equiv [\mathrm{U}(1)_2 \times \mathrm{U}(1)_\mathrm{F}]_\mathrm{diag}$ as a new flavour group. The hypermultiplet index for the new theory is obtained from (E.1) by keeping the relevant terms:

$$-\frac{e^{\frac{1}{2}(i\varepsilon_1+i\varepsilon_2)}}{2(1-e^{i\varepsilon_1})(1-e^{i\varepsilon_2})}\left(e^{i\hat{a}'-i\hat{m}'}+e^{-i\hat{a}'+i\hat{m}'}\right), \tag{E.2}$$

where $\hat{a}' \equiv \hat{a}_1$, and the Coulomb parameter $\hat{a}_2$ and the original mass parameter $\hat{m}$ have combined into a new mass parameter $\hat{m}' \equiv \hat{a}_2 - \hat{m}$. Thus we have derived the formula (6.28) for the spacial case of gauge group U(1) and a single charged hypermultiplet. Noting that a general complex irreducible representation of an arbitrary gauge group G can be thought of as embedding G into $\mathrm{U}(\dim R)$ whose maximal torus is $\mathrm{U}(1)^{\dim R}$, this U(1) result implies the formula (6.28) for any complex R.

Similarly any strictly real irreducible representation defines an embedding of G into $\mathrm{SO}(\dim R)$ with maximal torus $\mathrm{SO}(2)^{[\dim R/2]}$. Noting that the vector representation of $\mathrm{SO}(2) \simeq \mathrm{U}(1)$ gives the minimal real irreducible representation, the U(1) formula (E.2) also generalizes to (6.28) for any real representation R.

To treat the case where R is a pseudo-real representation, let us begin with $\mathcal{N}=4$ theory with gauge group SU(3) and perform a projection as follows. We pick a subgroup $G' = \mathrm{SU}(2)$ of SU(3) as a new gauge group, and denote its commutant by $\mathrm{U}(1)'$. We parametrize the maximal torus of $G' \times \mathrm{U}(1)'$ by $\mathrm{diag}(e^{i(\hat{a}+\hat{b})}, e^{i(-\hat{a}+\hat{b})}, e^{-2i\hat{b}})$. Let us keep only the vectormultiplet fields for G'. Under the embedding

$$\mathrm{SU}(3) \times \mathrm{SU}(2)_\mathrm{F} \supset \mathrm{SU}(2) \times \mathrm{U}(1)' \times \mathrm{U}(1)_\mathrm{F} \tag{E.3}$$

where $\mathrm{U}(1)_\mathrm{F}$ is the maximal torus of $\mathrm{SU}(2)_\mathrm{F}$, the hypermultiplet splits as

$$(\mathrm{adj}, \mathbf{2}) \to \ldots \oplus \mathbf{2}_{+1,+1} \oplus \mathbf{2}_{+1,-1} \oplus \mathbf{2}_{-1,+1} \oplus \mathbf{2}_{-1,-1} \oplus \ldots . \tag{E.4}$$

We project the hypermultiplets onto $\mathbf{2}_{-1,+1}$ and its conjugate $\mathbf{2}_{+1,-1}$. Picking the diagonal $\mathrm{U}(1)'_\mathrm{F} \equiv [\mathrm{U}(1)' \times \mathrm{U}(1)_\mathrm{F}]_\mathrm{diag}$, we get half-hypermultiplets in the pseudo-real representation $\mathbf{2}$ of gauge group SU(2) with flavour symmetry $\mathrm{SO}(2) \simeq \mathrm{U}(1)'$. This is the minimal case involving a pseudo-real representation. We can obtain the hypermultiplet index in the present case by keeping relevant terms in the adjoint formula (6.27):

$$\begin{aligned}
&-\frac{e^{\frac{1}{2}(i\varepsilon_1+i\varepsilon_2)}}{(1-e^{i\varepsilon_1})(1-e^{i\varepsilon_2})}\frac{e^{i\hat{m}}+e^{-i\hat{m}}}{2}\left(e^{2i\hat{a}}+e^{-2i\hat{a}}+e^{i\hat{a}+i\hat{b}}+e^{-i\hat{a}-i\hat{b}}+e^{-i\hat{a}+i\hat{b}}+e^{i\hat{a}-i\hat{b}}\right)\\
&\to -\frac{e^{\frac{1}{2}(i\varepsilon_1+i\varepsilon_2)}}{2(1-e^{i\varepsilon_1})(1-e^{i\varepsilon_2})}(e^{i\hat{a}'+i\hat{m}'}+e^{-i\hat{a}'+i\hat{m}'}+e^{i\hat{a}'-i\hat{m}'}+e^{-i\hat{a}'-i\hat{m}'}),
\end{aligned} \tag{E.5}$$

where the arrow indicates the projection and we have defined $\hat{m}' \equiv \hat{m} - \hat{b}$. The expression (E.5) is a special case of (6.28). For any gauge group G, a pseudo-real representation defines a homomorphism from G to the group $\mathrm{Sp}(\dim R)$ whose Cartan subalgebra is isomorphic to that of $\mathrm{Sp}(2)^{\frac{1}{2}\dim R} = \mathrm{SU}(2)^{\frac{1}{2}\dim R}$. The Cartan subalgebra of flavour $\mathrm{SO}(2N_\mathrm{F})$ is isomorphic to that of $\mathrm{SO}(2)^{N_\mathrm{F}} \simeq \mathrm{U}(1)^{N_\mathrm{F}}$. Thus this minimal case (E.5) implies the formula (6.28) for any pseudo-real representation R.

F Singular monopoles and instantons

Solutions of the Bogomolny equations are related to U(1) invariant instantons [14]. Let us consider a gauge field $\mathcal{A}$ in $\mathbb{R}^4 \simeq \mathbb{C}^2$. We regard the four-dimensional space $\mathbb{C}^2$ as a U(1) fibration over $\mathbb{R}^3$ using the map

$$(z_1, z_2) \mapsto (z_1, \overline{z}_2)\vec{\sigma} \begin{pmatrix} \overline{z}_1 \\ z_2 \end{pmatrix} =: \vec{x} \tag{F.1}$$

from $\mathbb{C}^2$ to $\mathbb{R}^3$. The right-hand side is invariant under $(z_1, z_2) \to (e^{-i\nu} z_1, e^{i\nu} z_2)$. We will denote this symmetry group by $\mathrm{U}(1)_K$. If ψ is a coordinate of the $\mathrm{U}(1)_K$ orbits, the four-dimensional metric is given by

$$ds^2_{\mathbb{C}^2} = \frac{1}{4x}\left(d\vec{x}^2 + 4x^2(d\psi + \omega)^2\right) , \tag{F.2}$$

where

$$x = |\vec{x}| \tag{F.3}$$

and ω is a 1-form on $\mathbb{R}^3$ such that

$$2d\omega = \mathrm{vol}(S^2) \tag{F.4}$$

is the volume form on the unit two-sphere.[77] In accord with this fibration structure, we decompose the four-dimensional gauge field as

$$\mathcal{A} = A + 2x(d\psi + \omega)\Phi\,, \tag{F.5}$$

where $A = A_i dx^i$ and Φ are the connection and a scalar on $\mathbb{R}^3$. If we assume that A is independent of ψ, or equivalently invariant under the $\mathrm{U}(1)_K$ action, the four-dimensional curvature $\mathcal{F} = d\mathcal{A} + \mathcal{A} \wedge \mathcal{A}$ decomposes as

$$\begin{aligned} \mathcal{F} &= d\mathcal{A} + \mathcal{A} \wedge \mathcal{A} \\ &= F - 2x(d\psi + \omega) \wedge D\Phi + \frac{\Phi}{x}\left(x^2 \mathrm{vol}(S^2) + 2x dx \wedge (d\psi + \omega)\right) , \end{aligned} \tag{F.6}$$

where $F = dA + A \wedge A$ and $D = d + [A, \cdot\,]$ are the three-dimensional curvature and covariant derivative. Its dual with respect to the four-dimensional metric (F.2) is given by[78]

$$*_4 \mathcal{F} = -(*_3 F) \wedge 2x(d\psi + \omega) - *_3 D\Phi - \frac{\Phi}{x}\left(x^2 \mathrm{vol}(S^2) + 2x dx \wedge (d\psi + \omega)\right) . \tag{F.7}$$

[77] For example, if we take angular parametrization $z_1 = x^{1/2} \cos\frac{\eta}{2} e^{-i\psi}$ and $z_2 = x^{1/2} \sin\frac{\eta}{2} e^{i\psi + i\varphi}$, then $\omega = \frac{1}{2}(1 - \cos\eta)d\varphi$.

[78] To compute the Hodge star, we need to know the orientation of $\mathbb{C}^2$ in terms of our coordinates. The standard orientation of $\mathbb{C}^2$ corresponds to the *sign* of the volume form $\mathrm{vol}(\mathbb{C}^2) \propto -dx d\eta d\varphi d\psi$ in the angular parametrization. Indeed, at $(x, \eta, \varphi, \psi) = (1, 0, 0, 0)$ we have $d\mathrm{Re} z_1 \supset dx, d\mathrm{Im} z_1 \supset -d\psi, d\mathrm{Re} z_2 \supset d\eta, d\mathrm{Im} z_2 \supset d\varphi$, so $\mathrm{vol}(\mathbb{C}^2) \propto -dx d\eta d\varphi d\psi$. The three-dimensional volume form is $\mathrm{vol}(\mathbb{R}^3) = x^2 \sin\eta\, dx d\eta d\varphi$.

Comparing (F.6) and (F.7) we see that the anti-self-duality equations $\mathcal{F}^+ = 0$ in four dimensions is equivalent to the Bogomolny equations

$$F = *_3 D\Phi \tag{F.8}$$

in three dimensions. Thus $\mathrm{U}(1)_K$-invariant instantons are in a one-to-one correspondence with solutions of the Bogomolny equations.

To be more precise, we need to specify the boundary conditions we impose in three and four dimensions. In three dimensions, we require that the Higgs field Φ vanishes at infinity. As we see from (F.5) this is indeed necessary if $\mathcal{A}$ at infinity becomes pure gauge $g^{-1}dg$ with $g : S^3 \to G$ depending only on the angular directions of $\mathbb{C}^2$.

To understand the appropriate boundary condition at the origin, let us consider the trivial background $\mathcal{A} = 0$ on $\mathbb{C}^2$. Let w be a coweight of the gauge group G. We recall that a coweight is an element of the Lie algebra, and discretely quantized in such a way that the exponential $e^{B\psi}$ is invariant under $\psi \to \psi + 2\pi$. A singular gauge transformation by $e^{B\psi}$ induces a non-trivial field

$$\mathcal{A} = e^{-B\psi} de^{B\psi} = -B\,\omega + 2x(d\psi + \omega)\frac{B}{2x}\,, \tag{F.9}$$

i.e.,

$$A = -B\,\omega\,, \quad F = -\frac{B}{2}\mathrm{vol}(S^2)\,, \quad \Phi = \frac{B}{2x}\,. \tag{F.10}$$

This is precisely the 't Hooft operator background in the transverse directions to the loop. If we start with a general gauge field, after the singular gauge transformation by $e^{B\psi}$, the group $\mathrm{U}(1)_K$ acts as an isometry that shifts ψ as well as a linear transformation on the fibers of the gauge bundle. In general, a smooth gauge field on $\mathbb{C}^2$ in variant under the $\mathrm{U}(1)_K$ group action becomes a field configuration in three dimensions that obeys the boundary condition appropriate for the 't Hooft loop. The linear transformation on the fiber at the origin encodes the magnetic charge of the 't Hooft operator. In fact, one can reverse the logic and use this connection with instantons to *define* the precise boundary conditions for singular solutions of the Bogomolny equations, which is otherwise difficult to specify. See for example [22], where this definition of boundary conditions was concretely used to compute the dimension of the moduli space by suitably applying the index theorem.

G Instanton partition functions for U(N)

For $G = \mathrm{U}(N)$, the localization calculation represents the instanton partition function Z_{inst} as a sum over the set of the $\mathrm{U}(1)_{\epsilon_1} \times \mathrm{U}(1)_{\epsilon_2} \times \mathrm{U}(1)^N$-fixed points on the moduli space of non-commutative instantons on $\mathbb{C}^2$. For each fixed point, we need to compute the equivariant Euler character of the self-dual complex

$$D^{\text{vm}} : \Omega^0 \otimes \mathrm{ad}(\mathfrak{g}) \xrightarrow{D} \Omega^1 \otimes \mathrm{ad}(\mathfrak{g}) \xrightarrow{D_+} \Omega^{2+} \otimes \mathrm{ad}(\mathfrak{g})\,. \tag{G.1}$$

Note that we can decompose the complexified spaces of differential forms as $\Omega^0_{\mathbb{C}} \simeq \Omega^{0,0}$, $\Omega^1_{\mathbb{C}} \simeq \Omega^{1,0} \oplus \Omega^{0,1}$, $\Omega^{2+}_{\mathbb{C}} \simeq \Omega^{2,0} \oplus \Omega^{0,0}\kappa \oplus \Omega^{0,2}$, where κ is the Kähler form. Using

Hodge duality, we also have the relations $\Omega^{2,2} \simeq \Omega^{0,0}$ and $\Omega^{2,1} \simeq \Omega^{1,0}$. It follows that the complexification of the self-dual complex (G.1) is isomorphic to the Dolbeault complex

$$\overline{D} : \Omega^{0,0} \otimes \mathrm{ad}(\mathfrak{g}) \xrightarrow{\overline{D}} \Omega^{0,1} \otimes \mathrm{ad}(\mathfrak{g}) \xrightarrow{\overline{D}} \Omega^{0,2} \otimes \mathrm{ad}(\mathfrak{g}) \tag{G.2}$$

twisted by $\Omega^{0,0} \oplus \Omega^{2,0}$. The index of the self-dual complex (G.1) differs from the index of the Dolbeault complex (G.2) by a factor accounting for complexification, and another computing the weights of the toric $\mathrm{U}(1)_{\epsilon_1} \times \mathrm{U}(1)_{\epsilon_2}$ action on the fiber of $\Omega^{0,0} \oplus \Omega^{2,0}$ at the origin:

$$\mathrm{ind}(D^{\mathrm{vm}}) = \frac{1 + t_1^{-1} t_2^{-1}}{2} \mathrm{ind}(\overline{D}) \,. \tag{G.3}$$

Mathematically it is sometimes more convenient to consider the torsion free sheaves, which are known to be in a one-to-one correspondence with non-commutative instantons. Deformations of the torsion free sheaves are captured by the Dolbeault complex (G.2). Each fixed point is labeled by an N-tuple of Young diagrams $\vec{Y} = (Y_1, \ldots, Y_N)$. Each partition Y_α defines an ideal sheaf of rank one $\mathcal{E}_Y$ in the standard way [36]. Let V_Y be the space of holomorphic sections of $\mathcal{E}_Y$. For $Y = (\lambda_1 \geq \lambda_2 \cdots \geq \lambda_{\lambda'_1})$, where λ_i and λ'_i are the number of squares in the i-th column and row respectively, the basis of V_Y is given by monomials $z_1^{i-1} z_2^{j-1}$ for all (i,j) such that $j > \lambda_i$. (The counting of squares in each Young diagram starts from $(i,j) = (1,1)$). In other words the basis in the V_Y is enumerated by the squares outside of the Young diagram Y. Each basis element $z_1^{i-1} z_2^{j-1}$ generates an eigenspace of the torus $T = \mathrm{U}(1)_{\epsilon_1} \times \mathrm{U}(1)_{\epsilon_2}$ with eigenvalue $t_1^{1-i} t_2^{1-j}$, where $(t_1, t_2) = (e^{i\varepsilon_1}, e^{i\varepsilon_2})$. Therefore the character of V_Y as a $\mathrm{U}(1)_{\epsilon_1} \times \mathrm{U}(1)_{\epsilon_2}$-module is

$$\mathrm{ch}(V_Y) = \sum_{(i,j) \notin Y} t_1^{1-i} t_2^{1-j} = \frac{1}{(1 - t_1^{-1})(1 - t_2^{-1})} - \chi(Y) \,, \tag{G.4}$$

where

$$\chi(Y) = \sum_{(i,j) \in Y} t_1^{1-i} t_2^{1-j} \,. \tag{G.5}$$

We also have

$$\chi^*(Y) = \sum_{(i,j) \in Y} t_1^{i-1} t_2^{j-1} \,. \tag{G.6}$$

For each fixed point $\vec{Y}$ we need to compute the equivariant index of the twisted Dolbeault complex (G.2) in the background of the connection defined by $\vec{Y}$. Since $\mathrm{ad}(\mathfrak{g}) = \overline{\mathbf{N}} \otimes \mathbf{N}$ where $\mathbf{N}$ is the fundamental representation, the adjoint-valued cohomology space of $\overline{D}$ is the tensor product of the V_Y^* and V_Y modules over the ring of holomorphic functions. Hence

$$\begin{aligned} \mathrm{ind}(\overline{D}) &= \mathrm{ch}(V_Y^* \otimes_{\mathcal{O}} V_Y) = \mathrm{ch}(V_Y^*) \mathrm{ch}(V_Y) / \mathrm{ch}(\mathcal{O}^*) \\ &= \sum_{i,j=1}^{N} s_i^{-1} s_j \left(\frac{1}{(1-t_1)(1-t_2)} - \chi^*(Y_i) \right) \left(\frac{1}{(1-t_1^{-1})(1-t_2^{-1})} - \chi(Y_j) \right) (1-t_1)(1-t_2) \end{aligned} \tag{G.7}$$

where $s_i = e^{i\hat{a}_i}$. We can extract the common infinite part independent of $\vec{Y}$

$$\mathrm{ind}(\overline{D})_{\text{1-loop}} = \sum_{i,j=1}^{N} s_i^{-1} s_j \frac{1}{(1-t_1^{-1})(1-t_2^{-1})}\,, \tag{G.8}$$

and denote the remainder in (G.7) by $\mathrm{ind}(\overline{D})_{\text{inst}}$:

$$\mathrm{ind}(\overline{D}) =: \mathrm{ind}(\overline{D})_{\text{1-loop}} + \mathrm{ind}(\overline{D})_{\text{inst}}\,. \tag{G.9}$$

To convert the index (Chern character) $\mathrm{ind}(D^{\text{vm}})$ to the fluctuation determinant (Euler character), we need to expand in powers of (t_1, t_2) and take the product of weights according to the rule

$$\sum_\alpha c_\alpha e^{w_\alpha(\varepsilon_1,\varepsilon_2,\hat{a})} \to \prod_\alpha w_\alpha(\varepsilon_1,\varepsilon_2,\hat{a})^{c_\alpha}\,. \tag{G.10}$$

Notice that $\mathrm{ind}(\overline{D})$ and $t_1^{-1}t_2^{-1}\mathrm{ind}(\overline{D})$ in (G.3) are exchanged by $(\varepsilon_1,\varepsilon_2,\hat{a}) \to (-\varepsilon_1,-\varepsilon_2,-\hat{a})$. For the common one-loop factor $Z_{\text{1-loop}}$ at the north pole, it is important to use

$$\mathrm{ind}(D^{\text{vm}})_{\text{1-loop}} = \frac{1+t_1^{-1}t_2^{-1}}{2}\mathrm{ind}(\overline{D})_{\text{1-loop}}\,, \tag{G.11}$$

rather than $\mathrm{ind}(\overline{D})_{\text{1-loop}}$, before expanding in positive powers of t_1, t_2 as we did in section 6.[79] For the finite instanton part Z_{inst} computed by the rule (G.10), however, the result obtained from

$$\mathrm{ind}(D^{\text{vm}})_{\text{inst}} = \frac{1+t_1^{-1}t_2^{-1}}{2}\mathrm{ind}(\overline{D})_{\text{inst}} \tag{G.12}$$

is identical to the result from $\mathrm{ind}(\overline{D})_{\text{inst}}$ because the signs that appear from $(\varepsilon_1,\varepsilon_2,\hat{a}) \to (-\varepsilon_1,-\varepsilon_2,-\hat{a})$ cancel out in the product. Thus the instanton partition function can be computed either from the self-dual complex or the Dolbeault complex.

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